24 April 2000

(First Draft: 1 March 2000)

Real National Income : the Hicks-Sraffa Approach

Takashi YAGI

No. 2001

Takashi YAGI (Associate Professor of Economics)

Faculty of Social and Information Studies, Gunma University

4-2 Aramaki-cho, Maebashi-shi, Gunma, 371-8510, JAPAN

Tel: +81-27-220-7524 (Office)

Fax: +81-27-220-7405 (Office)

E.mail: yagi@sta.att.ne.jp yagi@si.gunma-u.ac.jp

24 April 2000

Real National Income : the Hicks-Sraffa Approach*

Takashi YAGI

Associate Professor of Economics

Faculty of Social and Information Studies, Gunma University

* This paper was written during my stay at Wolfson College and the Faculty of Economics, University of Cambridge. I should like to thank Dr G.C.Harcourt. I also acknowledge the financial support (1999 fellowship) of the Ministry of Education of Japan.

Introduction

It is certainly true that real national income should be a basic concept of economic analysis, but, at the same time, to give an appropriate definition of the real national income is one of the most difficult problems in Economics, because national income is a collection of goods and services. One of the approaches to give a definition of real national income is to construct an index number. Index number is a method for aggregation of goods and services as well as a method for intertemporal or interspacial comparison of product. However, in general, the changes in distribution will affect the prices as weights for index number. This may be a troublesome thing in constructing an index.

It is well known that A.C.Pigou developed index number criteria as a measure of national dividend. J.R.Hicks followed Pigou's index number approach and he suggested the Utility approach for welfare comparison and the Cost approach for productivity comparison (Hicks [1940]). Although many economists criticized Hicks' approach[1], he continued to elaborate his index number approaches. The Utility approach was advanced by Hicks [1958], and the Cost approach appeared in Hicks [1981] over twenty years after his promise in Hicks [1958]. In Hicks [1981], he rejected prices as the weights for his index and adopted labour coefficients(or values). It is interesting that, while Hicks [1940] based on the Neoclassical framework, Hicks [1981] took the Classical approach. But the index number made with labour coefficients will be lacking the framework for considering the price changes caused by the distributional changes. We may say that Hicks [1981] failed in the qualification of the weights because of his adoption of labour coefficients (or values).

The Classical economists tried to explain the real value of heterogeneous products through finding out an invariable standard of value. It is well known that D.Ricardo devoted himself to the search for an invariable measure of value, but in vain. Sraffa [1960] followed Ricardo's quests and he discovered a marvelous device, the standard net product or the standard national income. While different rates of profit imply different sets of prices, the standard national income is independent of the price changes caused by the distributional changes. Thus the standard national income has been considered preferable as an invariable standard for the prices of commodities and the wage. Though many economists have been focused on the relation between the prices and distribution in the Sraffa system, we will make use of the standard national income for the definition of real national income. However, Sraffa left aside the problem of intertemporal comparison, which D.Ricardo might have strong interests on.

In order to give a definition of real national income by index number approach, we must give successful solutions to the problem of intertemporal and interspacial comparison and to the problem of aggregation independent of the price changes and the distributional changes. Index number of Hicks

[1981] is a device for productivity comparison. Sraffa's standard national income is independent of the changes in distribution. Both Hicks [1981] and Sraffa [1960] are considering in the Classical framework. Therefore, we can say that Hicks [1981] and Sraffa [1960] will be complementary to each other and provide significant ideas for the definition of real national income. My aim of this paper is to synthesize the index number approach of Hicks [1981] and the idea of standard national income introduced by Sraffa [1960]. I have already constructed a productivity index with Sraffa's standard national income, which I call Standard Productivity index (see Yagi [1998a][1998b]). In this paper, I will apply my Standard productivity index to the definition of real national income.

1. Hicks on Real National Income

1) the Criteria for the Real Income Comparison in Hicks [1940]

If an imaginary economy produces only one commodity, a comparison between two quantities produced in different situations will be straightforward. However, in the real world, an economy produces a heterogeneous bundle of commodities. Therefore, in order to measure the real value of national income, we should aggregate this heterogeneous bundle of commodities by some weights. Moreover, in order to make a comparison of real national income of different situations and measure the real changes in national income, we should illuminate what occurs in the real changes in quantities between two bundles of commodities, not in the changes in the weights for aggregation. Hicks suggested a method for this purpose by using index number approach.

Index number is a device to aggregate and compare two bundles of commodities of different situations with constant weights, i.e. constant prices. To construct an index number, the quantity data and the price data of different situations must be given. Let $\mathbf{q}^1$ be the quantity vector of Situation I (or base year) and $\mathbf{q}^2$ be that of Situation II (or given year). Let $\mathbf{p}^1$ be the price vector of Situation I and $\mathbf{p}^2$ be the price vector of Situation II. Then, the Laspeyres quantity index L_Q and the Laspeyres price index L_P can be represented as

$$L_Q = \frac{\mathbf{p}^1\mathbf{q}^2}{\mathbf{p}^1\mathbf{q}^1} \qquad L_P = \frac{\mathbf{p}^2\mathbf{q}^1}{\mathbf{p}^1\mathbf{q}^1} \tag{1}$$

The Paasche quantity index P_Q and the Paasche price index P_P can be represented as

$$P_Q = \frac{\mathbf{p}^2\mathbf{q}^2}{\mathbf{p}^2\mathbf{q}^1} \qquad P_P = \frac{\mathbf{p}^2\mathbf{q}^2}{\mathbf{p}^1\mathbf{q}^2} \tag{2}$$

The Money expenditure index E can be represented as

$$E = \frac{\mathbf{p}^2\mathbf{q}^2}{\mathbf{p}^1\mathbf{q}^1} \tag{3}$$

If the prices are chosen from statistical data, the values of these indexes are only statistical approximations. Hicks questioned as follows:

> The practical statistician has generally been content to take it for granted that every commodity entering into the aggregate should be at its market-price. I want to ask: what is the basis for this valuation. Are the prices we use for valuation prices in their own rights or do they stand for something else (say marginal utilities or marginal costs) to which prices are taken as approximations? These questions ought to be answered before we can have a clear idea of what National Income calculations mean. And they need to be answered before we can settle some of the disputed points about actual computation. (Hicks [1940], reprinted in Hicks[1981]pp.78-79)

The prices for constructing an index number should be qualified by some theoretical reasoning. There may be some approaches for explaining the theoretical interpretation of prices. To make the welfare comparison, Hicks introduces the index number criteria for welfare comparison by interpreting the prices for index number as marginal utility. This approach is called the Utility approach. Let us first show the Utility approach. If the prices are equal to the marginal utility, the double criteria as a measure of economic welfare are represented as the following proposition

[Proposition U (Hicks[1940])] If $E > L_P$ and $E > P_P$, then Situation II is preferred to Situation I .

It is well known that *Proposition U* is correct for a rational consumer in a competitive market. Hicks applied this proposition to the welfare comparison of group or real national income comparison.

On the other hand, for productivity comparison, Hicks introduced the index number criteria for productivity comparison by interpreting the prices as marginal costs. This approach is called the Cost approach. As for the Cost approach, the double criteria as a measure of productivity are represented as the following proposition.

[Proposition C (Hicks[1940])] If $E>L_P$ and $E>P_P$, then the productivity of Situation II is higher than that of Situation I .

It should be noted that, for *Proposition C* as well as *Proposition U*, the competitive market and rational agents are assumed. Although the structure of *Proposition U* is almost same as *Proposition C*, the implication of *Proposition U* is different from *Proposition C*.

2) the Productivity Index in Hicks[1981]

Let us proceed to the Cost Approach of Hicks [1981]. Hicks states, "The idea that the Social Income as a measure of productivity may be different from the Social Income as a measure of Economic Welfare comes as rather a shock" (Hicks[1940], reprinted in Hicks[1981]p.94). He gradually realized the importance of the Cost approach. Hicks [1940] based on the Neoclassical tradition. He considered prices as marginal costs, and assumes competitive market and rational agents. On the other hand, in Hicks [1981], Hicks stood on the Classical tradition and adopted labour coefficients as the weights for his index. He assumed the *given* production condition: the *constant* abour coefficients and the *given* total labour. On this assumption, he could obtain a unique value of index number as a measure of productivity, while in Hicks [1940] he explained only the criteria for productivity change.

In Hicks [1981], the productivity index was studied in two ways: one is called the Real cost approach, and the other the Opportunity cost approach. The index number of the Real cost approach is constructed by the actual bundles of commodities produced under the given production conditions. On the other hand, the index number of Opportunity cost approach is constructed by the bundle of commodities which can be potentially produced under the given production condition.

Let us explain Hicks' Opportunity cost approach[2]. Let **q** be the quantity vector of A-situation, and **q'** be the quantity vector of B-situation. Let **c** be the labour coefficient vector of A-situation, and **c'** be the labour coefficient vector of B-situation. Then the Paasche Productivity Index is defined by **cq' / c'q'**, the Laspeyres Output Index is defined by **cq' / cq**, and the input index is represented as **c'q' / cq** . For these three indexes, the following equation holds

$$\frac{\mathbf{cq'}}{\mathbf{c'q'}}=\frac{\mathbf{cq'}}{\mathbf{cq}}\cdot\frac{\mathbf{cq}}{\mathbf{c'q'}} \tag{4}$$

The left member of this equation is the Paasche productivity index. The right member of this equation is the *product* of the Laspeyres output index and the inverse of input index. Now let us first of all explain the calculation of the Laspeyres output index of the Opportunity cost approach.

In the Opportunity cost approach, Hicks calculates the Laspeyres output index by using the idea of opportunity cost. Hicks supposes that each q_i of the initial output q is divided into a large number of units. Let the number be m. m is to be the same for each commodity. Then take one unit of each commodity, and combine them into a 'bundle'. By this operation, in the initial situation (A-situation), total output consists of m such bundles (A-bundles). Each q_i of B-situation is also divided into m units. The total output of B-situation also consists of m bundles (B-bundles). Let x, y be the collection of quantities which is such that it can be put together by combining x A-bundles and y B-bundles.

With constant c-coefficients, the total labour of A-situation is represented as cq. Then x, y must be subject to the following equation

$$(x/m)\mathrm{cq} + (y/m)\mathrm{cq'} = \mathrm{cq} \tag{5}$$

From this, we have

$$x\mathrm{cq} + y\mathrm{cq'} = m\mathrm{cq} \tag{6}$$

In A-situation, the collection of commodities is

$$(x, y) = (m, 0) \tag{7}$$

On the other hand, under the condition (6), we can consider an imaginary or potentially producible collection

$$(x, y) = (0, \alpha) \tag{8}$$

This collection can be produced by the c-coefficients (the technique of A-situation) and the total labour of A-situation. Substituting (8) into (6), we obtain

$$\frac{\mathrm{cq'}}{\mathrm{cq}} = \frac{m}{\alpha} \tag{9}$$

This is Laspeyres output index.

Let us denote the ratio of labour inputs of different situations by L. Then the input index can be represented as

$$\frac{\mathrm{c'q'}}{\mathrm{cq}} = L \tag{10}$$

Then, from (4)(9)(10), the Paasche productivity index becomes

$$\frac{\mathbf{cq'}}{\mathbf{c'q'}} = \frac{m}{\alpha} \cdot \frac{1}{L} \tag{11}$$

This is the Paasche productivity index by the Opportunity cost approach[3].

It should be noted that, in constructing the index of Hicks' Cost approach, two factors must be given: the production conditions and the output compositions. The Real cost approach of Hicks based on the production conditions and the actual output composition of both situations. On the other hand, the Opportunity cost approach based on the production condition, the actual output composition of one situation and the *potentially producible* output of the other situation. The most significant contribution of Hicks lies in the introduction of the notion of opportunity cost. We should like to emphasize that the idea of the Opportunity cost approach will enlarge the applicability of the Cost approach and enable us to construct more various productivity indexes. However, the labour coefficients may have no relevance to the actual prices when an economy produces a surplus and the profit rate is positive. This may be the reason why the labour coefficients have been neglected in theoretical analyses. We will find in the following parts that this problem will be settled in the Sraffa system.

2. The Sraffa System and the Standard National Income

1) the Actual Quantity System

Now we turn to explain the Sraffa system (Sraffa [1960]). The Sraffa system can be modeled with the given production condition: the given total labour and the constant coefficients. Like Hicks' Cost approach, the Sraffa system based on the classical tradition. In this section, we will first show the basic structure of Sraffa system [4].

The Sraffa System can be easily formulated by using matrices and vectors. Let us consider the case of single product industry exclusively. There is no joint production. Land and fixed capital is excluded. Each industry produces a single commodity by using a certain quantity of labour and certain quantities of commodities as a means of production. Let n be the number of industries and thus the number of products. If we denote the total output vector by x, Leontief's input coefficient matrix by A, and the actual net product vector by y, then the quantity equation can be represented by

$$\mathbf{y} = \mathbf{x}(\mathbf{I} - \mathbf{A}) \tag{12}$$

where I is a unit matrix. This is the definition of the actual net product or the actual national income. The j th component of y represents the net product of industry j.

If each industry produces a surplus, y will be strictly positive: $y > 0$, and if there is no surplus in some industries, y will be semi-positive: $y \geq 0$. If x and A are given exogenously, y will be determined by them immediately. Therefore, we assume that x, A are given exogenously, and

[Assumption 1] $\quad x > 0, \quad A \geq 0$

Moreover, we assume, for simplicity, that all products are basic commodities, and therefore there is no *non-basic* commodity. When all products are basic, the input coefficient matrix A is indecomposable. Thus, we assume

[Assumption 2] A is indecomposable

Then let us explain the labour coefficient vector. If the labour input coefficient vector of Leontief type is denoted by l_L, then the Actual total labour L_A will be represented as

$$L_A = xl_L \tag{13}$$

The actual quantity system with the Leontief type labour coefficient vector is characterized by the triplet [x, A, l_L]. The Actual total labour is measured in terms of physical unit of labour.

The Actual total labour is measured in terms of natural unit of labour. However, Sraffa assumes that the total labour should be equal to unity. The normalization of total labour will be represented as

$$L_S = xl_L \,/\, xl_L = 1 \tag{14}$$

Let us call L_s the Standard total labour [5]. The Standard total labour is always set to be unity. Then if we define the Sraffa type labour coefficient vector as

$$l_S = (1 / xl_L)\, l_L \tag{15}$$

Then the Standard total labour is represented as follows

$$L_S = xl_S = 1 \tag{16}$$

We will call l_S the Standard labour coefficient vector [6]. We assume that l_S is also given exogenously and strictly positive. The assumption is

[Assumption 3] $\quad l_S > 0$

It should be noted that the Sraffa type labour coefficient vector is different from the Leontief type labour coefficient vector. The quantity system with the Standard labour coefficient vector is characterized by [x, A, l_S].

2) the Standard System

Let us proceed to the Standard System. The Standard system is defined as a virtual system whose output vector corresponds to the eigenvector of the input coefficient matrix **A**, and which has a uniform rate of surplus in physical terms throughout industries. If we denote the physical rate of surplus by Π, and the output vector of the Standard system by **h**, it can be represented as

$$\mathbf{h} = (1 + \Pi)\, \mathbf{hA} \tag{17}$$

If we denote the Standard commodity vector by **u**, it can be defined from (17) as

$$\mathbf{u} = \mathbf{h}\,(\mathbf{I} - \mathbf{A}) = \Pi\ \mathbf{hA} \tag{18}$$

The Standard national income is defined as the Standard commodity with an additional assumption: the total labour of the Standard system is equal to the total labour of the actual system. If we denote the vector of the total quantities produced in the Standard system corresponding to the Standard national income by **q** in place of **h**, the relation between **q** and **h** will be represented by

$$\mathbf{q} = t\,\mathbf{h} \tag{19}$$

where t is a positive scalar. Thus, like (17), the vector **q** is defined as a vector which can be represented as

$$\mathbf{q} = (1+\Pi)\, \mathbf{qA} \tag{20}$$

and, in addition, which satisfies the following assumption:

[Assumption 4] $\mathbf{q l_s} = \mathbf{x l_s}$

From (20), the Standard national income can be defined, like (18), as follows

$$\mathbf{s} = \mathbf{q}\,(\mathbf{I} - \mathbf{A}) = \Pi \mathbf{qA} \tag{21}$$

This is the definition of the *Standard net product* or the *Standard national income*. The vector **s** is uniquely determined if the matrix **A** and the vector $\mathbf{l_s}$ are given. In other words, **s** is independent of the distributional changes and the demand composition of the actual national income.

3) the Price Vector and the Value Vector

In Sraffa's model, the price equation depends only on the production conditions and the level of distribution. The prices can vary as the rate of profit rises: different rate of profits imply different sets of prices. It should be noticed that they are independent of the demand composition. Now we proceed to define the price equation.

In the Sraffa system, the prices imply the exchange-ratios that enable the system to be viable. Let us denote the price vector by **p**. We assume that the rate of profit is uniform throughout the economic system. Let us denote it by r. Similarly, uniform wage is assumed to be prevailing in the economy. Let us denote it by w. Then the price equation will be represented as

$$\mathbf{p} = (1+r)\,\mathbf{A}\mathbf{p} + w\,\mathbf{l}_s \qquad (22)$$

In (22), there are $(n + 2)$ unknowns and n equations. If we choose commodity j as unit and set the price of commodity j equal to unity,

$$p_j = 1 \qquad (23)$$

then we will have one degree of freedom in (22). If we make an additional assumption that one of the distributive variables is exogenous, then the degree of freedom will vanish and the equation (22) will become determinate. Let us consider the rate of profit as the exogenous distributive variable. *Assumptions 1-3* assure that the equation (22) may have a positive solution of **p** for all r $(0 \leqq r \leqq R)$. The price vector **p** is determined by the production condition and the level of distribution. It is independent of the demand composition of actual national income. It should be noted that the price vector is characterized by the data set $[\mathbf{A}, \mathbf{l}_s, r]$.

In the special case that the rate of profit is zero, i.e. $r = 0$, the wage becomes equal to unity, i.e. $w = 1$. Then, the price vector is equal to the value vector

$$\mathbf{v}_s = [\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{l}_s \qquad (24)$$

It should be noticed that $\mathbf{v}_s$ is given by the Standard labour coefficient vector $\mathbf{l}_s$. The value vector implies the quantity of labour used directly and indirectly in the economic system to obtain one physical unit of each commodity as a final commodity. We can consider that $\mathbf{v}_s$ is expressed in terms of the quantity of labour. The value vector $\mathbf{v}_s$ is characterized by the data $[\mathbf{A}, \mathbf{l}_s]$, instead of $[\mathbf{A}, \mathbf{l}_s, r]$. In other words, value vector $\mathbf{v}_s$ is independent both of the composition of actual national income and of distribution.

4) Equality between the Standard Total Labour and the Standard National Income

We are interested in the value of real national income. If the rate of profit is equal to zero, the problem we are concerned with become very easy. The actual national income is expressed as the *product* of the actual net product vector and the value vector, i.e. $\mathbf{y}\mathbf{v}_s$. The standard national income is also expressed as the *product* of the standard net product vector and the value vector, i.e. $\mathbf{s}\mathbf{v}_s$. Both the actual national income and the standard national income are equal to the Standard total labour, which is set to be unity. For the actual national income, this can be represented as

$$\mathbf{y}\mathbf{v}_s = \mathbf{x}\,[\mathbf{I}-\mathbf{A}][\mathbf{I}-\mathbf{A}]^{-1}\mathbf{l}_s = \mathbf{x}\mathbf{l}_s \qquad (25)$$

and for the standard national income, we can obtain

$$\mathbf{s}\mathbf{v}_s = \mathbf{q}\,[\mathbf{I}-\mathbf{A}][\mathbf{I}-\mathbf{A}]^{-1}\mathbf{l}_s = \mathbf{q}\mathbf{l}_s = \mathbf{x}\mathbf{l}_s \qquad (26)$$

Therefore, from (25)(26), we have

$$\mathbf{y}\mathbf{v}_s = \mathbf{s}\mathbf{v}_s \qquad (27)$$

Thus it should be noticed that the value of actual national income is equal to the value of standard national income when $r = 0$.

However, if the rate of profit is positive, the changes in distribution will affect the value of the actual national income and thus the actual national income is not necessarily equal to the Standard total labour. On the contrary, for the standard national income, we can obtain the equality between the Standard total labour and the standard national income. This result can be stated as the following Theorem. I have called the following theorem *Sraffa's Theorem* (Yagi[1999]).

[Theorem 1] Let $\mathbf{p} = (1 + r)\,\mathbf{Ap} + w\mathbf{l}_s$ be the price equation system under *Assumptions 1-4*.

Then the standard national income is equal to the Standard total labour if and only if

$$r = R\,(1 - w) \tag{28}$$

where $0 \leqq r < R$.

[Proof] Let R be the maximum rate of profits, then the price equation (22) will become

$$\mathbf{p} = (1 + R)\,\mathbf{Ap} \tag{29}$$

when $w = 0$. In this equation, $\mathbf{p}$ is the right-hand eigenvector of the matrix $\mathbf{A}$ and $1/(1 + R)$ is its eigenvalue. On the other hand, the vector $\mathbf{q}$ is the left-hand eigenvector of the matrix $\mathbf{A}$ and $1/(1+\Pi)$ is its eigenvalue. Hence, from the *Perron-Frobenius theorem*, we have

$$\Pi = R \tag{30}$$

Substituting (30) into (21), we have

$$\mathbf{s} = \mathbf{q}\,(\mathbf{I} - \mathbf{A}) = R\,\mathbf{qA} \tag{31}$$

On the other hand, pre-multiplying (22) by $\mathbf{q}$, we have

$$\mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} = (r / R)R\mathbf{qAp} + w\,\mathbf{ql}_s \tag{32}$$

Substituting (31) and *Assumption 4* into (32), we have

$$(1 - r / R)\,\mathbf{sp} = w\,\mathbf{xl}_s \tag{33}$$

then, we have

$$\mathbf{sp} = \mathbf{xl}_s \quad \Leftrightarrow \quad r = R\,(1 - w) \tag{34}$$

where $0 \leqq r < R$. Thus the theorem is verified.

Q.E.D.

This theorem asserts that, under the condition (28), the standard national income is equal to the Standard total labour:

$$\mathbf{sp} = \mathbf{xl}_s \tag{35}$$

The right member of this equation is measured in terms of unit of labour. Then we can consider that the left member of the equation (35) is measured in terms of unit of labour.

If we transform the price equation (22) and then substitute the condition (28) into it, then we have

$$\mathbf{p}_v = (1 - r/R)[\mathbf{I} - (1 - r)\mathbf{A}]^{-1}\ \mathbf{l}_s \qquad (36)$$

One of the members of the right-hand side of (36) is called the dated quantity of labour. It should be noted that the prices of the equation (36) are expressed in terms of unit of labour. The value vector (24) is also measured in terms of unit of labour. Therefore, both the price vector and the value vector are commesurable for all rates of profit ($0 \leqq r < R$). The fact that the price vector is expressed in terms of labour unit is very important for our real income comparison.

3 The Standard Productivity Index

1) the Quantity Vectors and the Weight Vectors for Productivity Index

So far we have been concerned with the Sraffa system in a given quantity system. Now, in order to construct productivity indexes, let us consider two different quantity systems of different situations.

Considering Hicks' idea of the Opportunity cost approach, the standard net product vectors as well as the actual net product vectors are eligible as the quantity vectors for productivity index. The actual net product vectors of Situation I and Situation II will be represented respectively as

$$\mathbf{y}^1 = \mathbf{x}^1[\mathbf{I} - \mathbf{A}^1] \qquad (37)$$

$$\mathbf{y}^2 = \mathbf{x}^2[\mathbf{I} - \mathbf{A}^2] \qquad (38)$$

where $\mathbf{x}^1$ and $\mathbf{x}^2$ are the total output vectors of the actual quantity system, and $\mathbf{A}^1$ and $\mathbf{A}^2$ are the input coefficient matrices. Vectors $\mathbf{y}^1$ and $\mathbf{y}^2$ are determined by $[\mathbf{x}^1, \mathbf{A}^1]$ and $[\mathbf{x}^2, \mathbf{A}^2]$ respectively.

The standard net product vectors are

$$\mathbf{s}^1 = \mathbf{q}^1[\mathbf{I} - \mathbf{A}^1] \qquad (39)$$

$$\mathbf{s}^2 = \mathbf{q}^2[\mathbf{I} - \mathbf{A}^2] \qquad (40)$$

where $\mathbf{q}^1$ and $\mathbf{q}^2$ are the total output vectors of the standard system. It is notable that $\mathbf{s}^1$ and $\mathbf{s}^2$ are dependent only on input coefficient matrices $\mathbf{A}^1$ and $\mathbf{A}^2$.

Then let us consider the weights to aggregate the net product. Let $\mathbf{l}_s{}^1$ be the Standard labour coefficient vector of Situation I, and $\mathbf{l}_s{}^2$ be that of Situation II, then the value vector of each situation is represented respectively as

$$\mathbf{v}_s{}^1 = [\mathbf{I} - \mathbf{A}^1]^{-1}\ \mathbf{l}_s{}^1 \qquad (41)$$

$$\mathbf{v}_s{}^2 = [\mathbf{I} - \mathbf{A}^2]^{-1}\ \mathbf{l}_s{}^2 \qquad (42)$$

The value vector is determined by the data $[\mathbf{A}^1, \mathbf{l}_s{}^1]$ or $[\mathbf{A}^2, \mathbf{l}_s{}^2]$. Unfortunately the value vectors are available only when $r = 0$.

If the profit rate becomes positive, the valuation system should be the price system. The price equation of each situation is respectively

$$\mathbf{p}_v{}^1 = (1 - r^1/R^1)\,[\mathbf{I} - (1 + r^1)\mathbf{A}^1]^{-1}\,\mathbf{l}_S{}^1 \tag{43}$$

$$\mathbf{p}_v{}^2 = (1 - r^2/R^2)\,[\mathbf{I} - (1 + r^2)\mathbf{A}^2]^{-1}\,\mathbf{l}_S{}^2 \tag{44}$$

The price vector is given by the data $[\mathbf{A}^1, \mathbf{l}_S{}^1, r^1]$ or $[\mathbf{A}^2, \mathbf{l}_S{}^2, r^2]$.

From above discussions, when we attempt to construct indexes, the following four cases may be possible.

Case I: Vectors $[\mathbf{y}^1, \mathbf{y}^2, \mathbf{p}_v{}^1, \mathbf{p}_v{}^2]$; Data $[\mathbf{x}^1, \mathbf{A}^1, \mathbf{l}_S{}^1, r^1]$ $[\mathbf{x}^2, \mathbf{A}^2, \mathbf{l}_S{}^2, r^2]$

Case II: Vectors $[\mathbf{s}^1, \mathbf{s}^2, \mathbf{p}_v{}^1, \mathbf{p}_v{}^2]$; Data $[\mathbf{A}^1, \mathbf{l}_S{}^1, r^1]$ $[\mathbf{A}^2, \mathbf{l}_S{}^2, r^2]$

Case III: Vectors $[\mathbf{y}^1, \mathbf{y}^2, \mathbf{v}_S{}^1, \mathbf{v}_S{}^2]$; Data $[\mathbf{x}^1, \mathbf{A}^1, \mathbf{l}_S{}^1]$ $[\mathbf{x}^2, \mathbf{A}^2, \mathbf{l}_S{}^2]$

Case IV: Vectors $[\mathbf{s}^1, \mathbf{s}^2, \mathbf{v}_S{}^1, \mathbf{v}_S{}^2]$; Data $[\mathbf{A}^1, \mathbf{l}_S{}^1]$ $[\mathbf{A}^2, \mathbf{l}_S{}^2]$

2) The Assumption on the Actual Total Labour

Before we set to discuss the construction of indexes, it will be better to make clear the assumptions. We will keep *Assumptions 1-4* in both situations. Moreover, for the time being, we will add an assumption that the Actual total labour of Situation I is equal to that of Situation II. This can be represented as

$$\mathbf{x}^1\,\mathbf{l}_L{}^1 = \mathbf{x}^2\,\mathbf{l}_L{}^2 \tag{45}$$

From this, we can obtain

$$\mathbf{x}^1\,\mathbf{l}_S{}^1 = \mathbf{x}^2\,\mathbf{l}_S{}^2 \tag{46}$$

Let us explain the implication of this equation before constructing productivity index.

Firstly, the actual national income of Situation I is equal to that of Situation II, when the rate of profit is zero. In each situation, the Actual total labour is equal to the value of the actual national income when the rate of profit is zero. This can be represented respectively as

$$\mathbf{x}^1\mathbf{l}_S{}^1 = \mathbf{y}^1\mathbf{v}_S{}^1 \tag{47}$$

$$\mathbf{x}^2\mathbf{l}_S{}^2 = \mathbf{y}^2\mathbf{v}_S{}^2 \tag{48}$$

From (46)(47)(48), we have

$$\mathbf{y}^1\mathbf{v}_S{}^1 = \mathbf{y}^2\mathbf{v}_S{}^2 \tag{49}$$

Secondly, the standard national income of Situation I is equal to that of Situation II, when the rate of profit is zero. This can be explained as follows.

$$\mathbf{x}^1\mathbf{l}_S{}^1 = \mathbf{s}^1\mathbf{v}_S{}^1 \tag{50}$$

$$\mathbf{x}^2\mathbf{l}_S{}^2 = \mathbf{s}^2\mathbf{v}_S{}^2 \tag{51}$$

From (46)(50)(51), we have

$$\mathbf{s}^1\mathbf{v}_S{}^1 = \mathbf{s}^2\mathbf{v}_S{}^2 \tag{52}$$

Finally, let r^1, r^2 be the rate of profit of each situation, let R^1, R^2 be the maximum rate of profits in each situation, and let w^1, w^2 be the wage in each situation. Then the conditions of *Theorem 1* will be represented as

$$r^1 = R^1 (1 - w^1) \tag{53}$$

$$r^2 = R^2 (1 - w^2) \tag{54}$$

if the conditions (53)(54) hold, we can obtain the following equations

$$\mathbf{s}^1 \mathbf{p}_v^1 = \mathbf{x}^1 \mathbf{l}_\mathbf{s}^1 \tag{55}$$

$$\mathbf{s}^2 \mathbf{p}_v^2 = \mathbf{x}^2 \mathbf{l}_\mathbf{s}^2 \tag{56}$$

The left members of the equations (55)(56), the standard national income of each situation, can be measured in terms of the same unit of labour. Thus, from equations (46)(55)(56), we can obtain

$$\mathbf{s}^1 \mathbf{p}_v^1 = \mathbf{s}^2 \mathbf{p}_v^2 \tag{57}$$

This equation means that, even if the rate of profit is positive, the standard national income of Situation I is equal to that of Situation II. It follows that we can compare the prices of Situation I with that of Situation II. This fact enables us to construct an index by the price vectors $\mathbf{p}_v^1$ and $\mathbf{p}_v^2$, under the conditions (53)(54). This is the reason why the price vectors are qualified for constructing indexes in our analysis.

3) Productivity Indexes Based on the Sraffa System [7)]

While Hicks consider a Laspeyres index or a Paasche index, we will be concerned only with Fisher type indexes in order to average the Laspeyres index and the Paasche index, because either the Laspeyres index or the Paasche index has a bias.

Let us denote the productivity index of *Case I* by Y_P, the output index by $Y_Q^{(p)}$, and the input cost index by $Y_C^{(p)}$. Then they are defined as

$$Y_P = \sqrt{\frac{\mathbf{y}^1\mathbf{p}_v^1}{\mathbf{y}^1\mathbf{p}_v^2} \cdot \frac{\mathbf{y}^2\mathbf{p}_v^1}{\mathbf{y}^2\mathbf{p}_v^2}} \qquad Y_Q^{(p)} = \sqrt{\frac{\mathbf{y}^2\mathbf{p}_v^2}{\mathbf{y}^1\mathbf{p}_v^2} \cdot \frac{\mathbf{y}^2\mathbf{p}_v^1}{\mathbf{y}^1\mathbf{p}_v^1}} \qquad Y_C^{(p)} = \frac{\mathbf{y}^2\mathbf{p}_v^2}{\mathbf{y}^1\mathbf{p}_v^1} \tag{58}$$

For these three indexes, the following equation holds

$$Y_P = Y_Q^{(p)} / Y_C^{(p)} \tag{59}$$

This means that the productivity index is equal to the quantity index divided by the input cost index.

The indexes of *Case II* will be much more interesting than those of *Case I*. For *Case II*, the productivity index, the output index, and the input cost index are defined as

$$S_V = \sqrt{\frac{\mathbf{s}^1\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^2} \cdot \frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^2\mathbf{p}_v^2}} \qquad S_Q^{(p)} = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{s}^1\mathbf{p}_v^2} \cdot \frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^1}} \qquad S_C^{(p)} = \frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{s}^1\mathbf{p}_v^1} \tag{60}$$

But if the conditions (53)(54) are satisfied, from the equations (57)(60), S_P, $S_Q{}^{(p)}$, $S_C{}^{(p)}$ will become much simpler forms as follows

$$S_P = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^2}} \qquad S_Q^{(p)} = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^2}} \qquad S_C^{(p)} = 1 \tag{61}$$

Moreover, from these, the following equation can be obtained.

$$S_P = S_Q{}^{(p)} \tag{62}$$

It should be noticed that, in this equation, the Fisher type productivity index is equal to the Fisher type output index. The productivity change measured by $S_p{}^{(p)}$ from the cost side becomes equal to the output changes measured by $S_Q{}^{(p)}$ from the quantity side. This is a remarkable property. However, S_P and $S_Q{}^{(p)}$ will vary as the rate of profit changes, because of the price changes. This fact is a troublesome thing.

Now let us proceed to *Case III*. As for the indexes of *Case III*, the value vectors are used as the weights to construct indexes. For this case, the productivity index, the output index, and the input cost index are defined as

$$Y_V = \sqrt{\frac{\mathbf{y}^1\mathbf{v}_s^1}{\mathbf{y}^1\mathbf{v}_s^2} \cdot \frac{\mathbf{y}^2\mathbf{v}_s^1}{\mathbf{y}^2\mathbf{v}_s^2}} \qquad Y_Q = \sqrt{\frac{\mathbf{y}^2\mathbf{v}_s^2}{\mathbf{y}^1\mathbf{v}_s^2} \cdot \frac{\mathbf{y}^2\mathbf{v}_s^1}{\mathbf{y}^1\mathbf{v}_s^1}} \qquad Y_C = \frac{\mathbf{y}^2\mathbf{v}_s^2}{\mathbf{y}^1\mathbf{v}_s^1} \tag{63}$$

From (49)(63), these indexes can be reduced to much simpler form as follows

$$Y_V = \sqrt{\frac{\mathbf{y}^2\mathbf{v}_s^1}{\mathbf{y}^1\mathbf{v}_s^2}} \qquad Y_Q = \sqrt{\frac{\mathbf{y}^2\mathbf{v}_s^1}{\mathbf{y}^1\mathbf{v}_s^2}} \qquad Y_C = 1 \tag{64}$$

From these, we have

$$Y_V = Y_Q \tag{65}$$

In this equation, we obtain again the equality between the productivity index and the output index. However, $\mathbf{x}^1$ and $\mathbf{x}^2$ are dependent upon the demand composition of each situations. Therefore, Y_V and Y_Q will vary as $\mathbf{x}^1$ and $\mathbf{x}^2$ change.

Then let us consider the indexes of *Case IV*. For this case, the productivity index, the output index, and the input cost index are defined as

$$S_V = \sqrt{\frac{\mathbf{s}^1\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^2} \cdot \frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^2\mathbf{v}_s^2}} \qquad S_Q = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_s^2}{\mathbf{s}^1\mathbf{v}_s^2} \cdot \frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^1}} \qquad S_C = \frac{\mathbf{s}^2\mathbf{v}_s^2}{\mathbf{s}^1\mathbf{v}_s^1} \tag{66}$$

From (52)(66), these indexes can be reduced to

$$S_V = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^2}} \qquad S_Q = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^2}} \qquad S_C = 1 \tag{67}$$

From these indexes, we have

$$S_V = S_Q \tag{68}$$

For *Case IV*, we also obtain the equality between productivity index and output index.

Which index should we choose for the definition of real national income? The indexes of *Case I*, Y_P, $Y_Q^{(p)}$, $Y_C^{(p)}$, are determined by three factors: 1) the production condition, 2) the price changes caused by the distributional changes, and 3) the demand composition of actual national income. The indexes of *Case II*, S_P, $S_Q^{(p)}$, are independent of the demand composition, but it is dependent upon the changes in distribution. The indexes of *Case III*, Y_V, Y_Q, are independent of the changes in distribution, but it is dependent upon the demand composition. The indexes of *Case IV*, S_V, S_Q, are independent both of the changes in distribution and of the demand composition. Furthermore, while the indexes of *Case I*, Y_P, $Y_Q^{(p)}$, $Y_C^{(p)}$, need the data sets of $[\mathbf{x}^1, \mathbf{A}^1, \mathbf{l}_s{}^1, r^1]$ and $[\mathbf{x}^2, \mathbf{A}^2, \mathbf{l}_s{}^2, r^2]$, the indexes of *Case IV*, S_V, S_Q, require only the data sets of $[\mathbf{A}^1, \mathbf{l}_s{}^1]$ and $[\mathbf{A}^2, \mathbf{l}_s{}^2]$, because the output vector of the Standard system is the eigenvector of the input coefficient matrix and the value vectors are independent of the rate of profit. If the techniques $[\mathbf{A}^1, \mathbf{l}_s{}^1]$ and $[\mathbf{A}^2, \mathbf{l}_s{}^2]$ are given, we can immediately calculate S_V or S_Q.

In the following, let us adopt S_V of *Case IV* as our productivity index. We will call S_V the *Standard productivity index* and S_Q the *Standard output index*. Like the productivity index of Hicks' Opportunity cost approach, the Standard productivity index is calculated on the given total labour, the constant input coefficients, and the potentially producible bundles of commodities of *both* situations. In our model, the following properties will hold. First, the Standard productivity index is equal to the Standard output index. Second, they are independent both of the changes in distribution and of the demand composition. Third, they require minimum data in our four possible cases. The equation (68) provides the double measuring rods for the productivity changes. We can say that the Standard productivity index is one of the invariable measures for productivity comparison.

4) Variation of the Actual Total Labour and the Standard Productivity Index

We have assumed that the Actual total labours of Situation II are equal to that of Situation I. Let us proceed to the case in which the Actual total labour of Situation II is different from that of Situation I. Let L_I be the labour quantity ratio or labour input index. Then L_I may be represented by

$$L_I = L_A^2 / L_A^1 \tag{69}$$

From the definition of the Actual total labour with vectors (the equation (13)), the equation (69) can be rewritten as

$$L_I = \mathbf{x}^2\mathbf{l}_L^2 / \mathbf{x}^1\mathbf{l}_L^1 \tag{70}$$

where $\mathbf{l}_L^1$, $\mathbf{l}_L^2$ are the Leontief type labour input coefficient vectors, and thus $\mathbf{x}^1\mathbf{l}_L^1$, $\mathbf{x}^2\mathbf{l}_L^2$ are the Actual total labour of each situations. We call L_I the *Indexed total labour* of Situation II. Let g_L be the growth rate of labour. Then the Indexed total labour can be represented as

$$L_I = 1+g_L \tag{71}$$

Then, let us define a new labour input coefficient vector $\mathbf{l}_I$ as

$$\mathbf{l}_I^2 = (1/\mathbf{x}^1\mathbf{l}_L^1) \cdot \mathbf{l}_L^2 \tag{72}$$

We will call $\mathbf{l}_I$ the *Indexed labour coefficient vector*. Since

$$\mathbf{l}_S^2 = (1/\mathbf{x}^2\mathbf{l}_L^2) \cdot \mathbf{l}_L^2 \tag{73}$$

dividing (72) by (73), we can obtain

$$\mathbf{l}_I^2 = (\mathbf{x}^2\mathbf{l}_L^2/\mathbf{x}^1\mathbf{l}_L^1) \cdot \mathbf{l}_S^2 \tag{74}$$

Moreover, from (70)(74), we have

$$\mathbf{l}_I^2 = L_I \cdot \mathbf{l}_S^2 \tag{75}$$

By the Indexed labour coefficient vector, the Indexed total labour of Situation II is represented as

$$L_I = \mathbf{x}^2\mathbf{l}_I^2 \tag{76}$$

Let us add an assumption as

$$\mathbf{q}^2\mathbf{l}_I^2 = \mathbf{x}^2\mathbf{l}_I^2 \tag{77}$$

This is the alternative expression of *Assumption 4*.

Let us define the value vector corresponding to $\mathbf{l}_I^2$. It will be defined as

$$\mathbf{v}_I^2 = [\,\mathbf{I} - \mathbf{A}^2\,]^{-1}\mathbf{l}_I^2 \tag{78}$$

The relation between $\mathbf{v}_I^2$ and $\mathbf{v}_S^2$ will be

$$\mathbf{v}_I^2 = L_I \cdot \mathbf{v}_S^2 \tag{79}$$

The value of standard national income can be represented as

$$\mathbf{s}^2\mathbf{v}_I^2 = L_I \cdot \mathbf{s}^2\mathbf{v}_S^2 \tag{80}$$

In case of $L_I \neq 1$, the value vector $\mathbf{v}_I$ can be available as the weights for the productivity index.

Let us show the indexes constructed with the set of vectors $[s^1, s^2, v_s{}^1, v_I{}^2]$. S_V^I, S_Q^I and S_C^I will be represented as

$$S_V^I = \sqrt{\frac{s^1 v_s^1}{s^1 v_I^2} \cdot \frac{s^2 v_s^1}{s^2 v_I^2}} \qquad S_Q^I = \sqrt{\frac{s^2 v_I^2}{s^1 v_I^2} \cdot \frac{s^2 v_s^1}{s^1 v_s^1}} \qquad S_C^I = \frac{s^2 v_I^2}{s^1 v_s^1} \tag{81}$$

Moreover, substituting (79)(80) into (81), we can obtain

$$S_V^I = \sqrt{\frac{s^2 v_s^1}{s^1 v_s^2}} \cdot \frac{1}{L_I} \qquad S_Q^I = \sqrt{\frac{s^2 v_s^1}{s^1 v_s^2}} \qquad S_C^I = L_I \tag{82}$$

Thus we have [8)]

$$S_V^I = S_Q^I / L_I \tag{83}$$

Both of the members of this equation express the productivity changes. The left member is the Standard productivity index, which represents the productivity change as the cost change. $S_Q{}^I$ of the right member is the Standard output index, which represents the output changes. L_I is the ratio of the actual total labour. Therefore, the right member express the productivity change as the changes in per capita output. The equation (83) also provides the double measuring rods for productivity changes.

3 Productivity and the Standard Ntaional Income

1) Definition of the Classical Effective Labour

We have explained the structure of the Standard productivity index and the Standard output index. The relation between the Standard productivity index and the Standard output index was represented by the equation (83). The equation (83) can be transformed into

$$S_Q^I = S_V^I \cdot L_I \tag{84}$$

By transforming (83) into (84), the implication of the equation (84) has changed from the equation (83). The left member of this equation is the Standard output index, which explains the output changes. The right member is the *product* of the Standard productivity index and the Indexed total labour. In other words, it is the total labour weighted by productivity. It follows that the right member of the equation (84) is measured in terms of unit of labour. It will be found convenient to represent the right member of equation (84) as

$$L_E = S_V^I \cdot L_I \tag{85}$$

Let us call this weighted total labour the *Classical effective labour*. By this definition, the equation

(84) becomes such a simple form as

$$S_Q = L_E \tag{86}$$

This implies that the Standard output index is equal to the Classical effective labour. The Classical effective labour is measured in terms of labour unit.

Cresponding to the Classical effective labour, a new labour coefficient vector can be defined as

$$\begin{aligned} \mathbf{l}_E{}^2 &= S_V^I \cdot \mathbf{l}_I{}^2 \\ &= S_V^I \cdot L_I \cdot \mathbf{l}_S{}^2 \end{aligned} \tag{87}$$

Then the Classical effective labour is represented as

$$\begin{aligned} L_E &= S_V^I \cdot \mathbf{x}^2 \mathbf{l}_I{}^2 \\ &= S_V^I \cdot L_I \cdot \mathbf{x}^2 \mathbf{l}_S{}^2 \\ &= \mathbf{x}^2 \mathbf{l}_E{}^2 \end{aligned} \tag{88}$$

We call $\mathbf{l}_E$ the Effective labour coefficient vector.

The Classical effective labour is determined by the Standard productivity index and the Indexed total labour. The Standard productivity index is given by $[\mathbf{A}^1, \mathbf{l}_S{}^1]$ and $[\mathbf{A}^2, \mathbf{l}_S{}^2]$. Moreover, as we have seen in (69) or (70), the Indexed total labour is given by $\mathbf{x}^1\mathbf{l}_L{}^1$ and $\mathbf{x}^2\mathbf{l}_L{}^2$, which are also exgonenously determined. Therefore, it should be noticed that the Classical effective labour is determined by the given variables of the Sraffa system and thus is independent of the price changes and the distributional changes.

2) Relation between the Classical Effective Labour and the Standard National Income

As we have seen in *Theorem 1*, in Situation I, the value of the standard national income was given by the Standard total labour and is equal to unity. On the contrary, in Situation II, the value of the standard national income remain unknown. We can explain it through the Classical effective labour. For the time being, we will confine ourselves to the case when the rate of profit is zero.

The value vector defined with $\mathbf{l}_E$ will be represented as

$$\mathbf{v}_E{}^2 = [\mathbf{I} - \mathbf{A}^2]^{-1}\mathbf{l}_E{}^2 \tag{89}$$

Considering the equation (87), the relationship between $\mathbf{v}_E{}^2$, $\mathbf{v}_I{}^2$ and $\mathbf{v}_S{}^2$ is represented as

$$\begin{aligned} \mathbf{v}_E{}^2 &= S_V^I \cdot \mathbf{v}_I{}^2 \\ &= S_V^I \cdot L_I \cdot \mathbf{v}_S{}^2 \end{aligned} \tag{90}$$

In Situation II, the value of the standard national income is the *product* of standard net product vector and the value vector. Thus it is represented as $\mathbf{s}^2\mathbf{v}_E{}^2$. Like (26), the standard national income is equal to the total labour employed in the Standard system. This can be shown as

$$\mathbf{s}^2\mathbf{v}_E{}^2 = \mathbf{q}^2 \ [\mathbf{I} - \mathbf{A}^2][\mathbf{I} - \mathbf{A}^2]^{-1}\mathbf{l}_E{}^2$$

$$=q^2 l_E{}^2 \tag{91}$$

Multiplying L_E to *Assumption 4*, we have

$$q^2 l_E{}^2 = x^2 l_E{}^2 \tag{92}$$

Therefore, from (91) (92), we have

$$s^2 v_E{}^2 = x^2 l_E{}^2 \tag{93}$$

This may be an interesting result. We can conclude that, from (93), the standard national income is equal to the Classical effective labour when the rate of profit is zero.

3) Real Income Comparison by the Opportunity Cost Approach When $r = 0$

The introduction of the new notion of the Classical Effective Labour will bring about a remarkable result into our analysis. It enables us to make an intertemporal comparison of real national income.

The Classical effective labour is measured in terms of labour unit. Therefore, the Classical effective labour of Situation II can be compared with the Standard total labour of Situation I . From (88), the Classical effective labour is represented as

$$x^2 l_E{}^2 = S_V^I \cdot L_I \cdot x^2 l_S{}^2 \tag{94}$$

Since the Standard total labour is always equal to unity, the relation between the Standard total labours of both situations is represented as

$$x^2 l_S{}^2 = x^1 l_S{}^1 \tag{95}$$

then, from (94)(95), we can obtain

$$x^2 l_E{}^2 = S_V^I \cdot L_I \cdot x^1 l_S{}^1 \tag{96}$$

This equation indicates that the Classical effective labour of Situation II can be compared with the Standard total labour of Situation I .

In Situation I , as is seen in (50), we have

$$s^1 v_S{}^1 = x^1 l_S{}^1 \tag{97}$$

Similarly, in Situation II , as is seen in (93), we have

$$s^2 v_E{}^2 = x^2 l_E{}^2 \tag{98}$$

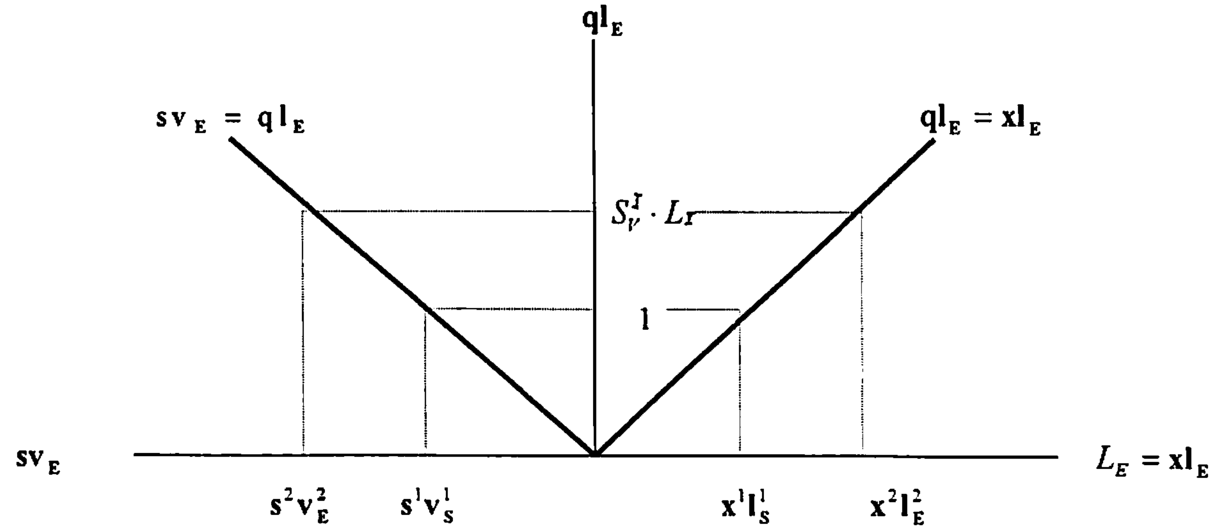
Substituting (97) (98) into (96), we can obtain

$$s^2 v_E{}^2 = S_V^I \cdot L_I \cdot s^1 v_S{}^1 \tag{99}$$

This will be a surprising result. Although they are composed of heterogenous commodities, the standard national incomes of different situations can be compared with each other by a common measure, the unit of effective labour.

These results are shown by *Figure 1* as follows.

*Figure*1



This graph shows the relationship between the Classical effective labour and the Standard national income. Moreover, let us consider Situation I as the base year, then the real income comparison can be performed through the unit of effective labour.

4) Real Productivity Comparison When $r = 0$

Now we will proceed to productivity comparison. If we divide (98) by (97), we can obtain

$$\frac{s^2 v_E^2}{s^1 v_S^1} = \frac{x^2 l_E^2}{x^1 l_S^1} \tag{100}$$

From (88)(100), the equation (100) can be transformed into

$$\frac{s^2 v_E^2}{x^2 l_I^2} = S_V^I \cdot \frac{s^1 v_S^1}{x^1 l_S^1} \tag{101}$$

This equation shows the relationship between the productivities of both situations. For convenience sake, let us denote the productivity of the Opportunity cost approach of each situation as

$$S_L^1 = \frac{s^1 v_S^1}{x^1 l_S^1} \qquad\qquad S_L^2 = \frac{s^2 v_E^2}{x^2 l_I^2} \tag{102}$$

From (102), the equation (101) can be rewritten as

$$S_L^2 = S_V^I \cdot S_L^1 \qquad (103)$$

From (97), the productivity in Situation I is reduced to

$$S_L^1 = 1 \qquad (104)$$

Then from (103)(104), the productivity of Situation II can be represented as

$$S_L^2 = S_V^I \qquad (105)$$

These results are shown in *Figure 2* as follows.

Figure2

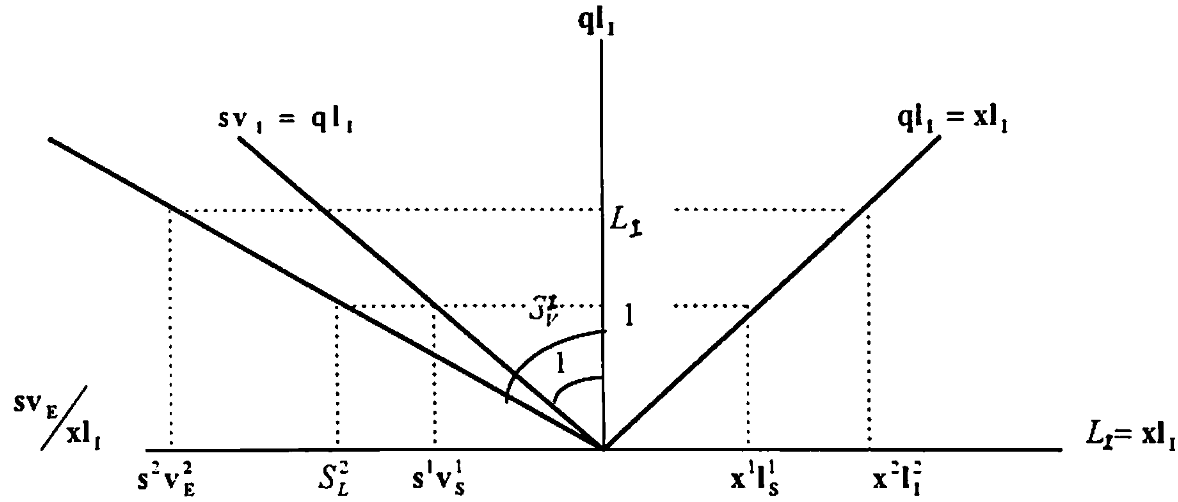


Figure 2 shows the productivity comparison as well as the income comparison.

5 Distribution and the Standard National Income

1) The Standard National Income When $0 \leqq r < R$

So far we have been concerned with the case when $r = 0$. Now let us consider the case when the rate of profit is positive, i.e. $0 \leqq r < R$. We should examine whether the value of standard national income does vary or not as the rate of profit increase. In Situation I (or base year), Theorem 1 asserts that the standard national income is equal to the Standard total labour under the condition of $r = R\,(1\text{-}w)$. Therefore, in Situation I, the value of the standard national income is constant as the rate of profit increases. This can be shown, from *Theorem 1* and the equations (26)(34), as

$$sv_S = sp \quad \Leftrightarrow \quad r = R\,(1 - w) \qquad (106)$$

In Situation Ⅱ, the price equation defined with the effective labour coefficient vector is represented as follows

$$\mathbf{p} = (1 + r)\,\mathbf{A}\mathbf{p} + w\mathbf{l}_E \tag{107}$$

Fortunately, the following theorem will hold,

[Theorem 2] Let $\mathbf{p} = (1 + r)\,\mathbf{A}\mathbf{p} + w\mathbf{l}_E$ be the price equation system under *Assumptions 1-4*. Then, the Standard national income is equal to the Classical effective labour if and only if

$$r = R\,(1-w) \tag{108}$$

where $0 \leqq r < R$.

[Proof] Multiplying L_E to the *Assumption 4*, we can obtain

$$\mathbf{q}\mathbf{l}_E = \mathbf{x}\mathbf{l}_E \tag{109}$$

As is seen in (32) (33), from (31) (109), the price equation can be transformed into

$$(1- r/R)\,\mathbf{s}\mathbf{p} = w\,\mathbf{x}\mathbf{l}_E \tag{110}$$

then, for $0 \leqq r < R$, we have

$$\mathbf{s}\mathbf{p} = \mathbf{x}\mathbf{l}_E \quad \Leftrightarrow \quad r = R\,(1 - w) \tag{111}$$

Thus the theorem is verified.

Q.E.D

This theorem asserts that, under the condition of $r = R(1 - w)$, the value of the standard national income is equal to the Classical effective labour, even when the rate of profit is positive. It follows that the value of standard national income remains constant even if the rate of profit increases. Under the condition of $r = R\,(1 - w)$, the value of the standard national income is independent of the changes in distribution. The value of the standard national income remains constant when the rate of profit is positive. This can be shown, from (98)(111), as

$$\mathbf{s}\mathbf{v}_E = \mathbf{s}\mathbf{p} \quad \Leftrightarrow \quad r = R(1 - w) \tag{112}$$

Therefore, if we follow the Opportunity cost approach, we can consider that the results of real income comparison or real productivity comparison will hold even if the rate of profit is positive.

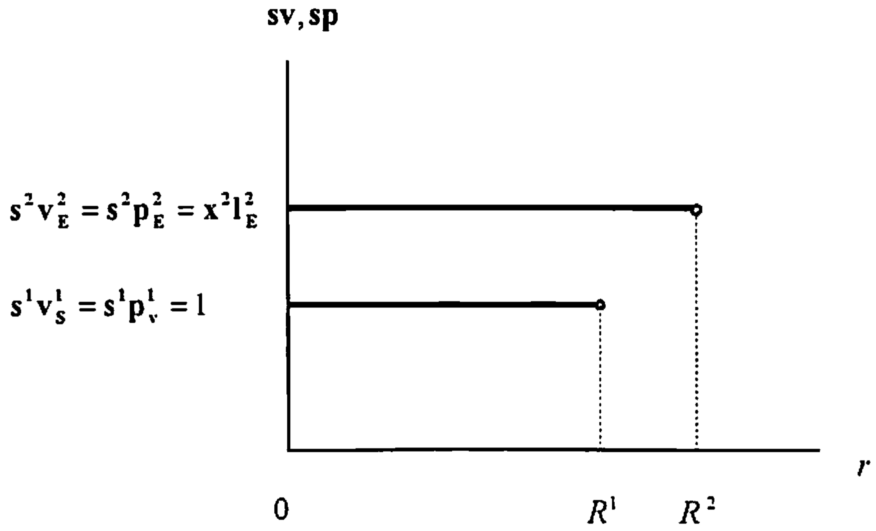
Let us denote the prices of (107) under the condition of $r = R\,(1 - w)$ $(0 \leqq r < R)$ by $\mathbf{p}_E$. Then, $\mathbf{p}_E$ is represented as

$$\mathbf{p}_E = (1- r/R)\,[\mathbf{I} - (1 - r)\,\mathbf{A}\,]^{-1}\,\mathbf{l}_E \tag{113}$$

It should be noted that the prices of this equation are expressed in terms of the unit of classical effective labour.

Let R^1, R^2 be the maximum rate of profit in each situation, and let us assume $R^2 > R^1$. Then, if the conditions of $r^1 = R^1 (1 - w^1)$ and $r^2 = R^2 (1 - w^2)$ are satisfied, from (106)(112), we can illustrate the relationship between the Standard national income and the rate of profit in *Figure 3.*

Figure 3.



2) The Actual National Income versus the Standard National Income

It will be better to make clear the relation between the actual national income and the standard national income. When the rate of profit is zero, the standard national income is equal to the actual national income. In Situation I, this result was represented in the equation (27). For Situation II, we can derive

$$y^2 v_E{}^2 = s^2 v_E{}^2 \tag{114}$$

Therefore, the actual national income becomes equal to the standard national income, when the rate of profit is equal to zero.

However, when the rate of profit is positive, the value of actual national income can vary as the profit rate varies, except the special case in which the capital-labour ratios (organic composition of capital) are uniform throughout industries. Now let us consider the difference between the actual national income and the standard national income. If we use the notation

$$k = xA \tag{115}$$

then the price equation (22) can be transformed into

$$yp = r\,kp + w\,xl_S \tag{116}$$

On the other hand, if we use the notation

$$k_S = qA \tag{117}$$

the price equation (22) can be rewritten as

$$\mathbf{sp} = r\,\mathbf{k_s p} + w\,\mathbf{ql_s} \qquad (118)$$

From Assumption 4, the total wages of the equations (116)(118) become

$$w\,\mathbf{ql_s} = w\,\mathbf{xl_s} \qquad (119)$$

Therefore, subtracting (118) from (116), we can obtain

$$\mathbf{yp} = \mathbf{sp} + r\,(\mathbf{kp} - \mathbf{k_s p}) \qquad (120)$$

This equation shows that the value of actual national income is divided into two parts. The first term of the right member is the constant standard national income, under the condition of $r = R\ (1 - w)$. The second term will vary as the rate of profit changes. It implies the profit stemming from the difference of the demand composition. To conclude, we may consider the standard national income as a reference of the actual national income.

Conclusion

In this paper, we have discussed the definition of real national income, by synthesizing the Opportunity cost approach of Hicks [1981] and the standard national income of Sraffa [1960]. The complex features in defining real national income may be connected to three factors: 1) the complex changes in the production technique, 2) the different levels of distribution, and 3) the various compositions of demand (or output). The Real cost approach of Hicks [1981] is considered on the given production conditions (the given total labours and the constant input coefficients) of different situations and the actual demand compositions of both situations. On the contrary, the Opportunity cost approach of Hicks [1981] can be calculated by the given production conditions of different situations, the actual demand composition of one situation, and the *potentially producible* composition of demand of the other situation. The most significant thing of the Opportunity cost approach lies in the separation of the problem of demand from the definition of real national inocme. My Standard productivity index is based on the idea of Hicks's Opportunity cost approach. It can be calculated by the given production techniques and the total labours of different situations. The remarkable property of our indexes is that the Standard productivity index and the Standard output index can provide the double measuring rods of productivity changes, because the Standard productivity index is equal to the Standard output index when the total labours of different situations are the same, or the Standard productivity index is equal to the Standard output index divided by the Indexed total labour when the total labours are different.

Hicks adopted the labour coefficients (values) as the weights for indexes. We also adopted the value vectors as the weight vectors for indexes. However, the value vectors can be applicable only when the rate of profit is zero. When the rate of profit is positive, they have no relevance in the real world. Therefore, we have applied the Standard productivity index to the definition of the Classical effective labor, which is the *product* of the Standard productivity index and the Indexed total labour. Then we explained the value of the standard national income through the Classical effective labour. The standard national income of Sraffa [1960] is a device whose value remains constant as the rate of profit varies in a special case. We have explained this by *Theorem 1* or *Theorem 2*. In Situation I , the value of standard national income is equal to the standard total labour (*Theorem 1*). In Situation II , the value of standard national income is equal to the Classical effective labour defined in this paper (*Theorem 2*).

Our definition of real national income is performed through labour amount. We distinct four notions of total labour: the Actual total labour, the Standard total labour, the Indexed total labour, and the Classical effective labour. Corresponding to these notions, four different labour coefficient vectors are defined: the Leontief type labour coefficient vector, the Standard labour coefficient vector, the Indexed labour coefficient vector, and the Effective labour coefficient vector. In my reasoning of the real national income, the definitions of total labour and the labour coefficient vectors are crucially important. The classical effective labours of different situations are measured in terms of unit of labour and can be compared with each other. Therefore, the standard national income given by the Classical effective labour can be compared with each other.

Both Hicks [1981] and Sraffa [1960] considered in the Classical framework where the total labour and the production technique are given. I will call our approach the *Hicks-Sraffa Approach*, the precedent ideas of which are those of A.C.Pigou or D.Ricardo. Its origin may be in the ideas of Adam Smith.

[Notes]

1) See Kuznets[1948], Little[1949], Samuelson[1950], Graaf[1957], Sen[1976], Sen[1979]. For the Cost approach, Hicks himself confesses insufficiencies. The reference of Sen[1979] will be helpful.
2) Hicks applied his reasoning of Utility approach developed by Hicks[1958] to his Opportunity cost approach of Hicks[1981].
3) If we take c'-coefficients (the technique of B-situation) and $\mathbf{c'q'}$ (the total labour of B-situation), instead of c-coefficients and $\mathbf{cq}$(the total labour of the A-situtation), the Laspeyres productiveity index, the Paasche output index can be calculated by the same way.
4) See Yagi [1999], Yagi [2000].
5) We will use, in this paper, four different notions of total labour: the Actual total labour, the Standard total labour, the Indexed total labour, and the Classical effective labour. The latter will be defined in the following pages.
6) Corresponding to the definition of total labour, we will use four different labour labour coefficient vectors: the Leontief type labour coefficient vector, the Standard labour coefficient vector, the Indexed lebour coefficient vector, and the Effective labour coefficient vector.
7) The productivity indexes based on Sraffa's system were introduced in Yagi [1998a], Yagi [1998b].
8) In the case of $L_I \neq 1$, if we use the Standard labour coefficient vector to construct indexes, we can obtain the equation (83) much easier. The indexes constructed by the value vectors with Standard labour coefficient vectors will be represented as

$$S_V^S = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^2}} \qquad S_Q^S = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_s^1}{\mathbf{s}^1\mathbf{v}_s^2}} \qquad S_C^S = 1$$

Thus

$$S_V^S = S_Q^S$$

Comparing S_Q^I with S_Q^S, we have

$$S_Q^I = S_Q^S$$

And comparing S_V^I with S_V^S, we have

$$S_V^I = S_V^S / L_I^2$$

Then we can derive, from above equations, the equation (83).

[Reference]

Graaf, J.de V.[1957] *Theoretical Welfare Economics*, Cambridge University Press.

Harcourt,G.C.[1972] *Some Cambridge Controversies in the Theory of Capital*, Cambridge University Press.

Hicks, J.R.[1940] "The Valuation of Social Income," *Economica*,NS,7(May), pp.105-124. Reprinted in Hicks [1981] pp. .

Hicks, J.R.[1963] "The Measurement of Capital in Relation to the Measurement of Other Economic Aggregates," in Lutz,F.A.and D.C.Hague(eds)[1963] pp.18-31, reprinted in Hicks,J.R. [1981] pp.189-203.

Hicks, J.R.[1981] "Valuation of Social Income Ⅲ—The Cost Approach, " in Hicks,J.R.[1981],pp.243-268.

Hicks, J.R.[1981] *Wealth and Welfare*, Basil Blackwell.

Hicks, J.R.[1983] *Classics and Modern*, Basil Blackwell.

Kuznets, S.[1948] "On the Valuation of Social Income-Reflections on Professor Hicks' Article," *Economica*, NS,15(February ,May), pp.1-16, pp.116-13.

Little, I.M.D.[1949] "The Valuation of Social Income," *Economica*, NS, 16(November), pp.11-26.

Pasinetti, L.L.[1977] *Lectures on the Theory of Production*, Columbia University Press.

Samuelson, P.A.[1950] "Evaluation of Real National Income," *Oxford Economic Papers*, 2(1), January, Oxford University Press, 1-29.

Sen, A.K.[1976] "Real National Income," *Review of Economic Studies*, 43.

Sen, A.K.[1979] "The Welfare Basis of Real Income Comparisons: A Survey," *Journal of Economic Literature*, Vol.XVII, March,pp.1-45.

Sraffa, P.[1960] *Production of Commodities by means of Commodities*, Cambridge University Press.

Yagi,T.[1998a] "Alternative Theories on the Evaluation of Real National Income (in Japanese)," in *Keizaigaku No Shosou: Essays in honor of Professor Toshinosuke Kashiwazaki*, Egawa,M., Tomono, N., Uemura,T., and T.Yagi (eds), Gakubunsha, Tokyo,March.

Yagi,T.[1998b] "Productivity Index and Capital Theory(in Japanese)," Presented at the Annual Conference of Japanese Economic Association held at Ritsumeikan University, *Mimeo*.

Yagi,T.[1999] "Sraffa's System and Alternative Standards." *Working Paper Series*, No.9907, Institute for Research in Contemporary Political and Economic Affairs, Waseda University.

Yagi,T.[2000] "The Structure of *Production of Commodities*(in Japanese)," in *Gendai Keizai Ronsou*, Kataoka,H. and M.Matusmoto(eds) Gakubunsha,Tokyo.