# Are You Tired of the Slutsky Equation?

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**Abstract**

The Slutsky equation has been evaluated to a considerable extent on the ground that it can give the rationale for a Giffen good. Since the evidence for such goods is still extremely limited, it is not so helpful from an actual point of view. This paper presents another equation that decomposes the effect of a change in its price on the demand for a good into two effects, the "ratio effect" and the "elasticity effect." The Slutsky equation and the new one are "complements," but the latter is much easier to understand intuitively. Examples of application are also provided.

Key words: The Slutsky equation, Price effect, Ratio effect, Elasticity effect

JEL classification: D11

## 1 Introduction

The very first mission of demand theory was, is, and will be to analytically answer the question of how the demand for a good responds to variations in its own price (the price effect). Since Slutsky (1915) and Hicks and Allen (1934a, b) independently discovered the "Slutsky equation," it is only the Slutsky equation that has been universally used for such an analysis.[1] The Slutsky equation teaches us, quite correctly, that the price effect can be decomposed into the substitution effect and the income effect (the Slutsky decomposition). It has been the most fundamental tool not only for pure demand theory but also for wide applications, microeconomic or macroeconomic. It is no exaggeration to say that without it

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[1]For the relationship between Slutsky (1915) and Hicks and Allen (1934a, b), see Allen (1936). See also Schultz (1935) for the first empirical application of the Slutsky equation.

economists could have been only half through their work. The contribution of the Slutsky equation to economics is immeasurable.

However, I cannot say that it leaves nothing to be desired, wondering if such a decomposition of the price effect is actually helpful at least to those who have started economics. For a long time it has been repeated proudly by microeconomists that the power of the Slutsky equation is that to explain why a Giffen good or equivalently an upward sloping demand curve can exist. If a Giffen good exists, it belongs to the category of inferior goods which involve a positive income effect. It is this positivity that attracted many theorists because it may dominate an always negative substitution effect and reverse the sign of the (intuitively negative) price effect. It is such a beautiful theory that all authors of microeconomics textbooks have been obliged to mention it.[2] Thus, as long as a Giffen good is assumed to exist, the Slutsky equation has a special meaning.

On the other hand, it has also been recognized among economists as the law of demand that demand curves are almost always downward sloping. In other words, there has been next to nothing like a Giffen good. It is well known that Stigler (1947, 1987) continued to negate the existence of a Giffen good all his life. Although persistent efforts to find Giffen goods have been made,[3] Hicks's (1939, p. 35) statement about the law of demand still makes sense: "Exceptions to it are rare and unimportant." Hence a strange feeling about the glorious Slutsky equation.

In my opinion, it is advisable at least at an introductory level to deal mainly with the *normal* case or the case of *normal* goods which are characterized by a negative income effect. The problem, if any, is that, when the demand for a normal good is analyzed through the Slutsky equation, the result is too ordinary to grab theorists. For the slopes of demand curves are always negative and the law of demand always holds. But it will do. The real world will be well described by the law.

By the way, I have noticed another way to decompose the price effect. According to it, the price effect is composed of the "ratio effect" and the "elasticity effect." The former effect is positive (negative) if the expenditure spent on a good under consideration increases

[2]For recent textbooks, see Perloff (2008), Pindyck and Rubinfeld (2009), and Whitehead (2010).

[3]For example, Battalio et al. (1991) confirmed the existence of a Giffen good for rats, whereas recently Jensen and Miller (2008) reported the "first real-world evidence of Giffen behavior." For the latter, see also footnote 24 below.

(decreases) when its own price goes up. The latter effect stands for a unitary price elasticity of demand. The new decomposition suggests that in many significant cases the latter effect governs the price effect because the former effect may be regarded as negligible, that is, the price elasticity of demand is not far from unity. It is easier to understand intuitively, isn't it? I do not intend to argue that this new decomposition is superior to the Slutsky decomposition. But I believe that it provides a fresh insight to demand theory at an advanced level and helps students of economics to familiarize themselves with demand theory at a basic level, too.

This paper is organized as follows. Reviewing the Slutsky equation, Section 2 presents a new equation which decomposes the price effect into the ratio effect and the elasticity effect. Applications of the two equations to examples often used in economics are given, too. Section 3 explains the new equation graphically for a better understanding. Section 4 applies it to the cross-price case, while Section 5 extends it to a case where a consumer holds initial endowments. Section 6 concludes the paper, quoting Pigou.

## 2 A New Decomposition of the Price Effect

A consumer under consideration has an ordinal utility function

$$v = u(q_1, q_2), \tag{1}$$

where $q_1$ and $q_2$ are the quantity of *normal* good 1 and that of *normal* good 2, respectively.[4] As usual, it has the following properties:

$$u_1 > 0, \; u_2 > 0,$$

and

$$\begin{aligned} |U| &\equiv \begin{vmatrix} u_{11} & u_{12} & u_1 \\ u_{21} & u_{22} & u_2 \\ u_1 & u_2 & 0 \end{vmatrix} \\ &= -u_2^2 u_{11} + 2u_1 u_2 u_{12} - u_1^2 u_{22} > 0, \end{aligned}$$

where $u_i = \partial u / \partial q_i$ and $u_{ij} = \partial^2 u / \partial q_i \partial q_j$, $i, j = 1, 2$. The positivity of bordered Hessian $|U|$ means the decreasing marginal rate of substitution.[5]

[4] The arguments below will proceed around good 1. Good 2 can be regarded as the rest of all goods.

[5] Utility must be ordinal. This is what Slutsky (1915) and Hicks and Allen (1934a, b) emphasized in order to say good-bye to the unrealistic assumption that utility is measurable. Economists should have replaced

The objective of the consumer is to maximize the utility represented by $u(q_1,\ q_2)$ under the budget constraint $p_1q_1 + p_2q_2 = y$ with prices $p_1$, $p_2$ of goods 1, 2 and income $y$ as given. This utility maximization problem can be written compactly as

$$\text{Problem I}: \quad \max_{q_1,\ q_2} \quad u(q_1,\ q_2)$$
$$\text{s.t.} \quad p_1q_1 + p_2q_2 = y.$$

The solutions, $q_1^*$, $q_2^*$, to this problem, i.e., the demands for goods 1 and 2, are the function of $p_1$, $p_2$, and $y$. Then, the Slutsky equation says that the price effect with respect to good 1 is expressed as the sum of the substitution effect and the income effect as follows:

$$\underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,y=\text{const}}}_{\text{price effect}} = \underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}}}_{\text{substitution effect}} + \underbrace{\left(-q_1\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{income effect}}, \tag{2}$$

where

$$\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}} = -\frac{u_1u_2^2}{p_1|U|} < 0 \quad \text{and} \quad -q_1\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}} = -q_1\frac{u_1(u_2u_{12} - u_1u_{22})}{p_1|U|} < 0,$$

all partial derivatives and $q_1$ being evaluated at $(q_1^*, q_2^*)$. The above arguments are too ordinary to need more explanation. Graphical representations of it can easily be found in every textbook of microeconomics.

Here, for concreteness, let us apply the Slutsky equation (2) to two familiar examples of (1). One is the Cobb-Douglas-type utility function

$$v = Aq_1^aq_2^b, a > 0, b > 0 \tag{3}$$

with $A$, $a$, and $b$ as positive parameters. The Slutsky equation for this utility function becomes

$$\underbrace{-\frac{a}{a+b}\frac{y}{p_1^2}}_{\text{price effect}} = \underbrace{-\frac{ab}{(a+b)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\left[-\frac{a^2}{(a+b)^2}\frac{y}{p_1^2}\right]}_{\text{income effect }\ominus} < 0. \tag{4}$$

Good 1 is certainly a *normal* good. The other example is the CES utility function

$$v = A\left(aq_1^{\frac{\sigma-1}{\sigma}} + bq_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{5}$$

a cardinal utility with an ordinal one. Nevertheless, the former did not die. Rather, it revived and is now prospering in economic theory such as game theory, expected utility theory, and optimal growth theory. Did they forget the great contribution by Slutsky et al.? Am I the only one who is perplexed at such a queer situation?

with $A$, $a$, $b$, and $\sigma$ as all positive parameters. $\sigma$ is referred to as the elasticity of substitution between good 1 and good 2.[6] The Slutsky equation in this case is written as

$$\underbrace{-\frac{1+\sigma c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{price effect}} = \underbrace{-\frac{\sigma c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\left[-\frac{1}{(1+c)^2}\frac{y}{p_1^2}\right]}_{\text{income effect }\ominus} < 0, \tag{6}$$

where

$$c = \left(\frac{b}{a}\right)^{\sigma}\left(\frac{p_1}{p_2}\right)^{\sigma-1} > 0.$$

Again good 1 is clearly a *normal* good. As seen from these examples, the Slutsky equation goes on justifying the law of demand as long as a good is a normal one.

Now let me propose another decomposition. It is derived from the solutions to the following problem which is obtained simply by rewriting Problem I just a little bit:

$$\text{Problem II}: \quad \max_{\theta} \quad u(q_1,\, q_2)$$
$$\text{s.t.} \quad q_1 = \frac{\theta y}{p_1},\ q_2 = \frac{(1-\theta)y}{p_2},\ 0 \leq \theta \leq 1.$$

$\theta$ is the ratio of income $y$ that goes to the purchase of good 1.[7] As is apparent, Problem I and Problem II are mathematically equivalent and the solutions to each problem are the same.[8] The difference is that the optimal quantities, $q_1^*$, $q_2^*$, of goods 1 and 2 are found in Problem I by adjusting those quantities $q_1$ and $q_2$ directly, whereas in Problem II by controlling the ratio $\theta$ indirectly. It seems trifling. But differentiating $q_1 = \frac{\theta y}{p_1}$ partially with respect to $p_1$ gives quite a new equation:

$$\underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,y=\text{const}}}_{\text{price effect}} = \underbrace{\frac{\frac{d\theta}{dp_1}y}{p_1}}_{\text{ratio effect}} + \underbrace{\left(\left.\frac{dq_1}{dp_1}\right|_{y,\theta=\text{const}}\right)}_{\text{elasticity effect}}, \tag{7}$$

where

$$\frac{\frac{d\theta}{dp_1}y}{p_1} = -\frac{u_1}{p_1}\frac{u_2^2}{|U|} + \frac{q_1}{p_1}\frac{u_1u_2u_{12} - u_2^2u_{11}}{|U|} \gtreqless 0 \text{ and } \left.\frac{dq_1}{dp_1}\right|_{y,\theta=\text{const}} = -\frac{q_1}{p_1} < 0,$$

all partial derivatives and $q_1$ being evaluated at $(q_1^*, q_2^*)$.

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[6]When $\sigma = 1$ and $a + b = 1$, the CES utility function (5) coincides with the Cobb-Douglas utility function (3). For the proof, see Henderson and Quandt (1980, p. 113).

[7]For other interpretations of $\theta$, see footnotes 18 and 22 below.

[8]The conditions for utility maximization are derived from Problem II in Appendix A.

According to (7), the price effect can be divided into the two effects which are called for convenience the "ratio" effect and the "elasticity" effect. The implication is very simple. It is this. When price $p_1$ of good 1 rises, a change in the demand for it is governed completely by the elasticity effect *unless the ratio $\theta$ changes, or in other words if the expenditure spent on it remains unchanged.* The magnitude of it can be calculated from the relation $p_1 q_1 = \theta y$ with $\theta$ and $y$ as given. Total differentiation of it yields the value of $-q_1/p_1$ at once. To put it differently, a unit increase in $p_1$ decreases income $y$ by $q_1$ (in nominal terms) which in turn implies a decrease in the demand for good 1 by $q_1/p_1$ (in real terms), *ceteris paribus*. In a word, the price elasticity of demand is unity. Hence the name "elasticity" effect.[9]

Of course, the price effect differs from the elasticity effect when the ratio $\theta$ is affected by variations in $p_1$. The "ratio" effect measures the difference. If the ratio changes by $d\theta/dp_1$, it means a variation in income by $(d\theta/dp_1)y$ (in nominal terms) which in turn leads to a change in the demand by $(d\theta/dp_1)y/p_1$ (in real terms). This is the ratio effect. The calculation of it is not so easy as that of the elasticity effect. Fortunately, however, the value of the price effect is already known in (2). Thus, the quickest way to obtain the ratio effect is to subtract $-q_1/p_1$ from it.[10] Anyway, it is assured that the elasticity effect always dominates the ratio effect as long as a normal good is analyzed.

In fact the ratio effect consists of the two effects:

$$\underbrace{\frac{\frac{d\theta}{dp_1}y}{p_1}}_{\text{ratio effect}} = \underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}}}_{\text{substitution effect}} + \underbrace{\frac{p_2}{p_1}q_1\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}}_{\text{transfer effect}}, \tag{8}$$

where

$$\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}} = -\frac{u_1 u_2^2}{p_1|U|} < 0 \quad \text{and} \quad \frac{p_2}{p_1}q_1\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}} = \frac{p_2}{p_1}q_1\frac{u_2}{p_2}\frac{u_1u_{12} - u_2u_{11}}{|U|} > 0,$$

all partial derivatives and $q_1$ being evaluated at $(q_1^*, q_2^*)$. The substitution effect of (8) is the same as that of the Slutsky equation (2). The second effect in the right-hand side of (8) is interpreted as follows. A unit increase in $p_1$ decreases income $y$ by $q_1$ (in nominal terms) which leads to a decrease in the demand for *good 2* by

$$q_1\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}} \quad \left(= q_1\frac{u_2}{p_2}\frac{u_1u_{12} - u_2u_{11}}{|U|} > 0\right)$$

[9] Note that the statement of this paragraph holds in the case of the n-good case, too. Particularly the way to derive the elasticity effect is always the same.

[10] Appendix B shows a direct method of calculating the ratio effect (8).

in real terms.[11] In nominal terms it has the value multiplied by $p_2$. That amount in turn is added to the expenditure on *good 1*. That is, a rise in $p_1$ caused the demand for good 2 to "transfer" to that for good 1. Correctly speaking, the demand for good 1 increases by the "transfer" effect indicated above.

The sign of the transfer effect is indeterminate because it is the sum of the two terms with opposite signs. When those two terms almost cancel out, the ratio effect is negligible. In such a situation, the elasticity effect dominates. This is what new equation (7) suggests.

In order to show the usefulness of it, let us apply it to the above examples. As for the Cobb-Douglas-type utility function (3), equation (7) becomes

$$\begin{aligned}\underbrace{-\frac{a}{a+b}\frac{y}{p_1^2}}_{\text{price effect}} &= \underbrace{0}_{\text{ratio effect}} + \underbrace{\left(-\frac{a}{a+b}\frac{y}{p_1^2}\right)}_{\text{elasticity effect}} \\ &= \underbrace{-\frac{ab}{(a+b)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{ab}{(a+b)^2}\frac{y}{p_1^2}}_{\text{transfer effect }\oplus} + \underbrace{\left(-\frac{a}{a+b}\frac{y}{p_1^2}\right)}_{\text{elasticity effect }\ominus} < 0. \quad (9)\end{aligned}$$

In this case the substitution effect and the transfer effect just cancel out. As a result, the ratio effect vanishes. The price effect is equal to the elasticity effect. So, the price elasticity of demand for good 1 is one, though this fact is well known.

Equation (7) for (5) is given by

$$\begin{aligned}\underbrace{-\frac{1+\sigma c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{price effect}} &= \underbrace{-\frac{(\sigma-1)c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{ratio effect}} + \underbrace{\left(-\frac{1}{1+c}\frac{y}{p_1^2}\right)}_{\text{elasticity effect}} \\ &= \underbrace{-\frac{\sigma c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{transfer effect }\oplus} + \underbrace{\left(-\frac{1}{1+c}\frac{y}{p_1^2}\right)}_{\text{elasticity effect }\ominus} < 0. \quad (10)\end{aligned}$$

The CES utility function is characterized by the elasticity of substitution $\sigma$. It is known that for $\sigma > 1$ ($< 1$) the expenditure on good 1 decreases (increases) as $p_1$ goes up. The above equation confirms it because for $\sigma > 1$ ($< 1$) the ratio effect is negative (positive) and therefore the price elasticity of demand is greater than (less than) one. And a new result: $\sigma$ is always the ratio of the substitution effect in absolute value to the transfer effect. If $\sigma$ is greater than unity, the substitution effect exceeds the transfer effect.[12]

[11] Remember that good 2 is assumed to be a normal good.

[12] Examples (3) and (5) were originally derived respectively by Cobb and Douglas (1928) and Arrow et al.

Finally, it should be noticed that the sum of the transfer effect and the elasticity effect is equal to the income effect in the Slutsky equation (2). This can be checked by comparing general formulations (2) and (7). It is also seen at once by looking at the above equations derived from the two examples.

# 3 Graphical Representations

In the previous section, a new equation decomposing the price effect was explained in detail by mathematical expressions and concrete examples. They made economists know well how new and how useful it is, I believe. But graphical representations are convenient for anyone. In this section, the new decomposition is explained further using three figures.

Figure 1. A Decomposition of the Price Effect: The Case of a Negative Ratio Effect.

Figure 1 shows how such a decomposition of the price effect is drawn on the $q_1q_2$ plane. When $p_1 = p_1'$ and $p_2 = p_2'$, this consumer chooses an optimal combination $(q_1^*, q_2^*)$ of goods 1 and 2. It is indicated by • as the point at which an upper budget line is tangent to a right indifference curve. Next assume that $p_1$ rises from $p_1'$ to $p_1''$ while $p_2 = p_2'$. Then, the budget line of this consumer pivots clockwise on the $q_2-$intercept. A new optimal combination $(q_1^{**}, q_2^{**})$ is found as the point where the lower budget line touches a left indifference curve.

As shown in the figure, the price effect ($PE$) is measured by the difference between $q_1^*$ and $q_1^{**}$. According to (7) the price effect is divided into the ratio effect and the elasticity effect. Such a relation is pictured by the coupling of two arrows. One is the *westward* arrow starting at the initial optimal point. Note that this arrow runs horizontally. It means that the ordinate is still $q_2^*$ at the head of it. Furthermore $p_2$ does not change, either. Thus, this arrow exactly represents the elasticity effect ($EE$). The other is the arrow running *northwestward*

(1961) as aggregate *production* functions. Despite the claim of the latter that empirically $\sigma$ (as the elasticity of substitution between capital and labor) is significantly less than unity, it is now believed by macroeconomists that there is the law that $\sigma$ is almost one. Solow (2005) calls such a situation the "addiction of theorists to Cobb-Douglas." In this sense the Cobb-Douglas production function is more than a mere special case of the CES. See also Douglas (1967) for the triumph. How about (3) and (5) as utility functions? Judging from very frequent use of (3) or a qualitatively equivalent log utility function, it seems that microeconomists also deem that in many cases it is appropriate to put it equal to unity. It applies to $\sigma$ as the intertemporal elasticity of substitution in consumption, too. But Hall (1988) estimates that it is close to zero.

along the lower budget line and reaching the new optimal point. It approximately shows the ratio effect ($RE$).[13]

Figure 2. Three Types of Indifference Curves.

In Figure 1 the ratio effect is negative since the demand for good 1 decreases along the northwestward arrow as $p_1$ goes up. The ratio effect may be zero or positive. As is easily seen, when the ratio effect is positive, the arrow starting at the head of the westward arrow runs *southeastward* along the lower budget line. The zero ratio effect means the disappearance of an arrow running along the budget line. In Figure 2, depending on the signs of the ratio effect, three types of indifference curves are depicted on the $q_1q_2$ plane. The Cobb-Douglas-type utility function corresponds exactly to type b. And the CES utility function with $\sigma$ greater than (less than) unity belongs to type a (type c).[14]

Figure 3. Three Types of Demand Curves.

So far the demand for good 1 was examined on the $q_1q_2$ plane taking into account the connection with the demand for good 2. No doubt it is a right way to analyze it. But people eventually want (visually more familiar) demand curves for good 1 on the $q_1p_1$ plane, don't they? Then, how do the above arguments based on a new decomposition (7) are related to ordinary demand curves? The answer is condensed into Figure 3, where three types of

[13]Strictly speaking, the ratio effect is represented by the arrow starting at the *initial* optimal point and running northwestward along the *upper* budget line. For $p_1$ in the denominator of the first term in the right-hand side of (7) takes a value before it rises. In this case it is $p'_1$, not $p''_1$. Such an arrow is also depicted in Figure 1. The two northwestward arrows almost coincide when variations in $p_1$ are small enough.

[14]Instead of the direction of an arrow, the sign of the ratio effect can also be discerned graphically with the help of "triangles." Look at Figure 1 again and call points $(y/p'_2, 0)$, $(0,0)$, $(0, y/p'_1)$, and $(q_1^*, q_2^*)$ respectively points $A$, $B$, $C$, and $D$. Then it is easy to understand that the ratio of the area of triangle $ABD$ to that of right triangle $ABC$ is equal to the ratio of income $y$ that goes to the purchase of good 1 for $p_1 = p'_1$, say, $\theta'$. In addition call points $(y/p''_1, 0)$ and $(q_1^{**}, q_2^{**})$ respectively points $C'$ and $D'$. Then the ratio of the area of triangle $ABD'$ to that of right triangle $ABC'$ is equal to the ratio of income $y$ that goes to the purchase of good 1 for $p_1 = p''_1$, say, $\theta''$. Note that right-angled triangles $ABC$ and $ABC'$ have side $AB$ in common. It follows that $\theta' > \theta''$, i.e., the ratio effect is negative, if and only if point $D'$ is above horizontal line $q_2 = q_2^*$ as in Figure 1 or Figure 2 (a). Similarly, the ratio effect is negative if and only if point $D'$ is below horizontal line $q_2 = q_2^*$ as in Figure 2 (c). It is only when points $D$ and $D'$ are both on the same horizontal line that the ratio effect is zero as in Figure 2 (b).

demand curves are drawn on the $q_1p_1$ plane, corresponding to three types of indifference curves in Figure 2.

For example, look at a demand curve of type a, the flattest one of three types. The demand for good 1 is $q_1^*$ for $p_1 = p_1'$. The price effect at this point is measured by the angle formed by the vertical axis and the tangential line of the demand curve at the same point. It is indicated by a bold arc *PE*. The elasticity effect, the magnitude of which is $q_1^*/p_1'$, is represented by the angle formed by the vertical axis and the straight line starting at the origin and passing through point $(q_1^*, p_1')$. The reason is straightforward. The arc *EE* shows the elasticity effect. Call the straight line characterized by the arc the *EE* line. To detect the ratio effect, draw a straight line from the origin such that the vertical axis and the straight line form the angle with the same size with the price effect. Such an angle is indicated by a bold arc near the origin. And call the straight line the *PE* line. Since the ratio effect is negative in type a, the price effect is greater than the elasticity effect in absolute value by the ratio effect also in absolute value.[15] Thus, on the $q_1p_1$ plane the slope of the *PE* line is less than that of the *EE* line by the ratio effect, as shown by the arc *RE*.

A demand curve of type c can be seen in a similar way. The only difference is that the slope of the *PE* line is greater than that of the *EE* line by the ratio effect as also shown by the arc *RE*. Finally the ratio effect vanishes and the *PE* line coincides with the *EE* line in a demand curve of type b. A demand curve which is always of type b is none other than a rectangular hyperbola.

As has been seen from Figure 3, a very convenient feature of this new decomposition is that it can be expressed *directly* on the $q_1p_1$ plane, using a demand curve in question.[16] This is not possible for the Slutsky decomposition which can be illustrated only on the $q_1q_2$ plane. Furthermore, the decompositions in Figure 3 basically obtain in the n-good case, too. The elasticity effect is always found with the help of a straight line connecting the origin and a relevant point like $(q_1^*, p_1')$. If the ratio effect is zero, the elasticity effect thus found is also the price effect even in the n-good case! The slope of a demand curve differs from that of a

[15]The relationship among the three effects are the same as that in Figure 1.

[16]Take as a familiar example a linear demand function $p_1 = -\alpha q_1 + \beta, \alpha, \beta > 0$ which is often used in monopoly theory. The price effect for good 1 is always $-1/\alpha$ in this case. But the demand curve belongs to all three types, i.e., it is of type a (b, c) for $p_1$ greater than (equal to, less than) $\beta/2$. It can be understood at once by drawing the graph on the $q_1p_1$ plane.

rectangular hyperbola according as the ratio effect is negative or positive. That is, the slope becomes flatter (less flat) for a negative (positive) ratio effect. And it is the ratio effect that such a difference means even in the n-good case!

## 4 Cross-Price Effect

It is interesting to examine the relation between the new decomposition and the Slutsky decomposition of the *cross*-price effect. Such a version of the Slutsky equation says the cross-price effect with respect to good 1 is expressed as the sum of the *cross*-substitution effect and the income effect as follows:

$$\underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1,y=\text{const}}}_{\text{cross-price effect}} = \underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}}}_{\text{cross-substitution effect}} + \underbrace{\left(-q_2\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{income effect}}, \tag{11}$$

where

$$\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}} = \frac{u_1u_2^2}{p_2|U|} > 0 \text{ and } -q_2\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}} = -q_2\frac{u_2(u_2u_{12}-u_1u_{22})}{p_2|U|} < 0,$$

all partial derivatives and $q_2$ being evaluated at $(q_1^*, q_2^*)$. Like (2) this equation is also derived by solving Problem I. The cross-substitution is always positive, while the income effect is always negative. Thus the sign of the cross-price effect is indeterminate. But this is a "normal" situation.

Next consider Problem II again. This time differentiate $q_1 = \frac{\theta y}{p_1}$ partially with respect to $p_2$. Then a new equation for the cross-price effect is obtained as follows:

$$\underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1,y=\text{const}}}_{\text{cross-price effect}} = \underbrace{\frac{\frac{d\theta}{dp_2}y}{p_1}}_{\text{ratio effect}}, \tag{12}$$

where

$$\frac{\frac{d\theta}{dp_2}y}{p_1} = \frac{u_2}{p_2}\frac{u_1u_2}{|U|} - q_2\frac{u_2}{p_2}\frac{u_2u_{12}-u_1u_{22}}{|U|} \gtreqless 0,$$

all partial derivatives and $q_2$ being evaluated at $(q_1^*, q_2^*)$. It is found from (12) that *there is no elasticity effect in the cross-price case.* Thus, the cross-price effect is governed completely by the ratio effect. But the ratio effect consists of the positive cross-substitution effect and the negative income effect just as the Slutsky equation (11) shows.[17] Therefore, the sign of the ratio effect is indeterminate, too.

Let us apply equation (12) to the Cobb-Douglas-type utility function (3). Then,

$$\begin{aligned}\underbrace{0}_{\text{cross-price effect}} &= \underbrace{0}_{\text{ratio effect}} \\ &= \underbrace{\frac{ab}{(a+b)^2}\frac{y}{p_1 p_2}}_{\text{cross-substitution effect } \oplus} + \underbrace{\left[-\frac{ab}{(a+b)^2}\frac{y}{p_1 p_2}\right]}_{\text{income effect } \ominus} = 0. \end{aligned} \tag{13}$$

In this case the cross-substitution effect and the income effect cancel out exactly. As a result, the ratio effect and the cross-price effect are zero. The demand for good 1 is unaffected by variations in price $p_2$ of good 2.

As for the CES utility function (5), equation (12) is written as follows:

$$\begin{aligned}\underbrace{\frac{(\sigma-1)c}{(1+c)^2}\frac{y}{p_1 p_2}}_{\text{cross-price effect}} &= \underbrace{\frac{(\sigma-1)c}{(1+c)^2}\frac{y}{p_1 p_2}}_{\text{ratio effect}} \\ &= \underbrace{\frac{\sigma c}{(1+c)^2}\frac{y}{p_1 p_2}}_{\text{cross-substitution effect } \oplus} + \underbrace{\left[-\frac{c}{(1+c)^2}\frac{y}{p_1 p_2}\right]}_{\text{income effect } \ominus} \gtreqless 0. \end{aligned} \tag{14}$$

It follows that for $\sigma > 1$ ($< 1$) the expenditure on good 1 increases (decreases) as $p_2$ rises. It is the ratio effect. And, since $p_1$ and $y$ remain constant, the demand for good 1 increases (decreases). It is the cross-price effect.[18]

Figure 4. The Cross-Price Effect.

In Figure 4 is illustrated the cross-price effect for good 1 when $p_2$ rises from $p_2'$ to $p_2''$ while $p_1 = p_1'$. The ratio effect is represented by an arrow starting the initial optimal point $(q_1^*, q_2^*)$

[17] Appendix C shows a direct method of calculating the ratio effect in the cross-price case.

[18] The saving decision problem is a typical economic example of Problem II here. Imagine that $q_1$ and $q_2$ are current consumption and future consumption, respectively. Moreover, put $p_1 = 1$ and $p_2 = 1/(1+r)$ with $r$ as the real interest rate. Then, saving $s$ is defined as $y - q_1$, while $1 - \theta$ can be regarded as the saving rate. And the response of saving to variations in the interest rate can be written as

$$\frac{\partial s}{\partial r} = \frac{1}{(1+r)^2}\frac{d\theta}{dp_2}y.$$

That is, analysis of saving is tantamount to that of the ratio effect. It is a well-known (but frequently ignored) fact that the influence of the interest rate on saving is unimportant. See, e.g., Stiglitz and Walsh (2006, p. 537).

and running southwestward along the budget line. Since the demand for good 1 increases, the cross-price effect is positive as in the case of the CES utility function with $\sigma$ greater than unity. Of course, such an arrow points northwestward when the cross-price effect is negative, and it disappears when the cross-price effect vanishes.[19]

Figure 5. The Relation with the Slutsky Decomposition.

Finally it is convenient to summarize the results obtained so far. Figure 5 shows the relation between the new decomposition and the traditional Slutsky decomposition for the price effect and the cross-price effect. First, the price effect can be divided into the ratio effect and the elasticity effect. The latter effect is always negative, whereas the sign of the former effect is indeterminate because it consists of the two effects with opposite signs, the negative (own-)substitution effect and the positive transfer effect. The sum of the transfer effect and the elasticity effect leads to the negative income effect of the Slutsky decomposition. Second, there is no elasticity effect in the cross-price case. The cross-price effect is equal to the ratio effect which in this case is the sum of the two effects of the Slutsky decomposition, i.e., the positive cross-substitution effect and the negative income effect. The sign of the ratio effect is indeterminate in this case, too.

# 5 A Case Where a Consumer Holds Initial Endowments

In this section a case in which a consumer holds initial endowments as in a pure exchange model is examined. Now the income of a consumer is not given but varies with prices $p_1$ and $p_2$.

## 5.1 Price Effect

Problem I in Section 2 must be rewritten as

$$\text{Problem I}': \quad \max_{q_1,\, q_2} \quad u(q_1,\, q_2)$$
$$\text{s.t.} \quad p_1 q_1 + p_2 q_2 = y,$$
$$y = p_1 \bar{q}_1 + p_2 \bar{q}_2,$$

[19]In fact, we have already seen these three patterns of the cross-price effect in Figure 2 as the relationship between price $p_1$ of good 1 and the demand for good 2.

where $\bar{q}_1$ and $\bar{q}_2$ are respectively non-negative initial endowments of goods 1 and 2. Solving it yields the Slutsky equation with initial endowments:

$$\underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2=\text{const}}}_{\text{price effect}} = \underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}}}_{\text{substitution effect}} + \underbrace{\left[(\bar{q}_1 - q_1)\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right]}_{\text{income effect}}, \tag{15}$$

where

$$\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}} = -\frac{u_1}{p_1}\frac{u_2^2}{|U|} < 0 \text{ and } (\bar{q}_1 - q_1)\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}} = (\bar{q}_1 - q_1)\frac{u_1}{p_1}\frac{u_2u_{12} - u_1u_{22}}{|U|} \gtreqless 0,$$

all partial derivatives and $q_1$ being evaluated at the optimal point $(q_1^*, q_2^*)$. Note that the assumption that good 1 is a normal good does not warrant the negativity of the price effect any more. It may be zero or positive for $q_1^* < \bar{q}_1$.

Next rewrite Problem II so as to deal with initial endowments $(\bar{q}_1, \bar{q}_2)$ as

$$\begin{aligned} \text{Problem II}'' : \quad & \max_{\theta} \quad u(q_1,\ q_2) \\ & \text{s.t.} \quad q_1 = \frac{\theta y}{p_1},\ q_2 = \frac{(1-\theta)y}{p_2},\ 0 \le \theta \le 1, \\ & \qquad\ \ y = p_1\bar{q}_1 + p_2\bar{q}_2. \end{aligned}$$

Differentiating $q_1 = \frac{\theta y}{p_1}$ partially with respect to $p_1$, taking account of $y = p_1\bar{q}_1 + p_2\bar{q}_2$, gives the following equation:

$$\underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2=\text{const}}}_{\text{price effect}} = \underbrace{\frac{\frac{d\theta}{dp_1}y}{p_1}}_{\text{ratio effect}} + \underbrace{\left(\left.\frac{dq_1}{dp_1}\right|_{\theta=\text{const}}\right)}_{\text{elasticity effect}}, \tag{16}$$

where

$$\frac{\frac{d\theta}{dp_1}y}{p_1} = -\frac{u_1}{p_1}\frac{u_2^2}{|U|} + \frac{q_1}{p_1}\frac{u_1u_2u_{12} - u_2^2u_{11}}{|U|} + \frac{\bar{q}_1q_1}{y}\frac{u_2^2u_{11} - u_1u_2u_{12}}{|U|} + \frac{\bar{q}_1q_2}{y}\frac{u_2^2u_{12} - u_1u_2u_{22}}{|U|} \gtreqless 0$$

and

$$\left.\frac{dq_1}{dp_1}\right|_{\theta=\text{const}} = -\frac{p_2\bar{q}_2}{y}\frac{q_1}{p_1} \le 0,$$

$q_1$, $q_2$, and all partial derivatives being evaluated at $(q_1^*, q_2^*)$.

According to (16), the price effect with initial endowments can also be divided into the "ratio" effect and the "elasticity" effect, though it is not the same as (7) with $y$ as given. The implication of (16) is still that, when price $p_1$ of good 1 goes up, a variation in the demand for it is governed by the elasticity effect if the ratio effect is small. It remains quite clear. The elasticity effect can be calculated from the relation $p_1 q_1 = \theta(p_1 \bar{q}_1 + p_2 \bar{q}_2)$ with $\theta$ as constant. Total differentiation of it gives the value of $-\frac{p_2 \bar{q}_2}{y}\frac{q_1}{p_1}$.[20] The price "elasticity" of demand in this case is $\frac{p_2 \bar{q}_2}{y}$ which lies between zero and one.

The price effect differs from the elasticity effect by the ratio effect. The ratio effect can be obtained by subtracting $-\frac{p_2 \bar{q}_2}{y}\frac{q_1}{p_1}$ from the right-hand side of (15) for the present.[21] The ratio effect consists of the four effects:

$$\begin{aligned}\underbrace{\frac{\frac{d\theta}{dp_1}y}{p_1}}_{\text{ratio effect}} &= \underbrace{\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}}}_{\text{substitution effect}} + \underbrace{\frac{p_2}{p_1}q_1\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{transfer effect}} \\ &\quad \underbrace{-\frac{p_2}{p_1}\left(\theta\frac{\partial y}{\partial p_1}\right)\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{first endowment effect}} + \underbrace{\left[(1-\theta)\frac{\partial y}{\partial p_1}\right]\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{second endowment effect}}, \qquad (17)\end{aligned}$$

where

$$\begin{aligned}\left.\frac{dq_1}{dp_1}\right|_{p_2,v=\text{const}} &= -\frac{u_1}{p_1}\frac{u_2^2}{|U|} < 0, \\ \frac{p_2}{p_1}q_1\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right) &= \frac{p_2}{p_1}q_1\frac{u_2}{p_2}\frac{u_1u_{12}-u_2u_{11}}{|U|} > 0, \\ -\frac{p_2}{p_1}\left(\theta\frac{\partial y}{\partial p_1}\right)\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right) &= -\frac{p_2}{p_1}\left(\frac{p_1q_1}{y}\bar{q}_1\right)\frac{u_2}{p_2}\frac{u_1u_{12}-u_2u_{11}}{|U|} \le 0, \text{ and} \\ \left[(1-\theta)\frac{\partial y}{\partial p_1}\right]\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right) &= \left(\frac{p_2q_2}{y}\bar{q}_1\right)\frac{u_1}{p_1}\frac{u_2u_{12}-u_1u_{22}}{|U|} \ge 0,\end{aligned}$$

$q_1$, $q_2$, and all partial derivatives being evaluated at $(q_1^*, q_2^*)$. The first two terms in the right-hand side of (17) are virtually the same as those in (8). The remaining two terms, which are called the "first endowment effect" and the "second endowment effect," are proper to a situation with initial endowments.

The way to interpret the latter two effects is similar to the case of the transfer effect in (8). The first endowment effect represented by the third term is interpreted as follows. A unit increase in $p_1$ actually increases the income of a consumer by $\bar{q}_1$ in nominal terms. The ratio $\theta$ of the increase in income leads to an increase in the demand for good 1 which is already

[20] Remember that $\theta = (p_1 q_1)/y$.

[21] It can directly be derived in the same way as in Appendix B except for the terms relating to $\bar{q}_1$.

reflected in the elasticity effect. On the other hand, the ratio $\theta$ of the increase in income also increases the demand for *good 2* by

$$\left(\theta\frac{\partial y}{\partial p_1}\right)\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right)\left[=\left(\frac{p_1q_1}{y}\bar{q}_1\right)\frac{u_2}{p_2}\frac{u_1u_{12}-u_2u_{11}}{|U|}\geq 0\right]$$

in real terms. In nominal terms it has the value multiplied by $p_2$. That amount is covered by part of the expenditure on *good 1.* That is, a rise in $p_1$ let the demand for good 1 "transfer" to that for good 2. The demand for good 1 decreases by what the first endowment effect means. The ratio $1-\theta$ of the increase in income coming from a unit increase in $p_1$ leads to an increase in the demand for good 1 by the second endowment effect.

The elasticity effect is non-positive, but the sign of the ratio effect is indeterminate. Thus, the price effect is ambiguous. Then, the applications of (17) to examples (3) and (5) will help. Equation (17) for the Cobb-Douglas-type utility function (3) becomes

$$\begin{aligned}
\underbrace{-\frac{a}{a+b}\frac{p_2\bar{q}_2}{p_1^2}}_{\text{price effect}} &= \underbrace{0}_{\text{ratio effect}} + \underbrace{\left(-\frac{a}{a+b}\frac{p_2\bar{q}_2}{p_1^2}\right)}_{\text{elasticity effect}} \\
&= \underbrace{-\frac{ab}{(a+b)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{ab}{(a+b)^2}\frac{y}{p_1^2}}_{\text{transfer effect }\oplus} \underbrace{-\frac{ab}{(a+b)^2}\frac{\bar{q}_1}{p_1}}_{\text{first endowment effect}} \\
&\quad + \underbrace{\frac{ab}{(a+b)^2}\frac{\bar{q}_1}{p_1}}_{\text{second endowment effect}} \underbrace{-\frac{a}{a+b}\frac{p_2\bar{q}_2}{p_1^2}}_{\text{elasticity effect}} \leq 0, \\
&\left[= \underbrace{-\frac{ab}{(a+b)^2}\frac{p_1\bar{q}_1+p_2\bar{q}_2}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{a}{(a+b)^2}\frac{bp_1\bar{q}_1-ap_2\bar{q}_2}{p_1^2}}_{\text{income effect}}\right]
\end{aligned}$$

where $y=p_1\bar{q}_1+p_2\bar{q}_2$ and the last line in big brackets is due to the Slutsky equation (15). The substitution effect and the transfer effect just cancel out as in (9). And, surprisingly, so do the first and the second endowment effects. Hence zero ratio effect. The price effect is governed entirely by the non-positive elasticity effect. The law of demand remains valid in the Cobb-Douglas-type utility function with initial endowments.

The CES utility function (5) is divided by (17) as follows:

$$\underbrace{-\frac{(\sigma-1)cp_1\bar{q}_1+(\sigma c+1)p_2\bar{q}_2}{(1+c)^2}\frac{1}{p_1^2}}_{\text{price effect}} = \underbrace{-\frac{(\sigma-1)c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{ratio effect}} + \underbrace{\left(-\frac{1}{1+c}\frac{p_2\bar{q}_2}{p_1^2}\right)}_{\text{elasticity effect}}$$

$$
\begin{aligned}
&= \underbrace{-\frac{\sigma c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{c}{(1+c)^2}\frac{y}{p_1^2}}_{\text{transfer effect }\oplus} \underbrace{-\frac{c}{(1+c)^2}\frac{\bar{q}_1}{p_1}}_{\text{first endowment effect}} \\
&+ \underbrace{\frac{c}{(1+c)^2}\frac{\bar{q}_1}{p_1}}_{\text{second endowment effect}} \underbrace{-\frac{1}{1+c}\frac{p_2\bar{q}_2}{p_1^2}}_{\text{elasticity effect}} \gtreqless 0, \\
&\left[= \underbrace{-\frac{\sigma c}{(1+c)^2}\frac{p_1\bar{q}_1+p_2\bar{q}_2}{p_1^2}}_{\text{substitution effect }\ominus} + \underbrace{\frac{1}{(1+c)^2}\frac{cp_1\bar{q}_1-p_2\bar{q}_2}{p_1^2}}_{\text{income effect}}\right]
\end{aligned}
$$

where $y = p_1\bar{q}_1 + p_2\bar{q}_2$ and the last line in big brackets is the Slutsky decomposition (15). Surprisingly again, the first and the second endowment effects cancel out exactly. Thus the ratio effect takes the same form as that in (10). The sign of the price effect is indeterminate, but if $\sigma > 1$ or if $\bar{q}_2 > 0$ and $\sigma$ is close enough to unity, the price effect is negative and the law of demand holds in the CES utility function with initial endowments, too.[22]

## 5.2 Cross-Price Effect

Solving Problem I′ also gives the Slutsky equation for the cross-price effect with intial endowments $(\bar{q}_1, \bar{q}_2)$:

$$
\underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1=\text{const}}}_{\text{cross-price effect}} = \underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}}}_{\text{cross-substitution effect}} + \underbrace{\left[(\bar{q}_2-q_2)\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right]}_{\text{income effect}}, \tag{18}
$$

where

$$
\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}} = \frac{u_1u_2^2}{p_2|U|} > 0 \quad\text{and}\quad (\bar{q}_2-q_2)\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}} = (\bar{q}_2-q_2)\frac{u_2}{p_2}\frac{u_2u_{12}-u_1u_{22}}{|U|} \gtreqless 0,
$$

[22] The labor supply decision problem of a worker can be solved by the application of (16). Put $p_2 = 1$ and $\bar{q}_2 = 0$, and think of $q_1$, $\bar{q}_1$, $q_2$, and $p_1$ respectively as leisure, total available time, consumption, and the (real) wage rate. Then, labor supply $n$ is the difference between $\bar{q}_1$ and $q_1$, i.e., $n = (1-\theta)\bar{q}_1$. $1-\theta$ can be regarded as the ratio of total available time used for labor. The response of the labor supply to variations in the wage rate can be written as

$$
\frac{\partial n}{\partial p_1} = -\frac{d\theta}{dp_1}\bar{q}_1.
$$

Since there is no elasticity effect in this case, labor supply is governed completely by the ratio effect. But I have not heard that it is affected by the wage rate significantly.

all partial derivatives and $q_2$ being evaluated at the optimal point $(q_1^*, q_2^*)$.

Instead a new equation for the cross-price effect with initial endowments can be derived from Problem II$'$. Differentiating $q_1 = \frac{\theta y}{p_1}$ partially with respect to $p_2$, taking account of $y = p_1\bar{q}_1 + p_2\bar{q}_2$, yields the following equation:

$$\underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1=\text{const}}}_{\text{cross-price effect}} = \underbrace{\frac{\frac{d\theta}{dp_2}y}{p_1}}_{\text{ratio effect}} + \underbrace{\left(\left.\frac{dq_1}{dp_2}\right|_{\theta=\text{const}}\right)}_{\text{elasticity effect}}, \tag{19}$$

where

$$\frac{\frac{d\theta}{dp_2}y}{p_1} = \frac{u_2}{p_2}\frac{u_1u_2}{|U|} - q_2\frac{u_2}{p_2}\frac{u_2u_{12} - u_1u_{22}}{|U|} + \frac{\bar{q}_2q_1}{y}\frac{u_2^2u_{11} - u_1u_2u_{12}}{|U|} + \frac{\bar{q}_2q_2}{y}\frac{u_2^2u_{12} - u_1u_2u_{22}}{|U|} \gtreqless 0$$

and

$$\left.\frac{dq_1}{dp_2}\right|_{\theta=\text{const}} = \frac{p_2\bar{q}_2}{y}\frac{q_1}{p_2} \geq 0,$$

$q_1$, $q_2$, and all partial derivatives being evaluated at $(q_1^*, q_2^*)$.

It should be noticed that there *is* the elasticity effect in (19) unlike (12) in which $y$ is constant. Even if the ratio effect is zero, the demand for good 1 can respond positively to a change in price $p_2$ of good 2. The elasticity effect can be calculated from the relation $p_1q_1 = \theta(p_1\bar{q}_1 + p_2\bar{q}_2)$. Differentiating it totally and putting $dp_1 = d\theta = 0$ gives the value of $\frac{p_2\bar{q}_2}{y}\frac{q_1}{p_2}$. The "cross-elasticity" of demand in this case is $\frac{p_2\bar{q}_2}{y}$ which is between zero and one.

The cross-price effect differs from the elasticity effect by the ratio effect. The ratio effect can be obtained by subtracting $\frac{p_2\bar{q}_2}{y}\frac{q_1}{p_1}$ from the right-hand side of (18).[23] The ratio effect is the sum of of the four effects:

$$\underbrace{\frac{\frac{d\theta}{dp_2}y}{p_1}}_{\text{ratio effect}} = \underbrace{\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}}}_{\text{cross-substitution effect}} \underbrace{-q_2\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{income effect}}$$
$$\underbrace{-\frac{p_2}{p_1}\left(\theta\frac{\partial y}{\partial p_2}\right)\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{first endowment effect}} + \underbrace{\left[(1-\theta)\frac{\partial y}{\partial p_2}\right]\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right)}_{\text{second endowment effect}}, \tag{20}$$

where

$$\left.\frac{dq_1}{dp_2}\right|_{p_1,v=\text{const}} = \frac{u_2}{p_2}\frac{u_1u_2}{|U|} > 0,$$

[23] Or it can directly be calculated in the same way as in Appendix C except for the terms relating to $\bar{q}_2$.

$$
\begin{aligned}
-q_2\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right) &= -q_2\frac{u_2}{p_2}\frac{u_2u_{12}-u_1u_{22}}{|U|} < 0, \\
-\frac{p_2}{p_1}\left(\theta\frac{\partial y}{\partial p_2}\right)\left(\left.\frac{dq_2}{dy}\right|_{p_1,p_2=\text{const}}\right) &= -\frac{p_2}{p_1}\left(\frac{p_1q_1}{y}\bar{q}_2\right)\frac{u_2}{p_2}\frac{u_1u_{12}-u_2u_{11}}{|U|} \leq 0, \text{ and} \\
\left[(1-\theta)\frac{\partial y}{\partial p_2}\right]\left(\left.\frac{dq_1}{dy}\right|_{p_1,p_2=\text{const}}\right) &= \left(\frac{p_2q_2}{y}\bar{q}_2\right)\frac{u_2}{p_2}\frac{u_2u_{12}-u_1u_{22}}{|U|} \geq 0,
\end{aligned}
$$

$q_1$, $q_2$, and all partial derivatives being evaluated at $(q_1^*, q_2^*)$. The substitution effect and the income effect in the right-hand side of (20) are virtually the same as those in (12). The first and second endowment effects can be interpreted similarly to those in (17).

To show the usefulness of (20), let us apply it to examples (3) and (5). Equation (20) for the Cobb-Douglas-type utility function (3) becomes

$$
\begin{aligned}
\underbrace{\frac{a}{a+b}\frac{\bar{q}_2}{p_1}}_{\text{cross-price effect}} &= \underbrace{0}_{\text{ratio effect}} + \underbrace{\frac{a}{a+b}\frac{\bar{q}_2}{p_1}}_{\text{elasticity effect}} \\
&= \underbrace{\frac{ab}{(a+b)^2}\frac{y}{p_1p_2}}_{\text{cross-substitution effect } \oplus} \underbrace{-\frac{ab}{(a+b)^2}\frac{y}{p_1p_2}}_{\text{income effect } \ominus} \underbrace{-\frac{ab}{(a+b)^2}\frac{\bar{q}_2}{p_1}}_{\text{first endowment effect}} \\
&\quad + \underbrace{\frac{ab}{(a+b)^2}\frac{\bar{q}_2}{p_1}}_{\text{second endowment effect}} + \underbrace{\frac{a}{a+b}\frac{\bar{q}_2}{p_1}}_{\text{elasticity effect}} \geq 0, \\
&\quad \left[= \underbrace{\frac{ab}{(a+b)^2}\frac{p_1\bar{q}_1+p_2\bar{q}_2}{p_1p_2}}_{\text{cross-substitution effect } \oplus} + \underbrace{\frac{a}{(a+b)^2}\frac{-bp_1\bar{q}_1+ap_2\bar{q}_2}{p_1p_2}}_{\text{income effect}}\right]
\end{aligned}
$$

where $y = p_1\bar{q}_1 + p_2\bar{q}_2$ and the last line in big brackets is due to the Slutsky equation (18). The cross-substitution effect and the income effect just cancel out as in (13). And so do the first and the second endowment effects. Thus the ratio effect vanishes and the cross-price effect is governed by the elasticity effect which is greater than or equal to zero.

As for the CES utility function (5), (19) is written as follows:

$$
\begin{aligned}
\underbrace{\frac{(\sigma-1)cp_1\bar{q}_1+(\sigma c+1)p_2\bar{q}_2}{(1+c)^2}\frac{1}{p_1p_2}}_{\text{cross-price effect}} &= \underbrace{\frac{(\sigma-1)c}{(1+c)^2}\frac{y}{p_1p_2}}_{\text{ratio effect}} + \underbrace{\frac{1}{1+c}\frac{\bar{q}_2}{p_1}}_{\text{elasticity effect}} \\
&= \underbrace{\frac{\sigma c}{(1+c)^2}\frac{y}{p_1p_2}}_{\text{cross-substitution effect } \oplus} \underbrace{-\frac{c}{(1+c)^2}\frac{y}{p_1p_2}}_{\text{income effect } \ominus} \underbrace{-\frac{c}{(1+c)^2}\frac{\bar{q}_2}{p_1}}_{\text{first endowment effect}}
\end{aligned}
$$

$$
+ \underbrace{\frac{c}{(1+c)^2}\frac{\bar{q}_2}{p_1}}_{\text{second endowment effect}} + \underbrace{\frac{1}{1+c}\frac{\bar{q}_2}{p_1}}_{\text{elasticity effect}} \gtreqless 0,
$$

$$
\left[ = \underbrace{\frac{\sigma c}{(1+c)^2}\frac{p_1\bar{q}_1 + p_2\bar{q}_2}{p_1 p_2}}_{\text{cross-substitution effect } \oplus} + \underbrace{\frac{1}{(1+c)^2}\frac{-cp_1\bar{q}_1 + p_2\bar{q}_2}{p_1 p_2}}_{\text{income effect}} \right]
$$

where $y = p_1\bar{q}_1 + p_2\bar{q}_2$ and the last line in brackets is the Slutsky decomposition for the cross-price effect (18). Since the first and the second endowment effects cancel out exactly, the ratio effect takes the same form as that in (14). The sign of the cross-price effect is indeterminate, but if $\sigma > 1$ or if $\bar{q}_2 > 0$ and $\sigma$ is close enough to unity, the price effect is positive and goods 1 and 2 are gross substitutes.

## 6 Conclusion

This paper tried to make a contribution to demand theory by proposing a new equation which decomposes the (own-)price effect into the ratio effect and the elasticity effect. The new equation becomes a powerful tool when the ratio effect is negligible and the elasticity effect dominates. In fact such situations are often encountered in economics. Thus, the new equation must play an important role along with the Slutsky equation.

It was also shown that there is no elasticity effect in the cross-effect case. If the demand for a good is affected by the price of other goods, the causal mechanism is based entirely on the ratio effect. Of course, we can not make a decisive statement on the magnitude of such a ratio effect. There can be cases in which it is large or small. But it is interesting to know that Pigou, the authentic successor to Marshall, attached importance to a case in which the ratio effect is negligible, though he did not use the term:

> "Broadly we may conclude that, as compared with the effect of variations in its own price on the quantity of any ordinary thing that is purchased, the effect of variations in the price of any one other ordinary thing is likely to be negligible. Mashall's picture of demand schedules relating the quantity of a thing purchased to its own price only is an adequate, though not, of course, a perfectly accurate, tool of analysis so long as the thing is an independent commodity." Pigou (1955,

p. 27)

It should be noticed here that a negligible ratio effect in the cross-price case may also mean a negligible ratio effect in the own-price case mentioned in the first paragraph of this section because an increase in the expenditure on one good necessarily causes a decrease in that on another good when income is considered to be given. Such being the case, the new decomposition proposed in this paper makes sense both in the cross-price case and in the own-price case. It is also applicable to a case where a consumer holds initial endowments. In such a case, too, the decomposition into the ratio effect and the elasticity effect proved to be useful.

Finally I like to add that, though this paper dealt exclusively with the "normal" case, the new equation sheds new light on a Giffen good, too. According to it, a Giffen good is something that has an *extraordinarily large positive* ratio effect. That is, when its price rises, the ratio of income that goes to the purchase of the good increases so rapidly that a negative elasticity effect is overturned. In a two-good case analyzed in this paper it is possible only by "crowding out" a normal (or superior) good considerably. An *inferior* good is chosen at the sacrifice of a *superior* one despite the *adverse* situation! A Giffen good does contradict intuition.[24]

## Appendix

[24] As far as I know, theorists have not given an example from which a demand function of a Giffen good can easily be derived. Why is it so difficult to do so? No doubt because they must follow a utility function (1) which is a function of the quantities of goods 1 and 2 only. If the rule is loosened, however, a desired example is easy to make. Think of a utility function $u = \log q_1 + (y/p_1)^2 q_2$. Then, good 1 turns out to be a Giffen good because its demand function is written as $q_1 = p_1 p_2 / y$. Note also that it is an inferior good as well. This utility function and the resulting demand function correspond well to economists' explanation for why Giffen goods exist as Whitehead (2010, p. 49) says, "This classification of commodities as inferior or Giffen typically fits the basic staples in foodstuffs consumed by the poorer sections of the society. As real income rises and consumers become more affluent, they tend to shift away from the more traditional food staples and towards commodities that were previously priced out of their reach." $y/p_1$ in my example can be regarded as "real income" the increase of which causes consumer's shift from an inferior good to a superior one. A typical consumer Jensen and Miller (2008) investigated theoretically has also a utility function which *apparently* depends on her income level. See Figure 1 therein.

## A Conditions for Utility Maximization in Problem II

Substituting $q_1 = \theta y/p_1$ and $q_2 = (1-\theta)y/p_2$ into utility function (1) yields

$$v = u\left(\frac{\theta y}{p_1}, \frac{(1-\theta)y}{p_2}\right). \tag{21}$$

A consumer is in a position to maximize $v$ by adjusting the ratio $\theta$ only. Differentiate (21) with respect to $\theta$ once and twice. Then,

$$\begin{aligned} \frac{dv}{d\theta} &= u_1\frac{y}{p_1} - u_2\frac{y}{p_2}, \\ \frac{d^2v}{d\theta^2} &= u_{11}\left(\frac{y}{p_1}\right)^2 - 2u_{12}\frac{y}{p_1}\frac{y}{p_2} + u_{22}\left(\frac{y}{p_2}\right)^2. \end{aligned}$$

Putting $dv/d\theta = 0$ gives the first-order condition

$$\frac{u_1}{u_2} = \frac{p_1}{p_2}. \tag{22}$$

(22) assures that $d^2v/d\theta^2$ takes a negative value because

$$\begin{aligned} \frac{d^2v}{d\theta^2} &= \left(\frac{y}{p_1}\right)^2\left[u_{11} - 2u_{12}\frac{p_1}{p_2} + u_{22}\left(\frac{p_1}{p_2}\right)^2\right] \\ &= \left(\frac{y}{p_1}\right)^2\left[u_{11} - 2u_{12}\frac{u_1}{u_2} + u_{22}\left(\frac{u_1}{u_2}\right)^2\right] \\ &= \left(\frac{y}{p_1}\right)^2\frac{u_2^2u_{11} - 2u_1u_2u_{12} + u_1^2u_{22}}{u_2^2} \\ &= -\left(\frac{y}{p_1}\right)^2\frac{|U|}{u_2^2} < 0. \end{aligned}$$

Therefore, the second-order condition is also satisfied.[25]

## B Direct Method of Calculating the Ratio Effect (8)

For the calculation of the ratio effect (8) it is necessary to notice that the first-order condition (22) always works. Thus the ratio $\theta$ can be written as

$$\theta = \frac{p_1q_1}{y}$$

[25]The extension of this two-good case to an $n-$good case is easy. When there are $n$ goods, (21) becomes

$$v = u\left(\frac{\theta_1 y}{p_1}, \frac{\theta_2 y}{p_2}, \ldots, \frac{[1-(\theta_1+\theta_2+\ldots+\theta_{n-1})]y}{p_n}\right),$$

where $\theta_i$ represents the ratio of income $y$ that goes to the purchase of good $i$, $i = 1, 2, \ldots, n-1$. The maximization conditions in this case are $dv/d\theta_i = 0$ and $d^2v/d\theta_i^2 < 0$, $i = 1, 2, \ldots, n-1$.

$$= \frac{\frac{p_1}{p_2}q_1}{\frac{p_1}{p_2}q_1 + q_2}$$

$$= \frac{\frac{u_1}{u_2}q_1}{\frac{u_1}{u_2}q_1 + q_2}$$

$$= \frac{u_1 q_1}{u_1 q_1 + u_2 q_2}. \tag{23}$$

(23) also means

$$u_1 q_1 + u_2 q_2 = u_1 \frac{y}{p_1}. \tag{24}$$

Differentiate (23) w.r.t. $p_1$. Then,

$$\frac{d\theta}{dp_1} = \frac{\frac{d(u_1q_1)}{dp_1}u_2q_2 - \frac{d(u_2q_2)}{dp_1}u_1q_1}{(u_1q_1+u_2q_2)^2}.$$

Multiplying both sides of the above equation gives

$$(u_1q_1+u_2q_2)^2\frac{d\theta}{dp_1} = \frac{d(u_1q_1)}{dp_1}u_2q_2 - \frac{d(u_2q_2)}{dp_1}u_1q_1.$$

Using (24) it can be rewritten as

$$\left(u_1\frac{y}{p_1}\right)^2\frac{d\theta}{dp_1} = \frac{d(u_1q_1)}{dp_1}u_2q_2 - \frac{d(u_2q_2)}{dp_1}u_1q_1. \tag{25}$$

By the way, it can easily be checked that the following calculations are correct:

$$\frac{d(u_1q_1)}{dp_1} = u_{11}\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right)q_1 - u_{12}\frac{\frac{d\theta}{dp_1}y}{p_2}q_1 + u_1\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right), \tag{26}$$

$$\frac{d(u_2q_2)}{dp_1} = u_{12}\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right)q_2 - u_{22}\frac{\frac{d\theta}{dp_1}y}{p_2}q_2 - u_2\frac{\frac{d\theta}{dp_1}y}{p_2}. \tag{27}$$

So substitute (26) and (27) into (25). Then,

$$\begin{aligned}\left(u_1\frac{y}{p_1}\right)^2\frac{d\theta}{dp_1} &= \left[u_{11}\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right)q_1 - u_{12}\frac{p_1}{p_2}\frac{\frac{d\theta}{dp_1}y}{p_1}q_1 + u_1\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right)\right]u_2q_2 \\ &\quad - \left[u_{12}\left(\frac{\frac{d\theta}{dp_1}y}{p_1} - \frac{q_1}{p_1}\right)q_2 - u_{22}\frac{p_1}{p_2}\frac{\frac{d\theta}{dp_1}y}{p_1}q_2 - u_2\frac{p_1}{p_2}\frac{\frac{d\theta}{dp_1}y}{p_1}\right]u_1q_1.\end{aligned}$$

In order to reach a desired result a few more steps are needed. That is,

$$\begin{aligned}&\left(u_1^2\frac{y}{p_1} - u_2u_{11}q_1q_2 + u_1u_{12}q_1q_2 - u_1u_2q_2 + u_1u_{12}q_1q_2 - \frac{u_1^2}{u_2}u_{22}q_1q_2 - u_1^2q_1\right)\frac{\frac{d\theta}{dp_1}y}{p_1} \\ &\qquad = \left[-\frac{1}{p_1}u_1u_2 + \frac{q_1}{p_1}(u_1u_{12} - u_2u_{11})\right]q_1q_2,\end{aligned}$$

$$\left(u_1^2\frac{u_1q_1+u_2q_2}{u_1}-u_2u_{11}q_1q_2+2u_1u_{12}q_1q_2-u_1u_2q_2-\frac{u_1^2}{u_2}u_{22}q_1q_2-u_1^2q_1\right)\frac{\frac{d\theta}{dp_1}y}{p_1}$$
$$=\left[-\frac{1}{p_1}u_1u_2+\frac{q_1}{p_1}(u_1u_{12}-u_2u_{11})\right]q_1q_2,$$
$$\left(-u_2u_{11}q_1q_2+2u_1u_{12}q_1q_2-\frac{u_1^2}{u_2}u_{22}q_1q_2\right)\frac{\frac{d\theta}{dp_1}y}{p_1}=\left[-\frac{1}{p_1}u_1u_2+\frac{q_1}{p_1}(u_1u_{12}-u_2u_{11})\right]q_1q_2,$$

and finally

$$(-u_2^2u_{11}+2u_1u_2u_{12}-u_1^2u_{22})\frac{\frac{d\theta}{dp_1}y}{p_1}=-\frac{u_1}{p_1}u_2^2+\frac{q_1}{p_1}(u_1u_2u_{12}-u_2^2u_{11}).$$

Dividing both sides of the last equation by the bordered Hessian $|U|$ gives the ratio effect (8)

$$\frac{\frac{d\theta}{dp_1}y}{p_1}=-\frac{u_1}{p_1}\frac{u_2^2}{|U|}+\frac{q_1}{p_1}\frac{u_1u_2u_{12}-u_2^2u_{11}}{|U|}.$$

# C Direct Method of Calculating the Ratio Effect (12) in the Cross-Price Effect

The derivation of (12) is similar to that of (8) shown in Appendix B. Differentiating (23) w.r.t. $p_2$ and considering (24) yields

$$\left(u_1\frac{y}{p_1}\right)^2\frac{d\theta}{dp_2}=\frac{d(u_1q_1)}{dp_2}u_2q_2-\frac{d(u_2q_2)}{dp_2}u_1q_1. \tag{28}$$

Notice here that the counterpart of equation (7) also obtains, that is,

$$\left.\frac{dq_2}{dp_2}\right|_{p_1,y=\text{const}}=-\frac{\frac{d\theta}{dp_2}y}{p_2}-\frac{q_2}{p_2}.$$

Keeping this in mind, one can obtain the following relations at once:

$$\frac{d(u_1q_1)}{dp_2}=u_{11}\frac{\frac{d\theta}{dp_2}y}{p_1}q_1+u_{12}\left(-\frac{\frac{d\theta}{dp_2}y}{p_2}-\frac{q_2}{p_2}\right)q_1+u_1\frac{\frac{d\theta}{dp_2}y}{p_1}, \tag{29}$$
$$\frac{d(u_2q_2)}{dp_2}=u_{12}\frac{\frac{d\theta}{dp_2}y}{p_1}q_2+u_{22}\left(-\frac{\frac{d\theta}{dp_2}y}{p_2}-\frac{q_2}{p_2}\right)q_2+u_2\left(-\frac{\frac{d\theta}{dp_2}y}{p_2}-\frac{q_2}{p_2}\right). \tag{30}$$

Substituting (29) and (30) into (28) and arranging the resulting terms yields

$$(-u_2^2u_{11}+2u_1u_2u_{12}-u_1^2u_{22})\frac{\frac{d\theta}{dp_2}y}{p_1}=\frac{1}{p_2}u_1u_2^2-q_2\frac{u_2}{p_2}(u_2u_{12}-u_1u_{22}).$$

Finally divide both sides of the above equation by the bordered Hessian $|U|$. Then the desired result, i.e., (12), follows:

$$\frac{\frac{d\theta}{dp_2}y}{p_1} = \frac{u_2}{p_2}\frac{u_1u_2}{|U|} - q_2\frac{u_2}{p_2}\frac{u_2u_{12} - u_1u_{22}}{|U|}.$$

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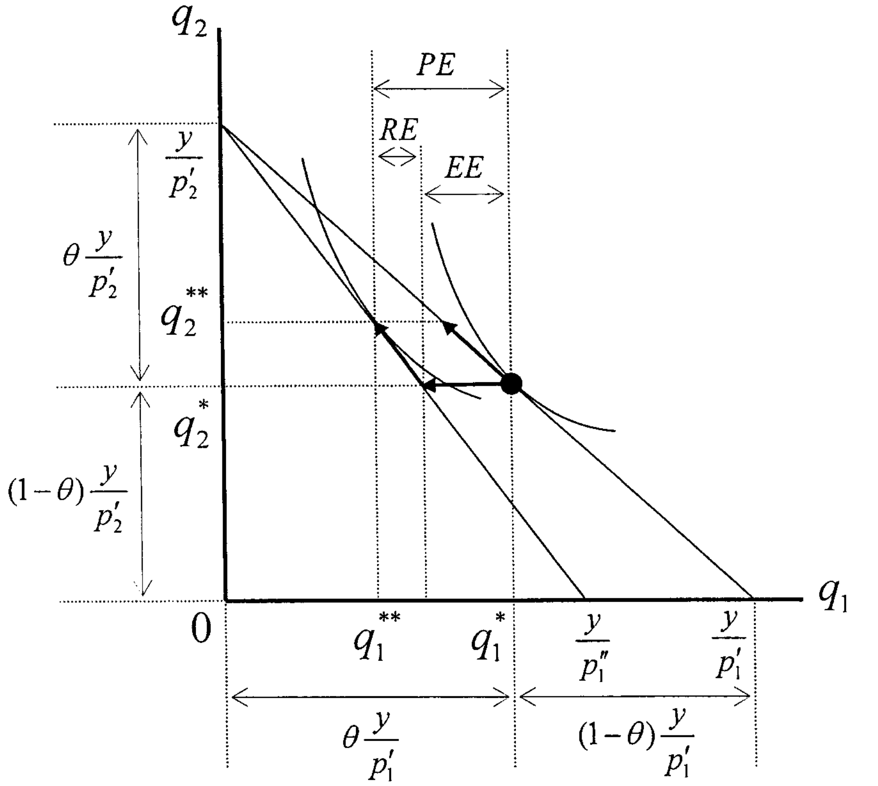
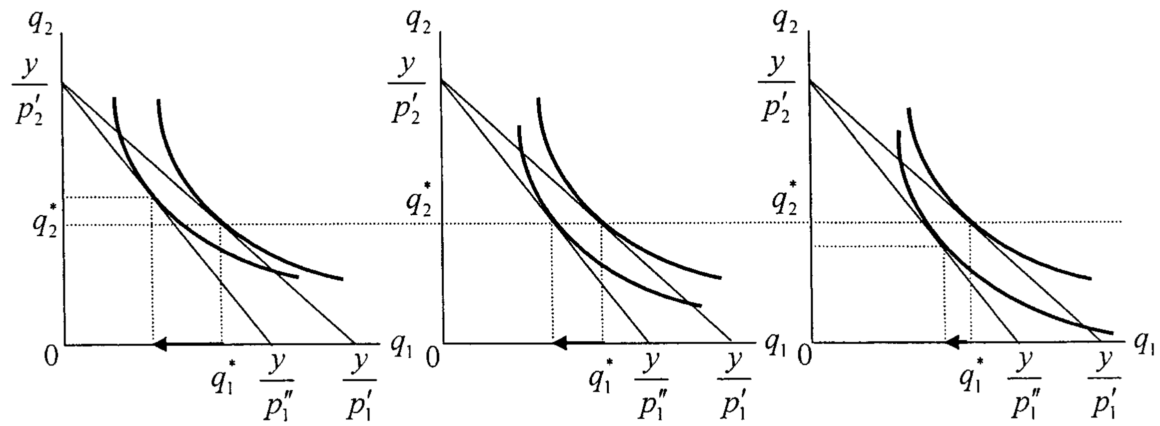
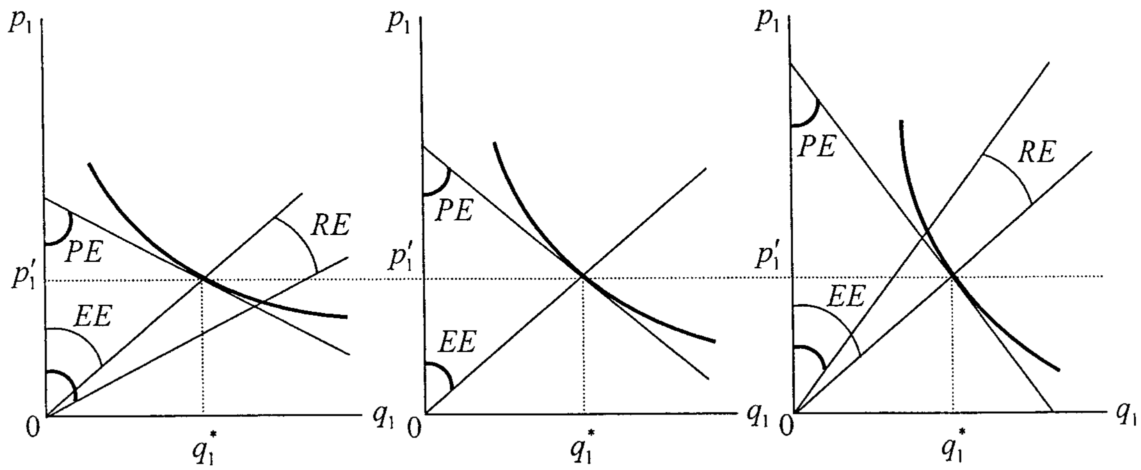



Figure 1. A Decomposition of the Price Effect: The Case of a Negative Ratio Effect.
Notes: *PE* < Price Effect, *RE* < Ratio Effect, *EE* < Elasticity Effect. *PE* = *RE* + *EE*.




(a) Negative Ratio Effect. (b) Zero Ratio Effect. (c) Positive Ratio Effect.

Figure 2. Three Types of Indifference Curves.




(a) Negative Ratio Effect. (b) Zero Ratio Effect. (c) Positive Ratio Effect.

Figure 3. Three Types of Demand Curves.

Notes: See Figure 1.

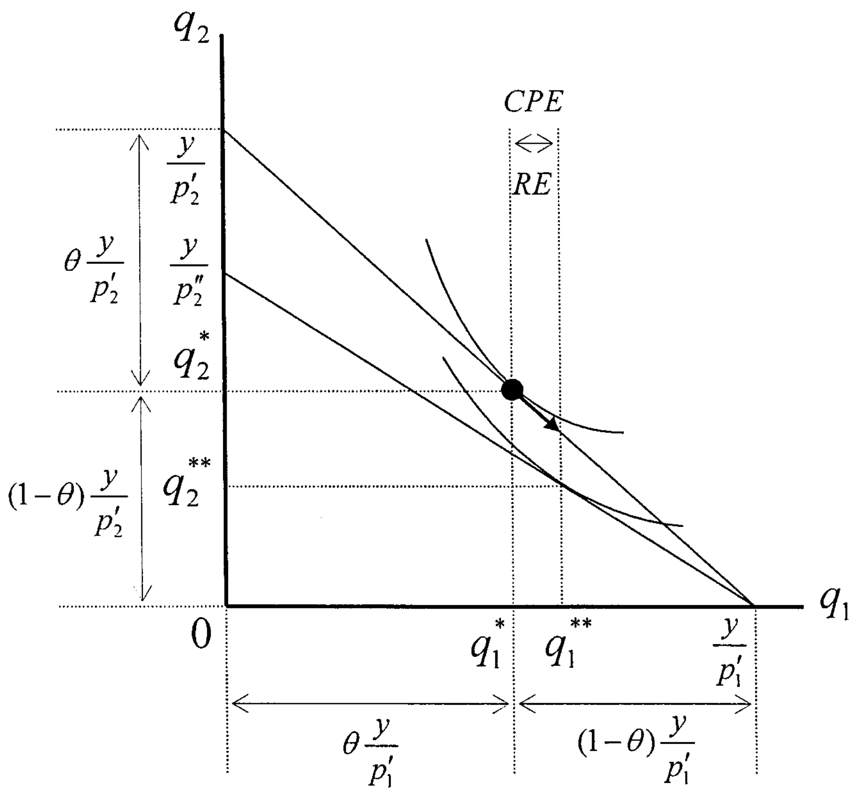



Figure 4. The Cross-Price Effect.

Notes: *CPE* < Cross-Price Effect, *RE* < Ratio Effect. *CPE* = *RE*.

$$\text{Price Effect}\left\{\begin{array}{l}\text{Ratio Effect}\left\{\begin{array}{l}\text{Substitution Effect}\\[1em] \text{Transfer Effect}\end{array}\right.\\[1em] \text{Elasticity Effect}\end{array}\right.\qquad\text{Income Effect}$$

$$\text{Cross-Price Effect} = \text{Ratio Effect}\left\{\begin{array}{l}\text{Cross-Substitution Effect}\\[1em] \text{Income Effect}\end{array}\right.$$

Figure 5. The Relation with the Slutsky Decomposition.