Axiomatizations of the core of assignment games

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Abstract

Sasaki (IJGT **24**, 1995) proposed axioms intended for characterizations of the core of assignment games. In the first part of this paper, we consider the set of zero-normalized assignment games having possibly different numbers of players on both sides. We show by a counter example that theorem 2 in Sasaki (1995) is not correct. We propose an amendment of the theorem replacing individual rationality by population monotonicity. In the second part, we also consider the set of all assignment games. On that domain, the core is the unique solution satisfying Pareto optimality, consistency, individual monotonicity and pairwise monotonicity.

1 Introduction.

Recently, Sasaki (1995) proposed axioms intended for characterizations of the core of assignment games in the zero-normalized class. In his paper, the core is assumed to determine the subset of matchings as well as the subset of pay-off vectors. This complicates the axioms, however, if we concentrate on the pay-off vectors, his theorem can be simplified as follows. The core is the unique solution on the set of zero-normalized assignment games satisfying individual rationality, Pareto optimality, consistency and pairwise monotonicity.

His arguments are quite ingenious but unfortunately the above statement is not correct. It is only valid on the special class of zero-normalized assignment games having the same number of agents on both sides of a market.

In this paper, we show by a counter example that theorem 2 in Sasaki (1995) is not true. We propose an amendment for the theorem by replacing individual rationality by population monotonicity[1].

Population monotonicity is one of the guiding principles in the recent axiomatic analysis. (See Thomson (1995) for an excellent survey of the literature.) It requires that in case of adding a new agent to a society, each existing agent suffers a loss if the resources remain unchanged. In an assignment game, there are two types of agents so that the effect of adding a new agent may be different depending on the agent's type. Then, the population monotonicity in the two-sided model requires that each existing agent of the same type as a new agent suffers a loss because in the perspective of agents on the opposite side, the arrival of a new agent increases their trading possibilities. This kind of population monotonicity was applied to the discrete model of Gale and Shapley (1962) by Toda (2001) but it is first applied to the continuous model in this paper.

From both practical and theoretical viewpoints, however, the set of zero-normalized games is restrictive. First, in real markets, trade gain and reservation value can be any real numbers, positive, negative or zero. Second, whether or not a characterization extends to the general case is not a simple question. Of course, it is not difficult to show that the core is invariant under the zero-normalization. Then, it may appear that by requiring the invariance under zero-normalization, a characterization in the general case follows from the zero-normalized case, but this may not be true. Besides that the assumption of zero-normalization invariance is rather technical and not so much economic meaningful, what is difficult is to show the independence of the axioms. Hence, the characterization of the core on the general domain of assignment games is a non-trivial problem which needs to be considered separately.

In the second part of this paper, on the domain of all assignment games, the core will be characterized by Pareto optimality, consistency, pairwise monotonicity and individual monotonicity. Individual monotonicity requires that if the reservation value of an agent increases, then the payoff of this agent should be increased. This axiom is similar to the monotonicity with respect to the disagreement point in the theory of bargaining. (See, e.g., Thomson (1987).) In the case of two agents, the both are equal.

Finally in this section, we mention briefly on a significance of our results. In general, axioms should not be explicit in the definition of the core. In assignment

[1]Toda (1993) also obtained a characterizations of the core of assignment games based on the Davis-Maschler type consistency axiom.

games, the core is defined by the set of stable pay-off vectors. A pay-off vector is stable if it has no blocking pair and no blocking individual. Hence, Pareto optimality is implicit but individual rationality is explicit in the definition of the core. It is remarkable that in our results individual rationality is not directly assumed.

The paper is constructed as follows. The next section gives notation and definitions employed in this paper. The third section proves our characterization theorems on the domain of zero-normalized games. The final section gives axiomatizations of the core without domain restrictions.

2 Definitions and axioms.

In this section, notation and definitions we employ will be given. We shall also introduce axioms which will be used in our characterization results and examples.

Let M and W be non-empty finite and disjoint sets of agents (or traders). We may consider an element $m \in M$ as a man or a buyer or whatever we imagine as a market participant and an element $w \in W$ as a woman or a seller or any component of the opposite side of a market. Each agent $a \in M \cup W$ has a reservation value denoted by $\pi(a)$. An agent is not willing to trade unless he or she is paid at least as much as $\pi(a)$. For each pair $(m, w) \in M \times W$, $\Pi(m, w)$ denotes the worth (or trade gain) that m and w can jointly achieve. Then, a list (M, W, Π, π) is called *an assignment game* and denoted for notational simplicity by γ. The set of all assignment games is denoted by Γ.

Definition 1. Given M and W, a *matching* is a bijection μ from $M \cup W$ into itself satisfying the following conditions.

(1) $\mu \circ \mu(a) = a$ for each $a \in M \cup W$.

(2) If $\mu(a) \neq a$, then $\mu(a) \in W$ for $a \in M$ and $\mu(a) \in M$ for $a \in W$.

For a give matching μ, let $c(\mu) = \{(m, w) \in M \times W \mid \mu(m) = w\}$ and $s(\mu) = \{a \in M \cup W \mid \mu(a) = a\}$, where $c(\mu)$ is the set of all pairs matched at μ and $s(\mu)$ is the set of all agents remaining single. Then, a matching μ will be denoted by such an expression

$$\mu = \{(m, w), \ldots, m', \ldots, w', \ldots\}$$

indicating all pairs in $c(\mu)$ and all agents in $s(\mu)$. A matching μ is *optimal* if it maximizes the sum

$$\sum_{(m,w)\in c(\mu)} \Pi(m, w) + \sum_{a\in s(\mu)} \pi(a)$$

of all trade gains over all matchings.

A vector $(u, v) \in \mathbb{R}^M \times \mathbb{R}^W$ is called *a pay-off vector*. A pay-off vector (u, v) is *feasible* if there exists a matching μ such that

(1) $u_m = \pi(m)$ and $v_w = \pi(w)$ for $m \in s(\mu)$ and for $w \in s(\mu)$, respectively.

(2) $u_m + v_w = \Pi(m, w)$ for each $(m, w) \in c(\mu)$.

In this case, (u, v) is called *compatible with* μ. A feasible pay-off vector (u, v) is *Pareto optimal* if it is compatible with an optimal matching. A pay-off vector (u, v) is *individually rational* if

$$u_m \geqq \pi(m) \text{ and } v_w \geqq \pi(w) \text{ for each } m \in M \text{ and for each } w \in W.$$

A feasible pay-off vector (u, v) is *stable* if it is individually rational and satisfies

$$u_m + v_w \geqq \Pi(m, w) \text{ for each } (m, w) \in M \times W.$$

The set of all stable pay-off vectors is called *the core* and denoted by $\mathcal{S}(\gamma)$.

Definition 2. *A solution* φ is a correspondence defined on a subset of Γ which associates a non-empty set of feasible pay-off vectors with each γ in its domain.

Now, we proceed to define axioms which will be used in the subsequent discussions.

Definition 3 (IR). A solution φ is *individually rational* if for each γ in its domain, each $(u, v) \in \varphi(\gamma)$ is individually rational.

Definition 4 (PO). A solution φ is *Pareto optimal* if for each γ in its domain, each $(u, v) \in \varphi(\gamma)$ is Pareto optimal.

Let $\gamma = (M, W, \Pi, \pi)$ be an assignment game and (u, v) a feasible pay-off vector which is compatible with a matching μ. Then, *an assignment game* $\gamma' = (M', W', \Pi', \pi')$ *is a subgame of* γ *at* (u, v) if the following conditions are satisfied.

(1) $M' \cup W' \subset M \cup W$ and $\mu(M' \cup W') = M' \cup W'$.

(2) For any $(m, w) \in M' \times W'$, $\Pi'(m, w) = \Pi(m, w)$.

(3) For any $a \in M' \cup W'$, $\pi'(a) = \pi(a)$.

Definition 5 (CONS). A solution φ is *consistent* if for any $(u, v) \in \varphi(\gamma)$ and for any subgame γ' of γ at (u, v),

$$(u_{M'}, v_{W'}) \in \varphi(\gamma')$$

where $(u_{M'}, v_{W'})$ is the projection of (u, v) to the subspace $\mathbb{R}^{M'} \times \mathbb{R}^{W'}$.

Let $\gamma, \gamma' \in \Gamma$ be such that,

(1) $\gamma = (M, W, \Pi, \pi)$, $\gamma = (M, W, \Pi', \pi)$.

(2) $\Pi'(m', w') \geqq \Pi(m', w')$ for some $(m', w') \in M \times W$ and $\Pi'(m, w) = \Pi(m, w)$ for any other $(m, w) \in M \times W$.

Definition 6 (PMON). A solution φ is *pairwise monotonic*[2] if for any $(u, v) \in \varphi(\gamma)$, there exists $(u', v') \in \varphi(\gamma')$ satisfying

$$u'_{m'} + v'_{w'} \geqq u_{m'} + v_{w'},$$

where (m', w') is the pair such that $\Pi'(m', w') \geqq \Pi(m', w')$.

[2] Sasaki (1995) distinguishes pairwise monotonicity and weak pairwise monotonicity. Our definition corresponds to his weaker notion, however, because in this paper, we do not need the stronger notion, we simply call it pairwise monotonicity.

Because all these axioms have appeared in Sasaki (1995), we do not need to repeat the economic interpretations of them.

In this and the next sections, we are concerned with the class of zero normalized assignment games.

The class Γ_0 of zero normalized assignment games is defined by the subset of assignment games $\gamma = (M, W, \Pi, \pi)$ satisfying the following conditions.

(i) $\Pi(m, w) \geqq 0$ for each $(m, w) \in M \times W$.

(ii) $\pi(a) = 0$ for each $a \in M \cup W$.

For the sake of simplicity, each $\gamma \in \Gamma_0$ can be written as (M, W, Π). By the arguments in Sasaki (1995), we can obtain the following result.

Theorem 2.1. *The core is the unique solution satisfying individual rationality, Pareto optimality, consistency and pairwise monotonicity if its domain is restricted to the subset of* Γ_0 *such that* $|M| = |W|$, *where* $|M|$ *and* $|W|$ *denote the cardinality of* M *and* W, *respectively and* $\Pi(m, w) > 0$ *for each* $(m, w) \in M \times W$.

If the domain of the core is extended to Γ_0, the assertion of theorem 2.1 is no longer valid. The following example shows this.

Example 2.1. Let φ^1 be the solution defined as follows [3].

(1) $\varphi(M, W, \Pi) = \mathcal{S}(M, W, \Pi)$ if $|M \cup W| \neq 3$.

(2) $\varphi(M, W, \Pi)$ is equal to the set of all Pareto optimal and individually rational pay-off vectors if $|M \cup W| = 3$.

Proposition 2.1. *On the domain of* Γ_0, *the solution* φ^1 *satisfies all axioms in Theorem 2.1.*

Proof. It suffices to prove that φ^1 satisfies consistency and pairwise monotonicity. To prove consistency, suppose that $(u, v) \in \varphi(\gamma)$ and γ' is a subgame of γ at (u, v). If the numbers of players in γ and in γ' are not equal to 3, then φ^1 satisfies the requirement of consistency because it equals to the core.

Otherwise, we need to distinguish two cases.

(Case 1). $|M \cup W| \neq 3$ in γ and $|M' \cup W'| = 3$ in γ'.

Since (u, v) is stable in γ, $(u_{M'}, v_{W'})$ is obviously individually rational and Pareto optimal in γ' and hence it is contained in $\varphi^1(\gamma')$.

(Case 2). $|M \cup W| = 3$ in γ and $|M' \cup W'| = 2$ in γ'.

$|M' \cup W'| = 2$ means that $M' = \{m\}$ and $W' = \{w\}$. Because $(m, w) \in c(\mu)$ for a matching μ compatible with (u, v), $u_m + v_w = \Pi(m, w)$ and $u_m \geqq 0, v_w \geqq 0$. Hence, it follows from the definition of φ^1 that $(u_m, v_w) \in \varphi^1(\gamma')$.

Next, we will see that φ^1 satisfies pairwise monotonicity. By the definition of φ^1, we only need to check this in case of $|M \cup W| = 3$. Let $\gamma = (M, W, \Pi)$ and $\gamma' = (M, W, \Pi')$ satisfy the conditions in the definition of pairwise monotonicity. Because $|M \cup W| = 3$, without loss of generality, we may assume that $M = \{m_1, m_2\}$, $W = \{w_1\}$, $\Pi'(m_1, w_1) = \Pi(m_1, w_1)$ and $\Pi'(m_2, w_1) \geqq \Pi(m_2, w_1)$.

[3] The construction is simplified in order to make verifications of the axioms easy but we can find a solution satisfying the axioms which is different from the core on a larger class of assignment games.

For an arbitrarily given $(u, v) \in \varphi^1(\gamma)$, let μ be a matching compatible with it. If we have

$$\mu = \mu' \equiv \{(m_2, w_1), m_1\},$$

then it is immediate to obtain the desired property. Otherwise, we distinguish two cases. At first, assume that the matching μ is still optimal in γ'. Then, the conclusion is also immediate. In the second case, the matching μ is no longer optimal in γ'. In this case, the matching μ' is the unique optimal matching in γ'. Therefore, we have,

$$\Pi'(m_2, w_1) > \Pi(m_1, w_1).$$

Because $\mu' \neq \mu$, if $(m_1, w_1) \in c(\mu)$, then

$$\Pi'(m_2, w_1) > \Pi(m_1, w_1) = u_{m_1} + v_{w_1} \geqq v_{w_1}$$

and $u_{m_2} = 0$. For the pay-off vector (u', v') defined by $u'_{m_1} = 0$, $u'_{m_2} = \Pi'(m_2, w_1) - v_{w_1}$ and $v'_{w_1} = v_{w_1}$, $(u', v') \in \varphi^1(\gamma')$ and $u'_{m_2} + v'_{w_1} \geqq u_{m_2} + v_{w_1}$.

If $(m_1, w_1) \notin c(\mu)$, then $(u, v) = (0, 0, 0)$ and the conclusion is obvious. This completes the proof.

□

3 Population monotonicity.

The example in the previous section constitutes a counter example of theorem 2 in Sasaki (1995). In this section, we propose an amendment of the theorem, in which the core is characterized on the domain Γ_0 replacing individual rationality by population monotonicity in the axioms of theorem 2.1. Population monotonicity requires that if a new agent is added to a group, then each of the existing agents suffers a loss when the resources remain unchanged. In our model, there are two types of agents so that it requires that each of the exiting agents having the same type of a new comer suffers a loss. For the mathematical expression of the idea, we need an additional definition.

An assignment game $\gamma' = (M', W', \Pi', \pi')$ *is an extension of* $\gamma = (M, W, \Pi, \pi)$ if $M \cup W \subset M' \cup W'$, $\Pi' \equiv \Pi$ on $M \times W$ and $\pi' \equiv \pi$ on $M \cup W$.

Definition 7 (POP). A solution φ is *population monotonic* if for any extension $\gamma' = (M \cup \{m\}, W, \Pi', \pi')$ of $\gamma = (M, W, \Pi, \pi)$ and for any $(u, v) \in \varphi(\gamma)$, there exists $(u', v') \in \varphi(\gamma')$ such that

$$u'_m \leqq u_m \text{ for each } m \in M$$

and the symmetric requirement is satisfied for any extension involving a new woman.

Proposition 3.1. *The core S satisfies population monotonicity.*

Proof. Obvious from proposition 8.17 in Roth and Sotomayer (1990). □

Proposition 3.2. *If a solution φ satisfies Pareto optimality and population monotonicity, then it satisfies individual rationality*[4].

[4]An analogous result has been obtained in the matching model of Gale and Shapley (1965) by Toda (2001).

Proof. At first, it is not difficult to show that population monotonicity implies the following : For any $\gamma = (M, W, \Pi)$ and its extension $\gamma' = (M', W, \Pi')$, if $(u, v) \in \varphi(\gamma)$, there exists $(u', v') \in \varphi(\gamma')$ such that $u'_m \leqq u_m$ for all $m \in M$.

By way of contradiction, suppose that for an assignment game $\gamma = (M, W, \Pi)$, $\varphi(\gamma)$ contains a pay-off vector (u, v) which is not individually rational. Without loss of generality, we may assume that $u_{m'} < 0$ for some $m' \in M$. Let us denote $W = \{w_1, w_2, \dots, w_n\}$. For each $j = 1, \dots, n$, we introduce a new man m_j^* and construct an extension $\gamma' = (M', W, \Pi')$ of γ satisfying the following conditions.

(1) $\Pi'(m_j^*, w_j) > \Pi(m, w_j)$ for each $m \in M$.

(2) $\Pi'(m_k^*, w_j) = 0$ for each $k \neq j$.

In the assignment game γ', the matching μ' such that $\mu'(m) = m$ for each $m \in M$ and $\mu'(w_j) = m_j^*$ for each $j = 1, \dots, n$ is the unique optimal matching. By Pareto optimality, for any $(u', v') \in \varphi(\gamma')$ and for any $m \in M$, $u'_m = 0$ and hence $u'_{m'} = 0 > u_{m'}$. This contradicts the population monotonicity of φ. □

The main result of this section is as follows.

Theorem 3.1. *The core is the unique solution on the domain Γ_0 satisfying Pareto optimality, consistency, pairwise monotonicity and population monotonicity.*

Proof. By proposition 3 in Sasaki (1995) and proposition 3.1, the core satisfies the axioms. Conversely, let φ be a solution satisfying the axioms. By proposition 2 in Sasaki (1995), $\varphi(\gamma) \subset \mathcal{S}(\gamma)$ for each $\gamma = (M, W, \Pi, \pi)$ such that $|M| = |W| = 2$. We show that the same inclusion holds for each γ with $|M| \leqq 2$ and $|W| \leqq 2$.

If $|M| \leqq 1$ and $|W| \leqq 1$, then by individual rationality and Pareto optimality, $\varphi(\gamma) = \mathcal{S}(\gamma)$.

Hence, we only need to consider the case of $|M \cup W| = 3$. Without loss of generality, we may assume that $M = \{m_1, m_2\}$ and $W = \{w_1\}$. Suppose that there exists $(u, v) \in \varphi(\gamma)$ such that $(u, v) \notin \mathcal{S}(\gamma)$. If $u_{m_1} = u_{m_2} = v_{w_1} = 0$, then by Pareto optimality, (u, v) must be stable. Then, without loss of generality, we may also assume that $u_{m_1} + v_{w_1} = \Pi(m_1, w_1) > 0$ and $u_{m_2} = 0$ for any $(u, v) \in \varphi(\gamma)$. For $(u, v) \in \varphi(\gamma)$ such that $(u, v) \notin \mathcal{S}(\gamma)$, it must be true that $u_{m_2} + v_{w_1} = v_{w_1} < \Pi(m_2, w_1)$. Then, let us introduce a new woman w_2 such that $\Pi(m, w_2) = \pi(w_2) = 0$ for any $m \in M$ and γ' denote the extension of γ obtained in this way. In γ', for any $(u', v') \in \varphi(\gamma')$, $u'_{m_1} + v'_{w_1} = \Pi(m_1, w_1) > 0$ and $u_{m_2} = 0$. Because in γ', there are two men and two women, we have already seen that $\varphi(\gamma') \subset \mathcal{S}(\gamma')$. Therefore, for any $(u', v') \in \varphi(\gamma')$, $u'_{m_2} + v'_{w_1} = v'_{w_1} \geqq \Pi(m_2, w_1) > v_{w_1}$. This contradicts population monotonicity.

We have shown that $\varphi(\gamma) \subset \mathcal{S}(\gamma)$ if $|M| \leqq 2$ and $|W| \leqq 2$. In order to prove the same inclusion for any γ, let $(u, v) \in \varphi(\gamma)$. Then, by consistency, $(u_{M'}, v_{W'}) \in \varphi(\gamma')$ for any subgame γ' of γ at (u, v) such that $|M'| \leqq 2$ and $|W'| \leqq 2$. Hence, $(u_{M'}, v_{W'}) \in \mathcal{S}(\gamma')$. By Proposition 3 in Sasaki (1995), the core satisfies converse consistency[5]. Therefore, we may conclude that $(u, v) \in \mathcal{S}(\gamma)$. By the same argument as in Sasaki (1995), we can show that the core has

[5] For the definition, see Sasaki (1995).

no proper subsolution satisfying consistency. Then, it follows that $\varphi(\gamma) = \mathcal{S}(\gamma)$ for each γ. This completes the proof. $\square$

The next task is to establish the logical independence of the axioms in theorem 3.1. We will show that deleting each axiom from the list in theorem 3.1 results in a solution different from the core. Because the solution in example 2.1 satisfies the axioms other than population monotonicity, we will consider the remaining three axioms.

Example 3.1 (Deleting PO). Let φ^1 be the solution which associates the set of all feasible pay-off vectors satisfying individual rationality with each γ. This is essentially equal to the solution in example 5 in Sasaki (1995). Then, we only need to check population monotonicity. Let $(u,v) \in \varphi^1(\gamma)$ for an arbitrary $\gamma = (M, W, \Pi)$ and $\gamma' = (M \cup \{m'\}, W, \Pi')$ an extension of γ. For a matching μ compatible with (u,v), we define $v' \in \mathbb{R}^W$ as follows.

$$v'_w = \begin{cases} \Pi(m,w) & \text{if } \mu(w) = m, \\ 0 & \text{if } \mu(w) = w. \end{cases}$$

From the definition of φ^1, it is obvious that $(0, v') \in \mathbb{R}^{M+1} \times \mathbb{R}^W$ is contained in $\varphi(\gamma')$. Because $u_m \geqq 0$ for each $m \in M$, φ^1 satisfies population monotonicity.

Example 3.2 (Deleting CONS). Let φ^2 be the solution which is different from the core only if $|M| = |W| = 1$ and in this case, $\varphi^2(\gamma)$ consists of $(u_m, 0)$ where $u_m = \Pi(m,w)$. This is essentially equal to the one in example 1 in Sasaki (1995). In order to show population monotonicity, we only need to consider the effect of adding a new man to γ such that $|M| = |W| = 1$. Let γ' be an extension of such a γ obtained by adding a new man m^*. Suppose that m is still matched with w at an optimal matching in γ'. Because $\varphi^2(\gamma') = \mathcal{S}(\gamma')$, for any $(u'_m, u'_{m^*}, v'_w) \in \varphi^2(\gamma')$, $u'_{m^*} = 0$ and $v'_w \geqq \Pi(m^*, w)$. Therefore, $u'_m = \Pi(m,w) - v'_w \leqq \Pi(m,w) - \Pi(m^*,w) \leqq \Pi(m,w) = u_m$. If m remains single at any optimal matchings in γ', then for any $(u'_m, u'_{m^*}, v'_w) \in \varphi^2(\gamma')$, $u'_m = 0 \leqq \Pi(m,w) = u_m$. Thus, the population monotonicity of φ^2 has been shown.

Example 3.3 (Deleting PMON). Let φ^3 be the solution which associates the set of all individually rational and Pareto optimal pay-off vectors with each assignment game. This corresponds to the one in example 6 in Sasaki (1995). We need to check population monotonicity. For any optimal matching μ, we consider the vectors $\overline{u} \in \mathbb{R}^M$ and $\overline{v} \in \mathbb{R}^W$ such that $u_m = v_w = \Pi(m,w)$ if $\mu(w) = m$ and $u_m = v_w = 0$ if $\mu(m) = m$ or $\mu(w) = w$. Because the pay-off vectors $(\overline{u}, 0)$ and $(0, \overline{v})$ are always individually rational and Pareto optimal, the population monotonicity of φ^3 is obvious.

Our characterization does not include individual rationality. As we argued in introduction, this is desirable because individual rationality is a part of the definition of the core. But, if we weaken the axiom of population monotonicity, we may obtain another characterization including individual rationality. This is a more simple-minded extension of theorem 2.1.

For an assignment game $\gamma = (M, W, \Pi)$, $m \in M$ is called *a null player in* γ if $\Pi(m,w) = 0$ for all $w \in W$. *A female null player* is analogously defined.

Now, let $\gamma' = (M \cup \{m\}, W, \Pi')$ be an extension of γ, in which m is a null player. Then, γ' is called *a male null player extension of* γ. *A female null player extension* is analogously defined.

Definition 8 (NPI). A solution φ satisfies *null player invariance* if the projection of the set $\varphi(\gamma')$ onto the subspace $\mathbb{R}^M \times \mathbb{R}^W$ is equal to the set $\varphi(\gamma)$ when γ' is a male (or female) null player extension of γ.

Then, we can obtain the following result.

Theorem 3.2. *The core is the unique solution on Γ_0 satisfying individual rationality, Pareto optimality, consistency, pairwise monotonicity and null player invariance.*

Proof. Almost obvious from the proof of theorem 3.1. □

4 Characterizations on Γ.

In this section, we shall characterize the core of assignment games on the domain of Γ. For an assignment game $\gamma = (M, W, \Pi, \pi) \in \Gamma$, $\Pi(m, w)$ and $\pi(a)$ can be any real numbers, positive, negative or zero for each $(m, w) \in M \times W$ and for each $a \in M \cup W$. As we discussed in introduction, characterization of the core on Γ is non-trivial.

In the following, we prove that the core is characterized by Pareto optimality, consistency, pairwise monotonicity and individual monotonicity. The axiom of individual monotonicity is defined as follows.

Definition 9 (IMON). A solution φ is *individually monotonic* if the following condition is satisfied.

Let $\gamma = (M, W, \Pi, \pi), \gamma' = (M, W, \Pi, \pi') \in \Gamma$ be such that $\pi'(a) \geqq \pi(a)$ for some $a \in M \cup W$ and $\pi'(a') = \pi(a')$ for any other $a' \in M \cup W$. For any $(u, v) \in \varphi(\gamma)$, there exists $(u', v') \in \varphi(\gamma')$ such that a is paid at (u', v') at least as much as at (u, v).

Proposition 4.1. *The core satisfies individual monotonicity.*

Proof. Let $\gamma = (M, W, \Pi, \pi)$ and $\gamma' = (M, W, \Pi, \pi')$ be such that $\pi'(m') \geqq \pi(m')$ for some $m' \in M$ and $\pi'(a) = \pi(a)$ for any $a \neq m'$. Let $(u, v) \in \mathcal{S}(\gamma)$ and μ a matching compatible with it. If $u_{m'} \geqq \pi'(m') \geqq \pi(m')$, then the conclusion is obvious. Suppose that $\pi'(m') > u_{m'}$. Then, for any $(u', v') \in \mathcal{S}(\gamma')$, $u'_{m'} \geqq \pi'(m') > u_{m'}$, which completes the proof. □

Proposition 4.2. *If φ satisfies Pareto optimality and individual monotonicity, then it satisfies individual rationality.*

Proof. Let $(u, v) \in \varphi(\gamma)$ and μ an optimal matching compatible with (u, v). Suppose that $u_m < \pi(m)$ for some $m \in M$. Then, there exists $w \in W$ such that $(m, w) \in c(\mu)$. Therefore, $u_m + v_w = \Pi(m, w)$. It follows from the optimality of μ, $\Pi(m, w) \geqq \pi(m) + \pi(w)$. For each $\varepsilon > 0$, we define π^ε by,

$$\pi^\varepsilon_w \equiv \Pi(m, w) - \pi(m) + \varepsilon > \pi(w)$$
$$\pi^\varepsilon_a \equiv \pi(a) \text{ for each } a \neq w.$$

Consider an assignment game $\gamma^\varepsilon = (M, W, \Pi, \pi^\varepsilon)$. Let μ' be an optimal matching in γ^ε. By construction,

$$\sum_{(m,w)\in c(\mu')} \Pi(m,w) + \sum_{a\in s(\mu')} \pi'(a) \geqq \sum_{(m,w)\in c(\mu)} \Pi(m,w) + \sum_{a\in s(\mu)} \pi(a) + \varepsilon.$$

If $w \notin s(\mu')$, then

$$\sum_{(m,w)\in c(\mu')} \Pi(m,w) + \sum_{a\in s(\mu)} \pi(a) > \sum_{(m,w)\in c(\mu)} \Pi(m,w) + \sum_{a\in s(\mu)} \pi(a),$$

which contradicts the optimality of μ. Therefore, for any optimal matching μ' in γ^ε, $\mu'(w) = w$. Then, for each $(u^\varepsilon, v^\varepsilon) \in \varphi(\gamma^\varepsilon)$,

$$v_w^\varepsilon = \Pi(m,w) - \pi(m) + \varepsilon.$$

Since $u_m < \pi(m)$ and $u_m + v_w = \Pi(m,w)$,

$$v_w = \Pi(m,w) - u_m > \Pi(m,w) - \pi(m).$$

Therefore, for sufficiently small $\varepsilon > 0$,

$$v_w > \Pi(m,w) - \pi(m) + \varepsilon = v_w^\varepsilon.$$

This contradicts individual monotonicity. Thus, it has been shown that $u_m \geqq \pi(m)$ for all $(u,v) \in \varphi(\gamma)$ and for all $m \in M$. By the symmetric argument, we can prove that $v_w \geqq \pi(w)$ for all $(u,v) \in \varphi(\gamma)$ and for all $w \in W$. □

Proposition 4.3. *If a solution φ satisfies Pareto optimality and individual monotonicity, then for any $\gamma = (M, W, \Pi, \pi)$ such that $|M \cup W| \leqq 3$, $\varphi(\gamma) \subset S(\gamma)$.*

Proof. If $|M \cup W| = 2$, the conclusion is obvious. Then, let us consider the case of $|M \cup W| = 3$. Without loss of generality, we may assume that $M = \{m_1, m_2\}$ and $W = \{w_1\}$. Let $(u,v) \in \varphi(\gamma)$ and μ a matching compatible with it. If $c(\mu) = \emptyset$, then for any (m,w), $\Pi(m,w) \leqq \pi(m) + \pi(w) = u_m + v_w$ and hence (u,v) is stable. Thus, we only need to consider the case of $c(\mu) \neq \emptyset$. Without loss of generality, we assume that $(m_1, w_1) \in c(\mu)$, which implies that $\mu(m_2) = m_2$. By way of contradiction, suppose that (u,v) is not stable. Because (u,v) is individually rational, we must have $u_{m_2} + v_{w_1} < \Pi(m_2, w_1)$. Let us define,

$$\sigma = \{\Pi(m_1, w_1) + \pi(m_2)\} - \{\Pi(m_2, w_1) + \pi(m_1)\} \geqq 0,$$

and for each $\varepsilon > 0$, $\pi^\varepsilon(m_1) = \pi(m_1) + \sigma + \varepsilon$ and $\pi^\varepsilon(a) = \pi(a)$ for any other $a \in M \cup W$. For each $\varepsilon > 0$, let $\gamma^\varepsilon = (M, W, \Pi, \pi^\varepsilon)$. By individual monotonicity, for each $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \varphi(\gamma^\varepsilon)$ such that $u_{m_1}^\varepsilon \geqq u_{m_1}$. In γ^ε, the matching $\mu' = \{(m_2, w_1), m_1\}$ is uniquely optimal and hence it is compatible with $(u^\varepsilon, v^\varepsilon)$. Therefore,

$$\begin{aligned} u_{m_2}^\varepsilon + v_{w_1}^\varepsilon &= \Pi(m_2, w_1) \text{ and} \\ u_{m_1}^\varepsilon + u_{m_2}^\varepsilon + v_{w_1}^\varepsilon &= \Pi(m_2, w_1) + \pi^\varepsilon(m_1) \\ &= \Pi(m_2, w_1) + \pi(m_1) + \sigma + \varepsilon \\ &= \Pi(m_1, w_1) + \pi(m_2) + \varepsilon \\ &= u_{m_1} + u_{m_2} + v_{w_1} + \varepsilon. \end{aligned}$$

Then, it follows that $u_{m_2} + v_{w_1} + \varepsilon \geqq u^{\varepsilon}_{m_2} + v^{\varepsilon}_{w_1} = \Pi(m_2, w_1)$ for any $\varepsilon > 0$. But, because $u_{m_2} + v_{w_1} < \Pi(m_2, w_1)$, for sufficiently small $\varepsilon > 0$,

$$\Pi(m_2, w_1) > u_{m_2} + v_{w_1} + \varepsilon \geqq \Pi(m_2, w_1),$$

which is a contradiction. This completes the proof. □

Proposition 4.4. *If a solution φ satisfies Pareto optimality, individual monotonicity and pairwise monotonicity, then for any $\gamma = (M, W, \Pi, \pi)$ with $|M| = |W| = 2$, $\varphi(\gamma) \subset \mathcal{S}(\gamma)$.*

Proof. Let $(u, v) \in \varphi(\gamma)$ and μ an optimal matching compatible with it. If $c(\mu) = \emptyset$, then it is obvious that $(u, v) \in \mathcal{S}(\gamma)$. Without loss of generality, we may assume that $(m_1, w_1) \in c(\mu)$. By way of contradiction, suppose that (u, v) is not stable. Because of the individual rationality of φ, there exists a pair $(m, w) \in M \times W$ such that $u_m + v_w < \Pi(m, w)$. If this pair (m, w) is (m_2, w_2),

$$\begin{aligned}\sum_{(m,w)\in c(\mu)} \Pi(m, w) + \sum_{a\in s(\mu)} \pi(a) &= u_{m_1} + v_{w_1} + u_{m_2} + v_{w_2} \\ &< \Pi(m_1, w_1) + \Pi(m_2, w_2),\end{aligned}$$

which contradicts the optimality of μ. Therefore, without loss of generality, let us assume that,

$$u_{m_1} + v_{w_2} < \Pi(m_1, w_2).$$

We define

$$\sigma = \sum_{(m,w)\in c(\mu)} \Pi(m, w) + \sum_{a\in s(\mu)} \pi(a) - \{\Pi(m_1, w_2) + \Pi(m_2, w_1)\}$$

and for each $\varepsilon > 0$, define an assignment game $\gamma^{\varepsilon} = (M, W, \Pi^{\varepsilon}, \pi)$ by,

$$\Pi^{\varepsilon}(m, w) = \begin{cases} \Pi(m, w) + \sigma + \varepsilon & \text{if } (m, w) = (m_2, w_1), \\ \Pi(m, w) & \text{otherwise.} \end{cases}$$

For each $\varepsilon > 0$, by pairwise monotonicity, there exists $(u^{\varepsilon}, v^{\varepsilon}) \in \varphi(\gamma^{\varepsilon})$ such that,

$$u^{\varepsilon}_{m_2} + v^{\varepsilon}_{w_1} \geqq u_{m_2} + v_{w_1}.$$

Since $\pi(m_1) + \pi(w_2) \leqq u_{m_1} + v_{w_2} < \Pi(m_1, w_2)$, the matching μ' given by $\{(m_1, w_2), (m_2, w_1)\}$ is uniquely optimal in γ^{ε}. Therefore, by Pareto optimality,

$$\begin{aligned} u^{\varepsilon}_{m_1} + v^{\varepsilon}_{w_2} &= \Pi^{\varepsilon}(m_1, w_2) = \Pi(m_1, w_2) \\ u^{\varepsilon}_{m_2} + v^{\varepsilon}_{w_1} &= \Pi^{\varepsilon}(m_2, w_1) = \Pi(m_2, w_1) + \sigma + \varepsilon \end{aligned}$$

Hence,

$$\begin{aligned} u^{\varepsilon}_{m_1} + v^{\varepsilon}_{w_2} + u^{\varepsilon}_{m_2} + v^{\varepsilon}_{w_1} &= \Pi(m_1, w_2) + \Pi(m_2, w_1) + \sigma + \varepsilon \\ &= \sum_{(m,w)\in c(\mu)} \Pi(m, w) + \sum_{a\in s(\mu)} \pi(a) + \varepsilon \\ &= u_{m_1} + v_{w_2} + u_{m_2} + v_{w_1} + \varepsilon \end{aligned}$$

Because $u^{\varepsilon}_{m_2} + v^{\varepsilon}_{w_1} \geqq u_{m_2} + v_{w_1}$, we must have,

$$u^{\varepsilon}_{m_1} + v^{\varepsilon}_{w_2} \leqq u_{m_1} + v_{w_2} + \varepsilon.$$

Since $u_{m_1} + v_{w_2} < \Pi(m_1, w_2)$, for sufficiently small $\varepsilon > 0$,

$$\Pi(m_1, w_2) = u^{\varepsilon}_{m_1} + v^{\varepsilon}_{w_2} \leqq u_{m_1} + v_{w_2} + \varepsilon < \Pi(m_1, w_2),$$

which is a contradiction. Therefore, it has been shown that $(u, v) \in \mathcal{S}(\gamma)$. □

Thus, we have proved

Proposition 4.5. *If a solution φ satisfies Pareto optimality, individual monotonicity and pairwise monotonicity, then for each $\gamma = (M, W, \Pi, \pi)$ with $|M| \leqq 2$ and $|W| \leqq 2$, $\varphi(\gamma) \subseteqq \mathcal{S}(\gamma)$.*

The following proposition is an immediate consequence of proposition 4.5.

Proposition 4.6. *If a solution φ satisfies Pareto optimality, consistency, individual monotonicity and pairwise monotonicity, then it is a subcorrespondence of the core.*

Proof. Let $(u, v) \in \varphi(\gamma)$. For each subgame γ' of γ at (u, v) such that $|M'| \leqq 2$ and $|W'| \leqq 2$, $(u_{M'}, v_{W'}) \in \varphi(\gamma')$ by consistency. Then, it follows from proposition 4.5 that $(u_{M'}, v_{W'}) \in \mathcal{S}(\gamma')$. Because the core satisfies converse consistency, we may conclude that $(u, v) \in \mathcal{S}(\gamma)$. This completes the proof. □

We are at last in a position to present the main result of this section.

Theorem 4.1. *The core is the unique solution satisfying Pareto optimality, consistency, individual monotonicity and pairwise monotonicity.*

Proof. Let $(u, v) \in \mathcal{S}(\gamma)$, where $\gamma = (M, W, \Pi, \pi)$ and μ a matching compatible with (u, v). We introduce a new man m' and a new woman w' into γ such that,

(1) $\Pi(m', w) = v_w$ for each $w \in W$,

(2) $\Pi(m, w') = u_m$ for each $m \in M$,

(3) $\Pi(m', w') = \pi(m') = \pi(w') = 0$.

Let γ' be the assignment game obtained in this way. Then, it is easy to see that $(u, 0, v, 0) \in \mathbb{R}^{M \cup \{m'\}} \times \mathbb{R}^{W \cup \{w'\}}$ is stable in γ' and the matching $\mu' = \mu \cup \{m', w'\}$ is compatible with $(u, 0, v, 0)$. Suppose that (u', v') is another stable pay-off vector in γ'. By corollary 8.7 in Roth and Sotomayor (1990), μ' is compatible with (u', v'). Therefore, $u'_{m'} = v'_{w'} = 0$ and $u'_m \geqq u_m$ for each $m \in M$ and $v'_w \geqq v_w$ for each $w \in W$. If one of these inequalities is strict, this contradicts the Pareto optimality of $(u, 0, v, 0)$. Thus, $(u', v') = (u, 0, v, 0)$. In other words, $(u, 0, v, 0)$ is the unique stable pay-off vector in γ'. If φ satisfies the axioms, then by proposition 4.6, it is a subcorrespondence of the core. Hence, $\varphi(\gamma') = \{(u, 0, v, 0)\}$. By consistency, $(u, v) \in \varphi(\gamma)$. Therefore, it has been shown that $\mathcal{S}(\gamma) \subset \varphi(\gamma)$ for each $\gamma \in \Gamma$. By proposition 4.6, we may conclude that $\mathcal{S} \equiv \varphi$. This completes the proof. □

Next, let us establish the logical independence of the axioms in theorem 4.1.

Example 4.1 (Deleting PO). Let φ^A be the solution which associates with each assignment game the set of all feasible and individually rational pay-off vectors. Because this is essentially equal to the solution in example 5 in Sasaki (1995), it is not difficult to check [CONS] and [PMON]. In order to check [IMON], let $\gamma = (M, W, \Pi, \pi)$ and $\gamma' = (M, W, \Pi, \pi')$ be such that $\pi'(m) \geqq \pi(m)$ and $\pi'(a) = \pi(a)$ for each $a \neq m$. For each $(u, v) \in \varphi^A(\gamma)$, if $u_m \geqq \pi'(m)$, then $(u, v) \in \varphi^A(\gamma')$ and thus the desired condition is satisfied. Hence, let us consider the case of $\pi'(m) > u_m$. In this case, for any $(u', v') \in \varphi^A(\gamma')$, $u'_m \geqq \pi'(m) > u_m$. Therefore, it has been shown that φ^A satisfies [IMON].

Example 4.2 (Deleting CONS). Let φ^B be the solution which differs from the core only for assignment games γ such that $|M| = |W| = 1$ and $\Pi(m, w) > \pi(m) + \pi(w)$. For such games γ, the set $\varphi^B(\gamma)$ consists of a single point $(u_m, \pi(w)) \in \mathbb{R} \times \mathbb{R}$ where $u_m = \Pi(m, w) - \pi(w)$. Because this is essentially equal to the solution in example 1 in Sasaki (1995), it is not difficult to show that [PO] and [PMON] are satisfied. In order to check [IMON], let $\gamma = (\{m\}, \{w\}, \Pi(m, w), \pi(m), \pi(w))$ and $\gamma' = (\{m\}, \{w\}, \Pi(m, w), \pi'(m), \pi(m))$ such that $\Pi(m, w) > \pi(m) + \pi(w)$ and $\pi'(m) \geqq \pi(m)$. If $\Pi(m, w) > \pi'(m) + \pi(w)$, then $(u, v) \in \varphi(\gamma')$ for any $(u, v) \in \varphi(\gamma)$. Hence, the desired condition is obtained. Suppose that $\pi'(m) + \pi(w) \geqq \Pi(m, w)$. Then, for any $(u', v') \in \varphi(\gamma')$, $u'_m = \pi'(m) \geqq \Pi(m, w) - \pi(w) = u_m$ for any $(u, v) \in \varphi(\gamma)$. This shows that φ^B satisfies [IMON].

Example 4.3 (Deleting PMON). Let φ^C be the solution which associates with each γ the set of all Pareto optimal and individually rational pay-off vectors. This is essentially equal to the solution in example 6 in Sasaki (1995) so that it is not difficult to show [PO] and [CONS]. Moreover, by an analogous argument as in example 4.1, it is easy to see that φ^C satisfies [IMON].

Example 4.4 (Deleting IMON). Let φ^D be defined as the one in example 2.1. Then, by an analogous way, we may prove that [PO] [CONS] and [PMON] are satisfied.

As the final remark, we will show that theorem 3.1 extends to Γ.

Theorem 4.2. *The core is the unique solution on Γ satisfying Pareto optimality, consistency, population monotonicity and pairwise monotonicity.*

Proof. It is easy to see that the core satisfies the axioms. By the same argument as in the proof of proposition 3.2, Pareto optimality and population monotonicity imply individual rationality. In proposition 4.4, individual monotonicity can be replaced by individual rationality. Then, by the same argument as in the proof of theorem 3.2, we may prove that a solution satisfying the axioms is a subcorrespondence of the core if $|M| \leqq 2$ and $|W| \leqq 2$. Hence, by consistency and the converse consistency of the core, the solution set is a subset of the core for all assignment games. Therefore, the conclusion follows from the consistency axiom. □

We thus have obtained

Theorem 4.3. *Under Pareto optimality, consistency and pairwise monotonicity, population monotonicity is equivalent to individual monotonicity.*

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