Difference Equations Solution of Exchange Rate Dynamics

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ABSTRACT

This paper focuses on one central question of an appropriate and consistent theoretical as well as empirical model of a reduced form of the rational expectations version of the asset market approach to the exchange rate determination. Using the "sticky-price" monetary model, and formulating the model by the difference equations system, the explicit solutions are obtained as functions of forcing variables extending to past dates only. This "backward-looking" characteristic of the solution is not only in stark contrast to the conventional "forward-looking" nature, but also alleviates empirical investigation because of requiring the past data only. Our discrete dynamic model is superior to the corresponding continuous model, because our solutions neither exhibit the empirically unfounded "overshooting" behaviour, nor have saddle-point stability. Rather, the exchange rate movement is shown to follow an oscillatory path in the case of asymptotic stability, and this seems to replicate the actual movements closely approximated by a random walk process.

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Difference Equations Solution of Exchange Rate Dynamics*

I. Introduction

This paper focuses on one central question of an appropriate and consistent theoretical as well as empirical model of a reduced form of the rational expectations version of the asset market approach to the exchange rate determination. There has existed a clear inconsistency between models in the asset market approach to the exchange rate determination in the literature. For example, according to the rational expectations version of the flexible-price monetary model, the reduced form of the exchange rate is theoretically shown to depend on the discounted present values of the vector of the expected forcing variables in *infinite future* (e.g., Isard (1995), chapter 7). However, the empirical versions of the model have usually been formulated by an AR or a VAR representation using only the *past* (or predetermined) forcing variables, without discussing any explicit rationale for using them instead of the *future* expected forcing variables (e.g., Isard (1995), chapter 8).

This inconsistency motivated me to reconsider rational expectations versions of the asset market approach to the exchange rate determination. In order to pursue my motivation, I took up the so-called "sticky-price" monetary model (which has also been known as the "overshooting" model in the exchange rate literature) because of the following two reasons. First of all, this essentially dynamic model is a simultaneous equations system with the price level and the exchange rate as

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the endogenous variables. Thus, once the original equilibrium point is disturbed by some shock, the time paths of adjustment towards the new equilibrium can be easily traced dynamically. Secondly, if the price level is "sticky" as this model explicitly assumes, then the simultaneous difference equations system is an alternative way for formulation to reconsider the problem at hand.

In economics, dynamic problems have been formulated either by a continuous or discrete form, but unfortunately, the relationships between them, particularly the characteristics of such issues as dynamic solution paths, stability properties, etc, have seldom been discussed seriously in the literature.

The "sticky-price" monetary model, which was originally formulated in a continuous system, is reformulated in a discrete system in this paper. Therefore, the relationship between the two alternative methods of formulation inevitably becomes another main concern. We will discuss and compare, explicitly or implicitly, the two methods of dynamic formulation and the consequential characteristics with respect to the solutions.

It will be shown that the "forward-looking" characteristic of the theoretical solution to the flexible-price monetary model is not replicated here,[1] but on the contrary, the "backward-looking" characteristic is derived from our solutions. This characteristic of the solution definitely alleviates empirical studies, simply because our solution validates the empirical models formulated to estimate the exchange rate with an AR or a VAR model using only the exogenous and predetermined variables.

Moreover, our discrete dynamic model is superior to the corresponding continuous "overshooting" model due to the following reasons: First of all, our solutions neither exhibit the empirically unfounded overshooting behaviour of the exchange rate, nor have saddle-point stability (which means that the model is

[1] This forward-looking characteristic, requiring observations of the expected future forcing variables, has been particularly troublesome for empirical studies, simply because expectations are unobservable. Thus, arbitrariness comes in when the asset market models are estimated by replacing those expectations with observed or proxy variables.

unstable). Secondly, the exchange rate movement is shown to follow an oscillatory path in the case of asymptotic stability. This characteristic seems to replicate the actual movement of the exchange rate that is closely approximated by a random walk process.

The organization of this paper is as follows. In section Ⅱ, an alternative formulation of the "sticky-price" monetary model is represented by a discrete dynamic system, and it is shown that quite a different but a larger set of solutions is derived from the formulation. In section Ⅲ, some implications for the characteristics of the model and also for the exchange rate are considered in detail. Section Ⅳ concludes the paper.

Ⅱ. The Model

In order to emphasize our purpose of this paper mentioned at the outset with greater clarification, let me begin with the so-called "sticky-price" monetary model of exchange rate determination due originally to Dornbusch (1976).[2] The model was extended by Frenkel and Rodoriguez (1982) and Akiba (1996) to accommodate different degrees of capital mobility. The model adopted here, however, looks more closely resemblant to Taylor's (1995a,b) version, simply because it is not "overshooting", but the exchange rate dynamics is our focal point.[3] Following them, let p, y, d, and m represent natural logarithms of the national price level, aggregate output, aggregate demand, and the money supply, respectively, and let i denote the nominal rate of interest. The logarithm of the spot exchange rate (defined as number of units of domestic currency needed in order to purchase one

[2] This model is also known as the "overshooting" monetary model (Frankel (1983), p.89), because it emphasized that sluggish adjustment of national price levels could help explain the phenomenon of exchange rate "overshooting" - that is, a tendency for the exchange rate to jump into one direction in response to news and subsequently to retreat at least part of the way back to its initial position (Isard (1995), p.118).

[3] Obstfeld and Rogoff (1995, p.644) wrote that "Given the lack of empirical support for the overshooting hypothesis,..., it is unclear that this should be regarded as an essential property of an exchange rate model." They also wrote in another place that "the evidence in support of overshooting is thin indeed" (Obstfeld and Rogoff (1996), p.678).

unit of foreign currency) is denoted by s. The foreign variables are identified with an asterisk (*), and the expectations by superscript (e). The model is succinctly summarized as follows:

$$d = \gamma y - \sigma (i - \dot{p}^{e}) + \delta (s + p^{*} - p) \qquad (1)$$

$$m - p = \phi y - i/\lambda \qquad (2)$$

$$\dot{p} = \theta (d - y) + \varepsilon_{p} \qquad (\theta > 0) \qquad (3)$$

$$\dot{s}^{e} = i - i^{*} + \varepsilon_{s} \qquad (4)$$

Equation (1) defines the aggregate demand for goods and services, which is formulated to depend on the aggregate production level, the real rate of interest, and the (log linearized) real exchange rate. γ, δ, and σ are all assumed as positive parameters, and $\delta > 0$ signifies that the so-called Marshall-Lerner condition is satisfied in our model. Equation (2) is a simplified portfolio balance schedule, where ϕ and λ are also assumed as positive parameters.[4] Equations (3) and (4) describe the dynamics of the model. The price level is sluggishly adjusted ($\infty > \pi > 0$) according to the excess demand for goods and services (d-y), while the spot exchange rate is instantaneously adjusted to the international interest rate differential.

ε_p and ε_s are i.i.d. disturbance terms representing the unanticipated shocks to the commodity and the capital market, respectively. To be more specific, they are assumed to be white noise. Assuming the perfect foresight version of rational expectations, the expected values of changes in p and s are replaced by the actually realized values, respectively, yields:

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[4] Because i is not logarithmically transformed, $(1/\lambda)$ is the *semi*-elasticity of demand for money with respect to the nominal rate of interest.

$$\begin{bmatrix} ds/dt \\ dp/dt \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ \dfrac{\theta\delta}{1-\theta\sigma} & \dfrac{-\theta(\delta+\sigma\lambda)}{1-\theta\sigma} \end{bmatrix} \begin{bmatrix} s \\ p \end{bmatrix} + \begin{bmatrix} Qs \\ Qp \end{bmatrix} \tag{5}$$

where $Q_s \equiv -\lambda m + \lambda \phi y - i^* + \varepsilon_s$ and $Q_p \equiv \theta(\gamma - \sigma \lambda \phi - 1)y + \theta \sigma \lambda m + \theta \delta p^* + \varepsilon_p$.

Denoting the Jacobian of equation (5) by H, $\alpha \equiv$ trace H and $\beta \equiv$ det H, the characteristic equation is expressed as:

$$\rho^2 - \alpha\rho + \beta = 0 \tag{6}$$

where ρ is the eigenvalue. Although Daniel (1989) imposed a plausible assumption of $1-\theta\sigma > \phi\delta$ (>0) from an economics point of view, we implicitly impose an even weaker condition of $1-\theta\sigma > 0$. This guarantees that both α and β are negative. The equilibrium point is *not* asymptotically stable, because β is negative, although α satisfies $\alpha < 0$ (Takayama (1994), p.407). However, the fact that $\beta < 0$ means that the equilibrium is a saddle point (Takayama (1994), theorem 7.4).[5]

The exchange rate is a "jump variable" (Taylor (1995a,b)) of the model that compensates for stickiness in other variables, including another endogenous variable, the price level. Thus, exchange rate expectations are related to expectations about other forcing variables in a forward-looking manner, while the price level behaves sticky in a backward-looking manner (Isard (1995), Taylor (1995a,b), MacDonald and Taylor (1992)). These two endogenous variables of the model are considered to be different in nature.

Therefore, it is not necessarily considered to be an appropriate way to formulate the model by a differential equations system (5). An alternative way for formulating the model is by a difference equations system. Although it could be argued that the exchange rate behaves rather smoothly compared to the price level, so that the discrete-time framework also has some drawbacks, an attempt to do so has been made at least for the exchange rate (Isard (1995) chapters 5,7, and 8). In fact, it is well known that the rational expectations solution of the discrete-time flexible price monetary model is expressed as the discounted present value of the vector of the expected forcing variables in infinite future (Isard (1995), p.127; Taylor (1995), p.22; MacDonald and Taylor (1992), p.5). However, if the exchange rate behavior is formulated with the price level behavior in a simultaneous equation system, this forward-looking characteristic of expectations seems inappropriate. The reason for it lies in a simple observed fact that the price level behaves quite sticky in the short and medium runs.

If the continuous planar system (5) is, disregarding $(Q_s, Q_p)'$, compactly denoted by vectors and a matrix as:

$$\dot{x} = Hx \tag{5'}$$

then it is known that the system (5') has an exact discrete analogue as:

$$x_{t+1} = e^H x_t , \qquad t \in N_0 \tag{7}$$

where x is the column vector (s, p)', N_0 is a set of non-negative integers, and e^H is called the exponential matrix.[6]

However, because the determinant of e^H is always positive (e.g., Jensen (1994)), p.223), the family of solutions to (7) is clearly only a subset of those to the discrete dynamic system:

$$x_{t+1} = Hx_t \tag{8}$$

where, e.g., the determinant of H, i.e., β , is not restricted to be positive.

[5] The two eigenvalues are shown to be $\rho_1<0<\rho_2$. The eigenvector for the stable eigenvalue ρ_1 is shown to be $c(1, \rho_1/\lambda)'$ where c is an arbitrary constant. Thus, the convergence path is represented by the stable arm $s-s_0=(\lambda/\rho_1)(p-p_0)$, where s_0 and p_0 are the equilibrium values.

[6] See, e.g., Cesari (1963), Sec.2.1 (pp.14-18), Guckenheimer and Holmes (1983), Sec. 1.4 (pp.16-22), Jensen (1994), Sec.13.4 (pp.288-291).

To achieve an analogy more general than that between (5') and (7), we must look for a certain coefficient matrix of differential equations system:

$$\dot{x} = Lx = (\log H)\, x \tag{5''}$$

that corresponds to the coefficient matrix of the difference equations system (8), viz.:

$$x_{t+1} = e^{L}x_t \equiv e^{\log H}x_t \equiv Hx_t \tag{8'}$$

This is the exact discrete analogue of the continuous planar system (5"). The matrix L, which is equal to the matrix log H, symbolizes the so-called logarithmic matrix of H (e.g., Cesari (1963), p.18, or Jensen (1994), p.288).

It should be noted at this stage that the coefficient matrices in equations (5) and (8) are deliberately selected to be the same for the following reasons. First of all, by doing so, we can directly compare the differences in the behaviour between the two systems, i.e., the continuous dynamic system (5) and the discrete dynamic system. Secondly, it can be shown that we will obtain even richer economic insights from a discrete dynamic system (8) than from the corresponding continuous dynamic system (5), as will be made clearer in the latter part of this and the next sections. For example, we will observe that the discrete dynamic system has an obvious advantage of being able to display asymptotic stability if the underlying parameters are restricted within a certain area. In striking contrast with this, the continuous dynamic system (5) exhibits saddle-point stability as mentioned earlier, and the system is not asymptotically stable, but unstable (e.g., Sanchez et al. (1988), p.493).

In describing the behavior of the solutions and the trajectory geometry of the discrete dynamic system (8), it is convenient to define the ratio variable:

$$r_t \equiv p_t/s_t\,, \qquad s_t \neq 0 \tag{9}$$

This ratio r_t is first used to determine invariant subspaces in the phase plane. Using the definition (9), the system of equations (8) is rewritten as:

$$s_{t+1}=s_t(0+\lambda r_t)=s_t f(r_t) \qquad (10.1)$$

$$p_{t+1}=s_t\left\{\frac{\theta\sigma}{1-\theta\sigma}-\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}r_t\right\}=s_t g(r_t) \qquad (10.2)$$

Next, the linear fractional recurrence equation of r_{t+1} is expressed as:

$$r_{t+1}=\frac{p_{t+1}}{s_{t+1}}=\frac{g(r_t)}{f(r_t)}=q(r_t),\quad f(r_t)\neq 0 \qquad (11)$$

Equation (11) represents rectangular hyperbolas with the center $(0, -\theta(\delta+\sigma\lambda)/\lambda(1-\theta\sigma))$ and the constant $-\beta/\lambda^2=\theta\delta/\lambda(1-\theta\sigma)$. Thus, a stationary ratio solution of the difference equation is derived from:

$$\Delta r_t = r_{t+1} - r_t = q(r_t) - r_t \qquad (12)$$

and is determined by the real roots of the polynomial:

$$\frac{\delta\theta}{1-\sigma\theta}-\frac{\theta(\delta+\sigma\lambda)}{1-\sigma\theta}r-\lambda r^2=0 \qquad (r\neq 0) \qquad (13)$$

Equation (13) corresponds exactly to the so-called director function of the corresponding continuous linear system. This second-order polynomial (parabola) has a discriminant, conveniently designated by $4\Delta^2$, and so Δ is defined by:

$$\Delta^2=\frac{1}{4}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]^2+\left(\frac{\lambda\theta\delta}{1-\theta\sigma}\right)=\frac{1}{4}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]^2-\beta \qquad (14)$$

It was shown that $\beta < 0$ because $1-\theta\sigma > 0$ as assumed earlier, and this guarantees that $\Delta^2 > 0$. Thus, the director roots (ρ_i) of (13) and directrix values ($f(\rho_i)$) of (10.1) and (10.2) are given by:

$$\mu_1 \equiv f(\rho_1) = \lambda\ \rho_1 = -\frac{1}{2}\left[\frac{\theta(\delta+\sigma\ \lambda)}{1-\theta\ \sigma}\right] + \Delta \tag{15.1}$$

$$\mu_2 \equiv f(\rho_2) = \lambda\ \rho_2 = -\frac{1}{2}\left[\frac{\theta(\delta+\sigma\ \lambda)}{1-\theta\ \sigma}\right] - \Delta \tag{15.2}$$

and thus, equation (14) implies that:

$$\mu_1 \cdot \mu_2 = \beta\ (\equiv \det H) < 0 \tag{15.3}$$

The general solutions $x_t = (s_t,\ p_t)$, $t \in N_0$, of the discrete linear dynamic system (8) for $\Delta^2 > 0$ are given as (e.g., Jensen (1994), p.283):

$$s_t = C_1 \mu_1^t + C_2 \mu_2^t \tag{16.1}$$

$$p_t = C_1 \rho_1 \mu_1^t + C_2 \rho_2 \mu_2^t \tag{16.2}$$

where C_1 and C_2 are arbitrary real constants. To determine these constants by an elimination procedure in a systematic way, let us introduce:

$$\zeta \equiv \frac{1}{2}\, ln\left|\frac{\mu_1}{\mu_2}\right| = \frac{1}{2}\, ln\left|\frac{-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\ \theta)}{1-\sigma\ \theta}\right] + \Delta}{-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\ \theta)}{1-\sigma\ \theta}\right] - \Delta}\right| \tag{17}$$

In the Appendix it is shown that ζ in equation (17) is negative. The negativity of ζ, together with the fact that det H $\equiv \beta < 0$ and trace H $\equiv \alpha < 0$, means that the entire family of solutions, $x_t = (s_t,\ p_t)'$, $t \in N_0$, solving the discrete linear dynamic system (8) with initial values $(s_0,\ p_0) \in R^2$ at t=0 is given by (Jensen (1994), pp.284-8):

$$\begin{pmatrix} s_t \\ p_t \end{pmatrix} = |\beta|^{\frac{t}{2}} \begin{pmatrix} \cosh(\zeta\ t) + \frac{\theta(\delta+\sigma\ \lambda)}{2\Delta(1-\theta\ \sigma)}\sinh(\zeta\ t) & \frac{\lambda}{\Delta}\sinh(\zeta\ t) \\ \frac{\theta\ \delta}{\Delta(1-\theta\ \sigma)}\sinh(\zeta\ t) & \cosh(\zeta\ t) - \frac{\theta(\delta+\sigma\ \lambda)}{2\Delta(1-\theta\ \sigma)}\sinh(\zeta\ t) \end{pmatrix} \begin{pmatrix} s_0 \\ p_0 \end{pmatrix} \tag{18.1}$$

for t=2n, $n \in N_0$, and:

$$\begin{pmatrix} s_t \\ p_t \end{pmatrix} = |\beta|^{\frac{t}{2}} \begin{pmatrix} \sinh(\zeta\ t) + \frac{\theta(\delta+\sigma\ \lambda)}{2\Delta(1-\theta\ \sigma)}\cosh(\zeta\ t) & \frac{\lambda}{\Delta}\cosh(\zeta\ t) \\ \frac{\theta\ \delta}{\Delta(1-\theta\ \sigma)}\cosh(\zeta\ t) & \sinh(\zeta\ t) - \frac{\theta(\delta+\sigma\ \lambda)}{2\Delta(1-\theta\ \sigma)}\cosh(\zeta\ t) \end{pmatrix} \begin{pmatrix} s_0 \\ p_0 \end{pmatrix} \tag{18.2}$$

for $t=2n+1$, $n \in N_0$, where:

$$cosh(\zeta\, t) \equiv \frac{1}{2}(e^{\zeta t}+e^{-\zeta t}) \text{ and } sinh(\zeta\, t) \equiv \frac{1}{2}(e^{\zeta t}-e^{-\zeta t}) \tag{19}$$

are the hyperbolic cosine and the hyperbolic sine functions.

Invoking Theorem 1 of Jensen (1994, p.289), the governing matrix , $L \equiv \log H$, of the correlated continuous linear dynamic system (5) which embeds the solution (18.1)-(18.2) into phase curves of the discrete non-singular dynamic system (8) is given as:

$$log\,H = \begin{pmatrix} \frac{1}{2}log|\beta| + \frac{\zeta\,\theta(\delta+\sigma\,\lambda)}{2\Delta(1-\theta\,\sigma)} & \frac{\zeta\,\lambda}{\Delta} \\ \frac{\zeta\,\theta\,\sigma}{\Delta(1-\theta\,\sigma)} & \frac{1}{2}log|\beta| - \frac{\zeta\,\theta(\delta+\sigma\,\lambda)}{2\Delta(1-\theta\,\sigma)} \end{pmatrix} \tag{20}$$

The system of trajectories obtained from the solutions $(s_t, p_t)'$ in (18.1)-(18.2) describes the global phase portrait of the discrete non-singular dynamic system (8). The traditional necessary and sufficient conditions for solutions of linear differential equations is the characteristic roots being less than one in absolute value, in which case the solutions converge to a stationary point from any initial values over time. The corresponding discrete counterpart, ensuring global asymptotic stability of the solutions (18.1)-(18.2) is (i) trace (log H) < 0, and (ii) det (log H) > 0.

According to Lemma 1 of Jensen (1994, p.291), these conditions are equivalent to: [7]

$$\text{(i) } \beta < 1, \text{ (ii) } \beta > -1+\alpha, \text{ and (iii) } \beta > -1-\alpha \tag{21}$$

[7] The derivation of the last two conditions (ii) and (iii) is briefly sketched as follows: Because of (15-1),(15-2) and (15-3), $|\log H| = \log|\mu_1| \cdot \log|\mu_2| = \log|-\theta(\delta\,\sigma+\lambda)/2(1-\theta\,\sigma)+\Delta| \cdot \log|\theta(\delta\,\sigma+\lambda)/2(1-\theta\,\sigma)-\Delta| = \log|\alpha/2+\Delta| \cdot \log|\alpha/2-\Delta| > 0$. To satisfy the last inequality together with (i) trace (log H) < 0 (which reduces to (i) $\beta < 1$), we only need to consider $|\mu_1| < 0$ and $|\mu_2| < 0$. After eliminating superfluous inequalities, these restrictions reduce to (ii) $\beta > -1+\alpha$, and (iii) $\beta > -1-\alpha$. For the complete proof, see Jensen (1994), pp.291-2.

In the discussion of saddle point stability of (5), we have confirmed that $\alpha < 0$ and $\beta < 0$, so that the first condition (i) in (21) is satisfied. Accordingly, the second and the third parts of (21) remain effective restrictions. Moreover, because both α and β are negative, only the last restriction (iii) turns out to be an effective restriction, which can be rewritten as:

$$1/\theta > (\lambda + 1)(\delta + \sigma) \tag{22}$$

Thus, we have derived an important restriction (22) for the parameters of the underlying model, if the discrete linear dynamic system (8) exhibits global asymptotic stability. Equation (22) can be regarded as an equation of hyperbola with respect to λ and δ, given θ and σ, with two asymptotes, $\lambda = -1$ and $\delta = -\sigma$. Asymptotic stability is assumed if the parameters happen to be restricted within an open set depicted in Figure 1 (area denoted as (a)).

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Insert Figure 1 around here
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It is straightforward to confirm that the larger the value of θ and/or σ, the narrower the area (a), so that the dynamic system tends not to exhibit global asymptotic stability. Because θ and σ represent the sensitivity of inflation with respect to aggregate excess demand and of investment with respect to the real rate of interest, respectively, the above finding has economically plausible implications for the stability of the underlying open economy macro model. For example, if an economy becomes more sensitive to excess aggregate demand than before (an increase in θ), the resultant degree of increase in the price level is higher. The larger increase in p_t has some destabilizing effects on exchange rate s_t, possibly through a decrease in exports but an increase in imports from abroad. Thus, the increase in θ makes the economy unstable through disequilibrium both in the internal and the external sectors.

As we have just seen, the signs of trace (log H) and det (log H) give crucially important information about whether the solutions (18.1)-(18.2) exhibit convergence to or divergence from a stationary value. In addition, the sign of Δ^2 distinguishes between oscillatory and non-oscillatory behavior of the solutions. Also the sign of trace H ($\equiv \alpha$) appears in the off diagonal elements of the log (H) matrices through the value of ζ, and thus affects the non-oscillatory values of the solutions. Finally, the sign of det (H)($\equiv \beta$) is also important for the phase portrait of non-oscillatory solutions.

If we combine the information of the signs of $\Delta^2 > 0$, $\alpha < 0$, and $\beta < 0$ in our model with the information of the signs of det (log H) and trace (log H) given in Figure 1, three possible behaviors of solutions can be deduced (e.g., Jensen (1994), p.293). The first possible case is a combination of det (log H)>0 and trace (log H)<0 that satisfies the global asymptotic stability condition mentioned earlier. The solutions are reflected in ρ_1-directrix and converging along the parabolic boundary curves. The solutions in this case are generated by the combination of parameters restricted in the area (a) of Figure 1. Although the exact shape of the trajectory depends on the parameter values, its general shape is exemplified in Figure 2.

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Insert Figure 2 around here
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In the second and the third possible cases, the solutions are also reflected in ρ_1-directrix but diverging. If det (log H) is negative, but trace (log H) is of either sign, the solutions diverge along the hyperbolic boundary curves. The parameters are restricted in the areas (b) and (c) in Figure 1.

On the other hand, other diverging solutions reflected also in ρ_1-directrix are obtained when both of det (log H) and trace (log H) are positive. The solutions diverge along the parabolic boundary curves in this case. A combination of parameter values restricted in the area (d) in Figure 1 gives rise to this solutions.[8]

Ⅲ. Implications for Exchange Rates

This section explores some important implications derived from the solutions (18.1)-(18.2), compared with those obtained from the differential equations version of the model.

First of all, s_t and p_t are jointly determined variables depending on the same sets of underlying variables, although their dynamic time paths are apparently different.

Secondly, there is a basic difference in stability property. Although the differential equation version is only saddle-point stable, and thus the equilibrium state is unstable (see Sanchez et al (1988), p.493), the difference equations version (8) is slightly more involved. We have seen that solutions of the price level and the exchange rate in our discrete nonsingular linear system are either converging or diverging, depending on the parameter values in our underlying model. This implies that, because there is no compelling evidence that the actual exchange rate series diverge from the stationary values in the recent floating period, the actual exchange rate as the relative price of two national monies tends to converge to the stationary value over time. Then, it is inferred that the solutions, given in (18.1)-(18.2) are oscillating, but the parameter values are restricted within the open region comprised by $(\lambda+1)(\delta+\sigma) < 1/\theta$, $\lambda > 0$, and $\delta > 0$, implying that the solutions have global asymptotic stability.

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[8] The exact curves cannot be depicted because the parameters (λ, δ, θ, and σ) are arbitrary constants, and therefore only the signs of α, β, μ_i, ρ_i, ζ, and Δ are important to determine the signs of det (log H) and trace (log H) for qualitative behavior of the solutions. However, the general forms of the trajectory in the three cases explained above are similar to Fig. 3.7, 3.8, and 3.9 in Jensen (1994, p.296).

Thirdly, while the exchange rate in the differential equations version (5) is known to exhibit the so-called "overshooting" phenomenon, the corresponding solution of the exchange rate in the difference equations version (8) behaves, when restricted by global asymptotic stability, to reflect in μ_1-directrix and to converge along the parabolic boundary curves.

The stability property and the "overshooting" phenomenon of the exchange rate mentioned above are not separate, but stem essentially from the same characteristics of the coefficient matrix H in equation (5). Element (1,1) of H is zero, which exhibits a quite peculiar characteristic of the "overshooting" model. It signifies that the current spot exchange rate has nothing to do with the change in the spot exchange rate, and this seems to be quite implausible from an economic point of view.

Fourthly, and related to the first point, the exchange rate in the differential equations version monotonically approaches to the stationary value after "overshooting". In contrast, in the difference equations version (8), it approaches to the stationary value with oscillation. These "alternating" solutions are admissible only in the discrete dynamic system.

One of the well-documented empirical regularities observed from the recent floating exchange rate system is the fact that the exchange rates have been very much volatile, with sporadic large jumps in the short-run. As far as the author knows, two explanations have been proposed to account for such a stylized fact. One is a more or less practically oriented compromise, and the other seems to be a theoretical curiosity.

On the one hand, it was demonstrated by Meese and Rogoff (1983a,b) that the out-of-sample forecasting performance of some structural models, including the "sticky-price" monetary model, was outperformed by a simple random walk model for a time horizon of up to twelve months. Although the exchange rate behaviour in the short-run seems to be closely approximated by a random walk process, the Meese-Rogoff conclusion does not mean that the exchange rates actually follow random walk processes. In fact, as Moose (2000, chapter 1) summarizes, the empirical evidence on the hypothesis that the exchange rate follows a random walk process is not completely supportive.

On the other hand, because of the observed fact that the expectations about the future exchange rates have not been formed rationally, De Grauwe and Dewachter (1992,1993) extended the "stick-price" monetary model with two classes of speculators in an *ad hoc* manner. The first are chartists who use past history of exchange rates to form expectations of future spot rates. The second group of speculators are fundamentalists who calculate the equilibrium exchange rate as given by the model and forecast the future spot rates based on the economic fundamentals. Based on this framework, De Grauwe and Dewachter theoretically demonstrated that the behaviour of the exchange rate becomes chaotic in the situation where the chartists dominate the market.

In contrast to these two explanations, our solutions are definitely superior for the following reasons. First, our solutions are derived from a celebrated "sticky-price" monetary model of the exchange rate determination and thus are based on a clear economic theory, while the random walk hypothesis provides no clear-cut explanation of where such randomness comes into the market in the first place. Secondly, our solutions do not depend on such an *ad hoc* assumption as the chaotic theory that there are two types of speculators, chartists and fundamentalists, in the market. Therefore, thirdly, depending on a clear economic model that is constructed on the behavioural equations of the price level and the exchange rate, our solutions offer a superior explanation of why the exchange rate approximately follow a random walk with little or no drift. Fourthly, because our solutions are deterministic in nature (see equation (18)), the empirical investigation is more easily implemented in either a form of time-series or more conventional regression techniques based on the fundamentals in the model. It is instructive to note that the reason why our exchange rate behaviour looks similar to a random walk process in the case of asymptotic stability seems to lie in the following fact. That is, the interaction between the adjustment in s_t and p_t is better treated in the discrete system (8) than in the corresponding continuous system (5), in the sense that the coefficient matrix in (20) is more symmetrical or more even, compared with that in (5).[9]

Finally, and, for our present purposes most importantly, the rational expectations solutions for s_t and p_t in the discrete dynamic model (8) depend on the underlying parameters of the model and the initial values, as clearly described in solutions (18.1)-(18.2) and (19). It should be recalled that the discrete dynamic system (8), compared with the continuous dynamic system (5), is constructed by approximating the UIP condition (4) with the aid of perfect foresight, a special case of rational expectations. As mentioned earlier, the rational expectations solution for the flexible-price monetary model depends on the discounted values of the vectors of the forcing variables in all future periods only.[10]

In striking contrast with it, our solutions for the "stick-price" monetary model with rational expectations were shown to depend on the contemporaneous parameter values as well as the initial values of the endogenous variables.

The implication of this finding is particularly important for empirical investigation. If the rational expectations solutions were to depend only on the discounted present values of the vectors of the forcing variables in all future periods which are essentially unobservable, i.e., "forward-looking" in nature, then

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[9] I owe this point to Professor Bjarne S. Jensen.

[10] Rational bubbles are assumed away here. For rational bubbles and its related topics, see Taylor (1995a, p.22; 1995b, pp.38-9), and MacDonald and Taylor (1992, pp.13-5).

any attempt to estimate and forecast the exchange rate is literally impossible unless such unobservable variables are somehow arbitrarily replaced by proxy variables. If one pushes such a regression ahead with approximated data, it in turn results in bringing in a classical but serious problem of errors in variable, and thus undermines the statistical tests involved. According to our solutions (18-1), (18-2), and (19), any attempt at empirical investigation should be more conventional, in the sense that both p_t and s_t can be estimated jointly by a VAR model with time-varying parameters. Thus, the "forward-looking" characteristic of the flexible-price monetary model with rational expectations is not replicated here, but a more conventional "backward-looking" characteristic is obtained.

Before leaving this section, one more comment is in order. Obstfeld and Rogoff (1995) showed that, making use of their two-country dynamic model based on the intertemporal approach and the sticky-price Keynesian approach, the exchange rate "jumps" immediately to its long-run level when prices are unable to adjust in the short-run. Their assumption of the short-run stickiness of the price level approximates θ in our model with a smaller real number. It can be easily observed that the smaller the value of θ, the wider the region (a) in Figure 1 that guarantees the global asymptotic stability. As our solutions (18.1)-(18.2)(and also Figure 2) make clear, the exchange rate in this case is still very likely to fluctuate, despite the inability of prices to adjust in the short-run. Thus, our difference equations model depicts more plausible short-run behavior of the exchange rate than theirs.

IV. Concluding Remarks

This article develops an appropriate and consistent model of rational expectations version of asset market approach to the exchange rate determination for the purposes of both theoretical and empirical analysis. Using the discrete dynamic formulation of the "sticky-price" monetary model, it was shown that both the exchange rate and the price level are a function of a set of forcing variables that extends from the present into the past only. This "backward-looking" characteristic of our solutions is in striking contrast to the "forward-looking" characteristic obtained from the modern theory of the asset market approach to exchange rate determination. In the latter, the exchange rate is considered as a "jump" variable (Isard (1995, p.118), Taylor (1995a, p.230; 1995b, p.39) in one direction, or a "forward-looking" variable, while the price level is not a jump but a "backward-looking" variable (Kawai and Murase (1990, p.57), Stansfield and Sutherland (1995, p.221)). In addition, our exchange rate solution with the backward-looking characteristic also alleviates empirical investigation, simply because the past data alone are sufficient for estimation.

According to Jensen (1994, section 13.6), the merits and factual choice between discrete and continuous time modeling must be determined from theoretical justifications as well as from their empirical performance. We come to the conclusion that, for exchange rate dynamics, our discrete dynamic representation of the "sticky-price" version of the monetary model with rational expectations is superior to the corresponding continuous version, known as the "overshooting" model, for the following reasons. (1) Since the exchange rate and the price level are different in nature, i.e., while the speed of adjustment of the former is sufficiently swift, that of the latter is "sticky", the exchange rate dynamics seem to be more legitimately formulated by a discrete system theoretically. Moreover, theoretically speaking, the discrete dynamic model is asymptotically stable, depending on the parameter values, whereas the continuous dynamic system is unstable (only saddle-point stable). And (2) while the continuous dynamic system generates the empirically unfounded overshooting behaviour of the exchange rate, the discrete dynamic system exhibits, in a certain case, oscillatory movement that seems to replicate the actual movements closely approximated by a random walk process. This also seems to be more realistic empirically.

APPENDIX

This section provides a proof that ζ defined in equation (17) is negative ($\zeta<0$). Using equations (15.1) and (14), it is straightforward to proceed:

$$\mu_1=-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\Delta=-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\sqrt{\frac{1}{4}\left[\frac{\theta(\delta+\sigma\theta)}{1-\theta\sigma}\right]^2+\frac{\lambda\theta\sigma}{1-\theta\sigma}} \qquad \text{(A.1)}$$

The inside of the square root is rewritten as:

$$\frac{1}{4}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]^2+\frac{\lambda\theta\sigma}{1-\theta\sigma}=\left\{\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}}\right\}^2-\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}}$$

$$<\left\{\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}}\right\}^2$$

and thus:

$$-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\Delta<-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\left\{\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}}\right\}$$

$$=\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}} \qquad \text{(A.2)}$$

Likewise, it can be shown that:

$$\mu_2=-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]-\Delta=-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]-\sqrt{\frac{1}{4}\left[\frac{\theta(\delta+\sigma\theta)}{1-\theta\sigma}\right]^2+\frac{\lambda\theta\sigma}{1-\theta\sigma}}$$

$$>-\frac{1}{2}\left[\frac{\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]-\left\{\frac{1}{2}\left[\frac{-\theta(\delta+\sigma\lambda)}{1-\theta\sigma}\right]+\sqrt{\frac{\lambda\theta\sigma}{1-\theta\sigma}}\right\}$$

$$=-\sqrt{\frac{\lambda\theta\delta}{1-\theta\sigma}} \qquad \text{(A.3)}$$

Thus, (A.2) and (A.3) imply that ζ in equation (17) is:

$$\zeta\equiv\frac{1}{2}\,ln\left|\frac{\mu_1}{\mu_2}\right|<\frac{1}{2}\,ln\left|\frac{\sqrt{\frac{\lambda\theta\delta}{1-\theta\sigma}}}{-\sqrt{\frac{\lambda\theta\delta}{1-\theta\sigma}}}\right|=\frac{1}{2}\,ln\,(1)=0$$

REFERENCES

Akiba, Hiroya, "Exchange-Rate-Sensitive Demand for Money and Overshooting," *International Economic Journal* 10 (1996): 119-29.

Cesari, Lamberto, *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations,* Berlin: Springer-Verlag, 1959.

Daniel, Betty C., "One-Sided Uncertainty about Future Fiscal Policy," *Journal of Money, Credit, and Banking* 21 (1989): 176-89.

De Grauwe. Paul and Dewachter, Hans. "Chaos in the Dornbusch Model of the Exchange Rate," *Kredit und Kapital* 25 (1992): 26-54.

_____ and _____. "A Chaotic Model of the Exchange Rate: The Role of Fundamentalists and Chartists," *Open Economy Review* 4 (1993): 351-379.

Dornbusch, Rudiger, "Expectations and Exchange Rate Dynamics." *Journal of Political Economy* 84 (1976): 1161-76.

Frankel, Jeffrey A., "Monetary and Portfolio - Balance Models of Exchange Rate Determination," in Jagdeep S. Bhandari and Bluford H. Putnam (eds.) *Economic Interdependence and Flexible Exchange Rates,* Cambridge, Mass.: The MIT Press, 1983, 84-115.

Frenkel, Jacob A. and Rodoriguez, Carlos A., "Exchange Rate Dynamics and the Overshooting Hypothesis," *IMF Staff Papers* 29(1982): 1-30.

Guckenheimer, John and Holmes, Philip, *Nonlinear Oscillations, Dynamical Systems, and Bifucations of Vector Fields* New York: Springer-Verlag, 1983.

Isard, Peter, *Exchange Rate Economics,* Cambridge: Cambridge University Press, 1995.

Jensen, Bjarne S., *The Dynamic Systems of Basic Economic Growth Models,* Dordrecht: Kluwer Academic Publishers, 1994.

Kawai, Masahiro and Murase, Hideaki, "Saikin-no Kawase Reito Kettei Riron: Tenbou Ronbun," (Recent Theories of Exchange Rate Determination: A Survey Article) *Financial Review* (The Ministry of Finance)(1990): 48-73 (in Japanese).

MacDonald, Ronald and Taylor, Mark P., "Exchange Rate Economics A Survey." *IMF Staff Papers* 39(1992): 1-57.

Meese, R. A. and Rogoff, K., "Empirical Exchange Rate Models of the Seventies: Do They Fit out of Sample ?," *Journal of International Economics* 14 (1983a):3-24.

_____ and _____, "The Out-of-Sample Failure of Empirical Exchange Rate Models: Sampling Error or Misspecification ?," in J. A. Frenkel (ed.), *Exchange Rates and International Macroeconomics*, Chicago: University of Chicago Press. 1983b: 67-105.

Moosa, Imad A., *Exchange Rate Forecasting: Techniques and Applications,* Houndsmills: Macmillan Press Ltd., 2000

Obstfeld, Maurice and Rogoff, Kenneth. "Exchange Rate Dynamics Redux," *Journal of Political Economy* 103(1995): 624-60.

_____ and _____. *Foundations of International Macroeconomics,* Cambridge. MA: The MIT Press, 1996.

Sanchez, David A., Allen, Richard C., Jr., and Kyner, Walter T., *Differential Equations* (2nd ed.), Reading, MA: Addison-Wesley Publishing Co., 1988.

Stansfield, Ed and Sutherland, Alan. "Exchange Rate Realignments and Realignment Expectations," *Oxford Economic Papers* 47(1995): 211-28.

Takayama, Akira, *Analytical Methods in Economics*. New York: Harvester Wheatsheaf, 1994.

Taylor, Mark P., "The Economics of Exchange Rate," *Journal of Economic Literature*, 33 (1995a): 13-47.

_____, "Exchange-Rate Behavior under Alternative Exchange-Rate Arrangements," in chapter 2 of Peter B. Kenen (ed.) *Understanding Interdependence: The Macroeconomics of the Open Economy*. Princeton. NJ: Princeton University Press, 1995b: 34-83.

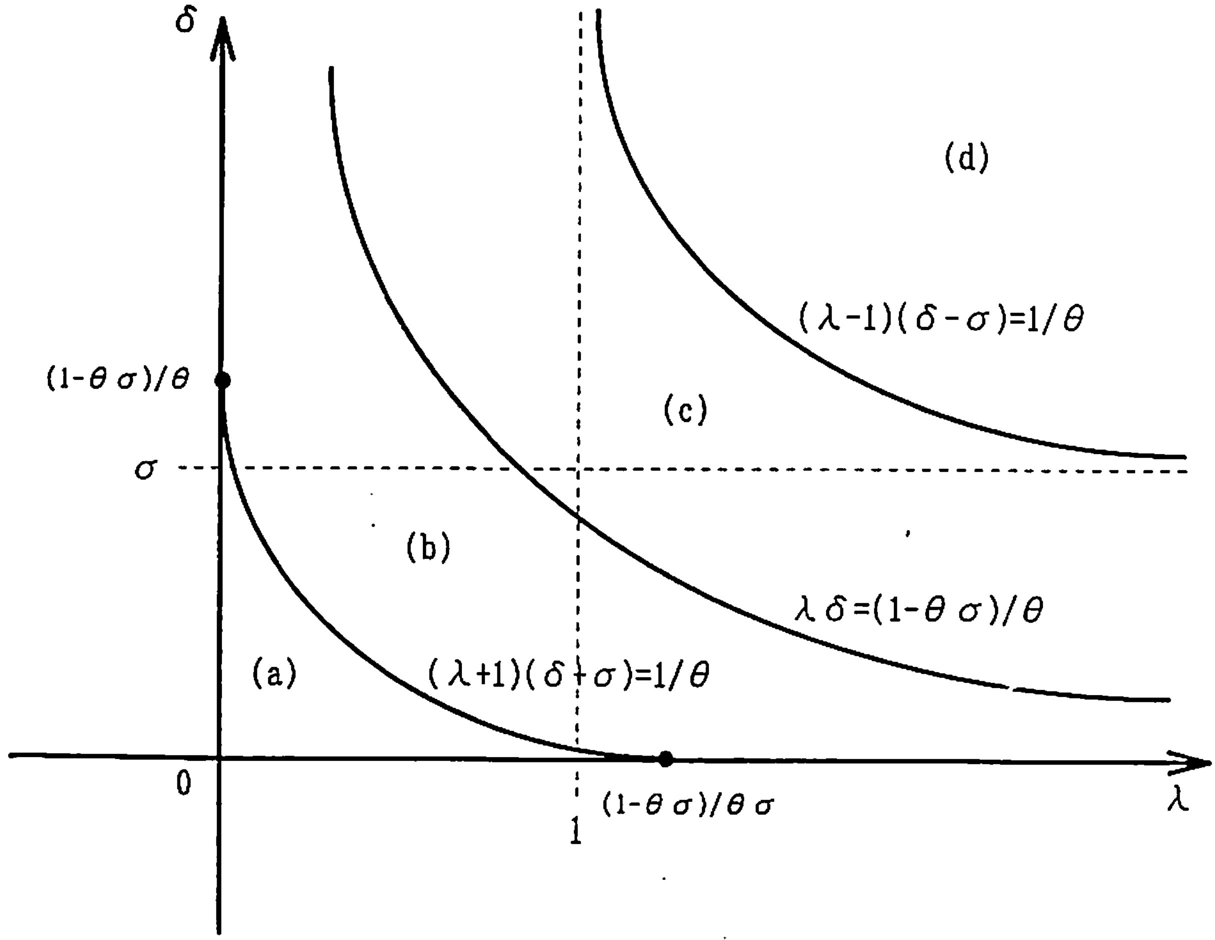

Figure 1 Parameter regions for stability

area (a) det (log H) > 0 and trace (log H) < 0

(b) det (log H) < 0 and trace (log H) < 0

(c) det (log H) < 0 and trace (log H) > 0

(d) det (log H) > 0 and trace (log H) > 0

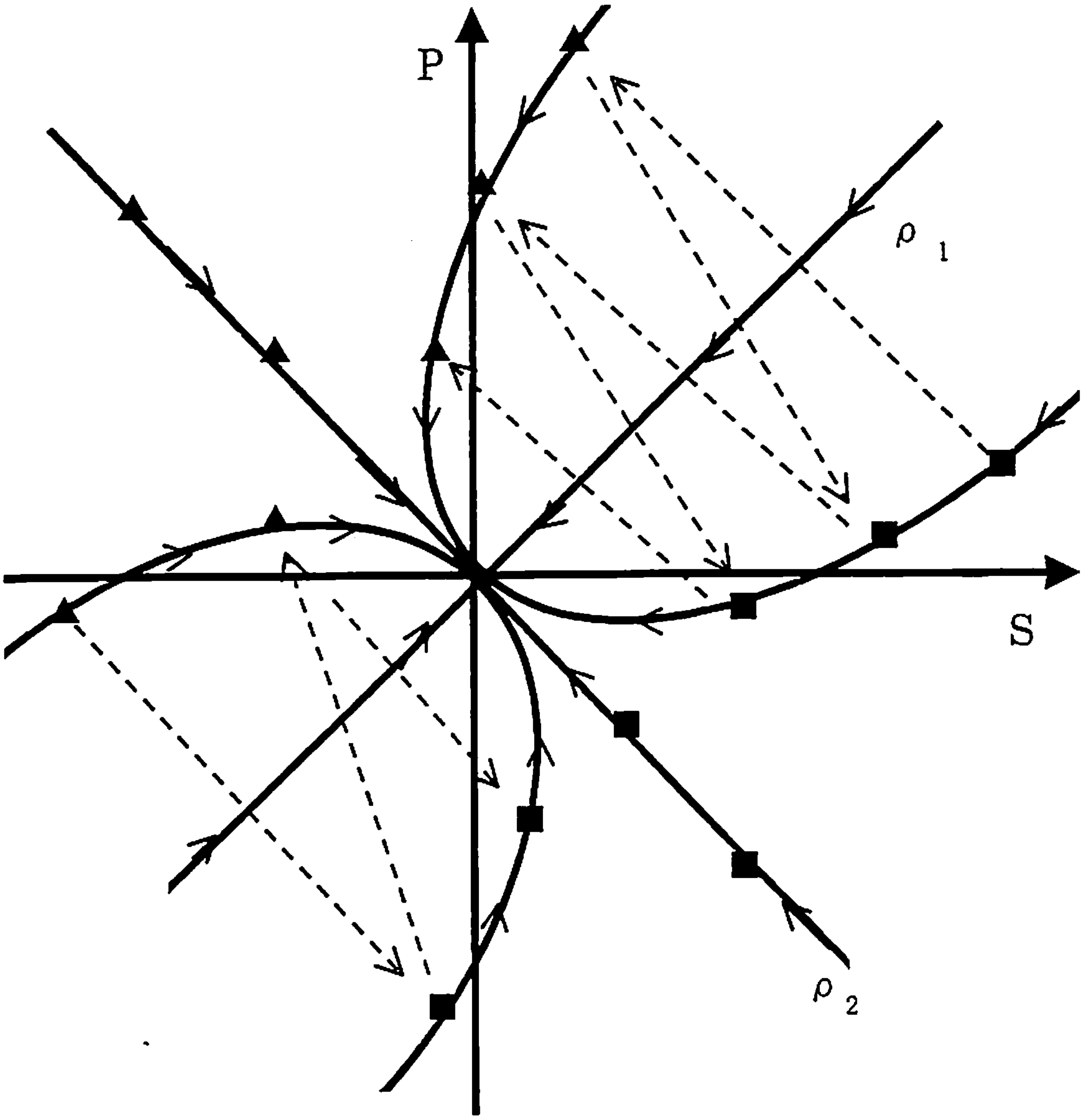

Figure 2 Stable Solutions

■: t even
▲: t odd