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Eliminations of Dominated Strategies and Inessential Players: an Abstraction Process[*] [†]

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Abstract

We study the process of iterated elimination of strictly dominated strategies and inessential players from a finite strategic game, abbreviated as the IEDI process A resulting finite sequence from this process is called a W-IEDI; and if all the dominated strategies and inessential players are eliminated at each step, it is called the IEDI. First, we show that any W-IEDI preserves Nash equilibrium (and many other solution concepts). The second result, an extension of the order-independence theorem, is that the IEDI is the shortest and smallest W-IEDI with the same resulting endgame. We have the third result about necessary and sufficient conditions on possible shapes and lengths for IEDS's to a given endgame. The conditions indicate a great variety of sequences possibly generated by the IEDI process. We interpret those results from the perspective of abstracting from social situations.

Key Words: Finite Strategic Form Games, Dominated Strategies, Inessential Players, Iterated Elimination, Order-Independence

1. Introduction

Elimination of dominated strategies is a basic notion in game theory, and its relationships to other solution concepts such as rationalizability have been extensively discussed (cf., Osborne-Rubinstein [15], and Maschler *et al.* [8]). Its nature, however, differs from other solution theories: It suggests negatively what would/should not be played, while other concepts suggest and predict what would/should be chosen in game situations. In this paper, we study elimination of dominated strategies and of *inessential players* whose unilateral changes of strategies do not affect any player's payoffs including

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his own. In this introduction, first, we describe this elimination process, and second consider its implications from the perspective of abstracting a social/game situation.

1.1. Elimination process of dominated strategies and inessential players

We consider elimination of strictly dominated strategies and inessential players in finite strategic form games, as well as their iterations. Elimination of dominated strategies has a long history from Gale *et al.* [4], but also has been studied extensively recently. We mention two results in the literature that are relevant to this paper.

One is the *preservation theorem,* presented in Maschler *et al.* [8], Theorem 4.35, that Nash equilibria are faithfully preserved in the elimination process. The other result is better known in the literature: The elimination process results in the same endgame regardless of the order of elimination of dominated strategies. This *order-independence theorem* was a kind of a folk theorem in the literature. A proof was given in Gilboa *et al.* [5], and also by T. Börgers and M. Stegman around 1990 (see Börgers [3]). Apt [1], [2] provide a comprehensive treatment of this theorem. First, we present generalizations of these two results in our framework, allowing elimination of inessential players, too; these are Theorem 2.1 (preservation) and Theorem 3.2 (smallest and shortest).

Then, we give another theorem on the possible shapes and lengths of sequences generated in the IEDI process leading to a given endgame, which is Theorem 4.1 (possible-shape).

Elimination of inessential players is newly introduced in this paper: A player is *inessential* iff his unilateral changes in strategies do not affect any players' payoffs including his own. We consider a possible finite sequence of games generated in the IEDI process from a given game. Such a sequence is called a *W-IEDI-sequence,* or *W-IEDI* for short, where each element of the sequence is generated from the previous element by eliminations of dominated strategies and of inessential players. When all dominated strategies and inessential players are eliminated at each step, we call such a sequence an *IEDI sequence*, or *IEDI* for short. We are interested in the shapes and lengths of such sequences as well as the resulting outcomes.

Elimination can be applied in several orders, but it is shown in Lemma 2.3 that one particular order is more effective than the others. We take the order of elimination of dominated strategies and then of inessential players.

The preservation result (Theorem 2.1) is simply obtained for any W-IEDI. Our smallest-shortest result (Theorem 3.2) states that from any given game, the IEDI is the shortest and smallest among all W-IEDI's, and the resulting endgames are identical, which is the order-independence mentioned above.

The third result (Theorem 4.1) describes possible shapes and lengths of the IEDI's. First, we derive a set of necessary conditions for a sequence being IEDI. It appears far from sufficient conditions, but actually, they are sufficient in the sense that when

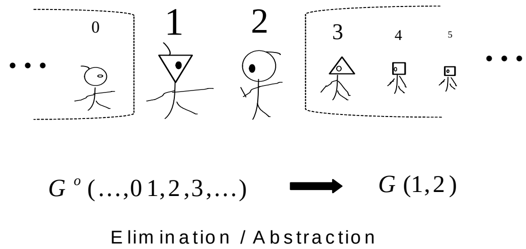

Figure 1.1: Relevant and Irrelevant People

those are satisfied, we can construct a game so that the IEDI generated from it meets them. For 2-person games, these conditions give specific information about the possible IEDI's, but for the games with more than 2 players, they are not very restrictive. Thus, we find a large variety of IEDI's (and W-IEDI's).

1.2. Choices of Relevant Actions/Players in an Abstraction Process

A social situation is a complex system containing a lot of seemingly relevant and/or irrelevant components, depending upon the perspective of a focus. We, social scientists, focus on a target situation, by choosing relevant components and eliminating irrelevant ones. The standard economics textbooks start with this methodological view:

> "*... An economic model or theory is a simplified representation of how the economy, or parts of the economy, behave under particular conditions. In building a model, economists do not try to explain every detail of the real world. Rather, they focus on the most important influences on behavior because the real world is so complex*" (Thompson [17], p.10).

An analysis in game theory/economic theory, however, starts after such abstraction is already completed. Our study can be viewed as a segment of this abstraction process; assuming that a social situation is abstracted as strategic games, we focus on the iterated elimination of dominated strategies and inessential players.

As described in Figure 1.1, our social world consists of many players, who are interdependent upon each other. Some interdependencies are significant but many are not. For example, the Battle of the Sexes is a 2-person game, while the corresponding social situation may include other boys and girls. When we have the Battle of the Sexes as an appropriate abstraction of the situation, we drop the other boys and girls. Let us consider a 3-person extension of the Battle of the Sexes.

Example 1.1 (Battle of the Sexes with the 2nd Boy). Consider the Battle of the Sexes situation including boys 1, 2, and a girl, 3. Each boy $i = 1, 2$ has two strategies, $\mathbf{s}_{i1}, \mathbf{s}_{i2}$, and girl 3 has four strategies, $\mathbf{s}_{31}, ..., \mathbf{s}_{34}$. Boy 1 and girl 3 date at the boxing arena ($\mathbf{s}_{11} = \mathbf{s}_{31}$) or the cinema ($\mathbf{s}_{12} = \mathbf{s}_{32}$), but make decisions independently. Now, another boy, 2, enters to this scene: Girl 3 can date with player 2, in a different arena ($\mathbf{s}_{21} = \mathbf{s}_{33}$) or cinema ($\mathbf{s}_{22} = \mathbf{s}_{34}$). When 1 and 3 consider their date, they would be happy even if they fail to meet; player 2' choice does not affect their payoffs at all. Also, we assume that when 2 thinks about the case that 3 chooses dating with boy 1, boy 2 is indifferent between the arena and cinema. The same indifference is assumed for boy 1 when 3 chooses dating with 2. This indifference assumption is assumed in this paper, but is possibly relaxed, which is discussed in Section 5.

A faithful description of the game is to have two 2×4 matrices, since each of 1 and 2 has two strategies and 3 has four. Due to the assumption that each's payoffs depend only upon the result of dating, the payoff matrices are described as Tables 1.1 and 1.2. The numbers in the parentheses in Table 1.1 are 2's payoffs. The dating situation for 2 and 3 is parallel to that for 1 and 3; only player 3 is much less happy than dating with player 1.

Now, player 3's two strategies $\mathbf{s}_{33}$ and $\mathbf{s}_{34}$ are dominated by $\mathbf{s}_{31}$ and $\mathbf{s}_{32}$. We eliminate those dominated strategies, and the resulting game is still a 3-person game. However, player 2 is inessential in the sense that 3 thinks only about dating with 1 and player 2's choice does not affect the players' payoffs at all. Thus, we can eliminate him as an inessential player, and get the Battle of the Sexes between 1 and 2.

Table 1.1

1\3 (2)	$\mathbf{s}_{31}$	$\mathbf{s}_{32}$
$\mathbf{s}_{11}$	15,10 (-10)	5,5 (-5)
$\mathbf{s}_{12}$	5,5 (-5)	10,15 (-10)

Table 1.2

2\3 (1)	$\mathbf{s}_{33}$	$\mathbf{s}_{34}$
$\mathbf{s}_{21}$	15,1 (-10)	5,0 (-5)
$\mathbf{s}_{22}$	5,0 (-5)	10,2 (-10)

Our study of the elimination process can be interpreted from the inside player's view as well as the outside analyst's view; the former interpretation requires a player's understanding of a situation, particularly, upon, experiential beliefs/knowledge on the situation, which can be understood from the viewpoint of inductive game theory (Kaneko-Kline [6]). However, this requires many restrictions on a player's experiences and understanding about the situation. In this paper, we do not consider such restrictions; instead, we take the outside analyst's perspective, i.e., consider no restrictions on iterations of eliminations of dominated strategies and inessential players.

The *preservation* result (Theorem 2.1) implies that eliminations of dominated strategies and inessential players do not affect the Nash equilibrium analysis after abstraction. The *smallest-shortest* result (Theorem 3.2) states that the iterated eliminations of the all the dominated strategies and then that of inessential players are the most efficient process from the perspective of the lengths of generated sequences and set-theoretical sizes of their components. However, this sequence is not necessarily the most efficient as far as the number of preference comparisons are concerned, which is briefly mentioned in Section 5. Finally, the *possible-shape* result (Theorem 4.1) suggests that behind the abstracted game, there are vast number of possible social situations which result in the same abstracted game.

The paper is organized as follows: Section 2 gives basic definitions of dominance, an inessential player and some game reduction concepts. Also, we present the preservation theorem. Section 3 defines the IEDI process, W-IEDI and IEDI sequences, and proves our version of the order-dependence theorem. Section 4 gives and proves the possible-shape theorem (Theorem 4.1). In Section 5, we return to our original motivation stated above, and discuss the difficulties raised by our considerations from the viewpoint of the outside analyst and that of an inside player.

2. Eliminations of Dominated Strategies and Inessential Players

We define three types of reductions of a game by elimination of dominated strategies and of inessential players, but we show that one type is more effective than the other two types. We show that the Nash equilibria is faithfully preserved in this process.

2.1. Basic definitions

Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a finite *strategic game*, where N is a finite set of *players*, S_i is a finite nonempty set of *strategies*, and $h_i : \Pi_{j\in N} S_j \to \mathbb{R}$ is a *payoff function* for player $i \in N$. We allow N to be empty, in which case the game is the *empty game*, denoted as $G_\emptyset$.

We use the following notation: Let I be a subset of N. Then, we may denote $s \in S_N := \Pi_{j\in N} S_j$ as $(s_I; s_{N-I})$, where $s_I = \{s_j\}_{j\in I}$ and $s_{N-I} = \{s_j\}_{j\in N-I}$. When $I = \{i\}$, we write S_{-i} for $S_{N-\{i\}}$ and $(s_i; s_{-i})$ for $(s_I; s_{N-I})$.

Let G be given, and $s_i, s_i' \in S_i$. We say that s_i' *dominates* s_i in G iff $h_i(s_i'; s_{-i}) > h_i(s_i; s_{-i})$ for all $s_{-i} \in S_{-i}$. When s_i is dominated by some s_i', we simply say that s_i is *dominated* in G.

We say that i *is* an *inessential player* in G iff for all $j \in N$,

$$h_j(s_i; s_{-i}) = h_j(s_i'; s_{-i}) \text{ for all } s_i, s_i' \in S_i \text{ and } s_{-i} \in S_{-i}. \tag{2.1}$$

A choice by i does not affect any player's payoffs including i's own, provided the others' strategies are arbitrarily fixed. We find a weaker version of this concept in Moulin [10]; he requires j to be i only. From the viewpoint of player i's own decision making, once i becomes inessential in this weak sense, he may stop thinking about his choice. However, his choice may still affect the others' payoffs; in this case, i's choice is still relevant to them. Some examples of inessential players are discussed below[1].

We should be careful about the domains of the payoff functions when the player set changes. In fact, (2.1) can be extended to an arbitrary set of inessential players, which is stated in the following lemma.

Lemma 2.1. Let I be a set of inessential players. Then, for all $j \in N$,

$$h_j(s_I; s_{N-I}) = h_j(s'_I; s_{N-I}) \text{ for all } s_I, s'_I \in S_I \text{ and } s_{N-I} \in S_{N-I}. \tag{2.2}$$

Proof. Let $I = \{i_1, ..., i_k\}$, and $I_t = \{i_1, ..., i_t\}$ for $t = 1, ..., k$. Also, let $s, s' \in S_N$ be arbitrarily fixed. We prove $h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$ by induction on $t = 1, ..., k$. Since $s, s' \in S_N$ are arbitrary, for $t = k$ this implies (2.2). The base case, i.e.., $h_j(s_{i_1}; s_{-i_1}) = h_j(s'_{i_1}; s_{-i_1})$, is obtained from (2.1). Suppose $h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$. Since $s = (s_{I_t}; s_{N-I_t}) = (s_{I_{t+1}}; s_{N-I_{t+1}})$, we have $h_j(s_{I_{t+1}}; s_{N-I_{t+1}}) = h_j(s_{I_t}; s_{N-I_t})$. Applying (2.1) to $h_j(s'_{I_t}; s_{N-I_t})$, we have $h_j(s'_{I_t}; s_{N-I_t}) = h_j(s'_{I_{t+1}}; s_{N-I_{t+1}})$. By the induction hypothesis, we now have $h_j(s_{I_{t+1}}; s_{N-I_{t+1}}) = h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$ $= h_j(s'_{I_{t+1}}; s_{N-I_{t+1}})$. Thus, we have the assertion for $t + 1$. ■

Let I be a set of inessential players in G, $N' = N - I$, and let i be any player in N'. The *restriction* h'_i of h_i to $\Pi_{j \in N'} S'_j$ with $\emptyset \neq S'_j \subseteq S_j$ for $j \in N'$ is defined by

$$h'_i(s_{N'}) = h_i(s_{N'}; s_{N-N'}) \text{ for all } s_{N'} \in S'_{N'} \text{ and } s_{N-N'} \in S_{N-N'}. \tag{2.3}$$

The well-definedness of h'_i is guaranteed by Lemma 2.1.

We say that $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ is a *D-reduction* of G iff the components of G' satisfy:

DR1: $N' \subseteq N$ and any $i \in N - N'$ is an inessential player in G;

DR2: for all $i \in N'$, $S'_i \subseteq S_i$ and any $s_i \in S_i - S'_i$ is a dominated strategy in G;

DR3: h'_i is the restriction of h_i to $\Pi_{j \in N'} S'_j$.

A D-reduction allows simultaneous eliminations of dominated strategies and inessential players. However, it would be easier to separate between these eliminations than to treat them at the same time.

First, we restrict a D-reduction as follows: Let G be a game, and G' a D-reduction of G. When $N' = N$ holds in DR1, G' is called a *ds-reduction* of G, denoted as $G \rightarrow_{ds} G'$.

[1] Also, the concept of an inessential player may look related to the condition on the payoff functions, called the *transference of decision maker indifference*, due to Marx-Swinkels [9], p.5, which states that if two strategies s_i, s'_i for player i have unilaterally the same effects for s_{-i}, the others have the same effects. This is relative to two strategies s_i, s'_i, and s_{-i}, while an inessential player i has no effects on his own and the others' payoffs by his own unilateral changes.

When $S_i - S'_i = \{s_i : s_i$ is a dominated strategy for i in $G\}$ in DR2, it is called the *strict* *ds*-reduction of G.

Returning to a D-reduction G' of G, when $S'_i = S_i$ for all $i \in N'$ in DR2, G' is called an *ip-reduction* of G, denoted by $G \to_{ip} G'$, and when $N - N' = \{i : i$ is an inessential player in $G\}$ in DR2, it is called the *strict ip*-reduction of G.

In this paper, we choose the order of applications of a *ds*-reduction and an *ip*-reduction. Hence, we have the following definition: We say that G' is a *DI-compound reduction,* or a *DI-reduction* of G for short, iff there is an *interpolating* game $\underline{G}$ such that $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$. This allows trivial cases, e.g., $G = \underline{G}$ or $\underline{G} = G'$. When $\underline{G}$ is the strict *ds*-reduction of G and G' is the strict *ip*-reduction of $\underline{G}$, we say that G' is the *strict DI*-reduction of G.

We have another compound reduction: We say that G' is an *ID-reduction* of G iff $G \to_{ip} \underline{G} \to_{ds} G'$ for some $\underline{G}$. Lemma 2.3. shows the equivalence between D-reductions and *ID*-reductions.

Lemma 2.2 states that if a dominated strategy in G remains in a subgame G' of G, then it is still dominated in G'. This is called *hereditarity* in the literature (cf. Apt [2]). A parallel fact holds for an inessential player. The third assertion states that eliminations of only inessential players do not generate new dominated strategies. On the other hand, Example 2.2 shows that only eliminations of inessential players may generate new inessential players.

Lemma 2.2. Let $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ be a D-reduction of G.

(1): If $s_i \in S'_i$ $(i \in N')$ is dominated in G, so is in G'.

(2): If $i \in N'$ is an inessential player in G, so is in G'.

(3): Suppose that $S'_i = S_i$ for all $i \in N'$. Let $s_i \in S_i$ and $i \in N'$. Then, a strategy s_i is dominated in G if and only if it is dominated in G'.

Proof. We prove (1); (2) is similarly proved. Suppose that s_i is dominated by s'_i in G. Then, $h_i(s'_i; s_{N-i}) > h_i(s_i; s_{N-i})$ for all $s_{N-i} \in S_{N-i}$. We can assume without loss of generality that s'_i is not a dominated strategy in G, so $s'_i \in S'_i$. We have, by (2.3), for all $s_{N-N'} \in S_{N-N'}$, $h'_i(s'_i; s_{N'-i}) = h_i(s'_i; s_{N'-i}; s_{N-N'}) > h_i(s_i; s_{N'-i}; s_{N-N'})$ $= h'_i(s_i; s_{N'-i})$ for all $s_{N'-i} \in S'_{N'-i}$. Thus, s_i is dominated by s'_i in G'.

(3): Let us show the *if* part. Suppose that s_i is dominated by s'_i in G'. Then, $h'_i(s'_i; s'_{N'-i})$ $> h'_i(s_i; s'_{N'-i})$ for all $s'_{N'-i} \in S'_{N'-i}$. By assumption, we have $S'_{N'-i} = S_{N'-i}$. Let $s'_{N'-i}$ be an arbitrary element in $S'_{N'-i} = S_{N'-i}$. We have, by (2.3), for all $s_{N-N'} \in S_{N-N'}$, $h_i(s'_i; s'_{N'-i}; s_{N-N'}) = h'_i(s'_i; s'_{N'-i}) > h'_i(s_i; s'_{N'-i}) = h_i(s_i; s'_{N'-i}; s_{N-N'})$ for all $s_{N'-i} \in S'_{N'-i}$. Thus, s_i is dominated by s'_i in G. ∎

Lemma 2.3 states that *ID*-reductions are equivalent to *D*-reductions, but *DI*-reductions are more effective than others. The converse of (2) does not hold.

Lemma 2.3.(1): G' is a D-reduction of G if and only if G' is an ID-reduction of G.

(2): If G' is a D-reduction of G, then G' is a DI-reduction of G.

Proof. (1):(Only-If): Let G' be a D-reduction of G. It follows from Lemma 2.2.(1) that we can postpone and separate eliminations of dominated strategies from eliminations of inessential players. Hence, G' can be an ID-reduction.
(If): Let G' be an ID-reduction of G, i.e., $G \rightarrow_{ip} \underline{G} \rightarrow_{ds} G'$ for some $\underline{G}$. Lemma 2.2.(3) states that $\underline{G}$ has the same set of dominated strategies as G. Hence, we can combine these two reductions to one, which yields the D-reduction G'.

(2): Since D is a set of dominated strategies in G, we can eliminate them from G, and we have $\underline{H}$, i.e., $G \rightarrow_{ds} \underline{H}$. By Lemma 2.2.(2), the inessential players in G remain inessential. Hence, we eliminate $N - \underline{N}$ from N in $\underline{H}$. This game is the same as G' and $\underline{H} \rightarrow_{ip} G'$. Hence, G' is a DI-reduction. ■

This is illustrated by the following example.

Example 2.1 (Large and Small Stores). Consider the game G_0 of Figure 2.1, which is interpreted as follows: 1 is a large supermarket, 2 is a small mart; and 1 ignores 2. Here, neither player is inessential, but $\mathbf{s}_{12}$ is dominated. By eliminating $\mathbf{s}_{12}$, we have the second $\underline{G}_0$, where player 1 is inessential, and by eliminating him, we get G_1. Since $\mathbf{s}_{22}$ is dominated there, we have $\underline{G}_1$. Finally, 2 is eliminated, and we get the empty game $G_\emptyset$. The game G_1 is a DI-reduction of G_0, but not a D-reduction. Also, the last $G_\emptyset$ is a DI-reduction of G_1. The game $\underline{G}_1$ is a D-reduction of $\underline{G}_0$.

$$\left(\underset{G_0}{\begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline \mathbf{s}_{11} & 30,1 & 30,0 \\ \hline \mathbf{s}_{12} & 20,0 & 20,1 \\ \hline \end{array}} \underset{ds}{\rightarrow} \underset{\underline{G}_0}{\begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline \mathbf{s}_{11} & 30,1 & 30,0 \\ \hline \end{array}}\right) \underset{ip}{\rightarrow} \left(\underset{G_1}{\begin{array}{|c|c|c|}\hline 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline & 1 & 0 \\ \hline \end{array}} \underset{ds}{\rightarrow} \underset{\underline{G}_1}{\begin{array}{|c|c|}\hline 2 & \mathbf{s}_{21} \\ \hline & 1 \\ \hline \end{array}}\right) \underset{ip}{\rightarrow} G_\emptyset$$

Figure 2.1

In this example, eliminations of dominated strategies generate new inessential players. Eliminations of inessential players may generate new inessential players, too.

Example 2.2 (Elimination of Inessential Players, only). The leftmost 2-person game has no dominated strategies, but player 1 is inessential. By eliminating 1, we have the second 1-person game, and by eliminating 2, we have the empty game.

$$\begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline \mathbf{s}_{11} & 4,6 & 2,6 \\ \hline \mathbf{s}_{12} & 4,6 & 2,6 \\ \hline \end{array} \underset{ip}{\rightarrow} \begin{array}{|c|c|c|}\hline 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline & 6 & 6 \\ \hline \end{array} \underset{ip}{\rightarrow} G_\emptyset.$$

Figure 2.2

2.2. Preservation of Nash equilibria

We study the IEDI-process to eliminate irrelevant players as well as irrelevant actions for some players. From the perspective of a negative criterion, it could be required that such eliminations should lose no essential features of the target social situation. Here, we show that this is the case with respect to the Nash equilibrium as well as many other solution concepts.

We say that $s \in S$ is a *Nash equilibrium* in a nonempty game G iff for all $i \in N$, $h_i(s) \geq h_i(s_i'; s_{-i})$ for all $s_i' \in S_i$. Let θ be the *null symbol,* i.e., for any $s \in S$, we set $(\theta; s) = s$, and stipulate that the restriction of s to the empty game $G_\emptyset$ is the null symbol θ. Also, we stipulate that θ is the Nash equilibrium in $G_\emptyset$.

We have the following basic theorem, stating that eliminations of dominated strategies and inessential players do not affect Nash equilibria. In the case of eliminations of only dominated strategies, the theorem is reduced to the one given in Maschler *et al.* [8], Theorem 4.35, p.109.

Theorem 2.1 (Preservation of Nash Equilibria). Let G' be a D-reduction of G. Then,

(1) if s_N is an NE in G, then its restriction $s_{N'}$ to G' is an NE in G';

(2) if $s_{N'}$ is an NE in G', $(s_{N'}; s_{N-N'})$ is an NE in G for any $s_{N-N'}$ in $\Pi_{j \in N-N'} S_j$.

Proof. **(1):** Let s be an NE in G. For any $i \in N$, we have $h_i(s_i; s_{-i}) \geq h_i(s_i'; s_{-i})$ for any $s_i' \in S_i$. Let $i \in N'$. Then, s_i is not dominated in G, and thus, $s_i \in S_i'$. Let $s_i' \in S_i'$. Since G' is a D-reduction, we have $h_i'(s_i; s_{N'-i}) = h_i(s_i; s_{N-i}) \geq h_i(s_i'; s_{-i}) = h_i'(s_i'; s_{N'-i})$. Thus, $s_{N'}$ is an NE in G'.

(2): Let $s_{N'}$ be an NE in G'. We choose any $s_{N-N'} \in S_{N-N'}$. We let $G^o = (N, \{S_i'\}_{i \in N}, \{h_i\}_{i \in N})$, where $S_j' = S_j$ for all $j \in N - N'$. First, we show that this $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Let $i \in N'$. We have $h_i'(s_{N'}') = h_i(s_{N'}'; s_{N-N'})$ for any $s_{N'}' \in S_{N'}'$ by Lemma 2.1, since the players in $N - N'$ are inessential in G. Since $s_{N'}$ is an NE in G', we have $h_i(s_i; s_{N'-i}; s_{N-N'}) = h_i'(s_i; s_{N'-i}) \geq h_i'(s_i'; s_{N'-i}) = h_i(s_i'; s_{N'-i}; s_{N-N'})$ for all $s_i' \in S_i'$. Let $i \in N - N'$. Then since i is inessential, we have $h_i^o(s_i; s_{N'-i}; s_{N-N'}) = h_i^o(s_i'; s_{N'-i}; s_{N-N'})$ for all $s_i' \in S_i^o$. Hence, $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Suppose that $i \in N'$ has a strategy s_i'' in G so that $h_i(s_i''; s_{N-i}) > h_i(s_i; s_{N-i})$. We can choose such an s_i'' giving the maximum $h_i(s_i''; s_{N-i})$. Then, this s_i'' is not dominated in G. Hence, s_i'' remains in G', which contradicts that $s_{N'}$ is an NE in G'. ■

Let $NE(G)$ and $NE(G')$ be the sets of Nash equilibria for a game G and its D-reduction G'. It follows from Theorem 2.1 that $NE(G)$ and $NE(G')$ are connected by:

$$NE(G) = \Pi_{j \in N-N'} S_j \times NE(G'). \tag{2.4}$$

When G' is the empty game $G_\emptyset$, Theorem 2.1.(1) states that the resulting outcome is the null symbol θ, and (2) states that any strategy profile $s = (\theta; s)$ in G is a Nash equilibrium in G. In Section 3.2, this will be used to make a comparison with the notion of d-solvability of Moulin [10], [11].

Let us apply (2.4) to the sequence of games in Figure 2.1. By (2.4), $(\mathbf{s}_{21})$ is the unique NE in $\underline{G}_1$, and so is in G_1. By (2.4) again, $(\mathbf{s}_{11}, \mathbf{s}_{21})$ is the NE in $\underline{G}_0$, and also is the NE in G_0. In the case of Figure 2.2, the set of NE's moves from θ to S_2 and from S_2 to $S_1 \times S_2$.

D-reduction preserves not only Nash equilibria, but also the Nash [12] noncooperative solution. We say that the set of NE's satisfies the *interchangeability* iff for any NE s and s', $(s'_i; s_{-i})$ is also an NE for all $i \in N$. He regards the game as *solvable* iff interchangeability holds for it. In fact, this is equivalent to that the set of NE's is expressed as the product of Nash strategies. In the case of game G' of (2.4), G' is solvable if and only if $NE(G')$ is expressed as $NE(G') = \Pi_{j \in N'} NE_j(G')$, where $NE_j(G') = \{s_j \in S'_j : (s_j; s_{N'-j}) \in NE(G')\}$ for $i \in N'$. By (2.4), G' is solvable if and only of so is G. Thus, Theorem 2.1 implies preservation for the Nash noncooperative theory.

The above theorem holds with respect to the mixed strategy Nash equilibrium, too. Preservations of other solution concepts such as rationalizability and correlated equilibrium hold. So far, we have only positive results as far as pure noncooperative solution concepts are concerned[2,3,4].

If we consider weak dominance rather than strict dominance, preservation does not hold. Table 2.1 is a counter example, where there are two Nash equilibria $(\mathbf{s}_{11}, \mathbf{s}_{21})$ and $(\mathbf{s}_{12}, \mathbf{s}_{22})$. If we adopt weak dominance, only $(\mathbf{s}_{12}, \mathbf{s}_{22})$ remains but $(\mathbf{s}_{11}, \mathbf{s}_{21})$ is eliminated.

Table 2.1

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$1, 1$	$1, 1$
$\mathbf{s}_{12}$	$1, 0$	$2, 2$

[2] The folk theorem may be regarded as a negative example for this. But if the above argument is applied to the repeated game itself, it would not be a coutner example.

[3] The solution concept called the *intraperson coordination equilibriun* in Kaneko-Kline [7] is regarded as a noncooperative solution concept, but it is incompatible with elimination of dominated strategies. An example for nonpreservation is a Prisoner's Dilemma.

[4] Theorem 2.1 my look related to the consistency property due to Peleg-Sudhöter's [16] for an axiomatization of Nash equilibria. They use the term "reduced game" to restrict a strategy profile to a subset of the player set by fixing the other players' strategies specified by the profile Thus, although the player sets vary, the concepts are very different.

3. The IEDI Process and Generated Sequences

Here, we consider the iterated elimination process of dominated strategies and inessential players (IEDS process). In Section 3.1, we provide definitions, while in Section 3.2, we present the smallest-shortest result on the IEDS sequence, which is also an extension of the order-independence theorem (cf., Apt [2]).

3.1. W-IEDI and IEDI sequences

We define a sequence generated by the IEDI process by means of DI-reductions. There are the other two alternative ways to define this process using D-reductions and ID-reductions. We consider these after Theorem 3.1.

Let G be a given finite game. We say that $\langle G^0, G^1, \ldots, G^\ell \rangle$ is a *W-IEDI sequence from* $G = G^0$ iff

$$G^{t+1} \text{ is a } DI\text{-reduction of } G^t \text{ and } G^{t+1} \neq G^t \text{ for each } t = 0, \ldots, \ell - 1; \tag{3.1}$$

$$G^\ell \text{ has no dominated strategies and no inessential players.} \tag{3.2}$$

We call ℓ the *length* of $\Gamma(G)$. We abbreviate a W-IEDI sequence simply as a W-IEDI. We denote the set of all W-IEDI's from G by $\mathbb{W}(G)$.

In particular, a W-IEDI $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ is said to be the *IEDI sequence* iff

$$G^{t+1} \text{ is the strict } DI\text{-reduction of } G^t \text{ for all } t = 0, \ldots, \ell - 1. \tag{3.3}$$

A W-IEDI may not be unique, but the IEDI is uniquely determined by a given G. The IEDI from G is denoted by $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$. We will show that this is the smallest and shortest W-IEDI generated from G.

In Example 2.1, Figure 2.1 shows the unique W-IEDI; *a fortiori,* it is the IEDI. It has the length 2 and is described as the sequence $\langle G^0, G^1, G^2 \rangle = \langle G^{*0}, G^{*1}, G^{*2} \rangle$, where $G^2 = G_\emptyset$. The game $\underline{G}^0$ is the interpolating game between G^0 and G^1. The other interpolating game is $\underline{G}^1$. For Example 2.2, the IEDI is represented in Figure 2.2. In this sequence, both steps involve the elimination of an inessential player. Here, there are no non-trivial interpolating games.

Let us return to the 3-person $G = (\{1, 2, 3\}, \{S_i\}_{i=1}^3, \{h_i\}_{i=1}^3)$ of Example 1.1.

Example 3.1 (Continued from Example 1.1). The IEDI is represented in Figure 3.1. Player 3's strategies $\mathbf{s}_{33}$ and $\mathbf{s}_{34}$ are dominated by both $\mathbf{s}_{31}$ and $\mathbf{s}_{32}$, and by eliminating $\mathbf{s}_{33}$ and $\mathbf{s}_{34}$, we get the second interpolating 3-person game. Now, players 1 and 3 focus on their dating, ignoring player 2 as inessential. By eliminating him, we get the

2-person battle of the Sexes with 1 and 3.

$$\left(G^0 \underset{ds}{\rightarrow} \begin{array}{|c|c|c|} \hline 1\backslash 2\backslash 3 & \mathbf{s}_{31} & \mathbf{s}_{32} \\ \hline \mathbf{s}_{11} & 15,-10,10 & 5,\text{-}5,\ 5 \\ \hline \mathbf{s}_{12} & 5,\ -5,\ 5 & 10,-10,15 \\ \hline \end{array} \right) \underset{ip}{\rightarrow} \begin{array}{|c|c|c|} \hline 1\backslash 3 & \mathbf{s}_{31} & \mathbf{s}_{32} \\ \hline \mathbf{s}_{11} & 15,10 & 5,\ 5 \\ \hline \mathbf{s}_{12} & 5,\ 5 & 10,15 \\ \hline \end{array}$$

$\mathbf{s}_{21}$ or $\mathbf{s}_{22}$

Figure 3.1

A W-IEDI sequence may be partitioned into two segments, $G^0, G^1, \ldots, G^{m_o}$ and $G^{m_o+1}, \ldots, G^\ell$ so that in the first segment, dominated strategies (and, maybe, inessential players) are eliminated, and in the second, only inessential players are eliminated, which is illustrated in (3.4).

$$\Gamma(G) = \langle \underbrace{G^0, G^1, \ldots, G^{m_o}}, \ \underbrace{G^{m_o+1}, ..., G^\ell} \rangle. \tag{3.4}$$

However, even if G^t has some dominated strategies, (3.1) may allow no dominant strategies to be eliminated. We exclude this possibility: We say that $\Gamma(G) = \langle G^0, G^t, \ldots, G^\ell \rangle$ is *proper* iff for all $t = 0, ..., \ell - 1$, if G^t has some dominated strategies and $G^t \rightarrow_{ds} \underline{G}^t \rightarrow_{ip} G^{t+1}$, then $G^t \neq \underline{G}^t$. When G^t has no dominated strategies, some inessential players are eliminated. The IEDI $\Gamma^*(G)$ is proper.

The following theorem holds for any proper W-IEDI.

Theorem 3.1 (Partition of a Proper W-IEDI). Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ be a proper W-IEDI from $G^0 = G$. There is an exactly one m_o $(0 \leq m_o \leq \ell)$ such that

(i) for any t $(0 \leq t < m_o - 1)$, at least one dominated strategy is eliminated in the step from G^t to G^{t+1};

(ii) for any t $(m_o \leq t \leq \ell - 1)$, no dominated strategies are eliminated but at least one inessential player is eliminated in the step from G^t to G^{t+1}.

Proof. Suppose that G^t has no dominated strategies. Then, G^{t+1} is obtained from G^t by eliminating inessential players. It follows from Lemma 2.2.(3) that G^{t+1} has no dominated strategies. Hence, we choose the smallest m_o among such t's for m_o. ■

We call the m_o given by Theorem 3.1 the *elimination divide*. Example 2.2, where $m_o = 0$, implies that the second segment may have a length greater than 1. The elimination divide m_o plays an important role in Section 4.

Let us apply Theorem 2.1 to a W-IEDI $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$. Then, $NE(G^0)$ can be written as:

$$NE(G^0) = \Pi_{j \in N^0 - N^1} S_j^0 \times \cdots \times \Pi_{j \in N^{\ell-1} - N^\ell} S_j^{\ell-1} \times NE(G^\ell). \tag{3.5}$$

Indeed, let $\underline{G}^t = (N^t, \{\underline{S}_i^t\}_{i \in N^t}, \{\underline{h}_i^t\}_{i \in N^t})$ be the interpolating game between G^t to G^{t+1} for each t, it follows from (2.4) that $NE(G^t) = NE(\underline{G}^t)$ and $NE(\underline{G}^t) = \Pi_{j \in N^t - N^{t+1}} \underline{S}_j^t \times NE(G^{t+1})$ for $t = 0, ..., \ell - 1$. Repeating this decomposition from $\ell - 1$, we have (3.5).

3.2. The Shortest and smallest: the IEDI sequence

The order-independence theorem states that when we restrict the reduction steps to eliminations of dominated strategies, any W-IEDI has the same endgame (cf., Gilboa *et al.* [5], and Apt [1], [2]). Here, we extend this result, allowing iterated eliminations of inessential players, too. We show that for any finite game G, the IEDI generated from G is the shortest-smallest among the W-IEDI's and has the same endgame.

To make comparisons between two finite games, we introduce the concept of a subgame. We say that $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ is a *subgame* of $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ iff (i) $N' \subseteq N$; (ii) $S'_i \subseteq S_i$ for all $i \in N'$; and (iii) for $i \in N'$, $h'_i : \Pi_{j \in N'} S'_j \to \mathbb{R}$ is given by (2.3). For the purpose of references, we state the following immediate result:

$$\text{if } G' \text{ is a } D\text{-reduction of } G, \text{ then } G' \text{ is a subgame of G.} \tag{3.6}$$

The subgame relation has the following property.

Lemma 3.1 (Partial Ordering). The subgame relation is a partial ordering over the set of all finite games.

Proof. It is reflexive, anti-symmetric, and transitive. Here, we consider only transitivity. Let G', G'' be subgames of G, G', respectively. It suffices to show that $h''_i(s_{N''}) = h_i(s_{N''}; s_{N-N''})$ for all $s_{N''} \in S''_{N'}$ and $s_{N-N''} \in S_{N-N''}$. Let s_N be an arbitrary strategy profile in S_N. We have $h''_i(s_{N''}) = h'_i(s_{N''}; s_{N'-N''}) = h'_i(s_{N'}) = h_i(s_{N'}; s_{N-N'}) = h_i(s_{N''}; s_{N-N''})$. The first and third equalities are due to (2.3) for G' and G'' and for G and G'. The second and fourth are simply changes of expressions. ■

When $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ is a W-IEDI, it follows from (3.6) and Lemma 3.1 that if $t < k$, then G^k is a subgame of G^t.

The following theorem states that the IEDI sequence is the smallest and shortest in $\mathbb{W}(G)$ and that its endgame is identical over those of W-IEDI sequences in $\mathbb{W}(G)$. We present the proof of the theorem in the end of this section.

Theorem 3.2 (shortest and smallest). Let G be a finite game, and let $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$ be the IEDI from G. Then, for any W-IEDI $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle \in \mathbb{W}(G)$,

(1): $\ell^* \leq \ell$; **(2)**: for each $t \leq \ell^*$, G^{*t} is a subgame of G^t; and **(3)**: $G^{\ell^*} = G^\ell$.

Thus, the IEDI $\Gamma^*(G)$ is the shortest and smallest in $\mathbb{W}(G)$. We have other elimination processes adopting different reductions such as *D*- and *ID*- reductions. The IEDI $\Gamma^*(G)$ based on *DI*- reductions is shorter and smaller than the sequences based on *D*- or *ID*- reductions by Lemma 2.3. Another possible process is to apply only *ds*-reductions up to the elimination divide m_0, and then apply *ip*-reductions: This may be strictly longer than the length of $\Gamma^*(G)$ as far as we count each compound *DI*-reduction step as one. We can take the IEDI as the benchmark case for a further analysis of iterated elimination of dominated strategies and inessential players; indeed, we use it in Section 4.

We adopt strict dominance for Theorem 3.2. Apt [1], [2] gives comprehensive discussions on order-independence theorems for various types of dominance relations[5].

It follows from Theorem 2.2 that if G^{ℓ^*} has a Nash equilibrium, then so does G. If G^{ℓ^*} is the empty game, which has the Nash equilibrium θ, then G has a Nash equilibrium. Moreover, these do not depend on the choice of a W-IEDI from G. Hence, the decomposition given in (3.5) is independent from the choice of a W-IEDI.

This is related to Moulin's [10], [11] d-solvability: We say that a game G is *d-solvable* iff there is a sequence $\langle G^0, ..., G^\ell \rangle$ with $G^0 = G$, $G^{t-1} \to_{ds} G^t$ for $t = 1, ..., \ell - 1$, and for all $i \in N$,

$$h_i^\ell(s_i; s_{-i}) = h_i^\ell(s_i'; s_{-i}) \text{ for all } s_i, s_i' \in S_i \text{ and } s_{-i} \in S_{-i}. \tag{3.7}$$

This requires constant payoffs for each player i with his unilateral deviation. Now, we have the following corollary.

Corollary 3.3. If a game G has a W-IEDI $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ with $G^\ell = G_\emptyset$, then G is d-solvable.

Proof. By Theorem 3.2, it suffices to consider the IEDI $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$ with $G^{\ell^*} = G_\emptyset$. Then, each G^{*t} is obtained from $G^{*(t-1)}$ so that $G^{*(t-1)} \to_{ds} \underline{G}^{(t-1)} \to_{ip} G^{*t}$ for some interpolating game $\underline{G}^{(t-1)}$. After the elimination divide m_0, only $\to_{ip}$ is applied to G^{*t}. Now, we can ignore eliminations of inessential players in this sequence. The resulting sequence is denoted by $\langle H^0, H^1, \ldots, H^{m_0} \rangle$: (1) $H^0 = G^{*0} = G$; and (2) for $t = 1, ..., m_0$, H^t is obtained from H^{t-1} by eliminating all the dominated strategies that are eliminated in $G^{*(t-1)} \to_{ds} \underline{G}^{(t-1)}$. Then, each H^t has the full set of players N. Since $G^{*\ell^*} = G_\emptyset$, G^{*m_0} has only inessential players. Hence, in H^{m_0}, all the players in N are inessential, so *a fortiori,* (3.7) holds. ■

The converse of Corollary 3.3 may not hold: Table 3.1, given in Moulin [11], is d-solvable, but this does not generate a W-IEDI to the empty game.

Table 3.1

1\2	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	1,1	0,1
$\mathbf{s}_{12}$	1,0	0,0

Now, let us prove Theorem 3.2. First, we refer to Newman's lemma (see also Apt [2]). An *abstract reduction system* is a pair $(X, \to)$, where X is an arbitrary nonempty

[5] It is well-known that the order-independence theorem does not hold for weak dominance, (cf., Myerson [14], p.60).

set and $\to$ is a binary relation on X. We say that $\{x_\nu : \nu = 0, ..., \lambda\}$ is a $\to$ *sequence* in $(X, \to)$ iff for all $\nu \geq 0$, $x_\nu \in X$ and $x_\nu \to x_{\nu+1}$ if $\nu < \lambda$. It may be the case that λ is infinite (i.e., $\lambda = \omega$). We use $\to^*$ to denote the transitive reflexive closure of $\to$.

Let an abstract reduction system $(X, \to)$ be given. We say that $y \in X$ is an *endpoint* for $\to$ iff there is no $z \in X$ such that $y \to z$. We say that $(X, \to)$ is *weakly confluent* iff for each $x, y, z \in X$ with $x \to y$ and $x \to z$, there is some $x' \in X$ such that $y \to^* x'$ and $z \to^* x'$.

Lemma 3.3 (Newman [13]): Let $(X, \to)$ be an abstract reduction system satisfying N1: each $\to$ sequence in X is finite; and N2: $(X, \to)$ is weakly confluent. Then, for any $x \in X$, there is a unique endpoint y with $x \to^* y$.

Proof of Theorem 3.2: First, we show (3). Let $\mathcal{G}$ be the set of all finite strategic form games. Then $(\mathcal{G}, \to_{DI})$ is an abstract reduction system, where we write $G \to_{DI} G'$ for $G \to_{ds} \underline{G}$ and $\underline{G} \to_{id} G'$ for some interpolating $\underline{G}$ and $G \neq G'$.Then, each $\to_{DI}$sequence is finite, i.e., we have N1, since at least one strategy or one inessential player is eliminated in each transition. We can see N2 as follows: Let $G, G', G'' \in \mathcal{G}$ with $G \to_{DI} G'$ and $G \to_{DI} G''$. Now, let G^* be the strict DI-reduction of G. Then, G^* is a DI-reduction of both G' and G''. Hence, $G' \to_{DI} G^*$ and $G'' \to_{DI} G^*$. Then it follows from Lemma 3.3 that for any $G \in \mathcal{G}$, there is a unique endpoint $G^{*\ell^*}$ with $G^\ell = G^{*\ell^*}$.

Now, we show (1) and (2). Let us see the following:

$$G^{*t} \text{ is a subgame of } G^t \text{ for each } t = 0, \ldots, \min(\ell, \ell^*). \tag{3.8}$$

We prove this by mathematical induction on t. When $t = 0$, this holds by definition. Suppose that it holds for $t < \min(\ell, \ell^*)$. Let $G^{*t} \to_{ds} \underline{G}^{*t} \to_{ip} G^{*t+1}$ and $G^t \to_{ds} \underline{G}^t \to_{ip} G^{t+1}$. Then if a strategy s_i in G^{*t} is dominated in G^t, it is also dominated in G^{*t}, since G^{*t} is a subgame of G^t. For the same reason, if a player i in G^{*t} is inessential in $\underline{G}^t$, then i is also inessential in G^{*t}, We obtain G^{*t+1} by eliminating all the dominated strategies in G^{*t} and all the inessential players in $\underline{G}^{*t}$. Hence it follows that G^{*t+1} is a subgame of G^{t+1}. Then (3.8) holds by the principle of mathematical induction.

Consider (1): By (3), $G^\ell = G^{\ell^*}$. If $\ell < \ell^*$, then $G^{*(\ell+1)}$ is a strict subgame of $G^\ell = G^{\ell^*}$ by (3.8). But by definition, $G^{*\ell^*}$is a subgame of $G^{*(\ell+1)}$; hence we get a contradiction. This proves (1). Then, we have (2) by (3.8). ■

4. Possible Shapes of IEDI Sequences

In Section 3, we studied the IDEI and W-IEDI's generated from a given game $G = G^0$. This approach is depicted in the top of Figure 4.1. In this section, we reverse the focus: We consider possible IEDI sequences to a given final game. In the bottom of Figure 4.1, H is a given final game, and we have a number of IEDI's that can lead to H.

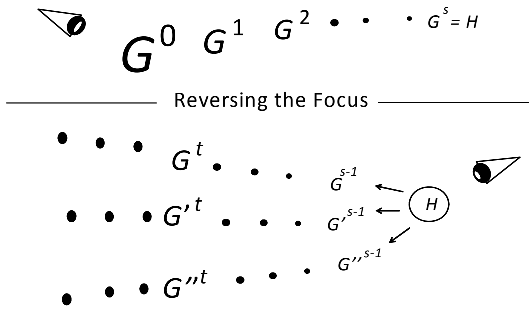

Figure 4.1: Start with the Final Game

We give necessary and sufficient conditions with respect to the player set and the set of players having dominated strategies. Throughout this section, we consider only the IEDI sequences.

4.1. Possible shapes of IEDI sequences

Consider the IEDI $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell} \rangle$ from $G = G^{*0}$, where $G^{*t} = (N^t, \{S_i^t\}_{i \in N^t}, \{h_i^t\}_{i \in N^t})$ for $t = 0, ..., \ell$. For each $t = 0, ..., \ell$, let

$$T^t := \{i \in N^t : \text{player } i \text{ has a dominated strategy in } G^{*t}\}. \tag{4.1}$$

We call T^t the *D-group* in G^{*t}. Recall the elimination divide m_o given by Theorem 3.1, i.e., $T^t \neq \emptyset$ if $t < m_o$ and $T^t = \emptyset$ if $t \geq m_o$. Then, we call $\left[(N^0, T^0), \ldots, (N^\ell, T^\ell)\right]$ the *player-configuration* of $\Gamma^*(G)$. This represents the structure of changes in players in $\Gamma^*(G)$. We focus only on the changes in the player sets and D-groups.

The following lemma gives simple observations, which turn out to be sufficient conditions to get an IEDI sequence from some game G.

Lemma 4.1(Necessary Conditions). Let $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell} \rangle$ be the IEDI with its elimination divide m_0 and player-configuration $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$. Then,

PC0: $T^t \subseteq N^t$ for $t = 0, ..., \ell$;

PC1: $N^0 \supseteq ... \supseteq N^{m_o} \supsetneq N^{m_o+1} \supsetneq ... \supsetneq N^\ell$ with $\left|N^\ell\right| \neq 1$;

PC2: for $t = 1, ..., m_o$, if $|T^{t-1}| = 1$, then $T^{t-1} \cap T^t = \emptyset$;

PC3: $T^{m_o} = T^{m_o+1} = \ldots = T^\ell = \emptyset$.

Proof. PC0 follows from the definition of T^t, and PC3 follows the definition of m_o.

PC1: Up to m_o, eliminations of inessential players may not occur; thus, we have weak inclusion relations up to m_o. After m_o, some inessential players are eliminated, and thus, we have strict inclusion after m_o. As remarked before, we have $|N^\ell| \neq 1$.

PC2: Let $T^{t-1} = \{i\}$. If $i \notin N^t$, then $i \notin T^t$, so *a fortiori*, $T^{t-1} \cap T^t = \emptyset$. Suppose $i \in N^t$. Then, let $G^{*(t-1)} \to_{ds} \underline{G}^{*(t-1)} \to_{ip} G^{*t}$. Then, all the dominated strategies for player i in $G^{*(t-1)}$ are eliminated in $\underline{G}^{*(t-1)}$. By Lemma 2.2.(3), player i has no dominated strategies in G^{*t}. Hence, $T^{t-1} \cap T^t = \emptyset$. ■

Lemma 4.1 derives four conditions on the player-configuration from the IDES $\Gamma(G)$ generated from $G = G^{*0}$. The following theorem reverses this lemma: A proof is given in Section 4.2.

Theorem 4.1 (Possible Shapes): Let $H = (N^H, \{S_i^H\}_{i \in N^H}, \{h_i^H\}_{i \in N^H})$ be a game with no dominated strategies and no inessential players. Let $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ be any sequence satisfying PC0-PC3 with $N^\ell = N^H$. Then, there exists a game G with the IEDI $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$ generated from $G = G^{*0}$ such that

(a) $G^{*\ell} = H$;

(b) $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ is the player-configuration of $\Gamma^*(G)$.

Conditions PC0-PC3 are sufficient for $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ to have the IEDI $\Gamma^*(G)$ so that its player-configuration coincides with $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$. Since PC0-PC3 are not really restrictive, the class of IEDI's leading up to a particular H can have a great variety of lengths and shapes.

Let us consider PC0-PC3 for the 2-person case. We get the following corollary.

Corollary 4.2. Let G be a 2-person game, and $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell} \rangle$ the IEDI with its elimination divide m_0 and player-configuration $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$. Then,

(1) $\ell - m_o \leq 2$;

(2) $N^0 = ... = N^{m_o - 2} = \{1, 2\}$;

and there is some $k_o \leq m_o - 2$ such that

(3) $T^t = \{1, 2\}$ if $t \leq k_o$; and $|T^t| = 1$ if $t = k_o + 1, ..., m_o - 1$;

(4) $T^t \cap T^{t+1} = \emptyset$ for $t = k_o + 1, ..., m_o - 1$.

G^{*0}	$\cdots$	G^{*k_o}	$G^{*(k_o+1)}$	$G^{*(k_o+2)}$	$\cdots$	$G^{*(m_o-1)}$	G^{*m_o}	$G^{*(m_o+1)}$
(N, N)	$\cdots$	(N, N)	$(N, \{i\})$	$(N, \{j\})$	$\cdots$	$(N, \{i\})$	$(\{i\}, \emptyset)$	$(\emptyset, \emptyset)$

Figure 4.2

Thus, up to G^{k_o}, the D-group remains $T^t = N = \{1, 2\}$, but once T^{k_o+1} becomes a singleton set, condition PC2 requires T^{k_o+2} to be empty or consists of the other player.

In the latter case, T^t starts alternating at G^{k_o+1} up to G^{m_o-1}. Then, the process continues for possibly two more steps, but may stop at G^{m_o}. Theorem 4.1 implies that there are no other possibilities for the 2-person case. Figure 4.2 describes one possibility.

The above monotonicity is observed only for the 2-person case. The following is a 3-person game, where T^0 is a singleton, but T^2 becomes the entire set.

Example 4.1 (Nonmonotonicity). Consider the following 3-person game G where each player has three strategies, and the payoffs are described by the following three tables. The IEDI from this game is as follows: In the game $G^0 = G$, player 3 has the dominated strategy $\mathbf{s}_{33}$, and in the resulting game G^1 from the elimination of $\mathbf{s}_{33}$, 1 and 2 have dominated strategies $\mathbf{s}_{13}$ and $\mathbf{s}_{23}$. Here, the game G^2 obtained from G^1 by eliminating $\mathbf{s}_{13}$ and $\mathbf{s}_{23}$ has still three players, each of whom has 2 strategies. Game G^2 is expressed by the northwest corner (PD) of each table, where each has a dominated strategy. Here, the player-configuration is $\left[(N^0,T^0),\ldots,(N^3,T^3)\right]$, where $N^0 = N^1 = N^2 = \{1,2,3\}$, $N^3 = \emptyset$, $T^0 = \{3\}, T^1 = \{1,2\}, T^2 = \{1,2,3\}, T^3 = \emptyset$, and $m_o = 3$.

Table 4.1, $\mathbf{s}_{31}$

1\2	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,2	1,6,2	3,0,1
$\mathbf{s}_{12}$	6,1,2	3,3,2	1,0,1
$\mathbf{s}_{13}$	0,3,2	0,1,1	0,0,2

Table 4.2, $\mathbf{s}_{32}$

1\2	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,0	1,6,0	3,0,2
$\mathbf{s}_{12}$	6,1,0	3,3,0	1,0,1
$\mathbf{s}_{13}$	0,3,1	0,1,1	0,0,2

Table 4.3, $\mathbf{s}_{33}$

1\2	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,0	1,6,0	3,9,0
$\mathbf{s}_{12}$	6,1,0	3,3,0	1,9,0
$\mathbf{s}_{13}$	9,3,0	9,1,0	9,9,0

4.2. Proof of Theorem 4.1

Consider a sequence $[(N^0,T^0),\ldots,(N^\ell,T^\ell)]$ and $H = (N^H, \{S_i^H\}_{i\in N^H}, \{h_i^H\}_{i\in N^H})$ in the theorem. We construct a sequence $G^{*\ell}, G^{*\ell-1},\ldots,G^{*0}$ from $G^{*\ell} = H$ along (N^ℓ,T^ℓ), ..., (N^0,T^0), and show that for each $t = \ell-1,...,0$, G^{*t+1} is the strict *DI*-reduction of G^{*t}; thus, $\langle G^{*0},...,G^{*\ell}\rangle$ is the IEDI generated from G^{*0}.

G^{*t}	$\rightarrow_{ds}$	$\underline{G}^{*t}$	$\rightarrow_{ip}$	G^{*t+1}
(N^t,T^t)	$\Longleftarrow$(construction)		$\Longleftarrow$(construction)	(N^{t+1},T^{t+1})
	Lemmas 4.3, 4.4		Lemma 4.2	

Figure 4.4

Lemma 4.2 is for the construction of the interpolating $\underline{G}^{*t}$ from G^{*t+1} in Figure 4.4.

Lemma 4.2. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with $|S_i| \geq 2$ for all $i \in N$, and let I' be a nonempty set of new players. Then, there is a $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ such that (1): $N' = N \cup I'$; (2): $|S'_i| \geq 2$ for all $i \in N'$; and (3): G is the strict *ip*-reduction of G'.

Proof. We choose the strategy sets $S_i, i \in N'$ so that $S'_i = S_i$ for all $i \in N$ and $S'_i = \{\alpha, \beta\}$ for all $i \in I'$, where α, β are new symbols not in G. Then, we define the payoff functions $\{h'_i\}_{i \in N'}$ so that the players in I' are inessential in G' but no players in N are inessential in G'. Let I be the set of inessential players in G. For each $i \in I$, we choose an arbitrary strategy, say $\mathbf{s}_{i1}$ from S_i. Then, we define $\{h'_i\}_{i \in N'}$ as follows:

(a): for any $j \in I'$, $h'_j(s_{N'}) = |\{i \in I : s_i = \mathbf{s}_{i1}\}|$ for $s_{N'} \in S_{N'}$;

(b): for any $j \in N$, $h'_j(s_{N'}) = h_j(s_N)$ for $s_{N'} \in S_{N'}$, where s_N is the restriction of $s_{N'}$ to N.

For any $j \in I'$, j's strategy s_j is nominal in (a) and (b) in the sense that s_j does not appear substantively in h'_i for any $i \in N \cup I'$. Thus, the players in I' are all inessential in G'. On the other hand, each $i \in I$, as far as such a player exists in G, affects j's payoffs for $j \in I'$ because of (a) and $|S_i| \geq 2$. This means that any $i \in I$ is not inessential in G'. Also, any $i \in N - I$ is not inessential in G' by (b). Thus, only the players in I' are inessential. In sum, G is the strict *is*-reduction of G'. ■

Now, we consider the step from $\underline{G}^{*t}$ to G^{*t} in Figure 4.4. For this construction, first we show the following lemma. In the following, we write s_j dom s'_j when s_j dominates s'_j in G.

Lemma 4.3. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be an n-person game, and $j \in N$ a fixed player. There are real numbers $\{\pi_j(s_j)\}_{s_j \in S_j}$ such that

$$\text{if } s_j \text{ dom } s'_j, \text{ then } \pi_j(s_j) < \pi_j(s'_j). \tag{4.2}$$

Proof: The relation dom is transitive and asymmetric. We call a sequence $\{s_j^1, ..., s_j^m\}$ a *descending chain* from s_j^1 to s_j^m iff s_j^k dom s_j^{k+1} for $k = 1, ..., m-1$.

We say that s_j is *maximal* in (S_j, dom) iff there is no $s'_j \in S_j$ such that s'_j dom s_j. Let $s_j^0, ..., s_j^k$ be the list of maximal elements in (S_j,dom). Then, we define the sets $A(s_j^0), ..., A(s_j^k)$ inductively by

$$A(s_j^0) = \{s_j^0\} \cup \{s_j \in S_j : s_j^0 \text{ dom } s_j\}; \tag{4.3}$$

$$A(s_j^l) = \{s_j^l\} \cup \{s_j \in S_j - \cup_{t=0}^{l-1} A(s_j^t) : s_j^l \text{ dom } s_j\} \text{ for } l \leq k. \tag{4.4}$$

That is, we classify each $s_j \in S_j - \{s_j^0, ..., s_j^k\}$ to the first $A(s_j^t)$ with s_j^t dom s_j, which implies

$$\text{if } s_j^t \text{ dom } s_j \text{ and } s_j \in A(s_j^{t'}), \text{ then } t' \leq t. \tag{4.5}$$

Thus, these sets $A(s_j^0), ..., A(s_j^k)$ form a partition of S_j.

Now, we define $\{\pi_j(s_j)\}_{s_j \in S_j}$ as follows: for $s_j \in A(s_j^t)$ and $t = 0, ..., k$,

$$\pi_j(s_j) = -t\,|S_j| + l_{s_j}, \tag{4.6}$$

where l_{s_j} is the maximum length of a descending chain from s_j^t to $s_j \neq s_j^t$, and is 0 if $s_j = s_j^t$. When $k = 0$, l_{s_j} may be equal to $|S_j|$, but when $k > 0$, l_{s_j} is smaller than $|S_j|$.

Now, we show (4.2). Let $s_j, s_j' \in S_j$ and s_j dom s_j'. Also, let $s_j \in A(s_j^t)$ and $s_j' \in A(s_j^{t'})$. Since s_j^t dom s_j, we have s_j^t dom s_j', which implies $t' \leq t$ by (4.5). Now, we consider two cases: $t' = t$ and $t' < t$. First, suppose $t = t'$. Let $l_{s_j}, l_{s_j'}$ be, respectively, the maximal lengths of descending chains from s_j^t to s_j and s_j'. Since s_j dom s_j', we have $l_{s_j} < l_{s_j'}$. Thus, $\pi_j(s_j) = -t\,|S_j| + l_{s_j} < \pi_j(s_j') = -t\,|S_j| + l_{s_j'}$. For the other case, suppose $t' < t$. Since $|S_j| > l_{s_j}, l_{s_j'}$ as remarked above, we have $\pi_j(s_j') - \pi_j(s_j) = -t'\,|S_j| + l_{s_j'} - (-t\,|S_j| + l_{s_j}) = (t - t')\,|S_j| + (l_{s_j'} - l_{s_j}) > 0$. ■

Now, we go to the step from $\underline{G}^{*t}$ to G^{*t} in Figure 4.4; in the lemma, G and G' are supposed to be $\underline{G}^{*t}$ and G^{*t} respectively.

Lemma 4.4. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with N, and let $T \subseteq N$ satisfying the following condition:

$$\text{if } T = \{i\}, \text{ then there are no } s_i, s_i' \in S_i \text{ with } s_i \text{ dom}_G\ s_i'. \tag{4.7}$$

Then, there is a game $G' = (N, \{S_i'\}_{i \in N}, \{h_i\}_{i \in N})$ such that T is the D-group for G' and G is the strict ds-reduction of G'.

Proof. First, let β_j be a new strategy symbol for each $j \in T$. We define $\{S_j'\}_{j \in N}$ as follows:

$$S_j' = \begin{cases} S_j \cup \{\beta_j\} & \text{if } j \in T \\ S_j & \text{if } j \in N - T \end{cases} \tag{4.8}$$

Then we extend h_j to $h_j' : \Pi_{i \in N} S_i' \to \mathbb{R}$ for each $j \in N$ so that the restriction of h_j' to $\Pi_{i \in N} S_i$ is h_j itself and G is the strict ds-reduction of G'.

To be precise, we define the payoff functions $\{h_j'\}_{j \in N}$. Let $j \in N$. First, h_j' is the same as h_j over $\Pi_{i \in N} S_i$, i.e., $h_j'(s) = h_j(s)$ if $s \in \Pi_{i \in N} S_i$. Now, let $s \in S' - S$, if $j \in N - T$, then

$$h_j'(s) = \pi_j(s_j); \text{ where } \pi_j(s_j) \text{ is given for } G \text{ in Lemma 4.3,} \tag{4.9}$$

and if $j \in T$, then

$$h_j'(s) = \begin{cases} \pi_j(s_j) & \text{if } s_j \neq \beta_j \\ \min\{\pi_j(t_j) : t_j \in S_j\} - 1 & \text{if } s_j = \beta_j. \end{cases} \tag{4.10}$$

First, let $j \in N - T$, and let $s_j, s'_j \in S_j = S'_j$. Suppose that s_j dominates s'_j in G. Then, consider $s, s' \in S' - S$ so that the j-th components of s and s' are s_j and s'_j. By (4.9), we get $h'_j(s) = \pi_j(s_j) < \pi_j(s'_j) = h'_j(s')$. Hence, s_j does not dominate s'_j in G', which implies that j has no dominated strategies in G'.

Second, let $j \in T$. We choose an $s^*_j \in S_j$ with $s^*_j \neq \beta_j$. By (4.10), we have, for any $s_{-j} \in S_{-j}$,

$$h'_j(\beta_j; s_{-j}) = \min\{\pi_j(t_j) : t_j \in S_j\} - 1 < \pi_j(s^*_j) = h'_j(s^*_j; s_{-j}).$$

This does not depend upon s_{-j}; thus, s^*_j dominates β_j in G'. From the analysis of the two cases, we can conclude that T is the D-group in G'.

It remains to show that s_j does not dominates s'_j in G' for any $s_j, s'_j \in S_j = S'_j - \{\beta_j\}$ and $j \in T$. If s_j does not dominate s'_j in G, then in G', s_j does not do s'_j. Now, we suppose that s_j dominates s'_j in G. By (4.7), we have $|T| > 1$. This guarantees that the existences of $s, s' \in S' - S$ such that their j-th components are s_j and s'_j. Then, by (4.10), we have $h'_j(s) = \pi_j(s_j) < \pi_j(s'_j) = h'_j(s')$. Hence, s_j does not dominates s'_j in G'. From these, we conclude that G is the strict ds-reduction of G'. ■

Now, we can prove the Theorem 4.1.

Proof of Theorem 4.1: Let $G^{*\ell} = H$. Since H has no dominated strategies and no inessential players, condition (3.2) holds.

Suppose that G^{*t+1} is already defined with $\left|S_i^{t+1}\right| \geq 2$ for all $i \in N^{t+1}$. Condition PC2 guarantees condition (4.7). By Lemma 4.2, we find an interpolating game $\underline{G}^{*t}$ so that G^{*t+1} is the strict ip-reduction of $\underline{G}^{*t}$ with its player set N^t and $\left|\underline{S}_i^t\right| \geq 2$ for all $i \in N^t$. By Lemma 4.4, we find another game G^{*t} so that $\underline{G}^{*t}$ is the strict ds-reduction of G^{*t} with its D-group T^t and satisfying $\left|S_i^t\right| \geq 2$ for all $i \in N^t$.

Now, we have an IEDI $\Gamma^*(G) = \left\langle G^{*0}, ..., G^{*\ell}\right\rangle$ such that $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ is the player-configulation of $\Gamma^*(G)$. ■

5. Conclusions

We have considered the process of iterated elimination of strictly dominated strategies and inessential players. Iterated elimination of inessential players is newly introduced in this paper, and is quite compatible with elimination of dominated strategies. The three main results given in this paper are: Theorem 2.2 (preservation), Theorem 3.1 (smallest and shortest) and Theorem 4.1 (possible shape).

The preservation theorem is a direct extension of the result given in Maschler *et al.* [8], and leads to the recovering result (3.5) on Nash equilibria. This result is important from the perspective of the abstraction process.

The second theorem is an extension of the order-independence theorem and states that any sequences generated from a given game by the IEDI process ends up with the same game and that the IEDI sequence is the shortest and smallest among the WIEDI's from any given game G.

The third theorem is entirely new. It gives necessary and sufficient conditions for possible shapes of IEDI's. They provide some specific structural information on the shapes of generated sequences, and imply that IEDI's have a vast variety of lengths and shapes.

In this paper, we have not touched upon the preference comparisons required to calculate the IEDI sequence or a W-IEDI sequence from a game G. The development given in this paper, however, facilitates this consideration, which may be interpreted as implying that the IEDI requires less than any other W-IEDI. In fact, we have an example of a game where some W-IEDI sequence can be calculated by a smaller number of preference comparisons than the IEDI. A detailed study is an open problem.

Finally, we return to the perspective of abstracting social situations. The preservation theorem is relevant for this. That is, the Nash equilibrium concept gives a positive criterion for decision making by a player and/or prediction by an outside analyst. In inductive game theory (cf., Kaneko-Kline [6]), an inside player takes this perspective.

From either perspective, we find an apparent restriction: The definition of an inessential player here is too stringent in that his unilateral changes have no effect at all on any player's payoffs. There are two directions to weaken this restriction. One direction is to consider one player's effects on some players separately, and the other is to introduce ε-inessential players. Both are captured together as follows: An ε-inessential player j against i is defined iff j's unilateral changes in strategies may affect only i's payoffs within ε-magnitudes for a given $\varepsilon > 0$. By this definition, we may capture "remote" players relative to a player in question such as players behinds the main players in Figure 1.1. This includes already a lot of aspects, each of which remains open. The study given in this paper is the benchmark for this consideration.

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