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Abstract

The Bondareva-Shapley condition is the most eminent necessary and sufficient condition for the core of a transferable utility game to be nonempty. In this paper, we provide a new necessary and sufficient condition. We show that a game has a nonempty core if and only if the game can be decomposed into some simple games. We demonstrate the decomposition through an example and the proof.

Keywords: cooperative game; core; decomposition

JEL Classification: C71

1 Introduction

The core probably is the most well-known stability concept in game theory and economics. The core of a cooperative game is defined as the set of allocations from which no coalition has an incentive to deviate.

The first necessary and sufficient condition for the core of a transferable utility game (TU-game) to be nonempty has been provided by Bondareva (1963) and Shapley (1967) as an application of linear programing problem. In their works, the notion of balanced collection features the nonemptiness of the core. After their result, a game that has a nonempty core is called a balanced game. As for TU-games, this Bondareva-Shapley condition has been the only necessary and sufficient condition that is applicable to all TU-games.[*1]

Scarf (1967) and Billera (1970) have extended the condition to nontransferable utility games (NTU-games). They call their conditions balancedness and π-balancedness respectively, while their conditions are sufficient conditions for the core to be nonempty in a NTU game. A necessary and sufficient condition has been recently provided by Predtetchinski and Herings (2004). In their work, they furthermore extend π-balancedness and establish the necessary and sufficient condition which they call Π-balancedness.

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*1 Abe and Funaki (2017) have extended the Bondareva-Shapley condition to the class of partition function form games.

As outlined above, the conditions for the core to be nonempty have been developed in line with the notion of balancedness. The purpose of this paper is to provide this context with a new approach. The approach we propose in this paper is to “decompose” a TU-game into some “easy” games. We show that it is necessary and sufficient for the core of a TU-game to be nonempty that decomposed games satisfy certain basic properties. The novelty is that our approach and proof are independent from linear programing and the concept of balancedness.

In the next section, we define TU-games and basic notions. We offer our condition in Section 3. An illustration of the decomposition is also provided. In Section 4, we conclude this paper with some remarks. The main proof is provided in Section 5.

2 Preliminaries

Let $N = \{1, \cdots, n\}$ be a finite set of players and a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ denote a characteristic function. A coalition of players is a nonempty subset of the player set: $S \subseteq N$. We denote the cardinality of coalition S by $|S|$. A cooperative game with transferable utility (a TU-game) is (N, v). We fix the player set N throughout this paper and typically use v instead of (N, v) to denote a game. Let $\mathcal{G}_N$ be the set of all TU-games with the player set N. For notational simplicity, we sometimes write, for example, coalition 12 instead of $\{1, 2\}$. We define the **core** as follows: for any game $v \in \mathcal{G}_N$,

$$C(v) = \left\{ x \in \mathbb{R}^N | \sum_{j \in S} x_j \geq v(S) \text{ for any } S \subseteq N, \sum_{j \in N} x_j = v(N) \right\}.$$

Below is the list of properties of a game that we use in this paper.

- A game v is **balanced** if $C(v) \neq \emptyset$.
- A game v is **simple** if $v(S) = 0$ or 1 for every $S \subseteq N$.
- A player i is a **veto player** in game v if $v(S) = 0$ for every $S \subseteq N$ with $i \notin S$.
- A game v is **veto-controlled** if there is a veto player in v.
- A game v is **N-monotonic** if $v(S) \leq v(N)$ for every $S \subseteq N$.
- A game v is **zero-normalized** if $v(\{i\}) = 0$ for every $i \in N$. For any game $v \in \mathcal{G}_N$, $\bar{v}$ is v's zero-normalized game if $\bar{v}(S) = v(S) - \sum_{j \in S} v(\{j\})$ for any $S \subseteq N$. In general, $C(\bar{v}) = \{\bar{x} | \bar{x}_j = x_j - v(\{j\})$ for every $j \in N, x \in C(v)\}$.
- A game v is **nonnegative** if $v(S) \geq 0$ for any $S \subseteq N$.
- A game v is called the **zero game** if $v(S) = 0$ for every $S \subseteq N$.

3 A necessary and sufficient condition

In this section, we provide the necessary and sufficient condition based on a decomposition and illustrate the decomposition process.

Proposition 3.1. A zero-normalized nonnegative game has a nonempty core if and only if it is a positive linear combination of simple N-monotonic veto-controlled games.

The following result readily follows.

Corollary 3.2. A game v has a nonempty core if and only if its zero-normalized game $\bar{v}$ is of the form $\bar{v} = v' + w$ for some v' and w, where v' is a positive linear combination of simple N-monotonic veto-controlled games, and w is given as follows: $w(N) = 0$, $w(\{i\}) = 0$ for every $i \in N$, and for any $S \subseteq N$ with $1 < |S| < n$,

$$w(S) = \begin{cases} \bar{v}(S) & \text{if } \bar{v}(S) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We begin with the only-if-part, since game v has a nonempty core, its zero-normalized game $\bar{v}$ has a nonempty core. Let $x \in C(\bar{v})$. Since $\bar{v}$ is zero-normalized, $x_i \geq 0$ for every $i \in N$. Now let $\mathcal{S} = \{S \subseteq N | \bar{v}(S) < 0,\ 1 < |S| < n\}$. We first assume that $\mathcal{S}$ is not empty. Then, we set $w(S) := \bar{v}(S)$ for any $S \in \mathcal{S}$, and $w(S) := 0$ for every $S \in 2^N \setminus \mathcal{S}$. Moreover, set $v' := \bar{v} - w$, namely, $v'(S) := 0$ for every $S \in \mathcal{S}$ and $v'(S) := \bar{v}(S)$ for every $S \in 2^N \setminus \mathcal{S}$. Note that $v'(N) := \bar{v}(N)$ as $N \in 2^N \setminus \mathcal{S}$. For any $S \in \mathcal{S}$, we have $\sum_{j \in S} x_j \geq 0 = v'(S)$; and for any $S \in 2^N \setminus \mathcal{S}$, by $x \in C(\bar{v})$, we have $\sum_{j \in S} x_j \geq \bar{v}(S) = v'(S)$; and for N, we have $\sum_{j \in N} x_j = \bar{v}(N) = v'(N)$. Hence, the core of v' is not empty. Moreover, by the construction of v', v' is zero-normalized because $\bar{v}$ is zero-normalized and $v'(\{i\}) = \bar{v}(\{i\})$ for any $i \in N$, and v' is nonnegative because for every $S \in \mathcal{S}$, $v'(S) = 0$ and for every $S \notin \mathcal{S}$, $v'(S) = \bar{v}(S) \geq 0$ by the construction of $\mathcal{S}$. Note that $v'(N) = \bar{v}(N) \geq 0$ holds because if $\bar{v}(N) < 0$, this contradicts $C(\bar{v}) \neq \emptyset$ and $\bar{v}(\{i\}) = 0$ for every $i \in N$. Hence, in view of Proposition 3.1, v' is a positive linear combination of simple N-monotonic veto-controlled games. If $\bar{v}(S) \geq 0$ for all $S \subseteq N$, *i.e.*, $\mathcal{S}$ is empty, then w is the zero game and $v' := \bar{v}$. Hence, $C(v') \neq \emptyset$ and $v' (= \bar{v})$ is nonnegative and zero-normalized. Again by Proposition 3.1, v' is a positive linear combination of simple N-monotonic veto-controlled games.

Now we prove the if-part. First $C(v') \neq \emptyset$ follows from Proposition 3.1. Moreover, it holds that $(0, ..., 0) \in C(w)$ because $w(N) = 0$, $w(S) \leq 0$ for all $S \subsetneq N$. Hence, superadditivity of the core and equality $\bar{v} = v' + w$ jointly imply that $C(\bar{v}) \neq \emptyset$. $\square$

The proposition shows that we can decompose a balanced game into some "easy" games, namely, simple N-monotonic veto-controlled games. The if-part is straightforward because each of such easy games also has a nonempty core.[*2] The only-if-part can be seen as a decomposition of a balanced game. To see this, we consider a game v that is given as Table 1.

Table 1 The first step

	1	2	3	12	13	23	123	coefficients
v	0	0	0	6	5	4	9	-
v^1	0	0	0	5	5	0	5	1
v^2	0	0	0	1	0	3	3	1
v^3	0	0	0	0	0	1	1	1

*2 Although formally stated in Section 5, every simple game with $v(N) = 1$ has a nonempty core if and only if it is veto-controlled.

Below, we describe the rough sketch of the proof thorough the example. The formal proof is provided in Section 5. The decomposition consists of two steps. In the first step, we first fix an arbitrary core allocation: $(5, 3, 1)$ in this example. Given this allocation, we decompose v into v^1, v^2, v^3, each of which is a N-efficient veto-controlled game as Table 1 shows. It holds that $v^1 + v^2 + v^3 = v$. Note that for each $i \in N$, i is a veto player in v^i.

Table 2 The second step

	1	2	3	12	13	23	123	coefficients
v^1	0	0	0	5	5	0	5	-
$\hat{v}^1_{12}$	0	0	0	1	1	0	1	5
v^2	0	0	0	1	0	3	3	-
$\hat{v}^2_{12}$	0	0	0	1	0	1	1	1
$\hat{v}^2_{23}$	0	0	0	0	0	1	1	2
v^3	0	0	0	0	0	1	1	-
$\hat{v}^3_{23}$	0	0	0	0	0	1	1	1

In the second step, each v^i is furthermore decomposed into some simple games so as to keep N-efficient. In view of Table 2, we consider, for example, v^2. We first focus on coalition 12, that has the smallest positive worth in v^2. Using this worth, we cut off $\hat{v}^2_{12}$ with coefficient 1. Now we focus on coalition 23, that has the second smallest positive worth in v^2. We derive $\hat{v}^2_{23}$ with coefficient 2. Although this finishes the decomposition in this example, we generally repeat this procedure until it reaches the zero game for every player. As a result, we obtain $v^i = \sum_{T \in \mathcal{T}^i} \eta^i_T \hat{v}^i_T$ for each $i \in N$, where $\mathcal{T}^i = \{T \subseteq N | v^i(T) > 0, \ \nexists T' \subseteq N \text{ such that } v^i(T) = v^i(T')\}$ and $\eta^i_T > 0$ for every $i \in N$ and $T \in \mathcal{T}^i$. In the end, we achieve the decomposition

$$v = \sum_{i \in N} \sum_{T \in \mathcal{T}^i} \eta^i_T \hat{v}^i_T.$$

4 Concluding remarks

In this paper, we provide a new necessary and sufficient condition for the core of a TU-game to be nonempty. We conclude this paper with the following two remarks.

Relationship with PMAS: Sprumont (1990) studies the concept of **population monotonic allocation schemes (PMAS)** that is a vector $(x^S)_{S \subseteq N}$ satisfying (i) $\sum_{j \in S} x^S_j = v(S)$ for any $S \subseteq N$, and (ii) $x^S_i \leq x^T_i$ for any $i \in N$ and any S, T with $i \in S \subseteq T \subseteq N$. Condition (i) is called coalitional efficiency. Condition (ii) is population monotonicity, which guarantees that as a coalition grows larger its members obtain more payoffs. In general, a game does not necessarily have a PMAS, while if it does then its allocation to the grand coalition N, x^N, lies in the core. Sprumont (1990) shows that a zero-normalized game has a PMAS if and only if it is a positive linear combination of simple monotonic veto-controlled games. Given that our condition is a positive linear combination of simple *N-monotonic* veto-controlled games, these two necessary and sufficient conditions are very contrastive. Note that monotonicity implies

N-monotonicity, which is consistent with the fact that if a game has a PMAS, it also has a nonempty core.

The core and the D-core: The domination core, which is also known as the D-core, is a superset of the core. An allocation x **dominates** y if there exists a coalition $S \subseteq N$ such that (i) $x_j > y_j$ for every $j \in S$ and (ii) $\sum_{j \in S} x_j \leq v(S)$. The **D-core** of a game v, $DC(v)$, is the set of efficient allocations that are not dominated by any other allocations. If the core is nonempty, then the core coincides with the D-core.[*3] This fact implies that as long as the core is nonempty, we can identify the D-core with the core. However, since there is a game that has a nonempty D-core and an empty core, our condition is a sufficient condition for the D-core to be nonempty.

5 Proof of Proposition 3.1

Proof. IF: This readily follows from the following three facts: (i) a simple game has a nonempty core if and only if the game is veto-controlled and N-monotonic. We provide the proof of this statement below;[*4] (ii) $C(v) + C(w) \subseteq C(v + w)$ for any $v, w \in \mathcal{G}_N$; (iii) $C(v) \neq \emptyset \Rightarrow C(av) \neq \emptyset$ for any $a \in \mathbb{R}_+$. Namely, every simple N-monotonic veto-controlled game has a nonempty core in view of (i), and their positive linear combination also has a nonempty core by (ii) and (iii).

Proof of (i) We begin with the if-part. Let i^* be a veto player. Consider x satisfying $x_{i^*} = 1$ and $x_i = 0$ for all $i \neq i^*$. This belongs to the core because for any S with $i^* \in S \subseteq N$, we have $\sum_{j \in S} x_j = 1$ and $v(S)$ is 0 or 1, which results in $\sum_{j \in S} x_j \geq v(S)$; and for any S with $i^* \notin S \subseteq N$, we have $\sum_{j \in S} x_j = 0 = v(S)$.

Now, we prove the only-if-part. Suppose that there is no veto player or v is not N-monotonic. If v is not N-monotonic, then there exists a coalition $S \subseteq N$ such that $v(S) > v(N)$. Hence, the core is empty. If there is no veto player, then we assume that some allocation x is in the core. Since every player is not a veto player, for any $i \in N$, there exists $S \not\ni i$ such that $v(S) = 1$. First, consider $i_1 \in N$ and $S_1 \not\ni i_1$ such that $v(S_1) = 1$. Note that the core element x must satisfy $\sum_{j \in S_1} x_j = 1$. Now, consider $i_2 \in S_1$ and $S_2 \not\ni i_2$ such that $v(S_2) = 1$. Then $S_1 \neq S_2$ and $\sum_{j \in S_2} x_j = 1$. Moreover $S_1 \cap S_2 \neq \emptyset$ and $\sum_{j \in S_1 \cap S_2} x_j = 1$ because otherwise, either one contradicts $\sum_{j \in N} x_j = 1$. Next, consider $i_3 \in S_1 \cap S_2$ and $S_3 \not\ni i_3$ such that $v(S_3) = 1$. In the same manner, we obtain $S_1 \cap S_2 \cap S_3 \neq \emptyset$ and $\sum_{j \in S_1 \cap S_2 \cap S_3} x_j = 1$. Repeating this procedure, we reach a coalition S_k and a player i^* such that $\cap_{h=1}^{k} S_h = \{i^*\}$ and $x_{i^*} = 1$. Hence, for any $j \in N \setminus \{i^*\}$, $x_j = 0$ (note that $x_j \geq 0$ for all $j \in N$ because of individual rationality). However, even for i^*, there exists $S^* \not\ni i^*$ such that $\sum_{j \in S^*} x_j = 1$. This contradicts the fact that $x_j = 0$ for any $j \in N \setminus \{i^*\}$. This completes the proof of (i).

ONLY IF: Since v is a balanced game, let $x \in C(v)$. Without loss of generality, let $x_1 \geq ... \geq x_n$. As v is zero-normalized, $x_n \geq 0$.

Step 1: Set $w^0(S) := v(S)$ for any $S \subseteq N$. Define $v^1(S) := \min\{x_1, w^0(S)\}$ for any S with $1 \in S \subseteq N$,

[*3] Since this simple fact is hardly mentioned in preceding works, we provide the proof here. We begin with the following condition: for any $S \subseteq N$, $v(N) \geq v(S) + \sum_{j \in N \setminus S} v(\{j\})$. If a game $v \in \mathcal{G}_N$ satisfies this condition, we have $C(v) = DC(v)$. Now, we assume that $C(v) \neq \emptyset$. From the definition of $C(v)$, it follows that for some $x \in C(v)$, $v(N) = \sum_{j \in N} x_j = \sum_{j \in S} x_j + \sum_{j \in N \setminus S} x_j \geq v(S) + \sum_{j \in N \setminus S} v(\{j\})$. Hence, the game satisfies the condition above, which implies $C(v) = DC(v)$. This implies that if $C(v) \neq \emptyset$, then $C(v) = DC(v)$.

[*4] See Peters (2008) Theorem 16.11 for another proof.

and $v^1(S) := 0$ for any S with $1 \notin S \subseteq N$. Now, set $w^1(S) := w^0(S) - v^1(S)$ for any $S \subseteq N$. Similarly, define $v^2(S) := \min\{x_2, w^1(S)\}$ for any S with $2 \in S \subseteq N$, and $v^2(S) := 0$ for any S with $2 \notin S \subseteq N$. In general, for any player $i = 1, ..., n$ and any $S \subseteq N$,

$$
\begin{aligned}
v^i(S) &:= \begin{cases} \min\{x_i, w^{i-1}(S)\} & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \\
w^i(S) &:= w^{i-1}(S) - v^i(S) \text{ for any } S \subseteq N.
\end{aligned}
\tag{5.1}
$$

This recursive process ends at player n with setting v^n.

We show that $v = v^1 + ... + v^n$. We arbitrarily fix $S \subseteq N$. We have

$$
\sum_{j \in N} v^j(S) \overset{(5.1)}{=} \sum_{j \in S} v^j(S) = \sum_{k=1}^{|S|} v^{j_k}(S), \tag{5.2}
$$

where players $j_1, ..., j_{|S|}$ are ordered to satisfy $x_{j_1} \geq ... \geq x_{j_{|S|}}$. Since $\sum_{j \in S} x_j \geq v(S)$, in view of (5.1), there exists a player j_{k^*} $(1 \leq k^* \leq |S|)$ such that

$$
\begin{aligned}
& v^{j_k}(S) = x_{j_k} \text{ for } k = 1, ..., k^* - 1, \\
& v^{j_{k^*}}(S) = w^{j_{k^*-1}}(S), \\
& v^{j_k}(S) = 0 \text{ for } k = k^* + 1, ..., |S|.
\end{aligned}
\tag{5.3}
$$

As for the second line of (5.3), we have

$$
\begin{aligned}
v^{j_{k^*}}(S) = w^{j_{k^*-1}}(S) & \overset{(5.1)}{=} w^{j_{k^*-2}}(S) - v^{j_{k^*-1}}(S) \\
& \overset{(5.1)}{=} w^{j_{k^*-3}}(S) - v^{j_{k^*-2}}(S) - v^{j_{k^*-1}}(S) \\
& \cdots \\
& = w^0(S) - \sum_{k=1}^{k^*-1} v^{j_k}(S) \\
& = v(S) - \sum_{k=1}^{k^*-1} x_{j_k},
\end{aligned}
\tag{5.4}
$$

where the last equality holds because $w^0(S) = v(S)$ and the first line of (5.3). Hence, we have

$$
\begin{aligned}
(5.2) \quad & : \quad \sum_{k=1}^{|S|} v^{j_k}(S) \\
& = \quad \left[\sum_{k=1}^{k^*-1} v^{j_k}(S)\right] + v^{j_{k^*}}(S) + \left[\sum_{k=k^*+1}^{|S|} v^{j_k}(S)\right] \\
& \overset{(5.3)(5.4)}{=} \left[\sum_{k=1}^{k^*-1} x_{j_k}\right] + v(S) - \sum_{k=1}^{k^*-1} x_{j_k} + \left[\sum_{k=k^*+1}^{|S|} 0\right] \\
& = \quad v(S).
\end{aligned}
$$

Thus, we obtain $v = v^1 + ... + v^n$.

From (5.1), it follows that for each v^i $(i = 1, ..., n)$, $v^i(S) = 0$ if $i \notin S$. Hence, every v^i is veto-controlled. Moreover, in view of $v(N) = \sum_{j \in N} x_j$ and $v(N) = v^1(N) + ... + v^n(N)$ and (5.1), for every

v^i $(i = 1, ..., n)$ we have

$$v^i(N) = x_i. \tag{5.5}$$

For any $S \subsetneq N$,

$$v^i(S) \overset{(5.1)}{=} \begin{cases} \min\{x_i, w^{i-1}(S)\} \leq x_i \overset{(5.5)}{=} v^i(N) & \text{if } S \ni i, \\ 0 \leq x_i \overset{(5.5)}{=} v^i(N) & \text{if } S \not\ni i. \end{cases}$$

Hence, every v^i is N-monotonic. This completes Step 1. We have decomposed v into the sum of $v^1, ..., v^n$ and shown that each v^i is veto-controlled and N-monotonic.

Step 2: Fix a player $i \in N$ such that v^i is not the zero game. Recall that a game v is called the zero game if $v(S) = 0$ for every $S \subseteq N$. Without loss of generality, let $T_1, ..., T_m \subseteq N$ satisfy

$$0 < v^i(T_1) < ... < v^i(T_m) \tag{5.6}$$

for some $m \leq 2^n - 1$. If there are some coalitions T, T' satisfying $0 < v^i(T) = v^i(T')$, we adopt one of them and remove the other to establish strict inequalities. In view of (5.6), if $T_m \neq N$, then either (i) there exists T_k such that $T_k = N$ and $k = 1, ..., m-1$ or (ii) $v^i(N) = 0$ holds. However, (i) immediately violates N-monotonicity because $v^i(T_k = N) < v^i(T_m \neq N)$ and (ii) is also incompatible with the fact that v^i is N-monotonic and is not the zero game. Hence we obtain $T_m = N$.

Let $\mathcal{T} = \{T_1, ..., T_m\}$.[*5] We recursively define $v^i_{T_1}, ..., v^i_{T_m}$ as follows: for any $S \subseteq N$,

$$\begin{aligned} v^i_{T_1}(S) &:= \begin{cases} v^i(T_1) & \text{if } S \in \mathcal{T}, \\ 0 & \text{if } S \notin \mathcal{T}, \end{cases} \\ v^i_{T_2}(S) &:= \begin{cases} v^i(T_2) - v^i(T_1) & \text{if } S \in \mathcal{T} \setminus \{T_1\}, \\ 0 & \text{if } S \notin \mathcal{T} \setminus \{T_1\}, \end{cases} \\ &\dots \\ v^i_{T_m}(S) &:= \begin{cases} v^i(T_m) - v^i(T_{m-1}) & \text{if } S \in \mathcal{T} \setminus \{T_1, ..., T_{m-1}\}, \\ 0 & \text{if } S \notin \mathcal{T} \setminus \{T_1, ..., T_{m-1}\}. \end{cases} \end{aligned} \tag{5.7}$$

By the construction (5.7), for any $S \in \mathcal{T}$, say $S = T_k$, we have

$$\begin{aligned} \sum_{T \in \mathcal{T}} v^i_T(T_k) &\overset{(5.7)}{=} [v^i(T_1)] + [v^i(T_2) - v^i(T_1)] \\ &\quad + ... + [v^i(T_k) - v^i(T_{k-1})] = v^i(T_k). \end{aligned}$$

For any $S \notin \mathcal{T}$, we readily have $\sum_{T \in \mathcal{T}} v^i_T(S) \overset{(5.7)}{=} 0 \overset{(5.6)(S \notin \mathcal{T})}{=} v^i(S)$. Hence, we obtain

$$v^i(S) = \sum_{T \in \mathcal{T}} v^i_T(S) \tag{5.8}$$

for any $S \subseteq N$, namely, $v^i = \sum_{T \in \mathcal{T}} v^i_T$.

In view of (5.6), for any $T_k \in \mathcal{T}$, $v^i(T_k) - v^i(T_{k-1}) > 0$. For any $T_k \in \mathcal{T}$ with $2 \leq k \leq m$, we define

$$\hat{v}^i_{T_k}(S) := \frac{1}{v^i(T_k) - v^i(T_{k-1})} v^i_{T_k}(S) \tag{5.9}$$

[*5] Strictly speaking, $\mathcal{T}$ depends on i. However, since the player $i \in N$ is fixed throughout this step, we omit i for simplicity.

for any $S \subseteq N$. For $k=1$, $\hat{v}^i_{T_1}(S) = \frac{1}{v^i(T_1)} v^i_{T_1}(S)$. Hence, for any $S \subseteq N$, we obtain

$$\begin{aligned} v^i(S) &\overset{(5.8)}{=} \sum_{k=1}^{m} v^i_{T_k}(S) \\ &\overset{(5.9)}{=} v^i(T_1) \cdot \hat{v}^i_{T_1}(S) \\ &\quad +(v^i(T_2) - v^i(T_1)) \cdot \hat{v}^i_{T_2}(S) \\ &\quad \ldots \\ &\quad +(v^i(T_m) - v^i(T_{m-1})) \cdot \hat{v}^i_{T_m}(S). \end{aligned}$$

In view of (5.7) and (5.9), every $\hat{v}^i_{T_k}$ is simple and N-efficient. By the construction of $\mathcal{T}$ and (5.7), every $\hat{v}^i_{T_k}$ is veto-controlled. By (5.6), the coefficients are positive. This completes the proof. □

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