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Abstract

This paper provides the conditions for a game with externalities to have a population monotonic allocation scheme (PMAS). We observe that the convexity defined by Hafalir (2007) does not guarantee the existence of a PMAS in the presence of externalities. We introduce a new convexity and show that while our convexity is not a stronger condition than Hafalir (2007)'s, it is a sufficient condition for a game to have a PMAS. Moreover, we show that the Aumann-Drèze value, which is defined for games with coalition structures, explicitly constructs a PMAS. In addition, we generalize Sprumont (1990)'s results and offer two necessary and sufficient conditions to guarantee a PMAS.

Keywords: Convexity; Core; Externalities; Population monotonicity

JEL Classification: C71

1 Introduction

Cooperative game theory has managed to answer the following two relevant questions: How should we distribute surplus? Which coalition is formed? To answer these questions, most traditional models have assumed that the worth of a coalition is the surplus obtained with no help from the players outside the coalition. This simplification has allowed us to provide a variety of solutions for the problem of surplus distribution and offer stability concepts to develop the theory of coalition formation. However, recent works attempt to answer these questions under more general settings, including models with mutual influence among coalitions. Such mutual influence is called *externalities* among coalitions. The notion of externalities divides the field of cooperative games into two classes. One is *games without externalities*, such as the class of traditional models. The other is *games with externalities*.[*1]

Many works address the problem of surplus distribution by generalizing the Shapley value to games with externalities. Albizuri *et al.* (2005), Bolger (1989), de Clippel and Serrano (2008), Macho-Stadler *et*

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[*1] Grafe *et al.* study *externality games*. Note that their externality games are different from our games with externalities. Their game is a coalition function form game, whereas ours is a partition function form game, which is introduced in Section 2.

al. (2007, 2010), Myerson (1977), and Pham Do and Norde (2007) propose and axiomatically characterize their Shapley values to include the effects of externalities. Dutta *et al.* (2010) generalize the potential function and characterize their Shapley value by extending Hart and Mas-Colell (1989)'s reduced game consistency. In addition to the Shapley value, the core is also extended to deal with externalities. Kóczy (2007) proposes recursive optimism and pessimism and defines the core for games with externalities. Abe (2017) provides a characterization result by extending consistency properties. Abe and Funaki (2017) analyze the core from the viewpoint of the Bondareva-Shapley condition.

As for the problem of coalition formation, its difficulty lies in the fact that some classic results do not apply for games with externalities. For example, Hafalir (2007) reveals that an extension of superadditivity is no longer sufficient for the grand coalition to be efficient in the presence of externalities. He extends convexity to games with externalities and shows that it is a sufficient condition for the efficient grand coalition.

As mentioned above, most of the preceding works separate the problem of coalition formation from surplus distribution by focusing on the grand coalition. However, as Greenberg (1994), Kóczy (2007, 2009), and Kóczy and Lauwers (2004) assumed, the process of coalition formation is supposed to depend on the share of surplus allocated to each member of each coalition. If the share a player receives in a partial coalition is more beneficial than that in the grand coalition, then the grand coalition is not necessarily formed. To describe this aspect of the coalition formation process, in this paper, we introduce surplus distributions into the analysis for coalition formation with externalities. Therefore, we need an idea by which we assign a surplus distribution to each coalition structure. To achieve this purpose, we employ the notion of the population monotonic allocation scheme (PMAS), which is originally defined by Sprumont (1990) for games without externalities.

The difficulty in analyzing a PMAS with externalities is ascribed to its existence condition. If there are no externalities, convexity is sufficient for a game to have a PMAS. However, in the presence of externalities, we show that Hafalir (2007)'s convexity does not guarantee the existence of a PMAS. This fact gives rise to the question, "What condition guarantees a PMAS in a game with externalities?"

In this paper, we provide a new extension of convexity. We show that our convexity is not a stronger condition than Hafalir (2007)'s, but it is a sufficient condition for a game to have a PMAS. Moreover, we show that the Aumann-Drèze value, which is defined for games with coalition structures, explicitly constructs a PMAS. In addition, we generalize Sprumont (1990)'s result and offer two necessary and sufficient conditions for a game to have a PMAS. The first condition is that we can decompose a game with a PMAS into easy games with a PMAS through a decomposition process different from Sprumont (1990)'s. The second condition is obtained by extending the Bondareva-Shapley condition to games with externalities.

The remainder of the paper is organized as follows. In the next section, we introduce games with externalities and a PMAS. In Section 3, we define our convexity and show that it guarantees a PMAS. We offer the necessary and sufficient conditions in Section 4. Section 5 concludes this paper with some remarks.

2 Preliminaries

Let N be the set of players. A *coalition* S is a subset of N. $|S|$ denotes the number of players in S. We use n to denote $|N|$. For any coalition $S \subseteq N$, a *partition* of S is given by $\{T_1, ..., T_h\}$, where $1 \leq h \leq |S|$, $T_i \cap T_j = \emptyset$ for $i, j = 1, ..., h$ $(i \neq j)$, $T_i \neq \emptyset$ for $i = 1, ..., h$ and $\bigcup_{i=1}^{h} T_i = S$. We typically use $\mathcal{P}, \mathcal{Q}$ to denote a partition. For any $i \in N$, $\mathcal{P}(i)$ denotes the coalition in $\mathcal{P}$ that contains player i. Assume that the partition of the empty set $\emptyset$ is $\{\emptyset\}$. For any coalition $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. We define an *embedded coalition* of N by $(S, \mathcal{P})$ satisfying $\emptyset \neq S \subseteq N$, $\mathcal{P} \in \Pi(N)$, and $S \in \mathcal{P}$. The set of all embedded coalitions of N is given by

$$EC(N) = \{(S, \mathcal{P}) \mid \emptyset \neq S \subseteq N,\ \mathcal{P} \in \Pi(N),\ \text{and } S \in \mathcal{P}\}.$$

A *game with externalities* (or simply a game) is a pair (N, v) in which a *partition function* v assigns a real number to each embedded coalition, namely, $v : EC(N) \to R$. By convention, we define $v(\emptyset, \mathcal{P}) = 0$ for all $\mathcal{P} \in \Pi(N)$. We use v to denote a game. Let $\mathcal{G}$ be the set of all games with externalities.

Now, we introduce a population monotonic allocation scheme for games with externalities.

Definition 2.1. Let v be a game. A *population monotonic allocation scheme* (*PMAS*) of v is a vector $x = (x_i^{\mathcal{P}})_{\mathcal{P} \in \Pi(N), i \in N}$ satisfying the following two conditions:

(i) For any $\mathcal{P} \in \Pi(N)$ and any $S \in \mathcal{P}$, $\sum_{j \in S} x_j^{\mathcal{P}} = v(S, \mathcal{P})$.

(ii) For any $i \in N$, any $S, T \subseteq N$ with $i \in S \subsetneq T$, and any $\mathcal{P}, \mathcal{Q} \in \Pi(N)$ with $S \in \mathcal{P}$ and $T \in \mathcal{Q}$, $x_i^{\mathcal{P}} \leq x_i^{\mathcal{Q}}$.

The first condition is coalitional efficiency. The second condition describes population monotonicity. The second condition is equal to the following condition. For any $i \in N$ and $S, T \subseteq N$ with $i \in S \subsetneq T$, $\max_{\mathcal{P} \ni S} x_i^{\mathcal{P}} \leq \min_{\mathcal{Q} \ni T} x_i^{\mathcal{Q}}$. This condition is an appropriate requirement for an allocation scheme to be population monotonic in the presence of externalities since if x violates (ii), then for some $i \in N$, there exist embedded coalitions $(S, \mathcal{P})$ and $(T, \mathcal{Q})$ with $i \in S \subsetneq T$ such that $x_i^{\mathcal{P}} > x_i^{\mathcal{Q}}$, even though $\mathcal{P}(i) = S \subsetneq T = \mathcal{Q}(i)$. If v has no externalities, namely, $v(S, \mathcal{P}) = v(S, \mathcal{P}')$ for any $S \subseteq N$ and $\mathcal{P}, \mathcal{P}' \in \Pi(N)$ with $S \in \mathcal{P}$ and $S \in \mathcal{P}'$, then Definition 2.1 is reduced to the definition of PMAS given by Sprumont (1990) for games without externalities.

3 Convex games

3.1 Convexity does not guarantee a PMAS.

Convexity for games with externalities is introduced by Hafalir (2007). A game v is *convex* if for any coalitions $S, T \subseteq N$ and any partition $\mathcal{P}' \in \Pi(N \setminus (S \cup T))$,

$$\begin{aligned} v(S \cup T, \{(S \cup T)\} \cup \mathcal{P}') + v(S \cap T, \{(S \setminus T), (S \cap T), (T \setminus S)\} \cup \mathcal{P}') \\ \geq v(S, \{(S), (T \setminus S)\} \cup \mathcal{P}') + v(T, \{(T), (S \setminus T)\} \cup \mathcal{P}'). \end{aligned} \tag{3.1}$$

Convexity is introduced as a sufficient condition for the grand coalition to be efficient and for some types of cores to be nonempty.*2 However, convexity does not guarantee the existence of a PMAS. To see this, we consider the following example. For notational simplicity, for example, we write $\{1, 23\}$ instead of $\{\{1\}, \{2, 3\}\}$.

Example 3.1. Let $N = \{1, 2, 3\}$.

$$v(S, \mathcal{P}) = \begin{cases} 0 & \text{if } S = \{i\} \text{ and } \mathcal{P} = \{i, j, k\}, \\ 2 & \text{if } S = \{i\} \text{ and } \mathcal{P} = \{i, jk\}, \\ 2 & \text{if } |S| = 2, \\ 5 & \text{if } |S| = 3. \end{cases}$$

This game satisfies convexity but does not have a PMAS. Suppose that there is a PMAS x. Coalitional efficiency requires $x_1^{\{12,3\}} + x_2^{\{12,3\}} = v(12, \{12, 3\}) = 2$, whereas population monotonicity requires $x_1^{\{12,3\}} \geq x_1^{\{1,23\}} = 2$ and $x_2^{\{12,3\}} \geq x_2^{\{2,13\}} = 2$. This results in $2 \geq 2 + 2$, which is a contradiction.

One might conjecture that the "tendency" of externalities matters, as Example 3.1 has positive externalities. However, Example 3.2 shows that the tendencies of externalities are not crucial for a game to have or lack a PMAS. We say that a game has *positive externalities* if for any disjoint coalitions $C, S, T \subseteq N$ and partition $\mathcal{P}' \in \Pi(N \setminus (S \cup T))$ with $C \in \mathcal{P}'$,

$$v(C, \{(S \cup T)\} \cup \mathcal{P}') > v(C, \{S, T\} \cup \mathcal{P}').$$

In the same manner, a game has *negative externalities* if the inequality is $<$. Moreover, a game has *nonnegative* (*nonpositive*) *externalities* if the inequality is $\geq$ ($\leq$).

Example 3.2. Let $N = \{1, 2, 3, 4\}$.

$$v(S, \mathcal{P}) = \begin{cases} 2 & \text{if } S = \{i\} \text{ and } \mathcal{P} = \{i, j, k, h\}, \\ 1 & \text{if } S = \{i\} \text{ and } \mathcal{P} = \{i, j, kh\}, \\ 0 & \text{if } S = \{i\} \text{ and } \mathcal{P} = \{i, jkh\}, \\ 4 & \text{if } S = \{ij\} \text{ and } \mathcal{P} = \{ij, k, h\}, \\ 2 & \text{if } S = \{ij\} \text{ and } \mathcal{P} = \{ij, kh\}, \\ 6 & \text{if } |S| = 3, \\ 10 & \text{if } |S| = 4. \end{cases}$$

This game has negative externalities and satisfies convexity, whereas it has no PMAS. Coalitional efficiency requires $x_1^{\{12,34\}} + x_2^{\{12,34\}} = v(12, \{12, 34\}) = 2$. Population monotonicity requires $x_1^{\{12,34\}} \geq x_1^{\{1,2,3,4\}} = 2$ and $x_2^{\{12,34\}} \geq x_2^{\{1,2,3,4\}} = 2$. This results in $2 \geq 2 + 2$, which is a contradiction.

3.2 Minmax-convexity guarantees a PMAS.

The observation in the previous subsection shows that convex games may lack a PMAS in the presence of externalities. What condition is sufficient for a game to have a PMAS? To answer this question, we

*2 An extension of superadditivity is not sufficient for the grand coalition to be efficient in the presence of externalities. See Example 1 of Hafalir (2007). We elaborate the plurality of the core for games with externalities in Subsection 3.2.

introduce a new convexity, minmax-convexity, and show that it guarantees a PMAS. Moreover, we show that minmax-convexity is *not* a stronger condition than the convexity given by (3.1). For simplicity, we write that for any coalition $S \subseteq N$,

$$\begin{array}{l} v(S, max) = \max_{\mathcal{P} \in \Pi(N): S \in \mathcal{P}} v(S, \mathcal{P}), \\ v(S, min) = \min_{\mathcal{P} \in \Pi(N): S \in \mathcal{P}} v(S, \mathcal{P}). \end{array} \tag{3.2}$$

Definition 3.3. A game is *minmax-convex* if for any $S, T \subseteq N$,

$$v(S \cup T, min) + v(S \cap T, max) \geq v(S, max) + v(T, max).$$

We can interpret Definition 3.3 as follows. Let S and T be coalitions whose members have an optimistic anticipation for the coalition structure that the other players form. Coalition $S \cap T$ keeps holding the optimistic view since the members of $S \cap T$ share the same original affiliation ($S \cap T$ is a subset of both S and T), whereas coalition $S \cup T$ does not since some members are from different affiliations (for example, $S \setminus T$ and $T \setminus S$).

This is equivalent to the following condition. For any $i, j \in N$ ($i \neq j$) and any (maybe empty) coalition $S \subseteq N \setminus \{i, j\}$,

$$v((S \cup \{j\}) \cup \{i\}, min) - v((S \cup \{j\}), max) \geq v(S \cup \{i\}, max) - v(S, max). \tag{3.3}$$

Minmax-convexity reduces to the traditional convexity for coalition function form games if a game has no externalities.

It is worth mentioning that minmax-convexity is independent of convexity. Example 3.2 is convex but not minmax-convex. The following example is minmax-convex but not convex.

Example 3.4. Let $N = \{1, 2, 3\}$.

$$v(S, \mathcal{P}) = \begin{cases} 0 & \text{if } \ S = \{i\} \text{ and } \mathcal{P} = \{i, j, k\}, \\ 1 & \text{if } \ S = \{i\} \text{ and } \mathcal{P} = \{i, jk\}, \\ 3 & \text{if } \ |S| = 2, \\ 5 & \text{if } \ |S| = 3. \end{cases}$$

In this example, the only difference between the two convexities lies in the case of $S = \{12\}$ and $T = \{13\}$. For convexity, $v(N, \{N\}) + v(1, \{1, 2, 3\}) = 5 \ngeq 3 + 3 = v(12, \{12, 3\}) + v(13, \{13, 2\})$, while the inequality holds for minmax-convexity, $v(N, \{N\}) + v(\{1\}, max = \{1, 23\}) = 5 + 1 \geq 3 + 3 = v(12, \{12, 3\}) + v(13, \{13, 2\})$.

Now, we provide our first main result. We begin by introducing an allocation scheme based on marginal contributions. Let σ be a permutation of players. Player i is the $\sigma(i)$-th player in σ. Let $\Psi(N)$ be the set of all permutations of N. For player $i \in N$, $L^{\sigma}(i)$ is the set of predecessors of i in σ, formally denoted as $L^{\sigma}(i) = \{j \in N | \sigma(j) < \sigma(i)\}$. Moreover, for any permutation $\sigma \in \Psi(N)$ and any partition $\mathcal{P} \in \Pi(N)$, we define

$$L^{\sigma, \mathcal{P}}(i) = L^{\sigma}(i) \cap \mathcal{P}(i) \text{ for any } i \in N, \tag{3.4}$$

where, as mentioned in Section 2, $\mathcal{P}(i)$ is the coalition in $\mathcal{P}$ that contains player i. For example, assuming $\sigma = (1, 2, 3, 4, 5)$ and $\mathcal{P} = \{12, 345\}$, we have $L^{\sigma, \mathcal{P}}(1) = \emptyset$, $L^{\sigma, \mathcal{P}}(2) = \{1\}$, $L^{\sigma, \mathcal{P}}(3) = \emptyset$, $L^{\sigma, \mathcal{P}}(4) = \{3\}$, and $L^{\sigma, \mathcal{P}}(5) = \{34\}$.

Definition 3.5. For a given permutation σ, the *allocation scheme of marginal contributions* (*ASMC*) of a game v is defined as follows. For any partition $\mathcal{P} \in \Pi(N)$ and any player $i \in N$,

$$ASMC_i^{\sigma,\mathcal{P}}(v) = v(L^{\sigma,\mathcal{P}}(i) \cup \{i\},\ f^*(v, L^{\sigma,\mathcal{P}}(i) \cup \{i\}, \mathcal{P})) - v(L^{\sigma,\mathcal{P}}(i),\ f^*(v, L^{\sigma,\mathcal{P}}(i), \mathcal{P})), \tag{3.5}$$

where for any $v \in \mathcal{G}$, $S \subseteq N$, and $\mathcal{P} \in \Pi(N)$, $f^*(v, S, \mathcal{P})$ is given by

$$f^*(v, S, \mathcal{P}) = \begin{cases} \mathcal{P} & \text{if } S \in \mathcal{P} \text{ or } S = \emptyset \\ \underset{\mathcal{P}' \in \Pi(N): S \in \mathcal{P}'}{\arg\max}\ v(S, \mathcal{P}') & \text{otherwise.} \end{cases} \tag{3.6}$$

We write $ASMC^\sigma(v) = (ASMC_i^{\sigma,\mathcal{P}}(v))_{i \in N, \mathcal{P} \in \Pi(N)}$ for every permutation $\sigma \in \Psi(N)$.

Proposition 1. If v is a minmax-convex game, then $ASMC^\sigma(v)$ is a PMAS for every $\sigma \in \Psi(N)$.

The proof is provided in the appendix. Below, we offer two remarks.

Expectation formation rules: The function given by (3.6) is a function assigning a partition of N to each pair $(v, S, \mathcal{P})$. Such functions are studied by Bloch and van den Nouweland (2014) in a more general framework. They call such functions *expectation formation rules*. They suppose that a coalition S tries to deviate from partition $\mathcal{P}$. In the presence of externalities, the reaction from the players in $N \setminus S$, which is given by a partition of $N \setminus S$, affects the worth of S. Assuming that S does not split after the deviation, $f(v, S, \mathcal{P})$ denotes the resulting partition. In general, an expectation formation rule is given by

$$f : \mathcal{G} \times 2^N \setminus \{\emptyset\} \times \Pi(N) \to \Pi(N), \tag{3.7}$$

which satisfies $S \in f(v, S, \mathcal{P})$ for any $S \in 2^N \setminus \emptyset$ and $\mathcal{P} \in \Pi(N)$ (but S is not necessarily in $\mathcal{P}$). The rule f^* given by (3.6) is similar to the optimistic rule defined later in (3.9). However, as described in the proof, the optimistic rule does not imply coalitional efficiency of an allocation scheme. Rule (3.6) might be seen as a variant of the optimistic rule modified to induce coalitional efficiency.

The core: As Sprumont (1990) shows, a PMAS (if any) is an element of the core in games without externalities. This is also true for games with externalities. To see this, we briefly recall the core of games with externalities. In the presence of externalities, the definition of the core is no longer unique. To define a core, we consider a more specific expectation formation rule g, formally,

$$g : \mathcal{G} \times 2^N \setminus \emptyset \to \Pi(N). \tag{3.8}$$

In other words, g can be seen as a function obtained by imposing $\mathcal{P} = \{N\}$ on f. Each type of expectation formation rule g generates its core $Core^g$ as follows. For any g, $Core^g(v) = Core(v^g)$, where $v^g(S) = v(S, g(v, S))$ for any $S \subseteq N$.[*3] For example, the optimistic rule given by

$$g(v, S) = \underset{\mathcal{P}' \ni S}{\arg\max}\, v(S, \mathcal{P}') \text{ for any } S \subseteq N \tag{3.9}$$

yields the optimistic core $Core^{opt}(v) = \{x \in \mathbb{R}^N | \text{for any } S \subseteq N,\ \sum_{j \in S} x_j \geq v(S, max)\}$. Now, let v be a game with a PMAS x. The following statement holds. For any rule g, $x^{\{N\}}$ lies in $Core^g(v)$. We offer the brief proof below. For any $S \subseteq N$, we have $\sum_{j \in S} x_j^{\{N\}} \geq \max_{\mathcal{P}' \ni S} \sum_{j \in S} x_j^{\mathcal{P}'} = \max_{\mathcal{P}' \ni S} v(S, \mathcal{P}')$ (=

[*3] The core for games without externalities, w, is given by $Core(w) = \{x \in \mathbb{R}^N | \sum_{j \in S} x_j \geq w(S) \text{ for any } S \subseteq N\}$.

$v(S, max)$), where the inequality holds by population monotonicity and the equality by coalitional efficiency. Hence, $x^{\{N\}}$ is in $Core^{opt}(v)$. Since $Core^{opt}(v) \subseteq Core^{g}(v)$ for any rule g, we obtain $x^{\{N\}} \in Core^{g}(v)$. From this statement and Proposition 1, the following holds:

$$v \text{ is minmax-convex} \Rightarrow Core^{g}(v) \text{ is nonempty for any rule } g.$$

Considering that minmax-convexity is independent of convexity, this is in contrast to Hafalir (2007)'s result:

$$v \text{ is convex} \Rightarrow Core^{g}(v) \text{ may be empty for some rule } g.$$[*4]

3.3 A convex combination of ASMCs

In this subsection, we show that a convex combination of ASMCs can be represented by an extension of the Aumann-Drèze value. The Aumann-Drèze value (AD value) is defined as a value for games with coalition structures (Aumann and Drèze (1974)). We begin by recalling the AD value. Let w be a coalition function form game or a game with no externalities. For any coalition $S \subseteq N$, the worth of S only depends on S, namely, $w(S) \in \mathbb{R}$. Let $w(\emptyset) = 0$ by convention. A coalition structure is a partition of N. We denote it by $\mathcal{P}$. For the given player set N, a game with coalition structure is $(N, w, \mathcal{P})$ or simply $(w, \mathcal{P})$. The AD value of $(w, \mathcal{P})$ is given as follows. For any $i \in N$,

$$AD_i(w, \mathcal{P}) = Sh_i(\mathcal{P}(i), w|_{\mathcal{P}(i)}), \tag{3.10}$$

where $(\mathcal{P}(i), w|_{\mathcal{P}(i)})$ is a coalition function form game with the player set $\mathcal{P}(i)$. Note that $\mathcal{P}(i)$ is the coalition to which player i belongs and $w|_{\mathcal{P}(i)}$ is the subgame of w to $\mathcal{P}(i)$, namely, for any $T \subseteq \mathcal{P}(i)$, $w|_{\mathcal{P}(i)}(T) = w(T)$. Function Sh is the Shapley value, which is generally given as $Sh_i(N, w) = \frac{1}{n!}\sum_{\sigma \in \Psi(N)}[w(L^{\sigma}(i) \cup \{i\}) - w(L^{\sigma}(i))]$.

The AD value is defined for games without externalities (with coalition structures), whereas our ASMC is defined for games with externalities. To connect these two different classes, we use the function defined by (3.6).[*5] Let v be a game with externalities. Fix a partition $\mathcal{P} \in \Pi(N)$ and consider the game with coalition structure $(v^*, \mathcal{P})$ given as follows. For any $S \subseteq N$,

$$v^*(S) = v(S, f^*(v, S, P)) = \begin{cases} v(S, \mathcal{P}) & \text{if } S \in \mathcal{P} \\ \max_{\mathcal{P}' \ni S} v(S, \mathcal{P}') & \text{otherwise.} \end{cases} \tag{3.11}$$

Now we claim that a convex combination of ASMCs coincides with the extended AD value.

Proposition 2. Let v be a game with externalities.

$$\frac{1}{n!} \sum_{\sigma \in \Psi(N)} ASMC^{\sigma}(v) = (AD(v^*, \mathcal{P}))_{\mathcal{P} \in \Pi(N)}.$$

*4 More specifically, $Core^{g}(v)$ is nonempty if the rule g is the pessimistic rule or the singleton rule. However, it can be empty if g is the merging rule, the optimistic rule, or the max rule. For the definitions, see Hafalir (2007).

*5 In many preceding papers, it is expectation formation rules g given as (3.8) that play a central role to establish a connection between games with externalities and games without externalities. However, in this paper, we can use f instead of g since we connect a game with externalities to a game with a coalition structure (without externalities). This allows us to consider $\mathcal{P}$ as given and use f.

The proof is provided in the appendix. Propositions 2 and 1 imply that if v is minmax-convex, then we can use the extended AD value to explicitly construct a PMAS of v. Moreover, from Proposition 2, we can derive a relationship between the Shapley value(s) and the core(s) for games with externalities. To see this, let v be a game with externalities. Let g be any function given by (3.8). We define $Sh^g(v)$ as

$$Sh^g(v) = Sh(v^g). \tag{3.12}$$

We consider the case of $\mathcal{P} = \{N\}$ in Proposition 2. For any game without externalities w, the AD value of $(w, \{N\})$ coincides with the Shapley value of w by (3.10). Moreover, the function $f(v, S, \{N\})$ defined by (3.6) coincides with the optimistic rule. Hence, we obtain $Sh^{opt}(v) \in Core^{opt}(v)$ as a corollary. However, this raises another question. Does $Sh^g(v) \in Core^{g'}(v)$ hold for any g, g'? Minmax-convexity solves this question.

Proposition 3. Let v be a game with externalities. If v is minmax-convex, then $Sh^g(v) \in Core^{opt}(v)$ for any expectation formation rule g.

Since the optimistic core is a subset of $Core^{g'}$, for any rule g', we obtain the following result.

Corollary 3.6. Let v be a game with externalities. If v is minmax-convex, then $Sh^g(v) \in Core^{g'}(v)$ for any expectation formation rules g and g'.

Note that the convexity given by (3.1) is not a sufficient condition for Proposition 3. As mentioned in the previous subsection, even if v is convex, $Core^g(v)$ can be empty for some rule g including the optimistic rule.

Many recent works employ the form of Sh^g to study values for games with externalities. Pham Do and Norde (2007) and de Clippel and Serrano (2008) study the "singleton" expectation function that assigns the partition of $N \setminus S$ into singletons to each S. Albizuri *et al.* (2005) and Macho-Stadler *et al.* (2007, 2010) employ the probabilistic expectation rule that assigns to each S a probability distribution over the partitions of $N \setminus S$.

4 Necessary and sufficient conditions

In the previous section, we focus on minmax-convex games as a sufficient class for a game to have a PMAS. In this section, we offer two necessary and sufficient conditions.

Let v be a game with externalities. A game v is *simple* if $v(S, \mathcal{P})$ is 1 or 0 for any $(S, \mathcal{P}) \in EC(N)$. A game v is *monotonic* if for any $S, T \subseteq N$ and any $\mathcal{P}, \mathcal{Q} \in \Pi(N)$ with $S \in \mathcal{P}$ and $T \in \mathcal{Q}$, if $S \subsetneq T$, then $v(S, \mathcal{P}) \leq v(T, \mathcal{Q})$. A game v is *veto-controlled* if there exists $i \in N$ such that $v(S, \mathcal{P}) = 0$ for any $S \subseteq N \setminus \{i\}$ and $\mathcal{P} \in \Pi(N)$ with $S \in \mathcal{P}$. We call such a player i a veto player.

Proposition 4. A game v has a PMAS if and only if v is a positive linear combination of simple monotonic veto-controlled games.

The proof is provided in the appendix. Proposition 4 states that any game with a PMAS can be decomposed into some "easy" games, each of which clearly has a PMAS. Note that our decomposition is different from Sprumont (1990)'s.*6 In our decomposition process and result, we exclude zero-normalized games due to the difficulty of utilizing zero-normalized games in the presence of externalities. We briefly illustrate our decomposition below, including the outline of the proof.

Example 4.1. Let $N = \{1, 2, 3\}$. Let v be a game with a PMAS to be decomposed, which is given as Table 1 (for example, $v(1, \{1, 23\}) = 0.2$). This game has some PMASs. We choose a PMAS as given in Table 2 and denote it by $x = (x_i^{\mathcal{P}})_{\mathcal{P} \in \Pi(N), i \in N}$. Our decomposition consists of two steps. The first step is decomposition into veto-controlled games, $v = v^1 + ... + v^n$. This step is completed by setting $v^i(S, \mathcal{P}) = \sum_{j \in \mathcal{P}(i)} x_j^{\mathcal{P}}$. In this example, we obtain $v = v^1 + v^2 + v^3$, as shown in Table 1.

We begin the second step by choosing a player, for example, 1. We decompose v^1 into some simple games with positive coefficients λ. In this example, $v^1 = \lambda^{1,\mathcal{P}_1} v^{1,\mathcal{P}_1} +, ..., +\lambda^{1,\mathcal{P}_5} v^{1,\mathcal{P}_5}$, as shown in Table 3. The partitions $\mathcal{P}_1, ..., \mathcal{P}_5$ are ordered with $v(\mathcal{P}_k(i), \mathcal{P}_k)$, namely, $v(\mathcal{P}_1(1), \mathcal{P}_1) < v(\mathcal{P}_2(1), \mathcal{P}_2) < ... < v(\mathcal{P}_5(1), \mathcal{P}_5)$. The positive coefficients are determined as follows. $\lambda^{1,\mathcal{P}_1} = v^1(\mathcal{P}_1(1), \mathcal{P}_1)$, $\lambda^{1,\mathcal{P}_2} = v^1(\mathcal{P}_2(1), \mathcal{P}_2) - v^1(\mathcal{P}_1(1), \mathcal{P}_1)$, ..., $\lambda^{1,\mathcal{P}_k} = v^1(\mathcal{P}_k(1), \mathcal{P}_k) - v^1(\mathcal{P}_{k-1}(1), \mathcal{P}_{k-1})$. Clearly, each game $v^{i,\mathcal{P}_k}$ is simple monotonic veto-controlled and, hence, has a PMAS. For any $j \in N$ and any $\mathcal{P}' \in \Pi(N)$,

$$y_j^{\mathcal{P}'} = \begin{cases} v^{i,\mathcal{P}_k}(\mathcal{P}'(j), \mathcal{P}') & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Applying this step to every player completes the decomposition.

Table 1 The original game v and the first step

	1	2	3	1	23	2	13	3	12	123
v	1	2	3	0.2	6	0	5	0	4.2	10
v^1	1	0	0	0.2	0	0	2	0	1.6	3
v^2	0	2	0	0	2	0	0	0	2.6	3
v^3	0	0	3	0	4	0	3	0	0	4

Table 2 A PMAS of v

$\mathcal{P} \setminus i$	1	2	3
$\{123\}$	3	3	4
$\{1, 23\}$	0.2	2	4
$\{2, 13\}$	2	0	3
$\{3, 12\}$	1.6	2.6	0
$\{1, 2, 3\}$	1	2	3

*6 Sprumont (1990)'s approach is to decompose a (coalition function form) game into an additive game, dictator games, and a zero-normalized game. Then, a zero-normalized game is furthermore decomposed into some simple monotonic veto-controlled games (see Theorem 1 and the Corollary to Theorem 1 of Sprumont (1990)). We note that an additive game is also composed of some simple monotonic veto-controlled games.

Table 3 Game v^1 and the second step

	1	2	3	1	23	2	13	3	12	123	$\lambda^{1,\mathcal{P}_k}$
v^1	1	0	0	0.2	0	0	2	0	1.6	3	
$v^{1,\mathcal{P}_1}$	1	0	0	1	0	0	1	0	1	1	0.2
$v^{1,\mathcal{P}_2}$	1	0	0	0	0	0	1	0	1	1	0.8
$v^{1,\mathcal{P}_3}$	0	0	0	0	0	0	1	0	1	1	0.6
$v^{1,\mathcal{P}_4}$	0	0	0	0	0	0	1	0	0	1	0.4
$v^{1,\mathcal{P}_5}$	0	0	0	0	0	0	0	0	0	1	1

As we have mentioned in Section 3, the definition of the core can be plural in the presence of externalities. Abe and Funaki (2017) investigate some different generalizations of the Bondareva-Shapley condition and balanced weights. We conclude this section by providing a generalization of the necessary and sufficient condition for a PMAS, which is given as a stronger form of the Bondareva-Shapley condition presented by Sprumont (1990).

Let $i \in N$ and $S \subsetneq N$ be a coalition containing i. Note that S is a proper nonempty subset of N. Let $\mathscr{P}$ be a set of partitions of N containing S such that $\mathscr{P} \subseteq \{\mathcal{P} \in \Pi(N) \setminus \{N\} | S \in \mathcal{P}\}$. Note that $\mathscr{P}$ can be an empty set. Example 4.2 offers an instance. Now, we fix i, S, and $\mathscr{P}$ and define $\chi^{i,S,\mathscr{P}}$ as follows. For any $\emptyset \neq T \subsetneq N$ and $\mathcal{Q} \in \Pi(N)$ with $T \in \mathcal{Q}$,

$$\chi^{i,S,\mathscr{P}}(T,\mathcal{Q}) = \begin{cases} 1 & \text{if } T \supsetneq S \text{ or } (T = S \text{ and } \mathcal{Q} \in \mathscr{P}), \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience, we define $EC^*(N) = EC(N) \setminus \{(N, \{N\})\}$. We say a vector $\delta = (\delta_{(T,\mathcal{Q})})_{(T,\mathcal{Q}) \in EC^*(N)}$ is an *extended vector of sub-balanced weights* if for any $i \in N$, $S \subsetneq N$ with $i \in S$, and $\mathscr{P} \subseteq \{\mathcal{P} \in \Pi(N) \setminus \{N\} | S \in \mathcal{P}\}$,

$$\sum_{(T,\mathcal{Q}) \in EC^*(N)} \delta_{(T,\mathcal{Q})} \chi^{i,S,\mathscr{P}}(T,\mathcal{Q}) \leq 1.$$

Proposition 5. A game has a PMAS if and only if for any extended vector of sub-balanced weights δ,

$$\sum_{(S,\mathcal{P}) \in EC^*(N)} \delta_{(S,\mathcal{P})} v(S,\mathcal{P}) \leq v(N, \{N\}).$$

The following example shows the configuration of χ and δ.

Example 4.2. Let $N = \{1, 2, 3\}$. Fix $i = 1$. Table 4 describes $\chi^{i,S,\mathscr{P}}$.

An obvious δ is 0 for each embedded coalition. A nontrivial δ is, for example, $\frac{1}{4}$ for each embedded coalition. Note that δ may be negative. Hence, the condition that only contains $\mathscr{P} = \{\mathcal{P} \in \Pi(N) \setminus \{N\} | S \in \mathcal{P}\}$ is not sufficient.

5 Concluding remarks

In this paper, we introduce new convexity and show that our convexity is a sufficient condition for a game to have a PMAS. Moreover, we extend the Aumann-Drèze value to constructs a PMAS. By

Table 4 An extended vector of sub-balanced weights

		$\chi^{i,S,\mathscr{P}}$								
S	$\mathscr{P}$	1	2	3	1	23	2	13	3	12
1	$\{\{1,2,3\},\{1,23\}\}$	1	0	0	1	0	0	1	0	1
1	$\{\{1,23\}\}$	0	0	0	1	0	0	1	0	1
1	$\{\{1,2,3\}\}$	1	0	0	0	0	0	1	0	1
1	$\emptyset$	0	0	0	0	0	0	1	0	1
12	$\{\{3,12\}\}$	0	0	0	0	0	0	0	0	1
13	$\{\{2,13\}\}$	0	0	0	0	0	0	1	0	0

generalizing Sprumont (1990)'s result, we offer two necessary and sufficient conditions to have a PMAS.

As briefly mentioned in Section 3, the functions taking the form of $f(v,s,\mathcal{P})$ bridge the class of games with externalities and the class of games with coalition structures. For a game with externalities v, we can obtain a game with coalition structures $(w,\mathcal{P})$ as $w(S)=v(S,f(v,s,\mathcal{P}))$ for any S. This allows us to analyze a connection between a solution for games with externalities and a solution for games with coalition structures. For example, we can propose a new solution concept for games with externalities by extending a solution concept for games with coalition structures through the function f. This new approach is left for our future research.

Appendix

Through the appendix, for notational simplicity, we omit v and $*$ and use $f(S,\mathcal{P})$ to denote $f^*(v,S,\mathcal{P})$ given by (3.6). Similarly, we write $g(S)$ instead of $g(v,S)$ for any rule g given by (3.8).

Proof of Proposition 1

Proof. We show that $ASMC^{\sigma}(v)$ satisfies (i) coalitonal efficiency and (ii) population monotonicity.
(i) For any $\mathcal{P}\in\Pi(N)$ and $S\in\mathcal{P}$, let $\{j_1,...,j_l,...,j_{l'},...,j_{|S|}\}=S$ and $l<l' \iff \sigma(j_l)<\sigma(j_{l'})$. We have

$$\begin{aligned}
\sum_{j\in S} ASMC_j^{\sigma,\mathcal{P}}(v) &\overset{(3.5)}{=} \sum_{k=1}^{|S|}\left[v(L^{\sigma,\mathcal{P}}(j_k)\cup\{j_k\},f(L^{\sigma,\mathcal{P}}(j_k)\cup\{j_k\},\mathcal{P}))-v(L^{\sigma,\mathcal{P}}(j_k),f(L^{\sigma,\mathcal{P}}(j_k),\mathcal{P}))\right]\\
&= v(\{j_1\},f(\{j_1\},\mathcal{P}))-0+\\
&\quad v(\{j_1,j_2\},f(\{j_1,j_2\},\mathcal{P}))-v(\{j_1\},f(\{j_1\},\mathcal{P}))+\\
&\quad ...+\\
&\quad v(S,f(S,\mathcal{P}))-v(\{j_1,...,j_{|S|-1}\},f(\{j_1,...,j_{|S|-1}\},\mathcal{P}))\\
&= v(S,f(S,\mathcal{P}))\\
&\overset{(3.6)}{=} v(S,\mathcal{P}).
\end{aligned}$$

(ii) Let $i\in N$ and $S,T\subseteq N$ with $i\in S\subsetneq T$. We define $\mathcal{Q}_*=\underset{\mathcal{Q}\in\Pi(N):T\in\mathcal{Q}}{\arg\min} ASMC_i^{\sigma,\mathcal{Q}}$ and $\mathcal{P}^*=$

$\underset{\mathcal{P}\in\Pi(N):S\in\mathcal{P}}{\arg\max}\; ASMC_i^{\sigma,\mathcal{P}}$. It suffices to show

$$ASMC_i^{\sigma,\mathcal{Q}_*} \geq ASMC_i^{\sigma,\mathcal{P}^*}.$$

We focus on $ASMC_i^{\sigma,\mathcal{Q}_*}$.

$$ASMC_i^{\sigma,\mathcal{Q}_*} = v(L^{\sigma,\mathcal{Q}_*}(i)\cup\{i\}, f(L^{\sigma,\mathcal{Q}_*}(i)\cup\{i\},\mathcal{Q}_*)) - v(L^{\sigma,\mathcal{Q}_*}(i), f(L^{\sigma,\mathcal{Q}_*}(i),\mathcal{Q}_*)). \tag{A.1}$$

For simplicity, let $\tilde{L} := L^{\sigma,\mathcal{Q}_*}(i)$. Note that $\tilde{L} = L^{\sigma,\mathcal{Q}_*}(i) = L^\sigma(i)\cap\mathcal{Q}_*(i) = L^\sigma(i)\cap T$. Let $T\setminus S = \{j_1,...,j_m\}$. We have

$$\begin{aligned}
(\text{A.1}) &= v(\tilde{L}\cup\{i\}, f(\tilde{L}\cup\{i\},\mathcal{Q}_*)) - v(\tilde{L}, f(\tilde{L},\mathcal{Q}_*))\\
&\overset{(3.2)}{\geq} v(\tilde{L}\cup\{i\}, min) - v(\tilde{L}, max)\\
&\overset{(3.3)}{\geq} v((\tilde{L}\setminus\{j_1\})\cup\{i\}, max) - v(\tilde{L}\setminus\{j_1\}, max)\\
&\geq v((\tilde{L}\setminus\{j_1\})\cup\{i\}, min) - v(\tilde{L}\setminus\{j_1\}, max)\\
&\overset{(3.3)}{\geq} v((\tilde{L}\setminus\{j_1,j_2\})\cup\{i\}, max) - v(\tilde{L}\setminus\{j_1,j_2\}, max)\\
&\geq v((\tilde{L}\setminus\{j_1,j_2\})\cup\{i\}, min) - v(\tilde{L}\setminus\{j_1,j_2\}, max)\\
&\geq ...\\
&\overset{(3.3)}{\geq} v((\tilde{L}\setminus\{j_1,...,j_m\})\cup\{i\}, max) - v(\tilde{L}\setminus\{j_1,...,j_m\}, max)\\
&= v((L^\sigma(i)\cap S)\cup\{i\}, max) - v(L^\sigma(i)\cap S, max)\\
&\geq v((L^\sigma(i)\cap S)\cup\{i\}, f((L^\sigma(i)\cap S)\cup\{i\},\ P^*)) - v(L^\sigma(i)\cap S, max)\\
&\overset{(3.6)}{=} v((L^\sigma(i)\cap S)\cup\{i\}, f((L^\sigma(i)\cap S)\cup\{i\},\mathcal{P}^*)) - v(L^\sigma(i)\cap S, f(L^\sigma(i)\cap S,\mathcal{P}^*)),
\end{aligned}$$

where the last equality holds because $S\in\mathcal{P}^*$ and $L^\sigma(i)\cap S\neq S$ imply $L^\sigma(i)\cap S\notin\mathcal{P}^*$. Now, in view of $S=\mathcal{P}^*(i)$, we have $L^\sigma(i)\cap S = L^{\sigma,\mathcal{P}^*}(i)$. Hence, the final equality results in

$$v(L^{\sigma,\mathcal{P}^*}(i)\cup\{i\}, f(L^{\sigma,\mathcal{P}^*}(i)\cup\{i\},\mathcal{P}^*)) - v(L^{\sigma,\mathcal{P}^*}(i), f(L^{\sigma,\mathcal{P}^*}(i),\mathcal{P}^*)) = ASMC_i^{\sigma,\mathcal{P}^*}.$$

This completes the proof. □

Proof of Proposition 2

Proof. Fix $i\in N$ and $\mathcal{P}\in\Pi(N)$. We use τ to denote a permutation of $\mathcal{P}(i)$, namely, $\tau\in\Psi(\mathcal{P}(i))$. For each τ, we can find $(n-|\mathcal{P}(i)|)!\frac{n!}{|\mathcal{P}(i)|!\cdot(n-|\mathcal{P}(i)|)!}$ permutations σ of N satisfying $\sigma(j)<\sigma(j') \iff \tau(j)<\tau(j')$ for $j,j'\in\mathcal{P}(i)$. Hence, we have

$$\begin{aligned}
&\frac{1}{n!}\sum_{\sigma\in\Psi(N)} ASMC_i^{\sigma,\mathcal{P}}(v)\\
\overset{(3.5)}{=}\ &\frac{1}{n!}\sum_{\sigma\in\Psi(N)} v(L^{\sigma,\mathcal{P}}(i)\cup\{i\}, f(L^{\sigma,\mathcal{P}}(i)\cup\{i\},\mathcal{P})) - v(L^{\sigma,\mathcal{P}}(i), f(L^{\sigma,\mathcal{P}}(i),\mathcal{P}))\\
=\ &\frac{1}{n!}\frac{(n-|\mathcal{P}(i)|)!\cdot n!}{|\mathcal{P}(i)|!\cdot(n-|\mathcal{P}(i)|)!}\sum_{\tau\in\Psi(\mathcal{P}(i))} v(L^\tau(i)\cup\{i\}, f(L^\tau(i)\cup\{i\},\mathcal{P})) - v(L^\tau(i), f(L^\tau(i),\mathcal{P}))\\
=\ &\frac{1}{|\mathcal{P}(i)|!}\sum_{\tau\in\Psi(\mathcal{P}(i))} v(L^\tau(i)\cup\{i\}, f(L^\tau(i)\cup\{i\},\mathcal{P})) - v(L^\tau(i), f(L^\tau(i),\mathcal{P}))
\end{aligned} \tag{A.2}$$

where $L^\tau(i)$ is the set of predecessors of i within $\mathcal{P}(i)$ with respect to τ, $L^\tau(i) = \{j \in \mathcal{P}(i) | \tau(j) < \tau(i)\}$. In view of (3.11), we have

$$\text{(A.2)} = \frac{1}{|\mathcal{P}(i)|!} \sum_{\tau \in \Psi(\mathcal{P}(i))} v^*(L^\tau(i) \cup \{i\}) - v^*(L^\tau(i)). \tag{A.3}$$

Let $v^*|_{\mathcal{P}(i)}$ be the subgame of v^* with the player set $\mathcal{P}(i)$. Since both $L^\tau(i) \cup \{i\}$ and $L^\tau(i)$ are subsets of $\mathcal{P}(i)$, we obtain

$$\begin{aligned} \text{(A.3)} &= Sh_i(\mathcal{P}(i), v^*|_{\mathcal{P}(i)}) \\ &\overset{(3.10)}{=} AD_i(v^*, \mathcal{P}). \end{aligned}$$

□

Proof of Proposition 3

Proof. We show that if v is minmax-convex, then $Sh^g(v) \in Core^{opt}(v)$ for any g.

Let $S \subseteq N$ ($S \neq \emptyset$). We have

$$\begin{aligned} \sum_{j \in S} Sh^g_j(v) &\overset{(3.12)}{=} \sum_{j \in S} Sh_j(v^g) \\ &= \frac{1}{n!} \sum_{j \in S} \sum_{\sigma \in \Psi(N)} [v^g(L^\sigma(j) \cup \{j\}) - v^g(L^\sigma(j))] \\ &= \frac{1}{n!} \sum_{\sigma \in \Psi(N)} \sum_{j \in S} [v^g(L^\sigma(j) \cup \{j\}) - v^g(L^\sigma(j))]. \end{aligned} \tag{A.4}$$

For the term parenthesized with $[\cdot]$, we have

$$\begin{aligned} v^g(L^\sigma(j) \cup \{j\}) - v^g(L^\sigma(j)) &= v(L^\sigma(j) \cup \{j\}, g(L^\sigma(j) \cup \{j\})) - v(L^\sigma(j), g(L^\sigma(j))) \\ &\geq v(L^\sigma(j) \cup \{j\}, min) - v(L^\sigma(j), max) \\ &\overset{(3.3)}{\geq} v((L^\sigma(j) \cap S) \cup \{j\}, max) - v(L^\sigma(j) \cap S, max). \end{aligned}$$

Hence, we have

$$\begin{aligned} \text{(A.4)} &\geq \frac{1}{n!} \sum_{\sigma \in \Psi(N)} \sum_{j \in S} [v((L^\sigma(j) \cap S) \cup \{j\}, max) - v(L^\sigma(j) \cap S, max)] \\ &= \frac{1}{n!} \sum_{\sigma \in \Psi(N)} \sum_{k=1}^{|S|} [v((L^\sigma(j_k) \cap S) \cup \{j_k\}, max) - v(L^\sigma(j_k) \cap S, max)] \\ &= \frac{1}{n!} \sum_{\sigma \in \Psi(N)} (S, max) \\ &= v(S, max). \end{aligned}$$

□

Proof of Proposition 4

Proof. If: Let $i \in N$ be a veto player and v^i be a simple monotonic game with veto player i. For any $\mathcal{P} \in \Pi(N)$ and $j \in N$, define

$$x_j^{\mathcal{P}} = \begin{cases} v^i(\mathcal{P}(j), \mathcal{P}) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

This constructs a PMAS. Moreover, for any two games v and v', if x is a PMAS of v and y is a PMAS of v', then $ax + by$ is a PMAS of $av + bv'$ for any nonnegative numbers a and b.

Only if: Let v be a game with a PMAS. Let x be the PMAS. Clearly, v is monotonic. For any player $j \in N$, we define v^j as follows: for any $(S, \mathcal{P}) \in EC(N)$,

$$v^j(S, \mathcal{P}) = \begin{cases} x_j^{\mathcal{P}} & \text{if } S = \mathcal{P}(j) \ (\iff j \in S), \\ 0 & \text{otherwise.} \end{cases} \tag{A.5}$$

Since x is a PMAS, v^j is monotonic. For any $(S, \mathcal{P}) \in EC(N)$, we have

$$v(S, \mathcal{P}) \overset{\text{(i) of Def 2.1}}{=} \sum_{j \in S} x_j^{\mathcal{P}} \overset{\text{(A.5)}}{=} \sum_{j \in S} v^j(S, \mathcal{P}) \overset{\text{(A.5)}}{=} \sum_{j \in N} v^j(S, \mathcal{P}),$$

equivalently,

$$v = \sum_{j \in N} v^j. \tag{A.6}$$

We fix a player $i \in N$ and define Π^i_{++} as the set of partitions in which the worth of player i's coalition is positive, formally, $\Pi^i_{++} = \{\mathcal{P} \in \Pi(N) | v^i(\mathcal{P}(i), \mathcal{P}) > 0\}$. For some $\mathcal{P}, \mathcal{P}' \in \Pi^i_{++}$, $v^i(\mathcal{P}(i), \mathcal{P})$ may be equal to $v^i(\mathcal{P}'(i), \mathcal{P}')$. We choose one from such partitions (if any) and define $\widehat{\Pi}^i_{++} \subseteq \Pi^i_{++}$. Hence, for any $\mathcal{P} \in \widehat{\Pi}^i_{++}$, we have $v^i(\mathcal{P}(i), \mathcal{P}) > 0$ and no $\mathcal{P}' \in \widehat{\Pi}^i_{++}$ satisfies $v^i(\mathcal{P}(i), \mathcal{P}) = v^i(\mathcal{P}'(i), \mathcal{P}')$. We define an ordering of partitions in $\widehat{\Pi}^i_{++}$ as $\mathcal{P}_1, ..., \mathcal{P}_m$, where $v^i(\mathcal{P}_k(i), \mathcal{P}_k) < v^i(\mathcal{P}_{k+1}(i), \mathcal{P}_{k+1})$ for every k $(1 \leq k \leq K-1)$, and $K = |\widehat{\Pi}^i_{++}|$. Note that $|\mathcal{P}_k(i)| \leq |\mathcal{P}_{k+1}(i)|$ follows from the monotonicity of v^i.

Now, for every k $(1 \leq k \leq K)$, we define $\lambda^{i,\mathcal{P}_k}$ as follows:

$$\begin{aligned}
\lambda^{i,\mathcal{P}_1} &= v^i(\mathcal{P}_1(i), \mathcal{P}_1), \\
\lambda^{i,\mathcal{P}_2} &= v^i(\mathcal{P}_2(i), \mathcal{P}_2) - v^i(\mathcal{P}_1(i), \mathcal{P}_1), \\
&\dots \\
\lambda^{i,\mathcal{P}_k} &= v^i(\mathcal{P}_k(i), \mathcal{P}_k) - v^i(\mathcal{P}_{k-1}(i), \mathcal{P}_{k-1}), \\
&\dots \\
\lambda^{i,\mathcal{P}_K} &= v^i(\mathcal{P}_K(i), \mathcal{P}_K) - v^i(\mathcal{P}_{K-1}(i), \mathcal{P}_{K-1}).
\end{aligned}$$

Moreover, we define $v^{i,\mathcal{P}_k}$ as follows: setting $u^{i,\mathcal{P}_0} = v^i$,

$$v^{i,\mathcal{P}_1}(S, \mathcal{P}) = \begin{cases} 1 & \text{if } u^{i,\mathcal{P}_0}(S, \mathcal{P}) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad u^{i,\mathcal{P}_1} = u^{i,\mathcal{P}_0} - v^{i,\mathcal{P}_1};$$

$$\dots$$

$$v^{i,\mathcal{P}_k}(S, \mathcal{P}) = \begin{cases} 1 & \text{if } u^{i,\mathcal{P}_{k-1}}(S, \mathcal{P}) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad u^{i,\mathcal{P}_k} = u^{i,\mathcal{P}_{k-1}} - v^{i,\mathcal{P}_k};$$

$$\dots$$

$$v^{i,\mathcal{P}_K}(S, \mathcal{P}) = \begin{cases} 1 & \text{if } u^{i,\mathcal{P}_{K-1}}(S, \mathcal{P}) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad u^{i,\mathcal{P}_K} = u^{i,\mathcal{P}_{K-1}} - v^{i,\mathcal{P}_K}.$$

We obtain $v^i = \sum_{k=1}^{K} \lambda^{i,\mathcal{P}_k} v^{i,\mathcal{P}_k}$. By the construction of $\lambda^{i,\mathcal{P}_k}$, each $\lambda^{i,\mathcal{P}_k}$ is positive. Moreover, each $v^{i,\mathcal{P}_k}$ is a simple monotonic veto-controlled game: simplicity follows from the construction, monotonicity from $|\mathcal{P}_k(i)| \leq |\mathcal{P}_{k+1}(i)|$ for any k $(1 \leq k \leq m-1)$, and i is a veto player because of (A.5). In view of (A.6), we have

$$v = \sum_{j \in N} \sum_{k=1}^{K} \lambda^{j,\mathcal{P}_k} v^{j,\mathcal{P}_k}.$$

This completes the proof. □

Proof of Proposition 5

Proof. The proof consists of the following equivalent statements.

1. A game v has a PMAS.
2. A game v is a nonnegative linear combination of monotonic simple veto-controlled games $v_1, ..., v_M$.
3. For any δ satisfying $\delta \cdot v_m \leq 0$ for any $m = 1, ..., M$, we have $\delta \cdot v \leq 0$.
4. For any δ satisfying $\delta \cdot v_m \leq 0$ for any $m = 1, ..., M$ and $\delta_{(N,\{N\})} = -1$, we have $\delta \cdot v \leq 0$.
5. For any extended vector of subbalanced weights δ,

$$\sum_{(S,\mathcal{P}) \in EC^*(N)} \delta_{(S,\mathcal{P})} v(S,\mathcal{P}) \leq v(N,\{N\}).$$

In view of Proposition 4, (1) $\Rightarrow$ (2) holds. For (1) $\Leftarrow$ (2), using Π^i_{++} instead of $\widehat{\Pi}^i_{++}$ in the only if part of the proof of Proposition 4 results in nonnegative coefficients $\lambda_1, ..., \lambda_M$ for some natural number M.

The Minkowski-Farkas lemma implies (2) $\Leftrightarrow$ (3). Statement (2) is formally written as follows: there exist nonnegative numbers $\lambda_1, ..., \lambda_M$ such that $v = \sum_{m=1}^{M} \lambda_m v_m$. From the Minkowski-Farkas lemma, it follows that no $\delta = (\delta_{(S,\mathcal{P})})_{(S,\mathcal{P}) \in EC(N)}$ satisfies both $\delta \cdot v < 0$ and $\delta \cdot v_m \geq 0$ for any $m = 1, ..., M$. Hence, for any δ satisfying $\delta \cdot v_m \geq 0$ for any $m = 1, ..., M$, we have $\delta \cdot v \geq 0$, which is equivalent to (3) by replacing $\geq$ with $\leq$.

Clearly, (3) implies (4). However, the opposite direction is not necessarily obvious. The proof of Claim (3) $\Leftarrow$ (4) is provided below.

For the equivalence between (4) and (5), δ satisfies $\delta \cdot v_m \leq 0$ for any $m = 1, ..., M$ and $\delta_{(N,\{N\})} = -1$ if and only if for any $i \in N$, $S \subsetneq N$ with $i \in S$, and $\mathscr{P} \subseteq \{\mathcal{P} \in \Pi(N) \setminus \{N\} | S \in \mathcal{P}\}$,

$$\sum_{(T,\mathcal{Q}) \in EC^*(N)} \delta_{(T,\mathcal{Q})} \chi^{i,S,\mathscr{P}}(T,\mathcal{Q}) \leq 1.$$

This completes the proof.

Claim (3) $\Leftarrow$ (4). We prove this by contraposition. We assume that there exists δ such that

$$\delta \cdot v_m \leq 0 \text{ for any } m = 1, ..., M, \text{ and} \tag{A.7}$$
$$\delta \cdot v > 0. \tag{A.8}$$

We construct δ' satisfying (i) $\delta' \cdot v_m \leq 0$ for any $m = 1, ..., M$, (ii) $\delta' \cdot v > 0$, and (iii) $\delta'_{(N,\{N\})} = -1$.

Case 1: $\delta_{(N,\{N\})} \neq 0$. We define $\delta'_{(S,\mathcal{P})} = -\frac{1}{\delta_{(N,\{N\})}}\delta_{(S,\mathcal{P})}$ for every $(S,\mathcal{P}) \in EC(N)$. Note that $\delta_{(N,\{N\})} < 0$ because, among the games $v_1, ..., v_M$, we can find the game v_{m^*} such that $v_{m^*}(S,\mathcal{P}) = 0$ for $(S,\mathcal{P}) \neq (N,\{N\})$ and $v_{m^*}(N,\{N\}) = 1$, and (A.7) implies $\delta \cdot v_{m^*} = \delta_{(N,\{N\})} \leq 0$. Since $\delta_{(N,\{N\})} \neq 0$, we have

$$\delta_{(N,\{N\})} < 0. \tag{A.9}$$

Hence, δ' satisfies (i) because for any $m = 1, ..., M$,

$$\delta' \cdot v_m = -\frac{1}{\delta_{(N,\{N\})}}\delta \cdot v_m \overset{\text{(A.7),(A.9)}}{\leq} 0,$$

(ii) because

$$\delta' \cdot v = -\frac{1}{\delta_{(N,\{N\})}}\delta \cdot v \overset{\text{(A.8),(A.9)}}{>} 0,$$

and (iii) because $\delta'_{(N,\{N\})} = -\frac{1}{\delta_{(N,\{N\})}}\delta_{(N,\{N\})} = -1$.

Case 2: $\delta_{(N,\{N\})} = 0$. The following construction is a slight variant of Sprumont (1990)'s. For simplicity, let $(\delta \cdot v)^* = \sum_{(T,\mathcal{Q})\in EC^*(N)} \delta_{(T,\mathcal{Q})} v(T,\mathcal{Q})$ for any δ and v. Note that

$$(\delta \cdot v_m)^* = \delta \cdot v_m - \delta_{(N,\{N\})} v_m(N,\{N\}) = \delta \cdot v_m \overset{\text{(A.7)}}{\leq} 0 \text{ for any } m = 1, ..., M, \tag{A.10}$$

$$(\delta \cdot v)^* = \delta \cdot v - \delta_{(N,\{N\})} v(N,\{N\}) = \delta \cdot v \overset{\text{(A.8)}}{>} 0. \tag{A.11}$$

We define

$$\delta'_{(S,\mathcal{P})} = \left(\frac{|v(N,\{N\})|}{(\delta \cdot v)^*} + \epsilon\right)\delta_{(S,\mathcal{P})} \text{ for every } (S,\mathcal{P}) \in EC^*(N), \tag{A.12}$$

where $\epsilon > 0$, and define $\delta'_{(N,\{N\})} = -1$. Hence, δ' satisfies (i) because for any $m = 1, ..., M$, we have

$$\begin{aligned}\delta' \cdot v_m = (\delta' \cdot v_m)^* + \delta'_{(N,\{N\})} v_m(N,\{N\}) &\overset{\text{(A.12)}}{=} \left(\frac{|v(N,\{N\})|}{(\delta \cdot v)^*} + \epsilon\right)(\delta \cdot v_m)^* - 1 \\ &\overset{\text{(A.10),(A.11)}}{\leq} 0 - 1,\end{aligned}$$

(ii) because

$$\begin{aligned}\delta' \cdot v = (\delta' \cdot v)^* + \delta'_{(N,\{N\})} v(N,\{N\}) &\overset{\text{(A.12)}}{=} \left(\frac{|v(N,\{N\})|}{(\delta \cdot v)^*} + \epsilon\right)(\delta \cdot v)^* - v(N,\{N\}) \\ &= |v(N,\{N\})| + \epsilon(\delta \cdot v)^* - v(N,\{N\}) \\ &\overset{\text{(A.11)}}{>} 0,\end{aligned}$$

and (iii) because of the construction. This completes the proof of Claim (3) $\Leftarrow$ (4). □

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