WINPEC Working Paper Series No.E1711
Aug 2017

Priority-regarding Shapley values

Takaaki Abe and Satoshi Nakada

Waseda INstitute of Political EConomy
Waseda University
Tokyo,Japan

Priority-regarding Shapley values

Takaaki Abe *   Satoshi Nakada†

April 14, 2017

Abstract

We offer and axiomatize a new allocation rule for cooperative games, the priority-regarding Shapley values, which takes into account both each player's contribution and a priority order among players to determine each player's allocation. The priority order is exogenously or endogenously determined. For the exogenous case, we introduce two new axioms, fair evaluation for contribution (FEC) and relative difference for null players (RDN), and show that the allocation rule satisfies these two axioms with three standard axioms, efficiency, monotonicity and symmetry if and only if it takes the convex combination of the Shapley value and the weighted division. For the endogenous case, we also show that an allocation rule satisfies two weak monotonicity axioms, efficiency, weak symmetry and additivity if and only if the rule is the priority-regarding Shapley value. We also characterize the egalitarian Shapley values, the Shapley value, the weighted divisions and the egalitarian division as special cases of our priority-regarding Shapley values by imposing additional axioms.

Keywords: Cooperative game, Priority, Contribution, Shapley value.
JEL Classification: C71, D63, H20.

1 Introduction

One of the main concerns in the cooperative game theory is to explore desirable allocation rules and its axiomatic characterization. The most eminent allocation rule may be the Shapley value introduced by Shapley (1953), which allocates the surplus depending on each player's contributions. Recently, some allocation rules which depend not only on each player's contributions are intensively studied. An example is the egalitarian Shapley values, which is the convex combinations of the Shapley value and the egalitarinan devision.

Our purpose is to characterize an allocation rule defined as convex combinations of the Shapley value and the weighted division, which we call the *priority-regarding Shapley*

*Graduate School of Economics, Waseda University, 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan. JSPS Research Fellow, E-mail: takatomo3639@asagi.waseda.jp

†Graduate School of Economics, Hitotsubashi University, 2-1, Naka, Kunitachi, Tokyo, 186-8601, Japan. JSPS Research Fellow, Email: s.nakada0911@gmail.com

values. In many studies for the cooperative game theory, we assume that players are identical. However, there are many applications in which players have an exogenous or endogenous priority order, or weights.[1] The exogenous priority is thought of as a priori order which is determined before their cooperation such as qualifications for ability, educational backgrounds and seniority orders. In contrast, the endogenous priority can be seen as an order which is determined after the cooperation such as a promotion in firms, which is often associated with one's contributions.

One of the most prominent allocation rules in this context is the weighted Shapley value due to Shapley (1953), which is an efficient and linear rule such that players' surplus is proportionally distributed according to the weights of the non-null players in each unanimity game. Although this rule is quite simple, each player's contribution is not always evaluated fairly: even if a player i increases his contribution as much as another player j, according to the weighted Shapley value, player i may get less than player j because of the difference of their weights. In other words, the evaluation for their additional contributions may not be impartial. Moreover, according to the rule, null players always obtain nothing, that is, it is a completely contribution-based allocation rule. In contrast, it might be natural that even null players can obtain some amounts like the minimal payment of the firms and social securities. Therefore, replacing the null player axiom with a weaker axiom will enhance the equity among players.

Motivated by these observations, first, we consider the exogenous case by introducing two new axioms: *fair evaluation for contribution* (FEC) and *relative difference for null players* (RDN). The axiom (FEC) says that we should take each player's contribution and priority into consideration separately. In other words, if a player additionally contributes, then a reward for his additional contribution should be evaluated impartially regardless of his priority. The axiom (RDN) is a minimal fairness requirements for null players. It describes that the payoff difference between two null players, i.e., the players whose marginal contributions are zero, depends only on the relative difference of their priorities. Combining these two axioms with efficiency, monotonicity, and symmetry, we obtain the priority-regarding Shapley values with exogenously determined priorities.

Second, we characterize the family of priority-regarding Shapley values, where a priority profile is endogenously determined. Our axioms feature two monotonicity axioms, which are weaker than the standard monotonicity studied by Young (1985) and weak monotonicity by Casajus and Huettner (2014). Our first monotonicity is *total monotonicity for null players* (TMN). The axiom (TMN) says that even a null player (i.e., a no-contributed player) should be allocated more if the total amount of resources to be distributed increases. The second one is *contribution monotonicity* (CM). The axiom (CM) states that a player gets more if his contributions increase with keeping the total amount unchanged. In addition to monotonicity, symmetry also plays an important role. By symmetry axiom, we usually mean *equal treatment property*, which requires that two symmetric players in the sense of their contributions should receive the same amount. Combining these monotonicity axioms and weaker symmetry axiom with efficiency and additivity, we characterize the family of priority-regarding Shapley values.

Our rules subsume many well-known allocation rules in the literature such as the egalitarian Shapley values, the Shapley value, the weighted divisions and the egalitarian division. By imposing additional axioms, we show that our allocation rule coincides with

[1]See, for example, Kalai and Samet (1987), Chun (1991), Nowak and Radzik (1995) for these arguments.

those rules.

1.1 Literature Review

As we stated above, our study is closely related to three allocation rules: the Shapley value, the weighted divisions and the egalitarian Shapley values. We briefly review the literature and mention a relationship with our study.

There are numerous axiomatizations of the Shapley value. Shapley (1953) initially characterizes the allocation rule with efficiency, symmetry (namely, the equal treatment property), additivity and the null player property. Young (1985) replaces additivity and the null player property with its marginality axiom. Young (1985) shows that the Shapley value allocates the surplus solely depending on each player's productivity measured by her marginal contribution.

Moreover, our paper is related to the weighted Shapley value, which uses weights to aggregate players' contribution. Kalai and Samet (1987) and Chun (1991) propose their axiomatizations for the endogenous weight case; Nowak and Radzik (1995) offer an axiomatization for the exogenous weight case. As for the weighted division, a leading work is provided by Béal et al (2016). The weighted division assigns the share of the total surplus to each player proportionally depending on their weights. The simplest example is the equal division, which regards all players as equal. The approach to the weighted division is technically close to Chun (1988) and Roth (1979). They study the proportional allocation for bankruptcy problems and bargaining problems, respectively. These studies assume that a weight profile is endogenously induced.

The egalitarian Shapley value is initially defined by Joosten (1996) as a convex combination of the Shapley value and the equal division. Many properties of the egalitarian Shapley value are clarified by Casajus and Huettner (2013, 2014) and van den Brink, Funaki and Ju (2013). van den Brink, Funaki and Ju (2013) propose three different axiomatizations and a non-cooperative implementation of the rule. Casajus and Huettner (2013) focus on the fact that the egalitarian Shapley value provides even null players with non-negative payoffs. They replace the null player property by the new axiom representing this fact (with slightly strengthening symmetry). Moreover, Casajus and Huettner (2014) drop additivity and characterize it with the three axioms: efficiency, weak monotonicity, and symmetry (the equal treatment property). Our set of axioms is closely related to that of Casajus and Huettner (2014). Nevertheless, this paper is different from their paper for two reasons. First, they consider only a cooperative game as primitive (and the equal priority is endogenously induced), whereas we also consider the case where priority profiles as primitives. Second, they study only the case where the priority is equal among agents, whereas we study general cases. Therefore, the egalitarian Shapley value can be seen as a corollary of our result when players have equal priorities.

The remainder of this paper is organized as follows. In Section 2, we provide basic definitions and notations. In Section 3, we offer the characterization in the case of exogenous priority profiles. In Section 4, we offer the characterization for the family of priority-regarding Shapley values. Section 5 is the concluding remarks of this paper.

2 Preliminaries

Let $N = \{1, \cdots, n\}$ be the set of players and a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ denote a characteristic function. A *coalition* of players is defined as a subset of the player set: $S \subseteq N$, or $S \in 2^N$. Let $|S|$ denote the cardinality of coalition S. We sometimes use n to denote $|N|$. We call (N, v) a cooperative game with transferable utility or a TU game. We denote $\mathcal{G}_N$ the set of all TU games with the player set N. For each $v \in \mathcal{G}_N$, we say that player $i \in N$ is a *null player* in v if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. For any $T \in 2^N \setminus \{\emptyset\}$, unanimity game $u_T \in \mathcal{G}_N$ is defined as follows: for any S,

$$u_T(S) = \begin{cases} 1 & \text{if } \; T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Any game $v \in \mathcal{G}_N$ is uniquely described as a linear combination of unanimity games:

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} \lambda_T(v) u_T,$$

where $\lambda_T(v)$ is called *dividend* of the coalition T and given by

$$\lambda_T(v) = \sum_{R \subseteq T} (-1)^{|T \setminus R|} v(R).$$

By using unanimity games, the Shapley value $Sh(N, v)$ is given as follows: for any player $i \in N$,

$$Sh_i(N, v) = \sum_{T \in 2^N \setminus \{\emptyset\}:\ i \in T} \frac{\lambda_T(v)}{|T|}.$$

Let $w = (w_i)_{i \in N} \in \mathbb{R}^N_+$ such that $\sum_{i \in N} w_i = 1$ be a priority profile and $\mathcal{W}$ be the set of all possible priority profiles. The weighted Shapley value $Sh^w_i(v)$ is defined as the unique linear solution (of cooperative games) satisfying, for each unanimity game u_T and a given weight $w \in \mathbb{R}^N_{++}$ such that $\sum_{j \in N} w_j = 1$,

$$Sh^w_i(u_T) = \begin{cases} \frac{w_i}{\sum_{j \in T} w_j} & \text{if } \; i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The weighted Shapley value allows us to allocate players' surplus based not only on their contributions but also on their weights. The rule's evaluation for contribution mainly depends on the weights. To see this, we fix an arbitrary player, say, player 1. Consider unanimity games u_N and $u_{N \setminus \{1\}}$. In the latter case, player 1 is a null player. Then, the difference of the amount the player 1 receives is given by $f_1(u_N) - f_1(u_{N \setminus \{1\}})$, which depends upon the players' priorities:

$$Sh^w_1(u_N) - Sh^w_1(u_{N \setminus \{1\}}) = \frac{w_1}{\sum_{j \in N} w_j} \neq \frac{w'_1}{\sum_{j \in N} w'_j} = Sh^{w'}_1(u_N) - Sh^{w'}_1(u_{N \setminus \{1\}}),$$

where $w_1 \neq w'_1$. That is, even if additional contribution for player 1 is same in the two situations (w and w'), the difference of the allocations depends on the player's priority, which seems unfair. In addition to this observation, any null-player obtains nothing in any situation. In this sense, this rule is a completely contribution-based in contrast to, say, the minimal payment of the firms and social securities.

To avoid such unfairness and enhance equity among players, we then introduce the following allocation rule *priority-regarding Shapley value* for the priority profile w:

$$f_i(v) = \delta \cdot Sh_i(v) + (1-\delta) \cdot w_i v(N) \text{ where } \delta \in [0,1].$$

Note that the allocation rule is specified by the parameter $\delta \in [0,1]$. If $\delta = 1$, the allocation rule coincides with the Shapley value and distribute the surplus $v(N)$ based only on the players' contribution. If $\delta = 0$, the assignment is determined by their prioritities. Moreover, if weight is symmetric, i.e, $w^{sym} = (\frac{1}{n}, ..., \frac{1}{n})$, the allocation rule is called the *egalitarian Shapley value* introduced by Joosten (1996).

3 Axiomatizations of the Priority-Regarding Shapley Values

In this section, we assume that a profile $w = (w_i)_{i \in N} \in \mathcal{W}$ is exogenously given. That is, we consider an allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$.

3.1 The Axiomatic Characterization

We first introduce three basic axioms. These are variants of the axioms used in Casajus and Huettner (2014).

Axiom 1 (Efficiency (E))**.** For any $v \in \mathcal{G}_N$ and $w \in \mathcal{W}$, $\sum_{i \in N} f_i(v,w) = v(N)$.

Axiom 2 (Monotonicity (M))**.** For any $v, v' \in \mathcal{G}_N$, $w \in \mathcal{W}$ and $i \in N$, if $v(N) \geq v'(N)$ and $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \setminus \{i\}$, then $f_i(v,w) \geq f_i(v',w)$.

Axiom 3 (Weak Symmetry (SYM-))**.** For any $v \in \mathcal{G}_N$, $w \in \mathcal{W}$ and $i, j \in N$, if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i,j\}$ and $w_i = w_j$, then we have $f_i(v,w) = f_j(v,w)$.

(SYM-) requires that two players whose contributions and priorities are same should receive the same amount. Note that (SYM-) is weaker than the symmetry used in Casajus and Huettner (2014) (see Axiom 6 of this paper for the formal definition) because, to be symmetric, players must have not only the same contribution but also the equal weight. If we fix w to the equal weight $(\frac{1}{n}, ..., \frac{1}{n})$, (SYM-) is equal to their symmetry.

Axiom 4 (Relative Difference for Null players (RDN))**.** For any $v \in \mathcal{G}_N$, $w \in \mathcal{W}$ and any null players $i, j \in N$, we have $w_i \cdot f_j(v,w) = w_j \cdot f_i(v,w)$.

(RDN) distinguishes our allocation rule from the Shapley value and the egalitarian Shapley value through its attitude to null players: the Shapley value assigns zero, the egalitarian Shapley value assigns $v(N)/n$, whereas our value gives $w_i v(N)$ with respect to their weights to null players.

Axiom 5 (Fair Evaluation for Contribution (FEC))**.** For any $v \in \mathcal{G}_N$, $w, w' \in \mathcal{W}$ and $i \in N$, $f_i(v,w) - f_i(v(N)u_{N \setminus \{i\}}, w) = f_i(v,w') - f_i(v(N)u_{N \setminus \{i\}}, w')$.

(FEC) states that player i's contribution should be evaluated separately from his priorities (w_i and w'_i). This axiom describes the independence between contributions and priorities. Taking an example in the payment in a firm, (FEC) says that if a boss's contribution is equal to that of a freshman, then their contribution must be equally evaluated.

Now we are ready to state our axiomatization result.

Theorem 3.1. Suppose that $n \geq 3$. An allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E), (M), (SYM-), (RDN) and (FEC) if and only if there exists a $\delta \in [0, 1]$ such that $f_i(v, w) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N)$.

Proof. See Appendix A. □

The independence of axioms is shown by examples in Appendix B. Note that the uniqueness of the Theorem does not hold for the case of $n = 2$. The counterexample is found in Appendix C.

To clarify the difference between our allocation rule and the egalitarian Shapley value, we consider the following axiom. Using the following strong axiom (together with (E) and (M)), we can obtain the egalitarian Shapley value for any $w \in \mathcal{W}$.

Axiom 6 (Strong Symmetry (SYM+))**.** For any $v \in \mathcal{G}_N$ and $w \in \mathcal{W}$, if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$, then we have $f_i(v, w) = f_j(v, w)$.

To characterize the egalitarian Shapley value, we should remove (RDN) and (FEC) because the value is independent of $w \in \mathcal{W}$. This result immediately follows from Casajus and Huettner (2014) because, regardless of w, the three axioms (E) (M) (SYM+) coincide with the axioms they used.

Theorem 3.2. Suppose that $n \geq 3$. An allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E), (M), (SYM+) if and only if it is the egalitarian-Shapley value.

3.2 Special Cases

As we mentioned above, if $\delta = 1$, our allocation rule coincides with the Shapley value and, if $\delta = 0$, the weighted division. In this section, we characterize the Shapley value and the weighted division as the special cases of the rule.

The Shapley Value

We consider the following additional axioms.

Axiom 7 (Constant amount for Null Player (C-Null))**.** For any $v \in \mathcal{G}_N$ and $w \in \mathcal{W}$, there exists $d \in \mathbb{R}$ such that $f_i(v, w) = d$ for any null player $i \in N$.

Axiom 8 (Null Player Property (Null))**.** For any $v \in \mathcal{G}_N$, $w \in \mathcal{W}$ and null player $i \in N$, $f_i(v, w) = 0$.

Note that (Null) implies (RDN) and (C-Null) respectively. Moreover, we have the following lemma.

Lemma 3.3. (Null) $\iff$ (RDN) and (C-Null).

Proof. Since (Null) implies (RDN) and (C-Null), it suffices to show that (RDN) and (C-Null) implies (Null). Let $i, j \in N$ be any null players. By (C-Null), for any v and w, there exists $d \in \mathbb{R}$ such that $f_i(v, w) = f_j(v, w) = d$. By (RDN), $w_i \cdot f_j(v, w) = w_j \cdot f_i(v, w)$. Hence, we have $(w_i - w_j)d = 0$. Since this must be satisfied for any $w \in \mathcal{W}$, we obtain $d = 0$. □

Theorem 3.1 states that, within the domain $\mathcal{G}_N \times \mathcal{W}$, our allocation rule $f_i(v, w) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N)$ is the only solution satisfying (E), (M), (SYM-), (RDN) and (FEC). The Shapley value, which is the special case of $\delta = 1$, is the unique solution satisfying these axioms and (Null), because if δ is less than 1, null players can receive the positive amount. Therefore, by this argument and Lemma 3.3, we obtain the following corollary.

Corollary 3.4. Suppose that $n \geq 3$. An allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E), (M), (SYM-), (RDN), (FEC), and (C-Null) if and only if it is the Shapley value.

The Weighted Division

We now consider the following axiom which is the counterpart of (Null).

Axiom 9 (No Priority Property (NP))**.** For any $v \in \mathcal{G}_N$, $w \in \mathcal{W}$ and $i \in N$, if $w_i = 0$, then we have $f_i(v, w) = 0$.

The weighted division, which is the special case of $\delta = 0$, is the unique solution satisfying (E), (M), (SYM-), (RDN), (FEC) and (NP) because if δ is greater than 0, a player whose priority is 0 can receive positive amount. Formally, we have the following corollary.

Corollary 3.5. Suppose that $n \geq 3$. An allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E), (M), (SYM-), (RDN), (FEC), and (NP) if and only if it is the weighted division $f_i(v, w) = w_i v(N)$.

Our axiomatization results are summarized as Table 1 in which P-Sh, Eg-Sh, Sh, and WD represent the priority-regarding Shapley value, the egalitarian Shapley value, the Shapley value and the weighted division, respectively. Symbol "+" means the solution satisfies the axiom. Symbol "⊕" shows that the axiom is used for the axiomatization.

	E	M	SYM-	SYM+	RDN	FEC	Null	C-Null	NP
P-Sh (Thm. 3.1)	⊕	⊕	⊕	-	⊕	⊕	-	-	-
Eg-Sh (Thm. 3.2)	⊕	⊕	+	⊕	-	+	-	+	-
Sh (Cor. 3.4)	⊕	⊕	⊕	+	⊕	⊕	+	⊕	-
WD (Cor. 3.5)	⊕	⊕	⊕	-	⊕	⊕	-	-	⊕

Table 1: The axioms and the solutions

4 Axiomatizations of the Family of Priority-Regarding Shapley Values

In this section, we characterize the family of the priority-regarding Shapley values. In contrast to the previous section, we do not assume that the priority profile $w \in \mathcal{W}$ is exogenously given and induce the profile from the imposed axioms below. That is, we consider the allocation rules $f : \mathcal{G}_N \to \mathbb{R}^N$.

4.1 The Axiomatic Characterization

Let us consider the following axioms. Some axioms are almost identical with these in the previous section.

Axiom 10 (Efficiency (E))**.** For each $v \in \mathcal{G}_N$, $\sum_{i \in N} f_i(v) = v(N)$.

Axiom 11 (Total Monotonicity for Null Players (TMN))**.** For each $v, v' \in \mathcal{G}_N$ and each player $i \in N$ such that i is a null player in both v and v', if $v(N) \geq v'(N)$ then $f_i(v) \geq f_i(v')$.

Axiom 12 (Contribution Monotonicity (CM))**.** For each $v, v' \in \mathcal{G}_N$ with $v(N) = v'(N)$, if $v(S) - v(S \backslash \{i\}) \geq v'(S) - v'(S \backslash \{i\})$ for all $S \subseteq N$ with $i \in S$, then $f_i(v) \geq f_i(v')$.

(TMN) states that if resources to be allocated become bigger, even null players should receive more. (CM) describes that if a player contributes more and the total resource remains unchanged, she gets more. It is important that both (TMN) and (CM) are weaker than weak monotonicity (Mo^-) defined by van den Brink et.al (2013).[2]

Clearly, our value does not obey symmetry properties.[3] Hence, one might conjecture that our allocation rule can be axiomatized only by weak monotonicity and efficiency. However, this is not true. For example, the weighted Shapley value satisfies these two axioms, but it is not represented by the priority-regarding Shapley value. The following symmetry axiom can be the property which distinguishes our rule from the egalitarian Shapley values.

Axiom 13 (Null Symmetry (NSYM))**.** For any $i, j \in N$ and $v, v' \in \mathcal{G}_N$ such that i, j are null players in v, if $v'(S \cup \{i\}) - v'(S) = v'(S \cup \{j\}) - v'(S)$ for all $S \subseteq N \setminus \{i, j\}$ and $v(N) = v'(N)$, then $f_i(v) - f_i(v') = f_j(v) - f_j(v')$.

This axiom states that if two players are initially null players in v and they contribute the same amount in v' with keeping the total amount unchanged, then their contributions should be evaluated impartially.[4]

Note that the weighted Shapley value does not satisfy (NSYM).

Axiom 14 (Additivity (AD))**.** For any $v, v' \in \mathcal{G}_N$, $f(v) + f(v') = f(v + v')$, where $(v + v')(S) = v(S) + v'(S)$ for any $S \subseteq N$.

[2] They define weak monotonicity (Mo^-) as follows: For each $v, v' \in \mathcal{G}_N$ with $v(N) \geq v'(N)$, if $v(S) - v(S \backslash \{i\}) \geq v'(S) - v'(S \backslash \{i\})$ for all $S \subseteq N$ with $i \in S$, then $f_i(v) \geq f_i(v')$.

[3] The formal definition of (usual) symmetry axiom is given in Section 4.2

[4] This axiom is similar to *differential marginality* defined by Casajus (2010, 2011). Formally, differential marginality is defined as follows: For any $i, j \in N$ and $v, v' \in \mathcal{G}_N$, if $v(S \cup \{i\}) - v(S \cup \{j\}) = v'(S \cup \{i\}) - v'(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $f_i(v) - f_j(v) = f_i(v') - f_j(v')$. In general, both two axioms are logically independent.

We admit that (AD) is a technically strong requirement. However, we can show that these axioms are mutually independent. For the independence among axioms, see Appendix E.

Next, we define some helpful axioms: (TR) and (NPE).

Definition 4.1 (Triviality (TR))**.** Let $\mathbf{0}$ be the null game. For any $i \in N$, $f_i(\mathbf{0}) = 0$.

Lemma 4.2. (AD) $\Rightarrow$ (TR).

Proof. Let f be a function satisfying (AD). Assume that f violates (TR). Since f violates (TR), there exists $i \in N$ such that $f_i(\mathbf{0}) \neq 0$. Define $x := f_i(\mathbf{0}) \neq 0$. We have

$$f_i(\mathbf{0}) + f_i(\mathbf{0}) = 2x \neq x = f_i(\mathbf{0}) = f_i(\mathbf{0} + \mathbf{0}).$$

This contradicts (AD). □

The following definition is due to Casajus and Huettner (2013).

Definition 4.3 (Null Player in a Productive Environment, (NPE))**.** For any null player $i \in N$ in $v \in \mathcal{G}_N$ with $v(N) \geq 0$, $f_i(v) \geq 0$.

Lemma 4.4. For any function satisfying (TR), we have (TMN) $\Rightarrow$ (NPE).

Proof. Let f be a function satisfying (TR) and (TMN). For any $i \in N$, let v be a game in which the player i is a null player and $v(N) \geq 0$. Hence, for the game v and the null game $\mathbf{0}$, we have $f_i(v) \overset{\text{(TMN)}}{\geq} f_i(\mathbf{0}) \overset{\text{(TR)}}{=} 0$. □

Corollary 4.5. For any function satisfying (AD), we have (TMN) $\Rightarrow$ (NPE).

Now, we are ready to offer our main axiomatization result as follows.

Theorem 4.6. An allocation rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM), (NSYM) and (AD) if and only if there exists a $\delta \in [0, 1]$ and a priority profile $w \in \mathcal{W}$ such that $f_i(v) = \delta \cdot Sh_i(v) + (1 - \delta) \cdot w_i v(N)$.

Proof. See Appendix D. □

4.2 Special Cases

In this section, we characterize some allocation rules as the special cases of our allocation rule. In particular, we consider the egalitarian Shapley values, the Shapley value, the weighted divisions and the egalitarian division. To this end, we consider the following axioms.

Axiom 15 (Symmetry (SYM))**.** For any $v \in \mathcal{G}_N$ and any $i, j \in N$, if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$, then $f_i(v) = f_j(v)$.

Axiom 16 (Null Player Property (Null))**.** For any $v \in \mathcal{G}_N$ and any $i \in N$, if i is a null player, then $f_i(v) = 0$.

The following definition is due to Casajus and Huettner (2014).

Axiom 17 (Grand Coalition Monotonicity (GMo))**.** For any $v, v' \in \mathcal{G}_N$ and any player $i \in N$, if $v(N) \geq v'(N)$ then $f_i(v) \geq f_i(v')$.

The Egalitarian Shapley Values

Replacing (NSYM) with (SYM), we obtain the egalitarian Shapley values.

Corollary 4.7. An allocation rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM), (AD) and (SYM) if and only if there exists a $\delta \in [0, 1]$ such that $f_i(v) = \delta \cdot Sh_i(v) + (1-\delta) \cdot \frac{1}{n} v(N)$.

Proof. Let a function f satisfy (E), (TMN), (CM), (AD) and (SYM). Since (SYM) implies (WSYM), Theorem 4.6 shows that f takes the following form:

$$f_i(v) = \delta Sh_i(v) + (1-\delta) w_i v(N).$$

Assume that $w \neq (\frac{1}{n}, ..., \frac{1}{n})$. There exist at least two players i, j such that $w_i \neq w_j$ because w is a weight. For a game v satisfying $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$, we must have $f_i(v) \neq f_j(v)$. This contradicts (SYM). □

Note that this axiomatization of the egalitarian Shapley values are tight in the sense that all axioms are logically independent. We show this point in Appendix B. Comparing with Casajus and Huettner (2014), this result implies that (Mo-) can be decomposed into (TMN), (CM) and (AD) under (E) and (SYM).

The Shapley Value

We additionally impose (Null) and obtain the Shapley value.

Corollary 4.8. An allocation rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM), (AD), (NSYM) and (Null) if and only if $f_i(v) = Sh_i(v)$

Proof. Let a function f satisfy these axioms. Since w is a weight, there exists a player i such that $w_i > 0$. Assume that $\delta < 1$. For a game $v \in \mathcal{G}_N$ such that $v(N) > 0$ and i is a null player, we have $f_i(v) > 0$. This contradicts (Null). □

The Weighted Divisions and The Egalitarian Division

When we strengthen (TMN) to (GMo), we obtain the weighted devisions.

Corollary 4.9. An allocation rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (CM), (NSYM), (AD) and (GMo) if and only if $f_i(v) = w_i \cdot v(N)$

Proof. Let a function f satisfy these axioms. Note that (GMo) implies (TMN), namely, we can apply Theorem 4.6 as well. Assume that $\delta > 0$. Let $i, j \in N$ $(i \neq j)$. Consider unanimity games $u_{\{i\}}$ and $u_{\{j\}}$. We have $Sh_i(u_{\{i\}}) = 1 \neq 0 = Sh_i(u_{\{j\}})$. The assumption $\delta > 0$ implies $f_i(u_{\{i\}}) \neq f_i(u_{\{j\}})$, which contradicts (GMo) because $u_{\{i\}}(N) = u_{\{j\}}(N)$. □

If we additionally impose (SYM) for the weighted divisions, we obtain the egalitarian division.

Corollary 4.10. An allocation rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (CM), (AD), (SYM), (GMo) if and only if $f_i(v) = \frac{1}{n} \cdot v(N)$

Proof. It follows from Corollaries 4.7 and 4.9. □

Since Casajus and Huettner (2014) axiomatize the egalitarian division by only (E), (SYM) and (GMo), (CM) and (AD) are not needed to obtain the tight axiomatization.

Our axiomatization results are summarized in Table 2, in which P-Sh, Eg-Sh, Sh,WD, and ED represent the priority-regarding Shapley value, the egalitarian Shapley values, the Shapley values, the weighted divisions and the egalitarian division, respectively. Symbol "+" means the solution satisfies the axiom. Symbol "⊕" shows that the axiom is used for the axiomatization.

	E	TMN	CM	NSYM	AD	SYM	Null	GMo	Mo^-
PR-Sh (Thm. 4.6)	⊕	⊕	⊕	⊕	⊕	-	-	-	+
Eg-Sh (Cor. 4.7)	⊕	⊕	⊕	+	⊕	⊕	-	-	+
Eg-Sh	⊕	+	+	+	+	⊕	-	-	⊕
Sh (Cor. 4.8)	⊕	⊕	⊕	⊕	⊕	+	⊕	-	+
WD (Cor. 4.9)	⊕	+	⊕	⊕	⊕	-	-	⊕	+
ED (Cor. 4.10)	⊕	+	⊕	+	⊕	⊕	-	⊕	+

Table 2: The axioms and the solutions

5 Concluding Remarks

In this paper, we propose and axiomatically characterize the new allocation rule called priority-regarding Shapley value. This allocation rule integrates two different measures, namely, contributions and priorities among players. This rule, moreover, can be seen as a generalization of the egalitarian Shapley value, which premises that each player has the equal priority.

In addition to the example of the payment system in a firm mentioned in Section 1, we believe that the idea of "two measures" would be useful for some problems as follows.

One example is a variant of the *polluted lake* initially studied by Shapley and Shubik (1969). There are n cities with different populations around a lake. It costs c_i for a city i to purify its own water supply from the lake.[5] It costs a constant amount for every city to purify its water before discharging it into the lake. To simplify the discussion, let the worth of a coalition (of cities) denote how much money the coalition saves through their cooperation, and the allocation represent the redistribution of the saved money. The allocation rule based on two measures, namely, costs and population in this example, respects each city's population, which exhibits the number of citizens who make use of the water.

The other example is a redistribution of wealth through taxes. Casajus (2015) considers the simple model, which is not a cooperative game, and axiomatizes his redistribution rule in the spirit of egalitarian Shapley value. In the model, each agent is identified with a real number (i.e., the agent's contribution to the total wealth) and, except for the real number, all agents are treated equally. However, in reality, how much tax people must pay should vary depending on their *social status*. For example, some people may have tax exempt status; resident tax may vary from city to city. Our priority-regarding value captures the difference which can not be measured by contribution or productivity. This topic is also studied by Abe and Nakada (2017).

[5]The technique to purify water (or, c_i) is independent of the population of a city.

Acknowledgment

We thank Yukihiko Funaki, Hideshi Itoh, Xue Jingyi, Shinsuke Kambe, Takashi Ui and Koji Yokote for valuable comments. We also thank seminar participants in Contract Theory Workshop East and Graduate Summer Workshop on Game Theory 2016 in Soul National University. Abe and Nakada acknowledge the financial support from Japan Society for Promotion of Science (JSPS). All remaining errors are our own.

Appendix A. Proof of Theorem 3.1

Let $\mathcal{G}_N^c \subset \mathcal{G}_N$ denote the set of games such that $v(N) = c$. Also, for any player $i \in N$ and $c \in \mathbb{R}$, let $\mathcal{G}_N^{c,i}$ denote the set of games such that $v(N) = c$ and i is null player. Note that $\mathcal{G}_N^{c,i} \subset \mathcal{G}_N^c$ for all $i \in N$ and $c \in \mathbb{R}$. Our proof is divided into 4 steps. In the step 1, we show that, by (M) and (FEC), the allocation rule $f_i(v, w)$ is decomposed into the sum of two function $\phi_i^c(v) : \mathcal{G}_N^{c,i} \to \mathbb{R}$ and $\psi_i^c(w) : \mathcal{G}_N^{c,i} \times \mathcal{W} \to \mathbb{R}$. In the step 2, by the result of Casajus and Huettner (2014), we show that, by (M), (FEC), (E) and (SYM-), $\phi_i^c(v)$ can be written as $\phi_i^{v(N)}(v) = \delta sh_i(v) + d_i^{v(N)}$ for some $\delta \in [0.1]$ and $d_i^{v(N)} \in \mathbb{R}$ when $w = w^{sym}$. In the step 3, by (RDN), $\psi_i^{v(N)}(w) = (1-\delta) \cdot w_i v(N) - d_i^{v(N)}$ for each $w \in \mathcal{W}$. In the step 4, we combine the above arguments and show that $f_i(v, w) = \delta \cdot Sh_i(v) + (1-\delta) \cdot w_i v(N)$.

As a benchmark, we first show that (E), (M), (SYM-) characterizes the egalitarian Shapley value when we fix $w = w^{sym}$. We use this result in the step 2 of the proof.

Theorem .1. Suppose that $\mathcal{W} = \{w^{sym}\}$ and $n \geq 3$. Then, an allocation rule $f : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ satisfies (E), (M), (SYM-) if and only if it is the Egalitarian-Shapley value.

Proof. This follows from the axioms and arguments in Casajus and Huettner (2014) if $w = (\frac{1}{n}, ..., \frac{1}{n})$. □ □

Proof of Theorem 3.1

Proof. Claim 1: For any $c \in \mathbb{R}$, $v \in \mathcal{G}_N^{c,i}$ and $w \in \mathcal{W}$, there are two functions $\phi_i^c(v) : \mathcal{G}_N^{c,i} \to \mathbb{R}$ and $\psi_i^c(w) : \mathcal{G}_N^{c,i} \times \mathcal{W} \to \mathbb{R}$ such that $f_i(v, w) = \phi_i^c(v) + \psi_i^c(w)$.

It follows from (FEC) that for any $c \in \mathbb{R}$ and $i \in N$, there is a function $\phi_i^c : \mathcal{G}_N^c \to \mathbb{R}$ such that

$$\phi_i^c(v) = f_i(v, w) - f_i(cu_{N\setminus\{i\}}, w) \tag{.1}$$

for any $v \in \mathcal{G}_N^c$ and $w \in \mathcal{W}$.

Moreover, it follows from (M) that for any $c \in \mathbb{R}$ and $i \in N$, there is a function $\psi_i^c : \mathcal{W} \to \mathbb{R}$ such that

$$\psi_i^c(w) = f_i(cu_{N\setminus\{i\}}, w) \tag{.2}$$

for any $w \in \mathcal{W}$, because for any $v' \in \mathcal{G}_N^{c,i}$, in which player i is null player, we have $\psi_i^c(w) := f_i(v', w) = f_i(cu_{N\setminus\{i\}}, w)$ by (M).

For any $i \in N$, $c \in \mathbb{R}$, $v \in \mathcal{G}_N^c$, $v' \in \mathcal{G}_N^{c,i}$ and $w \in \mathcal{W}$, we have

$$\begin{aligned} \phi_i^c(v) &\overset{(.1)}{=} f_i(v, w) - f_i(cu_{N\setminus\{i\}}, w) \\ &\overset{(.2)}{=} f_i(v, w) - \psi_i^c(w). \end{aligned}$$

Hence, we obtain $f_i(v,w) = \phi_i^c(v) + \psi_i^c(w)$.

Claim 2: For any $v \in \mathcal{G}_N$, there exists $\delta \in [0,1]$ and $d_i^{v(N)} \in \mathbb{R}$ such that $\phi_i^{v(N)}(v) = \delta sh_i(v) + d_i^{v(N)}$.

Let $w^* = (1/n, \cdots, 1/n) \in \mathcal{W}$, i.e., the equal weight. For any $c \in \mathbb{R}$ and any $v \in \mathcal{G}_N^c$, by Claim 1, we have

$$f_i(v,w^*) = \phi_i^c(v) + \psi_i^c(w^*), \tag{.3}$$

and, by Theorem .1, there exists $\delta \in [0,1]$ such that

$$f_i(v,w^*) = \delta sh_i(v) + (1-\delta)\frac{1}{n}c. \tag{.4}$$

Note that δ does not depend on $c \in \mathbb{R}$. For any $v' \in \mathcal{G}_N^{c,i}$, we have

$$\begin{aligned}\phi_i^c(v') + \psi_i^c(w^*) \overset{(.3)}{=} f_i(v',w^*) &\overset{(.4)}{=} \delta sh_i(v') + (1-\delta)\frac{1}{n}c \\ &= (1-\delta)\frac{1}{n}c. \end{aligned} \tag{.5}$$

Note that player i is null player in game $v' \in \mathcal{G}_N^{c,i}$. Hence, for any $v', v'' \in \mathcal{G}_N^{c,i}$, we have $\phi_i^c(v') + \psi_i^c(w^*) \overset{(.5)}{=} (1-\delta)\frac{1}{n}c \overset{(.5)}{=} \phi_i^c(v'') + \psi_i^c(w^*)$ and, so, denote $d_i^c := \phi_i^c(v') = \phi_i^c(v'')$. We obtain

$$\psi_i^c(w^*) \overset{(.5),d_i^c}{=} (1-\delta)\frac{1}{n}c - d_i^c. \tag{.6}$$

Therefore, for every $v \in \mathcal{G}_N^c$, we must have

$$\phi_i^c(v) \overset{(.3)(.4)(.6)}{=} \delta sh_i(v) + d_i^c. \tag{.7}$$

Setting $v(N) = c \in \mathbb{R}$, we obtain $\phi_i^{v(N)}(v) = \delta sh_i(v) + d_i^{v(N)}$ for all $v \in \mathcal{G}_N$.

Claim 3: $\psi_i^{v(N)}(w) = (1-\delta)\cdot w_i v(N) - d_i^{v(N)}$ for each $w \in \mathcal{W}$.

Case 1 There is a player $i \in N$ such that $w_i = 0$.

There is at least one player k^* such that $w_{k^*} > 0$, because w is a weight and $\sum_{j\in N} w_j = 1$. For a player $i \neq k^*$, consider an unanimity game u_i. By (RDN), for any $l \in N \setminus \{i\}$, $w_{k^*}(\psi_l^{v(N)}(w) + d_l^{v(N)}) = w_l(\psi_{k^*}^{v(N)}(w) + d_{k^*}^{v(N)})$. Hence, if $w_l = 0$, we must have $\psi_l^{v(N)}(w) = -d_l^{v(N)}$.

Case 2 For any $i \in N$, $w_i > 0$.

Consider any $w \in \mathcal{W}$ and player $k^* \in N$ such that $k^* \in \mathrm{argmin}_{i\in N} w_i$. For any player $i \neq k^*$ and any $c \in \mathbb{R}$, let $cu_i(S) := c \cdot u_i(S)$ for any $S \subseteq N$. By Claim 2, we have the following formula: for any $k \in N$,

$$f_k(cu_{\{i\}}, w) = \begin{cases} \delta c + d_k^c + \psi_k^c(w) & \text{if } \ k = i, \\ d_k^c + \psi_k^c(w) & \text{otherwise.} \end{cases}$$

Hence, we have

$$\sum_{k\in N}(\psi_k^c(w) + d_k^c) \overset{(E)}{=} c(1-\delta). \tag{.8}$$

Moreover, for any $i \neq k^*$, j $(j \neq i,\ j \neq k^*)$ and unanimity game u_j, we have

$$\psi_i^c(w) + d_i^c \overset{\text{(RDN)}}{=} \frac{w_i}{w_{k^*}}(\psi_{k^*}^c(w) + d_{k^*}^c), \tag{.9}$$

because i and k^* are null players in u_j.

For any $i \in N$, we have

$$\begin{aligned}
\psi_i^c(w) + d_i^c - (1-\delta)w_i c &\overset{(.9)}{=} w_i \cdot \left[(\psi_{k^*}^c(w) + d_{k^*}^c)\frac{1}{w_{k^*}} - (1-\delta)c\right] \\
&\overset{(.8)}{=} w_i \cdot \left[(\psi_{k^*}^c(w) + d_{k^*}^c)\frac{1}{w_{k^*}} - \sum_{k\in N}(\psi_k^c(w) + d_k^c)\right] \\
&\overset{(.9)}{=} w_i \cdot \left[(\psi_{k^*}^c(w) + d_{k^*}^c)\frac{1}{w_{k^*}} - \sum_{k\in N}\frac{w_k}{w_{k^*}}(\psi_{k^*}^c(w) + d_{k^*}^c)\right] \\
&\overset{\sum_k w_k = 1}{=} w_i \cdot \left[(\psi_{k^*}^c(w) + d_{k^*}^c)\frac{1}{w_{k^*}} - (\psi_{k^*}^c(w) + d_{k^*}^c) \cdot \frac{1}{w_{k^*}}\right] \\
&= 0.
\end{aligned}$$

Hence, setting $v(N) = c \in \mathbb{R}$, we obtain $\psi_i^{v(N)}(w) = (1-\delta)\cdot w_i v(N) - d_i^{v(N)}$ for all $v \in \mathcal{G}_N$.

Claim 4: For any $v \in \mathcal{G}_N$ and $w \in \mathcal{W}$, there exists $\delta \in [0,1]$ such that $f_i(v,w) = \delta \cdot Sh_i(v) + (1-\delta)\cdot w_i v(N)$.

For any $v \in \mathcal{G}_N$ and $w \in \mathcal{W}$, we have

$$\begin{aligned}
f_i(v,w) &\overset{\text{C1}}{=} \phi_i^{v(N)}(v) + \psi_i^{v(N)}(w) \\
&\overset{\text{C2}}{=} \delta sh_i(v) + d_i^{v(N)} + \psi_i^{v(N)}(w) \\
&\overset{\text{C3}}{=} \delta sh_i(v) + d_i^{v(N)} + (1-\delta)\cdot w_i v(N) - d_i^{v(N)} \\
&= \delta sh_i(v) + (1-\delta)\cdot w_i v(N).
\end{aligned}$$

This completes the proof. □ □

Appendix B. Independence of Axioms for Theorem 3.1

The independence of these axioms is shown in the examples listed below.

Example .2. Define $f^{\mathrm{E}} : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ by, for any v and any w,

$$f_i^{\mathrm{E}}(v,w) = 0.$$

Then, the function satisfies all axioms except (E).

Example .3. Define $f^{\mathrm{M}} : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ by

$$f_i^{\mathrm{M}}(v,w) = 2Sh_i(v) - w_i v(N).$$

Then, the function satisfies all axioms except (M).

Example .4. Let $f^{\text{SYM-}} : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ be given by

$$f_i^{\text{SYM-}}(v, w) = \delta \cdot Sh_i^z(v) + (1 - \delta) \cdot w_i v(N),$$

where $Sh_i^z(v)$ is the *weighted Shapley value* defined as the unique linear solution (of cooperative games) satisfying for each unanimity game u_T and a given weight $z \in R_{++}^N$ such that $\sum_{j \in N} z_j = 1$,

$$Sh_i^z(u_T) = \begin{cases} \frac{z_i}{\sum_{j \in T} z_j} & \text{if } \ i \in T, \\ 0 & \text{otherwise.} \end{cases} \tag{.10}$$

Note that $Sh_i^z(v)$ is defined with respect to a given weight profile z, which is different from the priority profile w. For example, we first fix the weight profile $z_1 = \frac{2}{3}$ and $z_j = \frac{1}{3(n-1)}$ for $j = 2, ..., n$ and define $Sh_i^z(v)$. Then, we define $f_i^3(v, w)$ as above. Since anonymity is defined over $\mathcal{G}_N$ and $\mathcal{W}$, the function satisfies all axioms except (SYM-).

Example .5. Define $f^{\text{RDN}} : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ by

$$f_i^{\text{RDN}}(v, w) = \delta \cdot \frac{v(N)}{|N|} + (1 - \delta) \cdot w_i v(N).$$

The function satisfies all axioms except (RDN).

Example .6. Define $f^{\text{FEC}} : \mathcal{G}_N \times \mathcal{W} \to \mathbb{R}^N$ by

$$f_i^{\text{FEC}}(v, w) = w_{\min} \cdot Sh_i(v) + (1 - w_{\min}) w_i v(N),$$

where $w_{\min} = \min_{j \in N} w_j$. This function satisfies all axioms except (FEC).

Appendix C. Counterexample to Theorem 3.1 for $n = 2$

Theorem 3.1 fails for $n = 2$. Consider the following allocation rule $f^\heartsuit$ on $N = \{1, 2\}$:

$$(f_1^\heartsuit(v, w), f_2^\heartsuit(v, w)) = \begin{cases} (Sh_1(v), Sh_2(v)), & Sh_1(v) \geq 0 \text{ and } Sh_2(v) \geq 0, \\ (0, v(N)), & Sh_1(v) < 0 \text{ and } Sh_2(v) > 0 \wedge v(N) \geq 0, \\ (v(N), 0), & Sh_1(v) < 0 \text{ and } Sh_2(v) > 0 \wedge v(N) < 0, \\ (Sh_1(v), Sh_2(v)), & Sh_1(v) \leq 0 \text{ and } Sh_2(v) \leq 0, \\ (0, v(N)), & Sh_1(v) > 0 \text{ and } Sh_2(v) < 0 \wedge v(N) \leq 0, \\ (v(N), 0), & Sh_1(v) > 0 \text{ and } Sh_2(v) < 0 \wedge v(N) > 0, \end{cases}$$

for all $v \in \mathcal{G}_N$ and any $w \in \mathcal{W}$. Note that this function does not depend on w. It is clear that $f^\heartsuit$ satisfies (E) and (M). It satisfies (SYM-) because if the players 1 and 2 are symmetric in the sense of marginal contribution and have the same weight, they receive $(Sh_1(v), Sh_2(v))$. It satisfies (RDN) because if the players 1 and 2 are null players, the game v is the null game: $v(12) = v(1) = v(2) = 0$. Since $f^\heartsuit$ does not depend on w, it clearly satisfies (FEC).

Appendix D. Proof of Theorem 4.6

Let $\Delta_i(v) = (v(S \cup \{i\}) - v(S))_{S \subseteq N \setminus \{i\}} \in \mathbb{R}^{2^{(N-1)}}$ be a vector of *marginal contributions* of i in v. By this notation, player $i \in N$ is a *null player* in v if $\Delta_i(v) = \mathbf{0}$.

Proof. It is clear that the rule satisfies all the axioms. We suppose that a rule $f : \mathcal{G}_N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM), (NSYM) and (AD).

For any $c \in \mathbb{R}$, let us define $\mathcal{G}_N^c = \{v \in \mathcal{G}_N | v(N) = c\}$. For any player i, we denote the set of games in which player i is a null player by $\mathcal{G}_N^i$. Define $\mathcal{G}_N^{c,i} := \mathcal{G}_N^c \cap \mathcal{G}_N^i$. Moreover, we use Λ_N^i to denote the set of marginal vectors of i: $\Lambda_N^i = \{\Delta_i(v) | v \in \mathcal{G}_N\}$.

Claim 1: For each $i \in N$, there exist functions $\phi_i : \Lambda_N^i \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(v) = \phi_i(\Delta_i(v), v(N)) + \alpha_i(v(N))$.

We first take any $c \in \mathbb{R}$. For any $i \in N$ and $v \in \mathcal{G}_N^c$, we have the following equation: for any $\bar{v} \in \mathcal{G}_N^c$ such that $\Delta_i(v) = \Delta_i(\bar{v})$,

$$f_i(v) \overset{\text{(CM)}}{=} f_i(\bar{v}) \; =: \alpha_i(c, \Delta_i(v)). \tag{.11}$$

Specifically, we denote

$$\alpha_i(c) = \alpha_i(c, \mathbf{0}). \tag{.12}$$

Moreover, for any $i \in N$ and $v, v' \in \mathcal{G}_N^c$, we have

$$\begin{aligned} f_i(v) - f_i(v') &\overset{(.11)}{=} \alpha_i(c, \Delta_i(v)) - \alpha_i(c, \Delta_i(v')) \\ &=: \phi_i(\Delta_i(v), \Delta_i(v'), c). \end{aligned} \tag{.13}$$

Hence, for any $i \in N$ and $v \in \mathcal{G}_N^c$, we obtain the following equation: for any $v' \in \mathcal{G}_N^{c,i}$,

$$\phi_i(\Delta_i(v), \Delta_i(v'), c) \overset{(.13)}{=} f_i(v) - f_i(v') \overset{(.11)}{=} f_i(v) - \alpha_i(c). \tag{.14}$$

Note that $f_i(v) - \alpha_i(c)$ is independent from $v' \in \mathcal{G}_N^{c,i}$. For any $i \in N$ and $v \in \mathcal{G}_N^c$ let

$$\phi_i(\Delta_i(v), c) := f_i(v) - \alpha_i(c). \tag{.15}$$

Hence, for any $i \in N$ and $v \in \mathcal{G}_N$, we obtain

$$f_i(v) \overset{(.15)}{=} \phi_i(\Delta_i(v), v(N)) + \alpha_i(v(N)). \tag{.16}$$

This completes Claim 1.

Before moving to Claim 2, note that for any null player $i \in N$, $\alpha_i(c)$ is nondecreasing with respect to c, i.e., $\alpha_i(v(N)) \geq \alpha_i(v'(N))$ for $v(N) \geq v'(N)$. To see this, we first confirm that for any $c \in \mathbb{R}$,

$$\phi_i(\mathbf{0}, c) \overset{(.15),(.14)}{=} \phi_i(\mathbf{0}, \mathbf{0}, c) \overset{(.13),(.11)}{=} \alpha_i(c, \mathbf{0}) - \alpha_i(c, \mathbf{0}) = 0. \tag{.17}$$

Thus, for any $v, v' \in \mathcal{G}_N^i$ with $v(N) \geq v'(N)$,

$$\begin{aligned} \alpha_i(v(N)) - \alpha_i(v'(N)) &= [0 + \alpha_i(v(N))] - [0 + \alpha_i(v'(N))] \\ &\overset{(.17)}{=} [\phi_i(\mathbf{0}, v(N)) + \alpha_i(v(N))] - [\phi_i(\mathbf{0}, v'(N)) + \alpha_i(v'(N))] \\ &\overset{(.15)}{=} f_i(v) - f_i(v') \\ &\overset{\text{(TMN)}}{\geq} 0. \end{aligned}$$

Similarly, for any $i \in N$, $\phi_i(\cdot, v(N))$ is nondecreasing with respect to the first argument with keeping the second argument unchanged, because of (CM).

Moreover, we show that ϕ is symmetric, namely, if $v(S\cup\{i\})-v(S) = v(S\cup\{j\})-v(S)$ for all $S \subseteq N\setminus\{i,j\}$, then $\phi_i(\Delta_i(v), v(N)) = \phi_j(\Delta_j(v), v(N))$. To see this, for any $i, j \in N$ and $v \in \mathcal{G}_N$ such that $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$, let $v' = v(N)u_{N\setminus\{i,j\}}$. We have

$$\phi_i(\Delta_i(v), v(N)) \overset{(.16)}{=} f_i(v) - f_i(v') \overset{\text{(NSYM)}}{=} f_j(v) - f_j(v') \overset{(.16)}{=} \phi_j(\Delta_j(v), v(N)). \tag{.18}$$

Claim 2: $\phi(\cdot,\cdot)$ does not depend on the value of second component and, for each $i \in N$, $v \in \mathcal{G}_N$ and $c \in \mathbb{R}$, $\phi_i(c\Delta_i(v)) = c\phi_i(\Delta_i(v))$, and $\alpha_i(cv(N)) = c\alpha_i(v(N))$.

For any nonempty $S \subseteq N$, consider the unanimity game u_S. In view of Claim 1, we have $f_k(u_S) = \phi_k(u_S, 1) + \alpha_k(1)$ for every $k \in N$. Since $\Delta_k(u_S) = \mathbf{0}$ for each $k \notin S$, each player's allocation is given as follows:

$$f_k(u_S) = \begin{cases} \phi_k(\Delta_k(u_S), 1) + \alpha_k(1) & \text{if } \ k \in S, \\ \alpha_k(1) & \text{otherwise.} \end{cases} \tag{.19}$$

The rest of Step 2 is similar to Casajus and Huettner (2014). Now, take any $S \subsetneq N$. For any $i \notin S$, by (.19), $f_i(u_S) = \alpha_i(1)$. By (AD), for any $q \in \mathbb{Q}$, $f_i(qu_S) = qf_i(u_S)$, namely, $\alpha_i(q) = q\alpha_i(1)$ for any $i \notin S$. Since $\alpha_i(\cdot)$ is monotonically nondecreasing, *i.e.*, $\alpha_i(q) \geq \alpha_i(q')$ for $q \geq q'$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, we obtain $\alpha_i(c) = c\alpha_i(1)$ for all $c \in \mathbb{R}$. This argument holds for each $i \in N$ because S is taken arbitrarily.

Note that ϕ is also additive, i.e.,

$$\phi_i(\Delta_i(v + v'), (v + v')(N)) = \phi_i(\Delta_i(v), v(N)) + \phi_i(\Delta_i(v'), v'(N)) \tag{.20}$$

because, for any $v \in \mathcal{G}_N$, we have $f_i(v) \overset{\text{(C1)}}{=} \phi_i(\Delta_i(v), v(N)) + \alpha_i(v(N))$ and f satisfies (AD). Moreover for any $v \in \mathcal{G}_N$ and any $c \in \mathbb{R}$ with $c \neq v(N)$, there exists $v'' \in \mathcal{G}_N$ satisfying $\Delta_i(v'') = \mathbf{0}$ and $v''(N) = c - v(N)$. We now define $v' := v + v''$ and then obtain $v = v' - v''$. Note that $v'(N) = c$. Hence, for any $v \in \mathcal{G}_N$,

$$\begin{aligned} \phi_i(\Delta_i(v), v(N)) &= \phi_i(\Delta_i(v' - v''), (v' - v'')(N)) \\ &\overset{(.20)}{=} \phi_i(\Delta_i(v'), v'(N)) - \phi_i(\Delta_i(v''), v''(N)) \\ &\overset{v'=v+v'', \Delta_i(v'')=\mathbf{0}}{=} \phi_i(\Delta_i(v), v'(N)) - \phi_i(\mathbf{0}, v''(N)) \\ &\overset{(.17)}{=} \phi_i(\Delta_i(v), v'(N)). \end{aligned}$$

Therefore, $\phi_i(\cdot,\cdot)$ only depends on the value $\Delta_i(v)$, and we write $\phi_i : \Lambda_N^i \to \mathbb{R}$. By (AD) of f and linearity of $\alpha_i(\cdot)$, it holds that $f_i(qv) = qf_i(v) \Leftrightarrow \phi_i(q\Delta(v)) + \alpha_i(qv(N)) = q\phi_i(\Delta(v)) + q\alpha_i(v(N)) \Leftrightarrow \phi_i(qv) = q\phi_i(v)$ for any $q \in \mathbb{Q}$. Moreover, the function $\Phi^{\Delta_i(v)} : \mathbb{R} \to \mathbb{R}$ defined as $\Phi^{\Delta_i(v)}(c) =: \phi_i(c\Delta_i(v))$ for each $v \in \mathcal{G}$ is monotonically increasing by (CM). Hence, we obtain $\Phi_i^{\Delta_i(v)}(c) = c\Phi_i^{\Delta_i(v)}(1)$ by density of $\mathbb{Q}$ in $\mathbb{R}$. Thus, we have $\phi_i(c\Delta_i(v)) = c\phi_i(\Delta_i(v))$ for all $i \in N$ and $c \in \mathbb{R}$. This completes Claim 2.

Claim 3: There exists $\delta \in \mathbb{R}$ such that $\phi_i(\Delta_i(v)) = \delta \cdot Sh_i(v)$ for each $i \in N$ and $v \in \mathcal{G}_N$.

For any nonempty $S \subseteq N$, consider the unanimity game u_S. In view of (.19), we have, for any $k \in N$,

$$f_k(u_S) = \begin{cases} \phi_k(\Delta_k(u_S)) + \alpha_k(1) & \text{if } \ k \in S, \\ \alpha_k(1) & \text{otherwise.} \end{cases}$$

Similarly, for any $i \in S$ and $u_{\{i\}}$, we have, for any $k \in N$,

$$f_k(u_{\{i\}}) = \begin{cases} \phi_k(\Delta_k(u_{\{i\}})) + \alpha_k(1) & \text{if } \ k = i, \\ \alpha_k(1) & \text{otherwise,} \end{cases}$$

and, by Claim 1, for u_N,

$$f_k(u_N) = \phi_k(\Delta_k(u_N)) + \alpha_k(1) \text{ for all } \ k \in N.$$

It follows from (E) and symmetry of ϕ that for any S,

$$|S| \cdot \phi_k(\Delta_k(u_S)) \overset{\text{(E)}}{=} \phi_k(\Delta_k(u_{\{i\}})) \overset{\text{(E)}}{=} n \cdot \phi_k(\Delta_k(u_N)) =: \delta \tag{.21}$$

and, then, obtain

$$\phi_k(\Delta_k(u_S)) \overset{(.21)}{=} \begin{cases} \frac{\delta}{|S|} & \text{if } \ k \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{.22}$$

Again, recall that any game v has an unique linear expression: $v = \sum_{T \subseteq N : \emptyset \neq T} [\lambda_T(v) \cdot u_T]$. By using (.22) and Claim 2, we obtain $\phi_i(\Delta_i(v)) = \delta \cdot Sh_i(v)$ for each $i \in N$ and $v \in \mathcal{G}_N$.

Claim 4: $\delta \geq 0$.

For any $j \in N$, we have $\delta = n\phi_k(\Delta_k(u_N)) \overset{\text{(CM)}}{\geq} n\phi_k(\Delta_k(u_{\{j\}})) = 0$ because $\Delta_k(u_{\{j\}}) = \mathbf{0}$ for any $k \neq j$.

Claim 5: There exists a priority profile $w \in \mathcal{W}$ such that for any $v \in \mathcal{G}_N$, $\alpha_i(v(N)) = (1 - \delta) \cdot w_i v(N)$.

By Corollary 4.5, f satisfies (NN). In Claim 2, we have already shown that $\alpha_i(c) = c\alpha_i(1)$ for all $c \in \mathbb{R}$. Hence, we have $\alpha_i(1) \overset{(.12)}{=} \alpha_i(1, \mathbf{0}) \overset{(.11)}{=} f_i(v) \overset{\text{(NN)}}{\geq} 0$ for every $i \in N$ and $v \in \mathcal{G}_N^i$.

Take any $i \in N$ and consider the unanimity game $u_{\{i\}}$. We have the following allocation:

$$f_k(u_{\{i\}}) \overset{\text{(C1)},(.21)}{=} \begin{cases} \delta + \alpha_k(1) & \text{if } \ k = i, \\ \alpha_k(1) & \text{otherwise.} \end{cases}$$

Hence, we have

$$\sum_{i \in N} \alpha_i(1) \overset{\text{(E)}}{=} 1 - \delta. \tag{.23}$$

If $\delta \neq 1$, then $\sum_{i\in N} \frac{\alpha_i(1)}{1-\delta} \overset{(.23)}{=} 1$. By setting $w_i := \frac{\alpha_i(1)}{1-\delta}$, we obtain a priority profile $w = (w_i)_{i\in N} \in \mathcal{W}$. Then, for any $v \in \mathcal{G}_N$, we have $\alpha_i(v(N)) \overset{(C2)}{=} v(N)\alpha_i(1) = (1-\delta)\cdot w_i v(N)$. If $\delta = 1$, similarly, $\alpha_i(v(N)) \overset{(C2),(.23),\alpha_i\geq 0}{=} 0$ for any $i \in N$. Hence, we have

$$\begin{aligned} f_i(v) &\overset{(C1)}{=} \phi_i(\Delta_i(v)) + 0 \\ &\overset{(C3)}{=} \delta Sh_i(v) + 0 \\ &\overset{\delta=1}{=} Sh_i(v). \end{aligned}$$

This is the desired form regardless of the setting of w.

Claim 6: $\delta \leq 1$.

The equality $\delta \overset{(.23)}{=} 1 - \sum_{i\in N} \alpha_i(1)$ and the inequalities from Claim 5, $\alpha_i(1) \geq 0$ for all $i \in N$, imply Claim 6.

From Claim 1 to Claim 6, we conclude that there exists $\delta \in [0,1]$ and a priority profile $w = (w_i)_{i\in N} \in \mathcal{W}$ such that for any $v \in \mathcal{G}_N$, $f_i(v) = \delta \cdot Sh_i(v) + (1-\delta)\cdot w_i v(N)$. This competes the proof. □

Appendix E. Independence among Axioms for Theorem 4.6

Example .7. Consider the function $f_i^{\mathrm{E}}(v) = 0$ for any $i \in N$ and $v \in \mathcal{G}_N$. This function satisfies all axioms except (E).

Example .8. Consider the following function: for any $i \in N$ and $v \in \mathcal{G}_N$,

$$f_i^{\mathrm{TMN}}(v) = 2Sh_i(v) - \frac{v(N)}{n}.$$

This function satisfies all axioms except (TMN).

Example .9. Consider the following function: for any $i \in N$ and $v \in \mathcal{G}_N$,

$$f_i^{\mathrm{CM}}(v) = 2\frac{v(N)}{n} - Sh_i(v).$$

This function satisfies all axioms except (CM).

Example .10. Let σ be a permutation of all players. A marginal vector with respect to σ is given as follows:

$$f_i^{\mathrm{NSYM}}(v) = v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma)$$

where P_i^σ is the set of predecessors of i in σ. This function satisfies all the axioms except (NSYM). Obviously, this function does not satisfy (SYM).

Example .11. Consider the following function: for any $i \in N$ and $v \in \mathcal{G}_N$,

$$f_i^{\mathrm{AD}}(v) = \begin{cases} sh_i(v) & \text{if } v(N) < 0. \\ \frac{v(N)}{n} & \text{if } v(N) \geq 0. \end{cases}$$

This function satisfies all axioms except (AD).

Reference

[1] Abe, T and Nakada, S. (2017). "Monotonic redistribution: Integrating performance-based allocation and equity," mimeo

[2] Béal, S., Casajus, A., Huettner, F., Rémila, E., and Solal, P. (2016). "Characterizations of weighted and equal division values," *Theory and Decision*, 80, 649–667.

[3] van den Brink, R., Funaki, Y., and Ju, Y. (2013). "Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values," *Social Choice and Welfare*, 40, 693–714.

[4] Casajus, A. (2010). "Another characterization of the Owen value without the additivity axiom," *Theory and decision*, 69, 523–536.

[5] Casajus, A. (2011). "Differential marginality, van den Brink fairness, and the Shapley value," *Theory and Decision*, 71, 163–174.

[6] Casajus, A. (2015). "Monotonic redistribution of performance-based allocations: A case for proportional taxation," *Theoretical Economics*, 10, 887–892.

[7] Casajus, A., and Huettner, F. (2013). "Null players, solidarity, and the egalitarian Shapley values," *Journal of Mathematical Economics*, 49, 58–61.

[8] Casajus, A., and Huettner, F. (2014). "Weakly monotonic solutions for cooperative games," *Journal of Economic Theory*, 154, 162–172.

[9] Chun, Y. (1988). "The proportional solution for rights problems," *Mathematical Social Sciences*, 15, 231–246.

[10] Chun, Y. (1991). "On the symmetric and weighted Shapley values," *International Journal of Game Theory*, 20, 183–190.

[11] Conrad, H. (2010). "From seniority to performance principle: The evolution of pay practices in Japanese firms since the 1990s," *Social Science Japan Journal*, 13, 115–135.

[12] Joosten, R (1996). "Dynamics, equilibria and values," dissertation, Maastricht University.

[13] Kalai, E., and Samet, D. (1987). "On weighted Shapley values," *International Journal of Game Theory*, 16, 205–222.

[14] Magoshi, E., and Chang, E. (2009). "Diversity management and the effects on employees' organizational commitment: Evidence from Japan and Korea," *Journal of World Business*, 44, 31–40.

[15] Nowak, A. S., and Radzik, T. (1995). "On axiomatizations of the weighted Shapley values," *Games and Economic Behavior*, 8, 389–405.

[16] Roth, A. E. (1979). "Proportional solutions to the bargaining problem," *Econometrica*, 47, 775–778.

[17] Shapley, L. S. (1953). "A Value for n-Person Games," in *Contributions to the Theory of Games II* (Annals of Mathematics Studies 28), ed. by H. W. Kuhn and A. W. Tucker. Princeton: Princeton University Press, 307–317.

[18] Shapley, L. S., and Shubik, M. (1969). "On the core of an economic system with externalities," *The American Economic Review*, 59, 678–684.

[19] Young, H. P. (1985). "Monotonic solutions of cooperative games," *International Journal of Game Theory*, 14, 65–72.