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Abstract

This paper considers the fiscal theory of the price level (FTPL) using a model which unifies the short-run and the long-run macro models. The short-run model is based on the principle of effective demand which means that investment determines saving. A difference from traditional Keynesian economics is the assumption of flexible prices even in the short run. The long-run model is constructed as a special case of the short-run model under several assumptions. This paper views the FTPL in the long-run steady state of the unified model. Then, it is concluded that the price level is not determined by the fundamental equation for the FTPL because it is an identity.

Key words: Unified macro model, Short-run equilibrium, Long-run equilibrium, Price determination, Fiscal theory of the price Level

JEL classification: E01, E31, E62

1 Introduction

The century before last Wicksell said, "I already had my suspicions ... that, as an alternative to the Quantity Theory, there is no complete and coherent theory of money. If the Quantity Theory is false —or to the extent that it is false— there is so far available only one false theory of money, and no true theory. ... It is no exaggeration to say that even to-day many of the most distinguished economists lack any real, logically worked out theory of money ...." (from the English translation (1936) of Wicksell (1898, p. iii)). Then, as is well-known, he argued that the price rises (falls, and remains unchanged) if the nominal rate of interest is less than (greater than, equal to) the natural rate of interest. Wicksell's challenge to the quantity theory of money had great influence on many contemporary economists.

Keynes was among them and wrote a *Treatise on Money* (1930) replacing the difference between the two rates of interest in Wicksell with that between investment and saving as the determinant of the price level. But it was a failure, as he admitted. Struggling to escape from the quantity theory Keynes took a different approach in his *General Theory* (1936). According to it, the price level is determined by supply and demand in the same way as individual prices. It is noteworthy that in both his books prices respond to market conditions and are flexible. This is contrary to the assumption of price rigidity in traditional Keynesians who regarded the price to be given and new Keynesians who regard it to be fixed by profit-maximizing firms.

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Despite these efforts, however, the quantity theory of money has held sway over macroeconomics. Needless to say, it is Friedman (1956) as a monetarist who revived it and it is Lucas who gave a more rigorous foundation called the rational expectations hypothesis. The long-lasting belief in the quantity theory may come from its theoretical simplicity and overwhelming evidence for the positive relationship between the rate of change in the money supply and the rate of inflation as Lucas (2007) stuck to it. The quantity theory continues to be a theoretical core at least in neoclassical theory. Solow's (1956) neoclassical growth model, a universal basis of modern macroeconomics, also relies on it.

Nevertheless, not everybody has been convinced by the quantity theory. Recently a new theory called the fiscal theory of the price level (the FTPL, hereafter) has been advocated by leading macroeconomists including Christiano and Fitzgerald (2000), Cochrane (1998, 2005), Leeper (1991), Sims (1994, 2013), and Woodford (1994, 1995, 1998).[1] The FTPL claims that the price level is so determined that the real value of government debt (i.e., nominal government debt divided by the price level) may become equal to the present discounted value of government primary surpluses. This equality between the two values is considered to be the equilibrium condition for the determination of the price level. Then the FTPL argues that the government budget constraint is satisfied *as a result of the determination of the equilibrium price level.*[2]

There is also quite a few criticism of it. Notably Buiter (2002) set up his own environment and concluded that the FTPL is fatally flawed because it confuses a budget constraint, which must always be satisfied, and an equilibrium condition, which is required to hold by a theory. Which is right? It seems that the disputes have not been decided yet. At least one lesson from these stimulating arguments is, I believe, that the government sector needs to be taken into consideration for the analysis of price determination. It is certainly meaningful to think what will become of the quantity theory if the government sector is incorporated explicitly. Then, how about the foreign sector? It would be more convincing if the foreign sector can be introduced too.

Motivated under these circumstances, I like to propose a macro model with the government sector and the foreign sector and consider the FTPL. The model can be used both in the short run and in the long run. In such a sense it is a unified model. It is assumed that prices are determined in the short run and the equilibrium prices hold true in the long run too. This paper is organized as follows. Section 2 analyzes the short-run equilibrium state by constructing the short-run model also called the Keynes model in which prices and outputs are determined through the adjustment in the goods markets as in the *General Theory*. Section 3 analyzes the long-run equilibrium state by turning the short-run model into the long-run model also called the Solow model in which outputs are determined by factor endowments. In the long run the economy finally reaches the steady state. Section 4 looks into the FTPL using the unified model also called the Keynes-Solow model. In my view the FTPL is a long-run theory or a neoclassical theory. So it is appropriate to consider it in the long-run steady state. Section 5 concludes that although the FTPL made a contribution in that it turned our attention to the role of government debt, the price level is not determined along the line of the FTPL from the perspective of the Keynes-Solow model.

[1] It is interesting to notice that Woodford (1994) starts with the description of modern monetary theory very similar to Wicksell (1898) nearly a century ago.

[2] Except academic journals, the fiscal theory of the price level appears in Ljungqvist and Sargent's (2012) advanced textbook and a dictionary of economics (Basseto (2008)), whereas neither Romer's (2012) advanced textbook nor De Vroey's (2016) history of macroeconomics includes it.

2 The Short-Run Model (The Keynes Model)

2.1 The Structure of the Model

It would be helpful to explain what the model looks like. It is based considerably on Keynes (1936), not on Keynes (1930) or traditional Keynesian economics with a sticky price. Prices are flexible and determined by supply and demand as in microeconomics. The principle of effective demand still holds in the sense that investment determines saving as in traditional Keynesian economics. On the other hand, the liquidity preference theory is not relied on explicitly since money is used only as a medium of exchange.

There are five sectors in this model, that is, the household sector, the production sector, the government sector, the central bank, and the foreign sector. The production sector consists of the investment-goods sector and the consumption-goods sector which produce respectively investment goods and consumption goods using labor and capital stock. In this sense it is a two-sector model.

Figure 1. The Time Structure of the Model

Time is discrete proceeding, e.g., from period $t-1$ to period t as in Figure 1.[3] Each period is divided into three sub-periods. The first sub-period is that of production, distribution, and expenditure as explained in usual macroeconomics. Value added generated during the production process is distributed to the household sector as wages, interests, and dividends and to the government sector as indirect taxes. Corresponding to the production sector, the goods market consists of the investment-goods market and the consumption-goods market. The household sector and the government sector use their incomes to consume or save, and the two goods markets clear at the same time through the adjustment of price and output. Newly produced investment goods are added to the existing capital stock.

The second sub-period is that of the portfolio selection. At the beginning of the second sub-period there are three types of assets, that is, capital stock, government bonds, and foreign bonds. The rates of interest on government bonds and foreign bonds are fixed. As for capital stock, the household sector can lend it to both the investment-goods sector and the consumption-goods sector. In each case, the household can choose the rate of return between the two kinds. One it the same as the rate of interest on government bonds, while the other is the expected rate of return on capital stock which is not known in the second sub-period.

The third sub-period is that of the plan for the first sub-period of the next period. There already exists capital stock in each production sector as a result of the portfolio selection during the previous sub-period. At the beginning of the third sub-period the nominal rate of wages is fixed through the negotiation between the production sector and the household sector. Given capital stock and the nominal rate of wages, each production sector calculates the amount of labor which maximizes the expected rate of return on capital stock. If the central bank promises to supply money required by the production sector, then the plan is realized in the first sub-period of the next period as said above. These processes repeat themselves over and over again.

[3] Symbols therein will be explained in detail below.

2.2 The Principle of Effective Demand

Suppose that the economy is at the end of the third sub-period of period $t-1$, and confirm the following definition about the system of national accounts for the first sub-period of period t:

$$\begin{aligned} p_{1t}^eQ_{1t}^e + p_{2t}^eQ_{2t}^e + F_t^e &= Y_t^e \\ &= C_t^e + G_{2t}^e + S_t^e + (T_t^e - G_{2t}^e), \end{aligned} \tag{1}$$

where all terms are denominated in domestic currency and a superscript e means an expected or planned value which is calculated in the third sub-period.[4] p_{1t}^e and Q_{1t}^e represent the price and output of domestic investment goods, while p_{2t}^e and Q_{2t}^e represent the price and output of domestic consumption goods. The investment-goods sector and the consumption-goods sector generate gross value added respectively by $p_{1t}^eQ_{1t}^e$ and by $p_{2t}^eQ_{2t}^e$. By definition nominal gross national product (GDP) is the sum of the two. By definition again nominal gross national income (GNI) Y_t^e is the sum of GDP and net receipts of factor income F_t^e from the rest of the world. This explains the first line of (1).

Gross private saving S_t^e is defined as

$$S_t^e = Y_t^e - T_t^e - C_t^e, \tag{2}$$

where T_t and C_t^e are total taxes and private consumption, respectively. The second line of (1) is derived from (2) and implies that GNI is used for either private consumption or private saving by the household sector, or for either government consumption G_{2t} or government saving $T_t - G_{2t}$ by the government sector. In other words, GNI is used for either national consumption $C_t^e + G_{2t}$ or national saving $S_t^e + (T_t - G_{2t})$. Since (1) and (2) are definitions, they hold true always, that is, both in the short run and in the long run.

Next consider equilibrium in the domestic consumption-goods market. It is written as

$$p_{2t}^eQ_{2t}^e + IM_{2t}^e = C_t^e + G_{2t}^e + EX_{2t}^e, \tag{3}$$

where EX_{2t}^e and IM_{2t}^e represent exports of domestic consumption goods and imports of foreign consumption goods. They are both denominated in domestic currency and assumed to be exogenous variables. The left-hand side of (3) implies the total nominal supply of consumption goods, while the right-hand side the total nominal expenditure on consumption goods.

As a consumption function I chose the simplest one:

$$C_t^e + G_{2t}^e = cY_t^e,\ 0 < c < 1, \tag{4}$$

where c is the ratio of national consumption to GNI. The household sector choose the value of c, but it is assumed to be constant throughout this paper. Substituting (4) into (3) and taking (1) into consideration yields equilibrium nominal output of domestic consumption goods with $p_{1t}^eQ_{1t}^e$, F_t^e, and NX_{2t}^e as given:

$$p_{2t}^eQ_{2t}^e = \frac{c(p_{1t}^eQ_{1t}^e + F_t^e) + NX_{2t}^e}{1-c}, \tag{5}$$

[4] As will be seen, a superscrit $*$ represents a realized value in the short run, and a superscrit $**$ represents a realized value in the long run.

where NX_{2t}^e is net exports of consumption goods defined as $NX_{2t}^e = EX_{2t}^e - IM_{2t}^e$.[5] At the same time equilibrium GNI is calculated as

$$Y_t^e = \frac{1}{1-c}\left(p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e\right). \tag{6}$$

When the consumption-goods market reaches equilibrium, private saving can be expressed using (2), (4), and (6) as follows:

$$\begin{aligned} S_t^e &= Y_t^e - T_t^e - C_t^e \\ &= (1-c)Y_t^e - T_t^e - G_{2t}^e \\ &= p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e - T_t^e + G_{2t}^e \\ &= I_t^e + BD_t^e + CA_t^e, \end{aligned} \tag{7}$$

where

$$\begin{aligned} I_t^e &= p_{1t}^e Q_{1t}^e - G_{1t}^e - EX_{1t}^e + IM_{1t}^e, \\ BD_t^e &= G_{1t}^e + G_{2t}^e - T_t^e, \\ CA_t^e &= EX_{1t}^e + EX_{2t}^e - (IM_{1t}^e + IM_{2t}^e) + F_t^e. \end{aligned}$$

Three terms I_t^e, BD_t^e, and CA_t^e need some explanations. G_{1t}^e in both I_t^e and BD_t^e is government investment. Then, I_t^e implies part of saving that goes to the purchase of newly produced domestic investment goods and imported foreign investment goods.

BD_t^e implies part of saving that goes to the purchase of a nominal increase in government bonds as will be shown below. One-period government bonds B_{t+1}^e are issued at the price $\tilde{p}_{Bt}$ in the first sub-period of period t following the budget constraint of the government sector:

$$G_{1t}^e + G_{2t}^e + (1+i_t)\tilde{p}_{Bt-1}B_t = IT_t^e + DT_t^e + \tilde{p}_{Bt}B_{t+1}^e. \tag{8}$$

Here IT_t^e and DT_t^e represent indirect taxes and direct taxes, respectively. i_t is the already fixed nominal rate of interest applied to government bonds B_t already issued at the price $\tilde{p}_{Bt-1}$ in the first sub-period of period $t-1$. Government bonds B_{t+1}^e bear a fixed nominal coupon d_{t+1} per unit of bond. Thus, the relationship between the price $\tilde{p}_{Bt}$ of B_{t+1}^e and the nominal interest rate i_{t+1} on B_{t+1}^e is written as

$$i_{t+1}\tilde{p}_{Bt} = d_{t+1}.$$

It says that i_{t+1} and $\tilde{p}_{Bt}$ move in the opposite direction, and that once one is set, the other is determined automatically. In what follow, it is assumed that the coupon d_{t+1} and the nominal rate i_{t+1} of interest are set respectively by the government sector and by the central

[5](5) is also expressed as

$$\begin{aligned} \frac{C_t^e + G_{2t}^e}{S_t^e + (T_t^e - G_{2t}^e)} &= \frac{c}{1-c} \\ &= \frac{p_{2t}^e Q_{2t}^e - NX_{2t}^e}{p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e}, \end{aligned}$$

which means that the household sector divides GNI into national consumption and national gross saving in the ratio of c to $1-c$ with $p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e$ as given.

bank at the beginning of the third sub-period of period $t-1$. As a result, the price $\tilde{p}_{Bt+1}$ is determined as $\tilde{p}_{Bt} = d_{t+1}/i_{t+1}$ at the same time.

Arranging (8) in terms of flow gives the equation for government budget deficits BD_t^e as

$$BD_t^e = G_{1t}^e + G_{2t}^e - T_t^e = \tilde{p}_{Bt}B_{t+1}^e - \tilde{p}_{Bt-1}B_t, \tag{9}$$

where T_t^e is total taxes defined as $T_t^e = IT_t^e + DT_t^e - d_tB_t$ with $d_t(= i_t\tilde{p}_{Bt-1})$ as an already fixed nominal coupon on B_t. T_t^e in (9) is just the same as T_t^e in (1) and (7). Indirect taxes may be specified as

$$IT_t^e = \mu(p_{1t}^eQ_{1t}^e + p_{2t}^eQ_{2t}^e + IM_{1t}^e + IM_{2t}^e), 0 \leq \mu < 1 \tag{10}$$

where μ is the rate of indirect taxes which is regarded to be a parameter.

CA_t^e implies part of saving that goes to the purchase of a nominal increase in foreign bonds which represent foreign assets as a whole. Foreign bonds B_{ft+1}^e are issued at the price $\tilde{p}_{Bft}$ in the first sub-period of period t following the "budget constraint of the foreign sector:"

$$EX_{1t}^e + EX_{2t}^e + e_{t-1}(1+i_{ft})\tilde{p}_{Bft-1}B_{ft} = IM_{1t}^e + IM_{2t}^e + e_t^e\tilde{p}_{Bft}B_{ft+1}^e. \tag{11}$$

Here i_{ft} is the already fixed nominal foreign rate of interest applied to foreign bonds B_{ft} already issued at the price $\tilde{p}_{Bft-1}$ in the first sub-period of period $t-1$. Foreign prices, $\tilde{p}_{Bft-1}$ and $\tilde{p}_{Bft}$, are multiplied respectively by the nominal rates of exchange, e_{t-1} and e_t^e, to denominate them in domestic currency. Arranging (11) in terms of flow yields the equation for the current account CA_t^e as

$$CA_t^e = EX_{1t}^e + EX_{2t}^e - (IM_{1t}^e + IM_{2t}^e) + F_t^e = e_t^e\tilde{p}_{Bft}B_{ft+1}^e - e_{t-1}\tilde{p}_{Bft-1}B_{ft}. \tag{12}$$

Here it is assumed for convenience that total interests $e_{t-1}i_{ft}\tilde{p}_{Bft-1}B_{ft}$ from foreign bonds is equal to net receipts of factor income F_t^e from the rest of the world in (1). Then, as is apparent, CA_t^e is to the "budget constraint" (12) what BD_t^e is to the budget constraint (9).

Finally, substituting (7) into (1) yields the familiar equation for equilibrium in the goods market:

$$Y_t^e = C_t^e + I_t^e + (G_{1t}^e + G_{2t}^e) + CA_t^e.$$

The theoretical developments so far are of the traditional Keynesian type. Particularly it should be noticed that the principle of effective demand obtains since investment $p_{1t}^eQ_{1t}^e$ determines saving S_t^e as (7) shows. That is why the short-run model is also called the Keynes model.

2.3 The Production Sector

However, there is a big difference. Prices are not rigid but flexible in the Keynes model here. To put it concretely, every third sub-period the price of investment goods of the next sub-period is expected. Such expected prices may vary over time. Moreover, corresponding prices of consumption goods may also change so that the consumption-goods market can clear every first sub-period. In order to understand how prices change it is necessary to introduce production functions which are ignored in traditional Keynesian economics.

Figure 2. Four Ways to Hold Capital Stock as an Asset

The technology of the investment-goods sector is given by the Cobb-Douglas production function:

$$Q_{1t} = K_{1t}^{\alpha}(A_t N_{1t})^{1-\alpha},\ K_{1t} = K_{1t}^{d} + K_{1t}^{h},\ 0 < \alpha < 1, \tag{13}$$

$$A_t = (1+g)A_{t-1}, \tag{14}$$

where Q_{1t}, K_{1t}, N_{1t}, and A_t are respectively output, capital stock, labor used, and the effectiveness of labor of the investment-goods sector in the first sub-period of period t. The effectiveness of labor or "knowledge" is assumed to grow at an exogenous rate g as in (14).[6] As Figure 2 shows, K_{1t} is held as either K_{1t}^{d} or K_{1t}^{h} as a result of the portfolio selection in the second sub-period of period $t-1$. During the portfolio selection the price $\tilde{p}_{1t-1}$ of K_{1t} is determined. In addition the holders of K_{1t}^{d} are promised to receive the fixed sum of interests and principal in the first sub-period of period t, whereas the holders of K_{1t}^{h} don't know what the return on it will be.

As said in Section 2.1, the nominal rate w_t of wage is determined at the beginning of the third sub-period of period $t-1$. After that the price of investment goods in the first sub-period of period t is expected. Then, the expected rate h_{1t}^{e} of return on capital stock K_{1t}^{h} can be calculated from the following definition:

$$\begin{aligned} & p_{1t}^{e}Q_{1t}^{e} + p_{1t}^{e}(1-\delta)K_{1t} \\ & = w_t N_{1t}^{e} + (1+i_t)\tilde{p}_{1t-1}K_{1t}^{d} + (1+h_{1t}^{e})\tilde{p}_{1t-1}K_{1t}^{h} + \mu p_{1t}^{e}Q_{1t}^{e}, \end{aligned} \tag{15}$$

where δ is the capital depreciation rate ($0 \leq \delta \leq 1$).

Rewriting (15) in terms of flow yields

$$p_{1t}^{e}Q_{1t}^{e} = w_t N_{1t}^{e} + i_t\tilde{p}_{1t-1}K_{1t}^{d} + h_{1t}^{e}\tilde{p}_{1t-1}K_{1t}^{h} + \mu p_{1t}^{e}Q_{1t}^{e} + p_{1t}^{e}(\delta - \pi_t^{e})K_{1t}, \tag{16}$$

where μ is the rate of indirect taxes as in (10), and $\pi_t^{e} = 1-(\tilde{p}_{1t-1}/p_{1t}^{e})$. For convenience' sake let us call π_t^{e} and $\delta - \pi_t^{e}$ respectively the "expected inflation rate" and the "inflation-adjusted depreciation rate."[7] (16) means that value added $p_{1t}^{e}Q_{1t}$ generated by the investment-goods sector is distributed as labor income, capital income, indirect taxes, or "inflation-adjusted capital depreciation."

The mission of the investment-goods sector is to maximize the expected rate h_{1t}^{e} of return in (16) subject to the production technology (13) and (14). Solving (16) for h_{1t}^{e} gives

$$h_{1t}^{e} = \frac{(1-\mu)p_{1t}^{e}Q_{1t}^{e} - w_t N_{1t}^{e} - i_t\tilde{p}_{1t-1}K_{1t}^{d} - p_{1t}^{e}(\delta - \pi_t^{e})K_{1t}}{\tilde{p}_{1t-1}K_{1t}^{h}}. \tag{17}$$

[6] g must be greater than -1. Its admissible value is specified in Section 3.2.

[7] An unfamiliar term π_t^{e} can be written as

$$\pi_t^{e} = \frac{\frac{p_{1t}^{e} - \tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}}{1 + \frac{p_{1t}^{e} - \tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}}.$$

Thus, π_t^{e} is approximately equal to a kind of expected inflation rate $\frac{p_{1t}^{e} - \tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}$ when it is close to zero. Another unfamiliar term $\delta - \pi_t^{e}$ can be understood by using the following expression:

$$p_{1t}^{e}(\delta - \pi_t^{e})K_{1t} = \tilde{p}_{1t-1}\delta K_{1t} - (p_{1t}^{e} - \tilde{p}_{1t-1})(1-\delta)K_{1t}.$$

$\tilde{p}_{1t-1}\delta K_{1t}$ on the right-hand side corresponds to what is usually called capital depreciation, i.e., the money necessary to restore depreciated capital to the original *nominal* value $\tilde{p}_{1t-1}K_{1t}$. The above expression states more correctly that such money can be decreased by $(p_{1t}^{e} - \tilde{p}_{1t-1})(1-\delta)K_{1t}$ when the price of investment goods rises from $\tilde{p}_{1t-1}$ to p_{1t}^{e} but it must be increased by $(\tilde{p}_{1t-1} - p_{1t}^{e})(1-\delta)K_{1t}$ when the price of investment goods falls from $\tilde{p}_{1t-1}$ to p_{1t}^{e}.

Since the right-hand side of (17) is a function of N_{1t}^e alone, the investment-goods sector has only to find the level of labor, N_{1t}^e, which maximizes h_{1t}^e. Substituting (13) into (17) and differentiating with respect to N_{1t}^e yields

$$\frac{dh_{1t}^e}{dN_{1t}^e} = \frac{(1-\mu)p_{1t}^e(1-\alpha)A_t^{1-\alpha}(N_{1t}^e)^{-\alpha}K_{1t}^\alpha - w_t}{\tilde{p}_{1t-1}K_{1t}^h}.$$

Then N_{1t}^e can easily be obtained by solving $dh_{1t}^e/dN_{1t}^e = 0$ and $d^2h_{1t}^e/d(N_{1t}^e)^2 < 0$ as follows:

$$N_{1t}^e = \left[(1-\alpha)A_t^{1-\alpha}\frac{(1-\mu)p_{1t}^e}{w_t}\right]^{\frac{1}{\alpha}} K_{1t}. \tag{18}$$

For N_{1t}^e in (18) the output of investment-goods is calculated as

$$\begin{aligned} Q_{1t}^e &= K_{1t}^\alpha(A_tN_{1t}^e)^{1-\alpha} \\ &= \left[(1-\alpha)A_t\frac{(1-\mu)p_{1t}^e}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t}. \end{aligned} \tag{19}$$

(19) is a supply curve of investment goods as a function of a price p_{1t}^e. In Figure 3 is drawn such a supply curve $Q_{1t}^S(= Q_{1t}^e)$.[8]

Figure 3. Supply Curve of Investment Goods

Now, let MPL_{1t} be the marginal productivity of labor in period t. Since $MPL_{1t} \equiv \partial Q_{1t}/\partial N_{1t}$, the usual profit maximization condition can be written as

$$MPL_{1t}^e = (1-\alpha)A_t^{1-\alpha}(N_{1t}^e)^{-\alpha}K_{1t}^\alpha = \frac{w_t}{(1-\mu)p_{1t}^e}, \tag{20}$$

which is equivalent to (18). It follows that the maximization of h_{1t}^e is equivalent to the usual profit maximization. Finally, the marginal productivity of capital in period t, MPK_{1t}, is

$$MPK_{1t}^e = \alpha K_{1t}^{\alpha-1}(A_tN_{1t}^e)^{1-\alpha}. \tag{21}$$

Similar explanations apply to the consumption-goods sector, too. The production function of the consumption sector is given by

$$Q_{2t} = K_{2t}^\alpha(A_tN_{2t})^{1-\alpha},\ K_{2t} = K_{2t}^d + K_{2t}^h,\ 0 < \alpha < 1, \tag{22}$$

where K_{2t} and N_{2t} are respectively capital and labor of the consumption-goods sector in the first sub-period of period t. K_{2t} is held as either K_{2t}^d or K_{2t}^h as a result of the portfolio selection in the second sub-period of period $t-1$. K_{2t}^d and K_{2t}^h in the consumption-goods sector correspond respectively to K_{1t}^d and K_{1t}^h in the investment-goods sector (See again Figure 2).

The expected rate h_{2t}^e of return on K_{2t}^h is calculated from the following definition:

$$\begin{aligned} &p_{2t}^eQ_{2t}^e + p_{1t}^e(1-\delta)K_{2t} \\ &= w_tN_{2t}^e + (1+i_t)\tilde{p}_{1t-1}K_{2t}^d + (1+h_{2t}^e)\tilde{p}_{1t-1}K_{2t}^h + \mu p_{2t}^eQ_{2t}^e. \end{aligned} \tag{23}$$

[8]For the derivation of it, see Appendix A.

Rewriting (23) in terms of flow gives

$$p_{2t}^eQ_{2t}^e = w_tN_{2t}^e + i_t\tilde{p}_{1t-1}K_{2t}^d + h_{2t}^e\tilde{p}_{1t-1}K_{2t}^h + \mu p_{2t}^eQ_{2t}^e + p_{1t}^e(\delta - \pi_t^e)K_{2t}, \tag{24}$$

The mission of the consumption-goods sector is to maximize the expected rate h_{2t}^e of return in (24) subject to the production technology (22) and (14). Solving (24) for h_{2t}^e yields

$$h_{2t}^e = \frac{(1-\mu)p_{2t}^eQ_{2t}^e - w_tN_{2t}^e - i_t\tilde{p}_{1t-1}K_{2t}^d - p_{1t}^e(\delta - \pi_t^e)K_{2t}}{\tilde{p}_{1t-1}K_{2t}^h}. \tag{25}$$

Substituting (22) into (25) and differentiating with respect to N_{2t}^e yields

$$\frac{dh_{2t}^e}{dN_{2t}^e} = \frac{(1-\mu)p_{2t}^e(1-\alpha)A_t^{1-\alpha}(N_{2t}^e)^{-\alpha}K_{2t}^\alpha - w_t}{\tilde{p}_{1t-1}K_{2t}^h}.$$

Then N_{2t}^e can be obtained by solving $dh_{2t}^e/dN_{2t}^e = 0$ and $d^2h_{2t}^e/d(N_{2t}^e)^2 < 0$ as follows:

$$N_{2t}^e = \left[(1-\alpha)A_t^{1-\alpha}\frac{(1-\mu)p_{2t}^e}{w_t}\right]^{\frac{1}{\alpha}} K_{2t}. \tag{26}$$

The output of consumption goods for N_{2t}^e in (26) is calculated as

$$\begin{aligned} Q_{2t}^e &= K_{2t}^\alpha(A_tN_{2t}^e)^{1-\alpha} \\ &= \left[(1-\alpha)A_t\frac{(1-\mu)p_{2t}^e}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t}. \end{aligned} \tag{27}$$

(27) is a supply curve of consumption goods as a function of a price p_2^e.

Figure 4. Equilibrium in the Consumption-Goods Market

The equilibrium price and output of consumption goods are obtained by substituting (27) into (5) as follows:

$$p_{2t}^e = \left[\frac{w_t}{(1-\mu)(1-\alpha)A_t}\right]^{1-\alpha}\left[\frac{1}{K_{2t}}\right]^{\alpha}\left[\frac{c(p_{1t}^eQ_{1t}^e + F_t^e) + NX_{2t}^e}{1-c}\right]^{\alpha}, \tag{28}$$

$$Q_{2t}^e = \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{1-\alpha} K_{2t}^\alpha \left[\frac{c(p_{1t}^eQ_{1t}^e + F_t^e) + NX_{2t}^e}{1-c}\right]^{1-\alpha}, \tag{29}$$

where $p_{1t}^eQ_{1t}^e$, F_t^e, and NX_{2t}^e are assumed to be given. In Figure 4 are shown a supply curve $Q_{2t}^S(= Q_{2t}^e)$ and a demand curve Q_{2t}^D of consumption goods.[9] Also note that the total expenditure on domestic investment goods in nominal terms can be written as $I_t^e + G_{1t}^e + EX_{1t}^e - IM_{1t}^e$ which is equal to $p_{1t}^eQ_{1t}^e$ (or $p_{1t}Q_{1t}^S$) due to (7). It follows that the supply of and the demand for domestic investment goods coincide if the consumption-goods market reaches equilibrium.

Finally denote the marginal productivity of labor in period t by MPL_{2t}. Because $MPL_{2t} \equiv \partial Q_{2t}/\partial N_{2t}$, the usual profit maximization condition becomes

$$MPL_{2t}^e = (1-\alpha)A_t^{1-\alpha}(N_{2t}^e)^{-\alpha}K_{2t}^\alpha = \frac{w_t}{(1-\mu)p_{2t}^e}, \tag{30}$$

which is equivalent to (26). Therefore, the maximization of h_{2t}^e is equivalent to the usual profit maximization. The marginal productivity of capital in period t, MPK_{2t}, is

$$MPK_{2t}^e = \alpha K_{2t}^{\alpha-1}(A_tN_{2t}^e)^{1-\alpha}. \tag{31}$$

[9] For the derivation of them, see Appendix B.

2.4 The Short-Run Equilibrium State

As said above, the price p^e_{1t} of investment goods in the first sub-period of period t is expected at the end of the third sub-period of period $t-1$. How? Although it is not discussed in detail in this paper, at least it is worth mentioning quite a possibility that the price $\tilde{p}_{1t-1}$ of capital stock realized through the portfolio selection during the second sub-period of period $t-1$ may strongly affect the value of p^e_{1t}. It would be easy to understand if you remember worldwide depression like the Great Depression of the last century and the Great Recession of this century both of which seem to have been triggered by plummeting asset prices. Anyway, once the value of p^e_{1t} is set, the expected or planned values of other variables such as Q^e_{1t}, p^e_{2t}, and Q^e_{2t} are also calculated on the basis of the Keynes model as in (19), (28), and (29).

But the realization of such a plan is not always warranted. Most importantly, the central bank needs to supply money by the amount which satisfies the following Fisher equation of exchange:

$$M_t V_t = p^e_{1t} Q^e_{1t} + p^e_{2t} Q^e_{2t}, \tag{32}$$

where V_t is the income velocity of money which is assumed to be exogenous. If the central bank promises (and keeps its promise) to supply money following (32), the third-sub-period plan is realized in the first sub-period of period t as

$$M_t V_t = p^*_{1t} Q^*_{1t} + p^*_{2t} Q^*_{2t}. \tag{33}$$

Here a superscript $*$ means a realized value. An economy is said to be in the short-run equilibrium state if expected or planned values are all realized. A superscript e attached to variables considered so far is replaced with a superscript $*$ in the short-run equilibrium state as F^*_t, EX^*_{1t}, IM^*_{1t}, EX^*_{2t}, IM^*_{2t}, e^*_t, etc.

Figure 5. The Short-Run Equilibrium State

(33) appears to suggest the quantity theory of money. But causality runs in the opposite direction in this case. That is, the nominal amount of aggregate outputs on the right-hand side determines the appropriate quantity of money as a means of payment on the left-hand side. On the other hand, the central bank may decide to supply less money than (32) requires. If so, the initial plan must be revised, i.e., p^e_{1t} must be lowered so that the corresponding planned value on the right-hand side of (32) becomes equal to the left-hand side. In this case, the money supply determines prices as the traditional quantity theory of money says. In either case the supply of and the demand for money coincide in the short-run equilibrium state in the form of (33). Anyway a macroeconomy in the short-run equilibrium state can be grasped at once by Figure 5.

The capital accumulation equation in the short-run equilibrium state is written as

$$K^*_{t+1} = (1-\delta)K^*_t + Q^*_{1t} - \left(\frac{EX^*_{1t}}{p^*_{1t}} - \frac{IM^*_{1t}}{e^*_t p^*_{f1t}} \right). \tag{34}$$

where p^*_{f1t} is the realized price of foreign investment goods in foreign currency. Write net exports of investment goods in the short-run equilibrium state as

$$NX^*_{1t} = EX^*_{1t} - IM^*_{1t}.$$

It should be noted that $\frac{EX^*_{1t}}{p^*_{1t}} - \frac{IM^*_{1t}}{e^*_t p^*_{f1t}}$ in (34) is not necessarily equal to $\frac{NX^*_{1t}}{p^*_{1t}}$ because $e^*_t p^*_{f1t}$ may be different from p^*_{1t}.

3 The Long-Run Model (The Solow Model)

3.1 The Characterization of the Long-Run Equilibrium State

An economy is said to be in the long-run equilibrium state if the following five conditions are all satisfied:

1. The short-run equilibrium state.
2. Full employment.
3. The equal rates of return.
4. The long-run price condition.
5. The proportionality condition.

The first condition says that *the long-run equilibrium state is a special case of the short-run equilibrium state*.

The second condition means equilibrium in the labor market holds every period as follows:

$$N_{1t}^{*} + N_{2t}^{*} = N_t, \tag{35}$$

where N_t is the natural level of employment which is assumed to grow at a constant rate n as[10]

$$N_t = (1+n)N_{t-1}. \tag{36}$$

The third condition means the following equality between all the rates of return:

$$h_{1t}^{*} = i_t = h_{2t}^{*} = i_{ft}. \tag{37}$$

The fourth condition means the following relations among various prices:

$$\frac{1}{1-\pi}\, p_{1t-1}^{**} = \frac{1}{1-\pi}\, \tilde{p}_{1t-1}^{**} = p_{1t}^{**} = e_t^{**} p_{f1t}^{**} = \frac{1}{1-\pi} \tilde{p}_{Bt-1}^{**} = \tilde{p}_{Bt}^{**}. \tag{38}$$

A superscript $**$ indicates a value in the long-run equilibrium state. (38) says that the price p_{1t-1}^{**} of investment goods as flow and the price $\tilde{p}_{1t-1}^{**}$ of investment goods as stock coincide in the same period, and that the expected and realized inflation rate is a constant π.[11] Equality $p_{1t}^{**} = e_t^{**} p_{f1t}^{**}$ implies that the theory of purchasing power parity obtains in the long-run equilibrium state. The price of government bonds rises at the same rate as the price of investment goods. The equality between p_{1t}^{**} and $\tilde{p}_{Bt}^{**}$ is possible by adjusting the unit of government bonds.

The fifth condition means the following proportional relations to capital stock:

$$\begin{aligned} \frac{G_{1t}^{**}}{p_{1t}^{**}} &= \beta_{G_1} K_t^{**}, \\ \frac{G_{2t}^{**}}{p_{1t}^{**}} &= \beta_{G_2} K_t^{**}, \end{aligned}$$

[10] n must be greater than -1. Its admissible value is specified in Section 3.2.

[11] Remember that the expected inflation rate was defined as $\pi_t^e = 1 - (\tilde{p}_{1t-1}/p_{1t}^e)$ in Subsection 2.3. See also footnote 7.

$$
\begin{aligned}
\frac{T_t^{**}}{p_{1t}^{**}} &= \beta_T\, K_t^{**}, \\
\frac{F_t^{**}}{p_{1t}^{**}} &= \beta_F\, K_t^{**}, \\
\frac{NX_{1t}^{**}}{p_{1t}^{**}} &= \beta_{NX_1}\, K_t^{**}, \\
\frac{NX_{2t}^{**}}{p_{1t}^{**}} &= \beta_{NX_2}\, K_t^{**},
\end{aligned}
$$

where K_t^{**} is capital stock in the long-run equilibrium state and coefficients $\beta's$ of K_t^{**} are constants. It can be said economically that $\beta_{G_1} \geq 0$ and $\beta_{G_2} \geq 0$. The signs of other $\beta's$ may be negative. As will be seen, it is convenient to define β_{NX} and β_{CA} as follows:

$$
\begin{aligned}
\frac{NX_{1t}^{**}}{p_{1t}^{**}} + \frac{NX_{2t}^{**}}{p_{1t}^{**}} &= \beta_{NX}\, K_t^{**}, \beta_{NX} = \beta_{NX_1} + \beta_{NX_2}, \\
\frac{NX_{1t}^{**}}{p_{1t}^{**}} + \frac{NX_{2t}^{**}}{p_{1t}^{**}} + \frac{F_t^{**}}{p_{1t}^{**}} &= \beta_{CA}\, K_t^{**}, \beta_{CA} = \beta_{NX_1} + \beta_{NX_2} + \beta_F.
\end{aligned}
$$

Now let us characterize the economy in the long-run equilibrium state using the above conditions. First, derive the difference between h_{1t}^e and i_t. Rewriting (16) gives

$$
(1-\mu)p_{1t}^e Q_{1t}^e = w_t N_{1t}^e + p_{1t}^e (r_t^e + \delta) K_{1t} + (h_{1t}^e - i_t)\tilde{p}_{1t-1} K_{1t}^h, \tag{39}
$$

where r_t^e is the real rate of interest defined as

$$
r_t^e = \frac{(1+i_t)\tilde{p}_{1t-1}}{p_{1t}^e} - 1. \tag{40}
$$

(40) is the Fisher equation. Substituting (18) and (19) into (39) and replacing a superscript e with $*$ yields the difference between h_{1t}^* and i_t:

$$
h_{1t}^* - i_t = \frac{p_{1t}^*(r_t^* + \delta)K_{1t}}{\tilde{p}_{1t-1} K_{1t}^h} \left\{ \left[\frac{(1-\mu)p_{1t}^*}{\hat{p}_{1t}^*} \right]^{\frac{1}{\alpha}} - 1 \right\}, \tag{41}
$$

where

$$
\begin{aligned}
\hat{p}_{1t}^* &= \left[\frac{(1+i_t)\tilde{p}_{1t-1} - (1-\delta)p_{1t}^*}{\alpha} \right]^{\alpha} \left[\frac{w_t}{(1-\alpha)A_t} \right]^{1-\alpha} \\
&= p_{1t}^* \left[\frac{r_t^* + \delta}{\alpha} \right]^{\alpha} \left[\frac{\frac{w_t}{p_{1t}^*}}{(1-\alpha)A_t} \right]^{1-\alpha}.
\end{aligned} \tag{42}
$$

A price $\hat{p}_{1t}^*$ seems strange. But I like to call it the "normal supply price" of investment goods since Keynes mentioned it in his explanation of production of investment goods.[12] Figure 6 shows the relationship among $\hat{p}_{1t}^*$, p_{1t}^*, h_{1t}^*, and i_t.

[12]In fact Keynes (1936, p. 228) said, "Now those assets [i.e., investment goods] of which the normal supply-price [$\hat{p}_{1t}^*$] is less than the demand-price [$(1-\mu)p_{1t}^*$] will be newly produced; and these will be those assets of which the marginal efficiency [h_{1t}^*] would be greater ... than the rate of interest [i_t]." (Notes in brackets are due to my interpretation.)

Figure 6. The Normal Supply Price of Investment Goods

The difference between h^*_{2t} and i_t can be calculated using (24) and (28) as

$$h^*_{2t} - i_t = \frac{p^*_{1t}(r^*_t + \delta)K_{2t}}{\tilde{p}_{1t-1}K^h_{2t}} \left\{ \left[\frac{(1-\mu)p^*_{2t}}{\hat{p}^*_{1t}} \right]^{\frac{1}{\alpha}} - 1 \right\}$$

$$= \frac{p^*_{1t}(r^*_t + \delta)}{\tilde{p}_{1t-1}K^h_{2t}} \left\{ \frac{c}{1-c} \left[\frac{(1-\mu)p^*_{1t}}{\hat{p}^*_{1t}} \right]^{\frac{1}{\alpha}} K_{1t} - K_{2t} + \left(\frac{c}{1-c} \frac{F^*_t}{p^*_{1t}} + \frac{1}{1-c} \frac{NX^*_{2t}}{p^*_{1t}} \right) \frac{(1-\mu)\alpha}{r^*_t + \delta} \right\}. \tag{43}$$

Taking account of (41) and the first line of (43), the third condition (37) and fourth condition (i.e., $\frac{1}{1-\pi}\, p^{**}_{1t-1} = p^{**}_{1t}$) imply that

$$\frac{1}{1-\pi}\, p^{**}_{1t-1} = p^{**}_{1t} = \frac{1}{1-\mu}\, \hat{p}^{**}_{1t} = p^{**}_{2t}. \tag{44}$$

It is found from (44) that the prices of investment goods and consumption goods coincide and change at the same rate in the long-run equilibrium state. Thus, it is convenient to use only p^{**}_{1t} for them. Then, a nominal value divided by p^{**}_{1t} can be interpreted as a real value in a usual sense.[13] For instance, real GDP Q^{**}_t is expressed as

$$Q^{**}_t = Q^{**}_{1t} + Q^{**}_{2t}, \tag{45}$$

due to (1). Then, using (45) the Fisher equation of exchange (33) in the short-run equilibrium state becomes

$$M_t V_t = p^{**}_{1t} Q^{**}_t. \tag{46}$$

(46) is of the form of the original quantity theory of money. But it should be remembered that (46) is just a special case of (33). As explained in Section 2.4, the money supply may or may not determine prices. Prices are basically determined on the basis of (28) with p^e_{1t} as given. Therefore, the quantity theory does not necessarily holds even in the long-run equilibrium state.

When $p^*_{1t} = p^*_{2t} = p^{**}_{1t}$, the labor demand (18) in the investment-goods sector becomes

$$N^*_{1t} = \left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K_{1t}, \tag{47}$$

and the labor demand (26) in the consumption-goods sector becomes

$$N^*_{2t} = \left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K_{2t}. \tag{48}$$

Since $K^*_{1t} + K^*_{2t} = K^*_t$, substituting (47) and (48) into the second condition (35) yields the following equality:

$$\left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K^*_t = N_t. \tag{49}$$

[13] Usually a "real value" means a nominal value divided by the price of consumption goods. And the "inflation rate" refers to the rate of change in the price of consumption goods.

(49) gives the equilibrium real rate of wage as

$$\frac{w_t^{**}}{p_{1t}^{**}} = (1-\mu)(1-\alpha)A_t \left(\frac{K_t^{**}}{A_t N_t}\right)^{\alpha}. \tag{50}$$

Let capital per effective labor in the right-hand side of (50) be designated by k_t^{**}, and capital per effective labor in the investment-goods sector and capital per effective labor in the consumption-goods sector respectively by k_{1t}^{**} and k_{2t}^{**}:

$$k_t^{**} = \frac{K_t^{**}}{A_t N_t},\ k_{1t}^{**} = \frac{K_{1t}^{**}}{A_t N_{1t}^{**}},\ \text{and}\ k_{2t}^{**} = \frac{K_{2t}^{**}}{A_t N_{2t}^{**}},$$

where $N_{1t}^{**} + N_{2t}^{**} = N_t$. Then, (50), (47), and (48) lead to the following equality between three kinds of capital stocks per effective labor:

$$\frac{w_t^{**}}{(1-\mu)p_{1t}^{**}} = (1-\alpha)A_t(k_t^{**})^{\alpha} = (1-\alpha)A_t(k_{1t}^{**})^{\alpha} = (1-\alpha)A_t(k_{2t}^{**})^{\alpha}. \tag{51}$$

(51) shows that in the long-run equilibrium state

$$k_t^{**} = k_{1t}^{**} = k_{2t}^{**}. \tag{52}$$

and $MPL_{1t}^{**} = MPL_{2t}^{**}$ from (20) and (30). Also (52) shows that $MPK_{1t}^{**} = MPK_{2t}^{**}$ from (21) and (31).

In the long-run equilibrium state the real rate of interest (40) is simplified as

$$\begin{aligned} r_t^{**} &= \frac{(1+i_t^{**})\tilde{p}_{t-1}^{**}}{p_{1t}^{**}} - 1 \\ &= (1+i_t^{**})(1-\pi) - 1, \end{aligned} \tag{53}$$

because of the fourth condition (38). Taking (44) and (51) into account, (42) leads to

$$\frac{r_t^{**}+\delta}{1-\mu} = \alpha(k_t^{**})^{\alpha-1} = \alpha(k_{1t}^{**})^{\alpha-1} = \alpha(k_{2t}^{**})^{\alpha-1}. \tag{54}$$

(53) and (54) require that the central bank should set the nominal rate of interest as

$$i_t^{**} = \frac{1}{1-\pi}\left[(1-\mu)\alpha(k_t^{**})^{\alpha-1} - (\delta-\pi)\right]$$

in order for the economy to be in the long-run equilibrium state. Note that a policy variable i_t^{**} is an increasing function of the inflation rate π and a decreasing function of capital stock k_t^{**} per effective labor.

(44) simplifies (43) with $h_{2t}^{*} = i_t$ as

$$\frac{c}{1-c}K_{1t}^{**} - K_{2t}^{**} + \left(\frac{c}{1-c}\frac{F_t^{**}}{p_{1t}^{**}} + \frac{1}{1-c}\frac{NX_{2t}^{**}}{p_{1t}^{**}}\right)\frac{(1-\mu)\alpha}{r_t^{**}+\delta} = 0. \tag{55}$$

Finally, applying (54) and the fifth condition to (55) yields the following ratios concerning labor and capital in the long-run equilibrium state:

$$\frac{N_{1t}^{**}}{N_t} = \frac{K_{1t}^{**}}{K_t^{**}} = 1 - c - (c\beta_F + \beta_{NX_2})(k_t^{**})^{1-\alpha}, \tag{56}$$

$$\frac{N_{2t}^{**}}{N_t} = \frac{K_{2t}^{**}}{K_t^{**}} = c + (c\beta_F + \beta_{NX_2})(k_t^{**})^{1-\alpha}, \tag{57}$$

where $K_{1t}^{**} + K_{2t}^{**} = K_t^{**}$.

3.2 The Long-Run Steady State

The dynamics of the economy in the long-run equilibrium state continues to be governed by the capital accumulation equation (34) in the short-run equilibrium state as

$$K_{t+1}^{**} = (1-\delta)K_t^{**} + Q_{1t}^{**} - \left(\frac{EX_{1t}^{**}}{p_{1t}^{**}} - \frac{IM_{1t}^{**}}{e_t^{**}p_{f1t}^{**}}\right).$$

But the fourth and fifth conditions above simplify it as

$$K_{t+1}^{**} = (1-\delta-\beta_{NX_1})K_t^{**} + Q_{1t}^{**}. \tag{58}$$

Dividing both sides of (58) by effective labor $A_{t+1}N_{t+1}$ in period $t+1$ and considering (14), (36), and (56) gives

$$k_{t+1}^{**} = \frac{1}{1+g_N}\left[(1-\delta-\beta_{CA}+s\beta_F)k_t^{**} + s(k_t^{**})^\alpha\right], \tag{59}$$

where s is the gross rate of saving defined as $s = 1-c$, and g_N is the natural rate of growth defined as $g_N = (1+g)(1+n)-1(>-1)$.[14]

The long-run model (59) is also called the Solow model here, as compared with the Keynes model as the short-run model in Section 2. The economy in the long-run equilibrium state is said to be in the long-run steady state if $k_{t+1}^{**} = k_t^{**}$ in the Solow model (59). So *the long-run steady state is a special case of the long-run equilibrium state.* Let a subscript S denote a value in the long-run steady state in what follows. Then, capital per effective labor k_S^{**} in the long-run steady state can easily be calculated from (59) as follows:

$$k_S^{**} = \left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}}. \tag{60}$$

In order to guarantee the existence of a unique positive steady state it is assumed that

$$\begin{aligned} g_N+\delta+\beta_{NX_1} &> 0, \\ (1-s)(g_N+\delta+\beta_{NX_1}+\beta_F)+\beta_{NX_2} &> 0, \\ g_N+\delta+\beta_{CA} &> 0. \end{aligned}$$

Note that the first and second assumptions imply that $g_N+\delta+\beta_{CA}-s\beta_F > 0$ which in turn warrants $k_S^{**} > 0$ in (60). Moreover, all three assumptions mean that k_S^{**} is an increasing function of the gross saving rate s since

$$\frac{\partial k_S^{**}}{\partial s} = \frac{1}{1-\alpha}\frac{g_N+\delta+\beta_{CA}}{(g_N+\delta+\beta_{CA}-s\beta_F)^2}\left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{\alpha}{1-\alpha}} > 0.$$

In the long-run steady state,

$$k_S^{**} = k_{S1}^{**} = k_{S2}^{**},$$

due to (52). Thus, the economy in the long-run steady state is characterized by k_S^{**} or K_S^{**}.

[14]Appendix C shows how to derive (59).

As for capital stock,

$$
\begin{aligned}
K_{St}^{**} &= \left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}} A_t N_t, \\
K_{S1t}^{**} &= \frac{s(g_N+\delta+\beta_{NX_1})}{g_N+\delta+\beta_{CA}-s\beta_F} K_{St}^{**}, && (61)\\
K_{S2t}^{**} &= \frac{(1-s)(g_N+\delta+\beta_{NX_1}+\beta_F)+\beta_{NX_2}}{g_N+\delta+\beta_{CA}-s\beta_F} K_{St}^{**}, && (62)
\end{aligned}
$$

because of (60), (56), and (57). K_{St}^{**}, K_{S1t}^{**}, and K_{S2t}^{**} are all positive by the assumptions.

As for output,

$$
\begin{aligned}
Q_{S1t}^{**} &= A_t N_{S1t}^{**}(k_{S1}^{**})^{\alpha} \\
&= A_t N_{S1t}^{**}(k_{S}^{**})^{\alpha} \\
&= (g_N+\delta+\beta_{NX_1})K_{St}^{**}, && (63)\\
Q_{S2t}^{**} &= A_t N_{S2t}^{**}(k_{S2}^{**})^{\alpha} \\
&= A_t N_{S2t}^{**}(k_{S}^{**})^{\alpha} \\
&= \frac{(1-s)(g_N+\delta+\beta_{NX_1}+\beta_F)+\beta_{NX_2}}{s} K_{St}^{**}, && (64)\\
Q_{St}^{**} &= Q_{S1t}^{**}+Q_{S2t}^{**} \\
&= \frac{g_N+\delta+\beta_{CA}-s\beta_F}{s} K_{St}^{**}, && (65)
\end{aligned}
$$

because of (13), (22), (56), (57), and (45). Q_{S1t}^{**}, Q_{S2t}^{**}, and Q_{St}^{**} are all positive by the assumptions too.

Finally the relationship between private disposable income and consumption is worthy to be examined in the long-run steady state. Real GNI in the long-run steady state is written as

$$
\begin{aligned}
\frac{Y_{St}^{**}}{p_{1t}^{**}} &= Q_{S1t}^{**}+Q_{S2t}^{**}+\frac{F_t^{**}}{p_{1t}^{**}} \\
&= \frac{g_N+\delta+\beta_{CA}}{s} K_{St}^{**}, && (66)
\end{aligned}
$$

due to (65). Since real private disposable income Y_{Dt}^e is defined as $Y_{Dt}^e = Y_t^e - T_t^e - p_{1t}^e(\delta-\pi_t^e)K_t$ on the basis of (1), (16), and (24), its value in the long-run steady state is calculated as

$$
\begin{aligned}
\frac{Y_{DSt}^{**}}{p_{1t}^{**}} &= \frac{Y_{St}^{**}}{p_{1t}^{**}} - \frac{T_t^{**}}{p_{1t}^{**}} - (\delta-\pi)K_{St}^{**} \\
&= \frac{g_N+\delta+\beta_{CA}-s(\delta-\pi+\beta_T)}{s} K_{St}^{**}, && (67)
\end{aligned}
$$

using (66). On the other hand, real private consumption in the long-run steady state is written as

$$
\begin{aligned}
\frac{C_{St}^{**}}{p_{1t}^{**}} &= Q_{S2t}^{**} - \frac{NX_{2t}^{**}}{p_{1t}^{**}} - \frac{G_{2t}^{**}}{p_{1t}^{**}} \\
&= \frac{(1-s)(g_N+\delta+\beta_{CA}+\beta_{G_2})-\beta_{G_2}}{s} K_{St}^{**}, && (68)
\end{aligned}
$$

due to (4) and (64). Dividing (68) by (67) leads to the average propensity to consume as

$$\frac{C_{St}^{**}}{Y_{DSt}^{**}} = \frac{(1-s)(g_N + \delta + \beta_{CA} + \beta_{G_2}) - \beta_{G_2}}{g_N + \delta + \beta_{CA} - s(\delta - \pi + \beta_T)}. \tag{69}$$

It follows from (69) that in the long-run steady state the average propensity to consume is a decreasing function of the inflation rate π. In other words, the average propensity to consume takes a constant value if the inflation rate remains unchanged.[15] [16]

4 The FTPL on the KS Model

The macro model constructed so far is the Solow model which is based on the Keynes model. Thus, it is called the Keynes-Solow model (the KS model, hereafter). Using the KS model, this section examines the FTPL in two respects under the assumption that the FTPL holds in the long-run equilibrium state or in the long-run steady state.

First let us consider the equilibrium condition (or the fundamental equation) for the FTPL. In the long-run equilibrium state the government budget constraint (8) can be written as

$$G_{1t}^{**} + G_{2t}^{**} + (1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**} = IT_t^{**} + DT_t^{**} + \tilde{p}_{Bt}^{**} B_{t+1}^{**}. \tag{70}$$

Rewriting (70) gives

$$(1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**} = PS_t^{**} + \tilde{p}_{Bt}^{**} B_{t+1}^{**}, \tag{71}$$

where PS_t^{**} is the primary surplus defined as

$$PS_t^{**} = G_{1t}^{**} + G_{2t}^{**} - (IT_t^{**} + DT_t^{**}).$$

Dividing both sides of (71) by p_{1t}^{**} yields

$$\frac{(1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**}}{p_{1t}^{**}} = \frac{PS_t^{**}}{p_{1t}^{**}} + \frac{\tilde{p}_{Bt}^{**} B_{t+1}^{**}}{p_{1t}^{**}}. \tag{72}$$

Forwarding (71) one period leads to

$$(1 + i_{t+1}^{**})\tilde{p}_{Bt}^{**} B_{t+1}^{**} = PS_{t+1}^{**} + \tilde{p}_{Bt+1}^{**} B_{t+2}^{**}. \tag{73}$$

Using (73) as well as the fourth condition (38) (i.e., $p_{1t}^{**} = \tilde{p}_{1t}^{**}$), the second term on the right-hand side of (72) is expressed as

$$\frac{\tilde{p}_{Bt}^{**} B_{t+1}^{**}}{p_{1t}^{**}} = \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})p_{1t}^{**}}{p_{1t+1}^{**}}} + \frac{\frac{\tilde{p}_{Bt+1}^{**} B_{t+2}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})p_{1t}^{**}}{p_{1t+1}^{**}}}$$

[15]Since real private saving S_{DSt}^{**} is defined and calculated as

$$\begin{aligned}\frac{S_{DSt}^{**}}{p_{1t}^{**}} &= \frac{Y_{DSt}^{**} - C_{St}^{**}}{p_{1t}^{**}} \\ &= (g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)K_{St}^{**},\end{aligned}$$

the net rate of saving becomes

$$\frac{S_{DSt}^{**}}{Y_{DSt}^{**}} = \frac{s(g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)}{g_N + \delta + \beta_{CA} - s(\delta - \pi + \beta_T)},$$

which is an increasing function of the inflation rate.

[16]For the analyses of the golden-rule state and the modified-golden-rule state, see Appendices D and E.

$$= \frac{\frac{PS^{**}_{t+1}}{p^{**}_{1t+1}}}{\frac{(1+i^{**}_{t+1})\tilde{p}^{**}_{1t}}{p^{**}_{1t+1}}} + \frac{\frac{\tilde{p}^{**}_{Bt+1}B^{**}_{t+2}}{p^{**}_{1t+1}}}{\frac{(1+i^{**}_{t+1})\tilde{p}^{**}_{1t}}{p^{**}_{1t+1}}}$$

$$= \frac{\frac{PS^{**}_{t+1}}{p^{**}_{1t+1}}}{1+r^{**}_{t+1}} + \frac{\frac{\tilde{p}^{**}_{Bt+1}B^{**}_{t+2}}{p^{**}_{1t+1}}}{1+r^{**}_{t+1}}, \tag{74}$$

where $PS^{**}_{t+1} = IT^{**}_{t+1} + DT^{**}_{t+1} - (G^{**}_{1t+1} + G^{**}_{2t+1})$ as defined above.

Substituting (74) into (72) yields

$$\frac{(1+i^{**}_t)\tilde{p}^{**}_{Bt-1}B^{**}_t}{p^{**}_{1t}} = \frac{PS^{**}_t}{p^{**}_{1t}} + \frac{\frac{PS^{**}_{t+1}}{p^{**}_{1t+1}}}{1+r^{**}_{t+1}} + \frac{\frac{\tilde{p}^{**}_{Bt+1}B^{**}_{t+2}}{p^{**}_{1t+1}}}{1+r^{**}_{t+1}}. \tag{75}$$

Similar calculations extend (75) to

$$\frac{(1+i^{**}_t)\tilde{p}^{**}_{Bt-1}B^{**}_t}{p^{**}_{1t}} = \frac{PS^{**}_t}{p^{**}_{1t}} + \frac{\frac{PS^{**}_{t+1}}{p^{**}_{1t+1}}}{1+r^{**}_{t+1}} + \frac{\frac{PS^{**}_{t+2}}{p^{**}_{1t+2}}}{(1+r^{**}_{t+1})(1+r^{**}_{t+2})} + \frac{\frac{\tilde{p}^{**}_{Bt+2}B^{**}_{t+3}}{p^{**}_{1t+2}}}{(1+r^{**}_{t+1})(1+r^{**}_{t+2})},$$

and finally to

$$\frac{(1+i^{**}_t)\tilde{p}^{**}_{Bt-1}B^{**}_t}{p^{**}_{1t}} = \frac{PS^{**}_t}{p^{**}_{1t}} + \sum_{j=1}^{\infty} \frac{\frac{PS^{**}_{t+j}}{p^{**}_{1t+j}}}{(1+r^{**}_{t+1})...(1+r^{**}_{t+j})} + \lim_{j\to\infty} \frac{\frac{\tilde{p}^{**}_{Bt+j}B^{**}_{t+1+j}}{p^{**}_{1t+j}}}{(1+r^{**}_{t+1})...(1+r^{**}_{t+j})}. \tag{76}$$

(76) seems to be the equilibrium condition for the FTPL. But the KS model says that it is just an artificial equation. In fact,

$$\frac{PS^{**}_{t+i}}{p^{**}_{1t+i}} = (1+r^{**}_{t+i})B^{**}_{t+i} - B^{**}_{t+1+i}, \ i = 0, 1, 2, ... ,$$

due to (72). Then, the right-hand side of (76) reduces to $(1+r^{**}_t)B^{**}_t$ which is exactly the value that the left-hand side takes. The important point is still that the prices are essentially determined by the Keynes model in the short-run equilibrium state as shown in Section 2.[17]

Second let us discuss the convergence of government debt. In the FTPL government debt must converge to a finite positive value. It is possible to find conditions for the convergence in the KS model. To do so, first dividing both sides of (70) by $p^{**}_{1t}(=\tilde{p}^{**}_{Bt})$ and remembering the definition of the government budget constraint (9) leads to

$$B^{**}_{t+1} = (1-\pi)B^{**}_t + \frac{BD^{**}_t}{p^{**}_{1t}}, \tag{77}$$

[17]As regards (76), there is an alternative expression as follows:

$$B^{**}_t = \sum_{j=0}^{\infty} \frac{\frac{PS^{**}_{t+j}}{p^{**}_{1t+j}}}{(1+r^{**}_t)...(1+r^{**}_{t+j})} + \lim_{j\to\infty} \frac{B^{**}_{t+1+j}}{(1+r^{**}_t)...(1+r^{**}_{t+j})}.$$

Although the two expressions are equivalent mathematically, it is (76) that is used in the FTPL as the equilibrium condition.

where $BD_t^{**} = G_{1t}^{**} + G_{2t}^{**} - T_t^{**}$ and $T_t^{**} = IT_t^{**} + DT_t^{**} - d_t^{**}B_t^{**}$. Next dividing both sides of (77) by effective labor $A_{t+1}N_{t+1}$ in period $t+1$ yields the equation for the evolution of government bonds as follows:

$$b_{t+1}^{**} = \frac{1}{1+g_N}\left[(1-\pi)b_t^{**} + \beta_{BD}k_t^{**}\right], \tag{78}$$

where b_{t+1}^{**} and b_t^{**} are government bonds per effective labor in periods $t+1$ and t defined as

$$b_{t+1}^{**} = \frac{B_{t+1}^{**}}{A_{t+1}N_{t+1}}, \text{ and } b_t^{**} = \frac{B_t^{**}}{A_tN_t},$$

and $\beta_{BD} = \beta_{G_1} + \beta_{G_2} - \beta_T$.

Since it is already known that in the KS model k_t^{**} tends to a constant k_S^{**} given by (60) independent of b_{t+1}^{**} and b_t^{**}, it is convenient to focus on the long-run steady state in which case (78) is simplified as

$$b_{t+1}^{**} = \frac{1}{1+g_N}\left[(1-\pi)b_t^{**} + \beta_{BD}k_S^{**}\right]. \tag{79}$$

As is easily seen, a solution to (79) depends on the values of parameters g_N, π, and β_{BD}. A "normal" one may be obtained if g_N and π take values such that $\left|\frac{1-\pi}{1+g_N}\right| < 1$ with β_{BD} as positive. In this situation, government bonds b_S^{**} per effective labor in the long-run steady state can be calculated as

$$b_S^{**} = \frac{\beta_{BD}}{g_N+\pi}k_S^{**}. \tag{80}$$

b_S^{**} in (80) is a stable positive solution to (79). That is, b_t^{**} approaches b_S^{**} with time, or it remains there if it starts from there.

From (80) the ratio of capital to government debt in the long-run steady state can be obtained at once as

$$\frac{K_{St}^{**}}{B_{St}^{**}} = \frac{g_N+\pi}{\beta_{BD}}.$$

In addition the ratio of government debt to GDP in the long-run steady state can be calculated as

$$\begin{aligned}\frac{B_{St}^{**}}{Q_{St}^{**}} &= \frac{B_{St}^{**}}{K_{St}^{**}}\frac{K_{St}^{**}}{Q_{St}^{**}} \\ &= \frac{\beta_{BD}}{g_N+\pi}\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F},\end{aligned}$$

because of (65).

A debt situation represented by (80) may be "normal" from a theoretical point of view, and the FTPL also makes such an assumption. But, as mentioned above, the convergence of government debt is not necessary for the KS model. Even if government debt b_t^{**} explodes, the existence and uniqueness of the long-run steady state k_S^{**} is still warranted in the KS model.

5 Conclusion

In the FTPL discussion concentrates on one simple equation:

$$\frac{\text{Nominal government debt}}{\text{Price level}} = \text{Present value of primary surpluses},$$

where the price level is so determined that both the sides coincide. This logic is parallel to that of the quantity theory of money in which the price level is so adjusted that the following Fisher equation of exchange holds:

$$(\text{Quantity of money})(\text{Income velocity of money}) = (\text{Price level})(\text{Real income}).$$

Which one is right? Are both of them right or wrong in any sense? Is government indispensable for the determination of the price level as the FTPL insists?

In order to answer these questions I constructed a macro model which includes the government sector as well as the foreign sector. It is based on the *General Theory* of Keynes who tried to generalize the quantity theory using flexible prices and the Solow model which does not work without the quantity theory. Such a model is called the KS model. It is a unified model because it is able to analyze both the short run and the long run. In the short run prices and outputs are determined by the principle of effective demand. As a result, goods markets clear every period. In the long run the economy is described by the Solow model and there exists a stable steady state. The Fisher equation of exchange can always be written as above. But whether the quantity of money influences the price level depends on the magnitude of nominal outputs on the right-hand side compared with the money supply on the left-hand side. In sum the quantity theory of money cannot be discarded as a "false theory" (Wicksell), but it does not always hold even in the long run.

In the FTPL literature it is often assumed that a single kind of output is given to a consumer every period by a fixed amount and the real rate of interest is also fixed. Thus, I regarded the FTPL as a neoclassical theory and analyzed it in the long-run equilibrium state or in the long-run steady state. And using the KS model it has been proved that the equilibrium condition for the FTPL as above reduces to an identity and it is not related to the determination of the price level. It has also been shown that in the KS model there are conditions under which government debt converges to a positive value as shown in the FTPL. Although the FTPL made a contribution in that it turned our attention to the role of government debt, the price level is not determined along the line of the FTPL from the perspective of the KS model.

Appendix

A Supply Curve of Domestic Investment Goods

As said in Section 2.2, (19) is a supply curve of domestic investment goods. To express it in a usual way, replace Q^e_{1t} and p^e_{1t} in (19) respectively with Q^S_{1t} and p_{1t}. Then,

$$Q^S_{1t} = p_{1t}^{\frac{1-\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t}.$$

To examine the shape of the graph on the Q_{1t}-p_{1t} plane, differentiate Q_{1t}^S w.r.t. p_{1t} once and twice. Then,

$$\frac{dQ_{1t}^S}{dp_{1t}} = \frac{1-\alpha}{\alpha} p_{1t}^{\frac{1-2\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t} > 0,$$

and

$$\frac{d^2Q_{1t}^S}{dp_{1t}^2} = \frac{1-\alpha}{\alpha}\frac{1-2\alpha}{\alpha} p_{1t}^{\frac{1-3\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t} \begin{cases} > 0 & \text{if } 0 < \alpha < \frac{1}{2} \\ = 0 & \text{if } \alpha = \frac{1}{2} \\ < 0 & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

The shape of a supply curve in Figure 3 is based on a usual assumption that α is around one third.

B Supply and Demand Curves of Domestic Consumption Goods

As said in Subsectoin 2.2, (27) is a supply curve of domestic consumption goods. To express it in a usual way, replace Q_{2t}^e and p_{2t}^e in (27) respectively with Q_{2t}^S and p_{2t}. Then,

$$Q_{2t}^S = p_{2t}^{\frac{1-\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t}.$$

As is obvious, the argument on Q_{1t}^S in Appendix A applies to that on Q_{2t}^S in the same fashion.

Then, let us move on to the consumption-goods demand curve with $p_{1t}^e Q_{1t}^e$, F_t^e, and NX_{2t}^e as given. The total expenditure on domestic consumption goods in nominal terms can be written as

$$\begin{aligned} C_t^e + G_{2t} + EX_{2t}^e - IM_{2t}^e &= cY_t^e + NX_{2t}^e \\ &= c(p_{1t}^e Q_{1t}^e + p_{2t}^e Q_{2t}^e + F_t^e) + NX_{2t}^e, \\ &= c(p_{1t}^e Q_{1t}^e + p_{2t} Q_{2t}^S + F_t^e) + NX_{2t}^e \end{aligned}$$

because of (4), (1), and the supply curve Q_{2t}^S above. The demand for domestic consumption goods Q_{2t}^D is obtained by dividing the above expenditure by the price p_{2t}:

$$Q_{2t}^D = cQ_{2t}^S + \frac{c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e}{p_{2t}}.$$

In order to know the shape of the demand curve on the Q_{2t}-p_{2t} plane, differentiate Q_{2t}^D w.r.t. p_{2t} once and twice. Then,

$$\frac{dQ_{2t}^D}{dp_{2t}} = c\frac{1-\alpha}{\alpha} p_{2t}^{\frac{1-2\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t} - \frac{c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e}{p_{2t}^2},$$

and

$$\frac{d^2Q_{2t}^D}{dp_{2t}^2} = c\frac{1-\alpha}{\alpha}\frac{1-2\alpha}{\alpha} p_{2t}^{\frac{1-3\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t} + \frac{2[c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e]}{p_{2t}^3} > 0.$$

It follows from these results that demand curve Q^D_{2t} is bending forward and that it changes the sign of the slope at $p_{2t} = \bar{p}_{2t}$, where

$$\bar{p}_{2t} = \left[\frac{\alpha}{1-\alpha}\frac{1-c}{c}\right]^{\alpha}\left[\frac{w_t}{(1-\mu)(1-\alpha)A_t}\right]^{1-\alpha}\left[\frac{1}{K_{2t}}\right]^{\alpha}\left[\frac{c(p^e_{1t}Q^e_{1t}+F^e_t)+NX^e_{2t}}{1-c}\right]^{\alpha}.$$

If $\alpha < c$ as assumed in usual macroeconomics, $\bar{p}_{2t}$ is smaller than p^e_{2t} in (28) as Figure 4 shows.

C Derivation of Capital Accumulation Equation (59)

$$\begin{aligned}
k^{**}_{t+1} &= \frac{1-\ \delta-\beta_{NX_1}}{(1+g)(1+n)}k^{**}_t + \frac{A_t N^{**}_{1t}}{A_{t+1}N_{t+1}}(k^{**}_{1t})^{\alpha} \\
&= \frac{1-\delta-\beta_{NX_1}}{(1+g)(1+n)}k^{**}_t + \frac{1}{(1+g)(1+n)}\frac{N^{**}_{1t}}{N_t}(k^{**}_{1t})^{\alpha} \\
&= \frac{1-\delta-\beta_{NX_1}}{1+g_N}k^{**}_t + \frac{1}{1+g_N}[1-c-(c\beta_F+\beta_{NX_2})(k^{**}_t)^{1-\alpha}](k^{**}_t)^{\alpha} \\
&= \frac{1-\delta-\beta_{NX_1}-(1-s)\beta_F-\beta_{NX_2}}{1+g_N}k^{**}_t + \frac{1}{1+g_N}s(k^{**}_t)^{\alpha} \\
&= \frac{1}{1+g_N}\left[(1-\delta-\beta_{NX}-\beta_F+s\beta_F)k^{**}_t + s(k^{**}_t)^{\alpha}\right] \\
&= \frac{1}{1+g_N}\left[(1-\delta-\beta_{CA}+s\beta_F)k^{**}_t + s(k^{**}_t)^{\alpha}\right].
\end{aligned}$$

D Analysis of the Golden-Rule State

The golden-rule state is defined as the long-run steady state in which real national consumption $\frac{C^{**}_{St}+G^{**}_{2St}}{p^{**}_{1t}}$ is maximized every period. Remembering the consumption function (4) and using (66) gives real national consumption as a function of the gross rate s of saving:

$$\begin{aligned}
\frac{C^{**}_{St}+G^{**}_{2St}}{p^{**}_{1t}} &= \frac{(1-s)Y^{**}_{St}}{p^{**}_{1t}} \\
&= (1-s)\left(\frac{g_N+\delta+\beta_{CA}}{s}\right)K^{**}_{St} \\
&= (1-s)\left(\frac{g_N+\delta+\beta_{CA}}{s}\right)\left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}}A_t N_t. \qquad (81)
\end{aligned}$$

Let s_G be the saving rate which maximizes the real national consumption. It can be obtained from the differentiation of (81) with respect to s.[18] Hence,

$$s_G = \frac{\alpha(g_N+\delta+\beta_{CA})}{g_N+\delta+\beta_{NX}+\alpha\beta_F}$$

[18] $\frac{C^{**}_{St}+G^{**}_{2St}}{p^{**}_{1t}}$ is equal to $Q^{**}_{S2t}-\beta_{NX_2}K^{**}_{St}$ because of (3). Then, s_G can also be calculated using the following relation:

$$Q^{**}_{S2t}-\beta_{NX_2}K^{**}_{St} = [(k^{**}_S)^{\alpha}-(g_N+\delta+\beta_{NX})\ k^{**}_S\ ]A_t N_t,$$

which is derived from (63) - (65).

$$= \alpha \left[1 + \frac{(1-\alpha)\beta_F}{g_N + \delta + \beta_{NX} + \alpha\beta_F} \right]. \quad (82)$$

The corresponding ratio c_G of national consumption to GNI is as follows:

$$\begin{aligned} c_G &= \frac{(1-\alpha)(g_N + \delta + \beta_{NX})}{g_N + \delta + \beta_{NX} + \alpha\beta_F} \\ &= (1-\alpha) \left[1 - \frac{\alpha\beta_F}{g_N + \delta + \beta_{NX} + \alpha\beta_F} \right]. \end{aligned}$$

Let a subscript G in place of S denote a value in the golden-rule state. Then, k_S^{**} in (60) is written as

$$k_G^{**} = \left(\frac{\alpha}{g_N + \delta + \beta_{NX}} \right)^{\frac{1}{1-\alpha}}. \quad (83)$$

by substituting (82) into (60).

As for capital stock,

$$\begin{aligned} K_{Gt}^{**} &= \left(\frac{\alpha}{g_N + \delta + \beta_{NX}} \right)^{\frac{1}{1-\alpha}} A_t N_t, & (84) \\ K_{G1t}^{**} &= \left(\alpha - \frac{\alpha\ \beta_{NX_2}}{g_N + \delta + \beta_{NX}} \right) K_{Gt}^{**}, & (85) \\ K_{G2t}^{**} &= \left[(1-\alpha) + \frac{\alpha\ \beta_{NX_2}}{g_N + \delta + \beta_{NX}} \right] K_{Gt}^{**}, & (86) \end{aligned}$$

because of (83), (61), and (62) along with (82).

As for output,

$$\begin{aligned} Q_{G1t}^{**} &= (g_N + \delta + \beta_{NX_1}) K_{Gt}^{**}, & (87) \\ Q_{G2t}^{**} &= \frac{(1-\alpha)(g_N + \delta + \beta_{NX}) + \alpha\ \beta_{NX_2}}{\alpha} K_{Gt}^{**}, & (88) \\ Q_{Gt}^{**} &= \frac{g_N + \delta + \beta_{NX}}{\alpha} K_{Gt}^{**}, & (89) \end{aligned}$$

because of (63) - (65) and (84) along with (82).[19]

Real GNI in the golden-rule state is written as

$$\frac{Y_{Gt}^{**}}{p_{1t}^{**}} = \frac{g_N + \delta + \beta_{NX} + \alpha\beta_F}{\alpha} K_{Gt}^{**},$$

because of (89). Then, real private disposable income is calculated by definition as

$$\frac{Y_{DGt}^{**}}{p_{1t}^{**}} = \frac{g_N + \delta + \beta_{NX} - \alpha(\delta - \pi + \beta_T - \beta_F)}{\alpha} K_{Gt}^{**}.$$

[19] As to the ratio of the consumption-goods sector to the investment-goods sector,

$$\frac{K_{G2t}^{**}}{K_{G1t}^{**}} = \frac{Q_{G2t}^{**}}{Q_{G1t}^{**}} = \frac{1-\alpha}{\alpha} + \frac{\beta_{NX_2}}{\alpha(g_N + \delta + \beta_{NX_1})},$$

because of (85) - (88). And in terms of the capital-output ratio, (89) becomes

$$\frac{K_{Gt}^{**}}{Q_{Gt}^{**}} = \frac{\alpha}{g_N + \delta + \beta_{NX}}.$$

Because real private consumption is written as

$$\frac{C_{Gt}^{**}}{p_{1t}^{**}} = \frac{(1-\alpha)(g_N+\delta+\beta_{NX})-\alpha\beta_{G_2}}{\alpha}K_{Gt}^{**},$$

the average propensity to consume in the golden-rule becomes

$$\begin{aligned}\frac{C_{Gt}^{**}}{Y_{DGt}^{**}} &= \frac{(1-\alpha)(g_N+\delta+\beta_{NX})-\alpha\beta_{G_2}}{g_N+\delta+\beta_{NX}-\alpha(\delta-\pi+\beta_T-\beta_F)} \\ &= (1-\alpha)+\frac{\alpha(1-\alpha)(\delta-\pi+\beta_T-\beta_F-\beta_{G_2})}{g_N+\delta+\beta_{NX}-\alpha(\delta-\pi+\beta_T-\beta_F)} \gtreqless 1-\alpha,\end{aligned}$$

which is a decreasing function of the inflation rate.

Similarly, since real private saving is written as

$$\frac{S_{DGt}^{**}}{p_{1t}^{**}} = (g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)K_{Gt}^{**},$$

the net rate of saving is calculated as

$$\frac{S_{DGt}^{**}}{Y_{DGt}^{**}} = \frac{\alpha(g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)}{g_N+\delta+\beta_{NX}-\alpha(\delta-\pi+\beta_T-\beta_F)} \lesseqgtr \alpha.$$

Finally the ratio of national consumption to GDP becomes

$$\frac{C_{Gt}^{**}+G_2^{**}}{p_{1t}^{**}Q_{Gt}^{**}} = 1-\alpha,$$

while the real rate of interest is calculated as

$$r_G^{**} = (1-\mu)(g_N+\delta+\beta_{NX})-\delta.$$

E Analysis of the Modified-Golden-Rule State

The modified-golden-rule state here is defined as the long-run steady state in which the sum of discounted present values of utility of the household sector is maximized as follows:

$$\begin{aligned}&\max_s \quad \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \frac{(c_{St}^{**})^{1-\gamma}}{1-\gamma}N_t,\ \gamma>0,\\ &\text{s.t.} \quad \sum_{t=0}^{\infty}\frac{c_{St}^{**}N_t}{(1+r_{S0}^{**})\ldots(1+r_{St}^{**})}+\lim_{t\to\infty}\frac{K_{St}^{**}+B_{fSt}^{**}}{(1+r_0^{**})\ldots(1+r_t^{**})} = K_{S0}^{**}+B_{fS0}^{**}+\sum_{t=0}^{\infty}\frac{\frac{w_{St}^{**}}{p_{1t}^{**}}N_t+\mu Q_{St}^{**}}{(1+r_0^{**})\ldots(1+r_t^{**})},\end{aligned}$$

where

$$\begin{aligned}c_{St}^{**} &= \frac{\frac{C_{St}^{**}+G_{2t}^{**}}{p_{1t}^{**}}}{N_t} \\ &= (1-s)\left(\frac{g_N+\delta+\beta_{CA}}{s}\right)\left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}}A_t,\end{aligned}$$

due to (81). The intertemporal budget constraint is obtained by the iteration of the budget constraint in period t:

$$K_{t+1}^{**} + B_{ft+1}^{**} = (1 + r_t^{**})(K_t^{**} + B_{ft}^{**}) + \left[\frac{w_t^{**}}{p_{1t}^{**}} N_t + \mu Q_t^{**} - \left(\frac{C_t^{**}}{p_{1t}^{**}} + \frac{G_{2t}^{**}}{p_{1t}^{**}} \right) \right].$$

A difference from an ordinary setting is that the utility maximization problem above is confined to the long-run steady state. This idea is based on two reasons. The first is that the analysis becomes much easier. As far as the saving rate s is fixed, the existence and uniqueness of a stable steady state of the KS model is guaranteed as in (60). No transversality condition is required, or no complicated explanation of solution paths on the phase plane is not needed. The second is that a dynamically inefficient economy can emerge in addition to a dynamically efficient economy. As is well known, the former can be examined only by the overlapping-generations model. The Ramsey-Cass-Koopmans model with an infinite horizon can deal with the latter only. The KS model with an infinite horizon is able to analyze both the situations at least theoretically.

The Euler equation related to the abobe utility maximization problem is written as

$$\frac{c_{St+1}^{**}}{c_{St}^{**}} = \left(\frac{1 + r_{St+1}^{**}}{1 + \rho} \right)^{\frac{1}{\gamma}},$$

where

$$r_{St+1}^{**} = (1 - \mu) \frac{\alpha(g_N + \delta + \beta_{CA} - s\beta_F)}{s} - \delta,$$

because of (54) and (60). Solving it gives the utility maximizing rate s_{MG} of saving as

$$s_{MG} = \frac{(1 - \mu)\alpha(g_N + \delta + \beta_{CA})}{(1 + \rho)(1 + g)^\gamma - 1 + \delta + (1 - \mu)\alpha\beta_F}. \tag{90}$$

$s_G \gtreqless s_{MG}$ for $(1 + \rho)(1 + g)^\gamma - 1 + \delta \gtreqless (1 - \mu)(g_N + \delta + \beta_{NX})$, because

$$s_G - s_{MG} = \frac{\alpha(g_N + \delta + \beta_{CA})[(1 + \rho)(1 + g)^\gamma - 1 + \delta - (1 - \mu)(g_N + \delta + \beta_{NX})]}{(g_N + \delta + \beta_{NX} + \alpha\beta_F)[(1 + \rho)(1 + g)^\gamma - 1 + \delta + (1 - \mu)\alpha\beta_F]}.$$

Let a subscript MG in place of S denote a value in the modified-golden-rule state. Then, k_S^{**} in (60) is written as

$$k_{MG}^{**} = \left[\frac{(1 - \mu)\alpha}{(1 + \rho)(1 + g)^\gamma - 1 + \delta} \right]^{\frac{1}{1-\alpha}}. \tag{91}$$

by substituting (90) into (60). $k_G^{**} \gtreqless k_{MG}^{**}$ for $(1+\rho)(1+g)^\gamma - 1 + \delta \gtreqless (1-\mu)(g_N + \delta + \beta_{NX})$, because

$$k_G^{1-\alpha} - k_{MG}^{1-\alpha} = \frac{\alpha[(1 + \rho)(1 + g)^\gamma - 1 + \delta - (1 - \mu)(g_N + \delta + \beta_{NX})]}{(g_N + \delta + \beta_{NX})[(1 + \rho)(1 + g)^\gamma - 1 + \delta]}.$$

As for capital stock,

$$\begin{aligned} K_{MGt}^{**} &= \left[\frac{(1 - \mu)\alpha}{(1 + \rho)(1 + g)^\gamma - 1 + \delta} \right]^{\frac{1}{1-\alpha}} A_t N_t, \qquad (92) \\ K_{MG1t}^{**} &= \frac{(1 - \mu)\alpha(g_N + \delta + \beta_{NX_1})}{(1 + \rho)(1 + g)^\gamma - 1 + \delta} K_{MGt}^{**}, \\ K_{MG2t}^{**} &= \left[1 - \frac{(1 - \mu)\alpha(g_N + \delta + \beta_{NX_1})}{(1 + \rho)(1 + g)^\gamma - 1 + \delta} \right] K_{MGt}^{**}, \end{aligned}$$

because of (91), (61), and (62) along with (90).

As for output,

$$\begin{aligned} Q^{**}_{MG1t} &= (g_N + \delta + \beta_{NX_1})K^{**}_{MGt}, \\ Q^{**}_{MG2t} &= \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(g_N + \delta + \beta_{NX_1})]}{(1-\mu)\alpha}K^{**}_{MGt}, \\ Q^{**}_{MGt} &= \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta}{(1-\mu)\alpha}K^{**}_{MGt}, \end{aligned} \tag{93}$$

because of (63) - (65) and (92) along with (90).[20]

Real GNI in the modified-golden-rule state is written as

$$\frac{Y^{**}_{MGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta + (1-\mu)\alpha\beta_F}{(1-\mu)\alpha}K^{**}_{MGt},$$

because of (93). Then, real private disposable is calculated as

$$\frac{Y^{**}_{DMGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(\delta - \pi + \beta_T - \beta_F)}{(1-\mu)\alpha}K^{**}_{MGt}.$$

Because real private consumption is written as

$$\frac{C^{**}_{MGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(g_N + \delta + \beta_{NX} + \beta_{G_2})}{(1-\mu)\alpha}K^{**}_{MGt},$$

the average propensity to consume in the modified-golden-rule becomes

$$\frac{C^{**}_{MGt}}{Y^{**}_{DGt}} = \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(g_N + \delta + \beta_{NX} + \beta_{G_2})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(\delta - \pi + \beta_T - \beta_F)},$$

which is a decreasing function of the inflation rate.

Since real private saving is written as

$$\frac{S^{**}_{DMGt}}{p^{**}_{1t}} = (g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)K^{**}_{MGt},$$

the net rate of saving is calculated as

$$\frac{S^{**}_{DMGt}}{Y^{**}_{DMGt}} = \frac{(1-\mu)\alpha(g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)}{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(\delta - \pi + \beta_T - \beta_F)}.$$

Finally the ratio of national consumption to GDP becomes

$$\frac{C^{**}_{MGt} + G^{**}_2}{p^{**}_{1t}Q^{**}_{MGt}} = 1 - \frac{(1-\mu)(g_N + \delta + \beta_{NX})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta}\alpha,$$

while the real rate of interest is calculated as

$$\begin{aligned} r^{**}_{MG} &= (1+\rho)(1+g)^{\gamma} - 1 \\ &\approx \gamma g + \rho, \end{aligned}$$

which is the natural rate of interest.

[20] In terms of the capital-output ratio, (93) becomes

$$\frac{K^{**}_{MGt}}{Q^{**}_{MGt}} = \frac{(1-\mu)\alpha}{(1+\rho)(1+g)^{\gamma} - 1 + \delta}.$$

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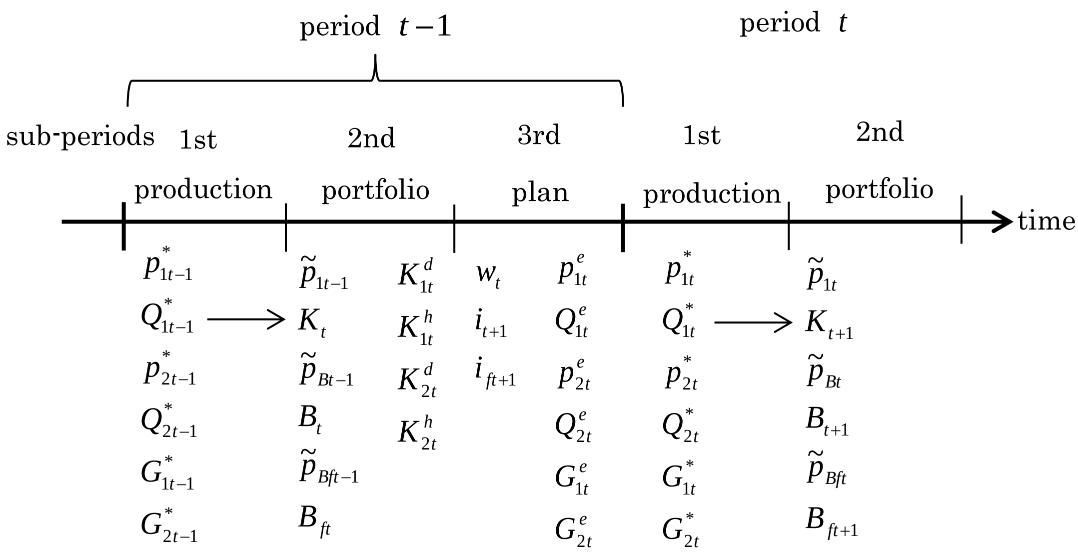

Figure 1. The Time Structure of the Model

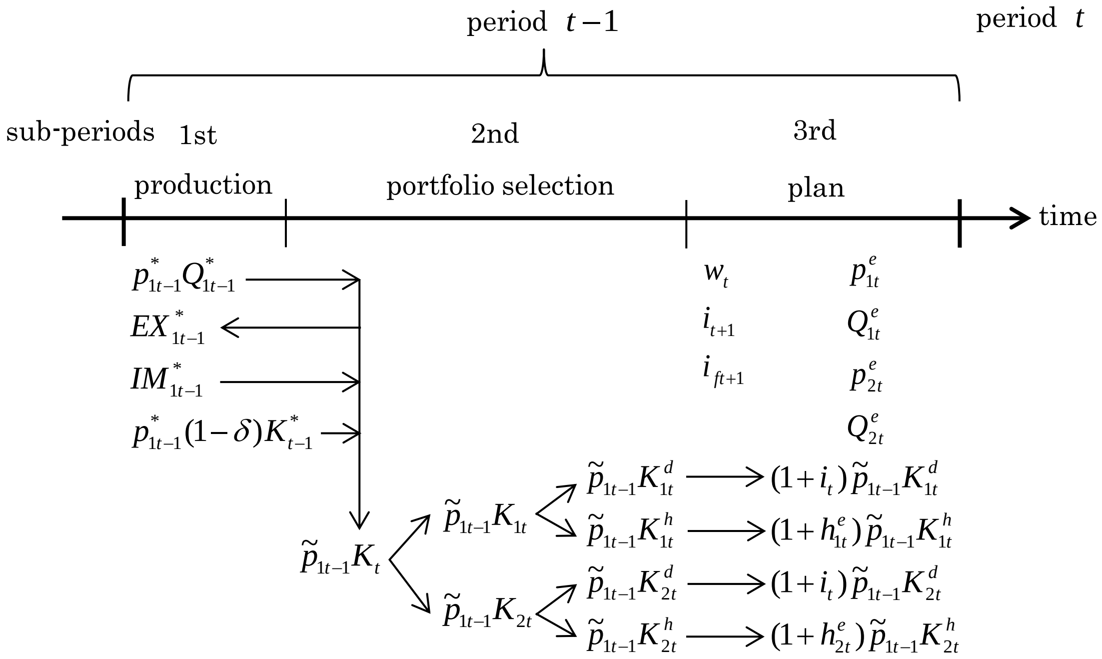

Figure 2. Four Ways to Hold Capital Stock as an Asset

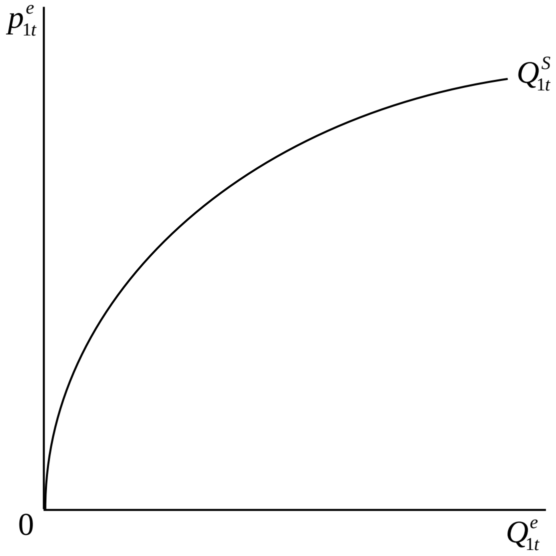

Figure 3. Supply Curve of Investment Goods

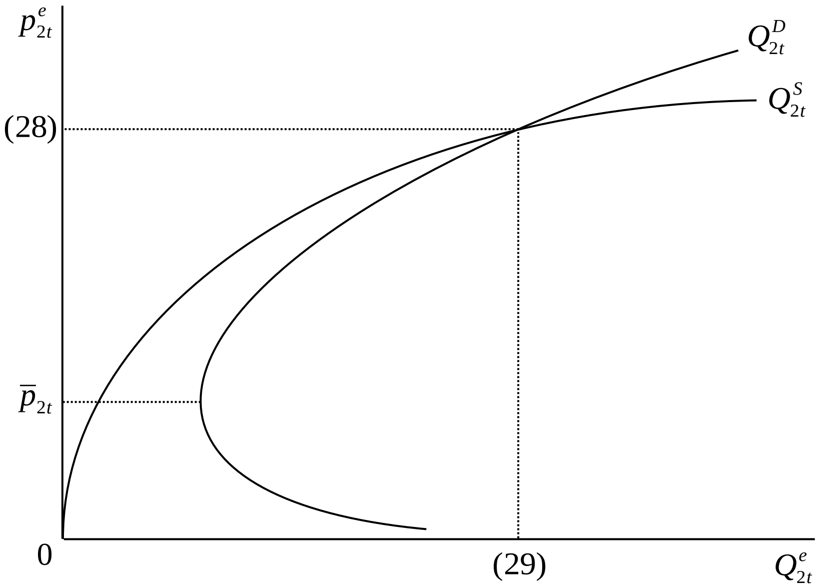

Figure 4. Equilibrium in the Consumption-Goods Market

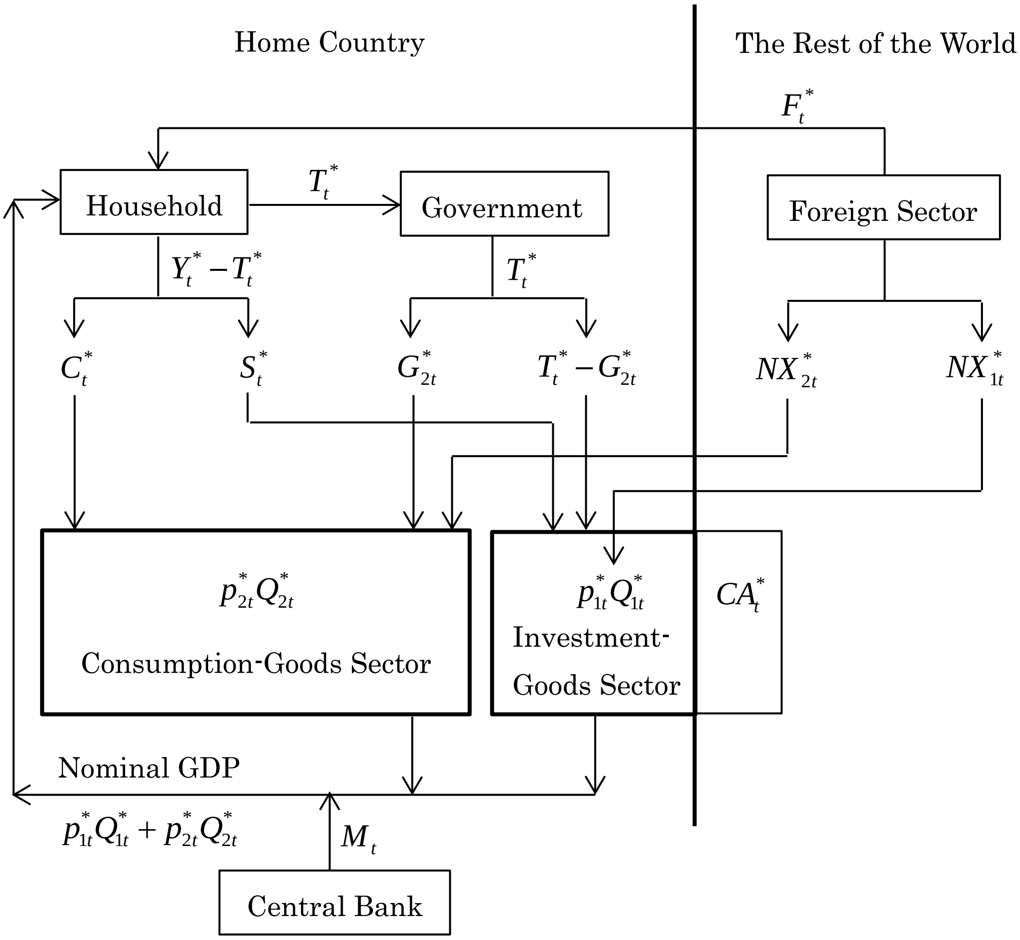

Figure 5. The Short-Run Equilibrium State

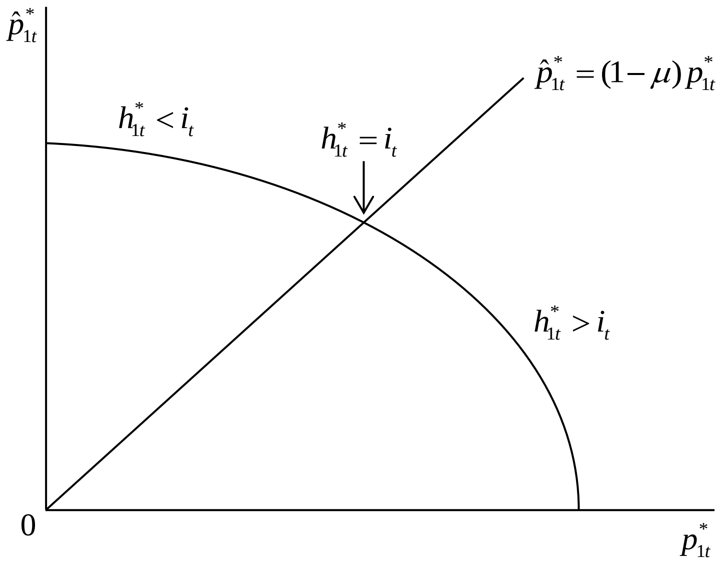

Figure 6. The Normal Supply Price of Investment Goods