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Abstract

We present a series of infinitary propositional epistemic logics $\{\mathrm{GL}(\boldsymbol{L}_\alpha) : \alpha$ is an ordinal up to $\alpha^o\}$ (α^o is a given ordinal with $\omega \leq \alpha^o$) so that $\mathrm{GL}(\boldsymbol{L}_0)$ is finitary KD^n with n agents, while $\mathrm{GL}(\boldsymbol{L}_\alpha)$ ($\alpha \geq 1$) allows certain countably infinite conjunctions of formulae. Those conjunctions are constructed in a recursive manner for each α. $\mathrm{GL}(\boldsymbol{L}_\alpha)$ is small in that the language is kept countable and constructive. The set of formulae $\boldsymbol{L}_\alpha$ is increasing up to $\alpha = \omega$, but it stops at ω. We present Kripke-completeness for $\mathrm{GL}(\boldsymbol{L}_\alpha)$ for each $\alpha \leq \omega$, which is proved using the Rasiowa-Sikorski lemma and Tanaka-Ono lemma. $\mathrm{GL}(\boldsymbol{L}_\alpha)$ has sufficient expressive power to discuss intrapersonal and interpersonal beliefs with infinite lengths. For example, in $\mathrm{GL}(\boldsymbol{L}_\alpha)$, we discuss the expressibility of Axioms T (truthfulness), 4 (positive introspection), and 5 (negative introspection). Also, we evaluate expressibility of the concept of common knowledge in $\mathrm{GL}(\boldsymbol{L}_\alpha)$.

Key Words: Infinitary Epistemic Logic, Completeness, Rasiowa-Sikorski Lemma, Tanaka-Ono Lemma, Common knowledge

1 Introduction

We develop a series of small infinitary epistemic logics to study deductive inference involving intra/inter-personal beliefs/knowledge in social situations. In these situations, people's beliefs may include infinitary components such as common knowledge, common beliefs, and infinite regress of beliefs. To approach such situations, we extend the finitary epistemic logic KD^n with n agents to a series of infinitary logics; each extension is small in that the set of formulae is kept countable and constructive. We also show that each is complete with respect to Kripke semantics.

Our logical system can be compared with two literatures on infinitary epistemic concepts: the *infinitary logic* literature since Karp [10] (for epistemic logics, Kaneko-Nagashima [11], Tanaka-Ono [23], Tanaka [21], Heifetz [6]), and the *fixed-point logic* literature (for epistemic logics, Fagin *et al.* [2], Meyer-van der Hoek [16]). The languages in the first literature are typically very large

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(at least continuum) in terms of sets of formulae. The languages in the second are finitary and infinitary concepts are indirectly defined by fixed-point arguments. In contrast to the second approach, our system allows for explicit expressions of targeted infinitary concepts and enables us to evaluate the depths required for an epistemic axiom or a discourse to derive a solution concept in game theory.

Our base logic KD^n requires the agents have classical logical abilities and contradiction-free beliefs. Keeping those requirements, we develop a series of infinitary logics illustrated as

$$\mathrm{KD}^n = \mathrm{GL}(\boldsymbol{L}_0) \rightrightarrows \mathrm{GL}(\boldsymbol{L}_1) \rightrightarrows \cdots \rightrightarrows \mathrm{GL}(\boldsymbol{L}_\omega) = \mathrm{GL}(\boldsymbol{L}_{\omega+1}) = ... = \mathrm{GL}(\boldsymbol{L}_{\alpha^o}), \tag{1}$$

where α^o is a given ordinal with $\alpha^o \geq \omega := \{0, 1, ...\}$. The series starts with $\mathrm{KD}^n = \mathrm{GL}(\boldsymbol{L}_0)$, and extends each by adding one more layer of infinitary expressions: For $\alpha < \omega$, $\mathrm{GL}(\boldsymbol{L}_\alpha)$ allows the nested occurrences of permissible infinitary conjunctions up to α times. The extension stops at $\mathrm{GL}(\boldsymbol{L}_\omega)$ ($\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha$), which is closed under permissible infinitary conjunctions within $\boldsymbol{L}_\omega$. The language $\boldsymbol{L}_\alpha$ of each $\mathrm{GL}(\boldsymbol{L}_\alpha)$ is kept countable for all $\alpha \leq \alpha^o$.[1]

The proof systems in the series (1) are uniform: they share the same logical axiom schemata and inference rules only with the restriction to $\boldsymbol{L}_\alpha$. In the same sense, the Kripke semantics is uniform over α. Each $\mathrm{GL}(\boldsymbol{L}_\alpha)$ is proved to be sound and complete with respect to the corresponding Kripke semantics. For this completeness result, extension from $\boldsymbol{L}_\alpha$ to $\boldsymbol{L}_{\alpha+1}$ should take some suitable form; it requires certain closure properties to guarantee that $\mathrm{GL}(\boldsymbol{L}_\alpha)$ respects the Barcan axiom and $\wedge$-rule. It follows from the soundness-completeness result that $\mathrm{GL}(\boldsymbol{L}_{\alpha+1})$ is a conservative extension of $\mathrm{GL}(\boldsymbol{L}_\alpha)$, i.e., for any formula $A \in \boldsymbol{L}_\alpha$, A is provable in $\mathrm{GL}(\boldsymbol{L}_\alpha)$ if and only if it is provable in $\mathrm{GL}(\boldsymbol{L}_{\alpha+1})$. In (1), the double arrow describes the relation of conservative extension in the hierarchy of these logics.

To prove Kripke-completeness, we adopt the $\mathcal{Q}$-filter method developed in Tanaka-Ono [23].[2] We use $\mathcal{Q}$-filters in our construction of counter-models, which play the same role as maximal consistent sets of formulae in the standard construction of a canonical model. This method is crucial, since $\mathrm{GL}(\boldsymbol{L}_\alpha)$ deals with both particular infinitary conjunctions and modality. To treat these aspects, our proof relies upon two lemmas, the Rasiowa-Sikorski lemma and Tanaka-Ono lemma. In fact, our completeness theorem can be modified to systems including additional epistemic axioms, Axioms T (truthfulness - - $\mathbf{B}_i(A) \supset A$, here $\mathbf{B}_i(\cdot)$ is the belief operator), 4 (positive introspection - - $\mathbf{B}_i(A) \supset \mathbf{B}_i\mathbf{B}_i(A)$), and/or 5 (negative introspection - - $\neg\mathbf{B}_i(A) \supset \mathbf{B}_i(\neg\mathbf{B}_i(A))$).

Our choice of base logic KD^n is deliberate; it enables us to evaluate Axioms T, 4, and 5 in terms of the required epistemic depths. It is known that KD^n+ T can be faithfully embedded into KD^n. We extend this result to $\mathrm{GL}(\boldsymbol{L}_\alpha)$ for any α ($0 \leq \alpha \leq \omega$). In contrast, to embed the finitary logic KD^n+ 4 into $\mathrm{GL}(\boldsymbol{L}_\alpha)$, it requires $\alpha = \omega$; Axiom 4 is finitary but its entirety can be captured only by an infinitary formula. For Axiom 5, faithful embedding is simply impossible. The results are summarized in (2), where $\rightrightarrows_\psi$ means faithful embedding with a

[1] Below KD^n, a hierarchy of logics of shallow epistemic depths is developed in Kaneko-Suzuki [12]. Each system is a fragment of KD^n with a finite epistemic structure, .and continues to KD^n.

[2] Tanaka [21], [22] used the $\mathcal{Q}$-filter method to obtain completeness of common knowledge (predicate as well as propositional) logics and other epistemic logics.

suitable translation ψ.

$$\begin{aligned}
[\mathrm{KD}^n + \mathrm{T} \rightrightarrows_\psi \mathrm{GL}(\boldsymbol{L}_\alpha)]\,\text{is possible for all } \alpha(0 &\leq \alpha \leq \omega); \\
[\mathrm{KD}^n + 4 \rightrightarrows_\psi \mathrm{GL}(\boldsymbol{L}_\alpha)]\ \text{ is possible only for } \alpha &= \omega; \\
[\mathrm{KD}^n + 5 \rightrightarrows_\psi \mathrm{GL}(\boldsymbol{L}_\alpha)]\ \text{ is impossible for any } \alpha(0 &\leq \alpha \leq \omega).
\end{aligned} \tag{2}$$

We also consider the faithful embedding of the fixed-point extension of KD^n expressing common knowledge logic, denoted $\mathrm{CK}(\boldsymbol{L}^{\mathbf{C}})$, to $\mathrm{GL}(\boldsymbol{L}_\omega)$. We show that $\mathrm{CK}(\boldsymbol{L}^{\mathbf{C}})$ is faithfully embedded into $\mathrm{GL}(L_\omega)$ using translation ψ. However, if we focus on a particular formula $A \in \boldsymbol{L}^{\mathbf{C}}$, we can use the hierarchy in (1) to identify the smallest logic $\mathrm{GL}(\boldsymbol{L}_\alpha)$ with $\psi(A) \in \boldsymbol{L}_\alpha$, where such α is finite. The provability of A in $\mathrm{CK}(\boldsymbol{L}^{\mathbf{C}})$ is equivalent to the provability of $\psi(A)$ in $\mathrm{GL}(\boldsymbol{L}_\alpha)$. A discourse involving common knowledge requires a much smaller infinitary language than typically assumed.

The paper format is as follows: Section 2 gives the definition of the sets of formulae. Section 3 formulates the system $\mathrm{GL}(\boldsymbol{L}_\alpha)$ and the Kripke semantics, and states the completeness result. In Section 4, we give discussions on applications of our framework and the completeness result. A proof of Kripke-completeness is given in Section 5. Section 6 concludes the paper.

2 Small Infinitary Languages $\boldsymbol{L}_\alpha$

We fix an ordinal α^o with $\omega \leq \alpha^o$, where $\omega = \{0, 1, ...\}$. We define the class of infinitary languages $\{\boldsymbol{L}_\alpha : \alpha \leq \alpha^o\}$. For each α, $\boldsymbol{L}_\alpha$ is constructed from $\cup_{\beta<\alpha}\boldsymbol{L}_\beta$ in a recursive manner, and we will show that $\boldsymbol{L}_\alpha$ becomes constant after $\alpha = \omega$. We also evaluate the depths of formulae in $\boldsymbol{L}_\alpha$, and show that the depth of the entire set $\boldsymbol{L}_\omega$ is ω^2. In the end of this section, we make brief comparisons with the set of formulae in the literature of infinitary logics since Karp [10].

We adopt the following list of primitive symbols:

propositional variables: $\mathbf{p}_0, \mathbf{p}_1, ...$; *logical connectives*: $\neg$ (not), $\supset$ (implies), $\wedge$ (and);

unary belief operators: $\mathbf{B}_1(\cdot)$, ..., $\mathbf{B}_n(\cdot)$ $(1 \leq n < \omega)$; *parentheses*: (,); *brackets*: $\langle$, $\rangle$.

The conjunction symbol $\wedge$ will be applied to a finite set of formulae and some infinite sequences of formulae. An infinitary conjunction will typically be written as $\wedge\langle C^\nu : \nu \geq 0\rangle$. We denote $\mathcal{P}_0 = \{\mathbf{p}_0, \mathbf{p}_1, ...\}$, and the set of agents (the subscripts for the beliefs operators) by $N = \{1, ..., n\}$.

Let α be an ordinal with $\alpha \leq \alpha^o$. Suppose that the set of formulae $\boldsymbol{L}_\beta$ is already defined for all $\beta < \alpha$, and define $\mathcal{P}_\alpha$ by $\mathcal{P}_\alpha = \cup_{\beta<\alpha}\boldsymbol{L}_\beta$. Let $\mathcal{F}_\alpha$ be a given set of formulae with $\mathcal{F}_0 = \emptyset$, which is the source of infinitary conjunctions and is given later. We define the set $\boldsymbol{L}_\alpha$ by the following three steps:

Iα0: all formulae in $\mathcal{P}_\alpha \cup \mathcal{F}_\alpha$ belong to $\boldsymbol{L}_\alpha$;

Iα1 (finitary extension): if A, B are formulae in $\boldsymbol{L}_\alpha$, so are $(A \supset B)$, $(\neg A)$, $\mathbf{B}_i(A)$ $(i \in N)$; and if Φ is a nonempty finite set of formulae in $\boldsymbol{L}_\alpha$, then $(\wedge\Phi)$ is a formula in $\boldsymbol{L}_\alpha$;

Iα2 (infinitary extension): if $\wedge\langle C^\nu : \nu \geq 0\rangle, \wedge\langle D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ and $A \in \boldsymbol{L}_\alpha$, then

- *(i)* $\wedge\langle A \supset C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$;
- *(ii)* $\wedge\langle \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ for all $i \in N$;
- *(iii)* $\wedge\langle\wedge\{C^\nu, D^\nu\} : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.

Like Iα1, Iα2 requires α-permissible sequences to be closed under the logical connectives and the belief operators. Iα2 is interactive with Iα1 since formulae generated by Iα2 may be used in Iα1, e.g., $A \in \boldsymbol{L}_\alpha$ may already include infinitary conjunctions in Iα2.(i). When $\alpha = 0$, step Iα2 is vacuous since $\mathcal{F}_0 = \emptyset$; that is, $\boldsymbol{L}_0$ is the set of all finitary formulae. For $\alpha > 0$, we specify $\mathcal{F}_\alpha$ in the following. Incidentally, our completeness result can be obtained without **I**α2.(iii); but we add it to discuss one faithful embedding result in Section 4.2.

We write $A \wedge B$, $A \wedge B \wedge C$ for $\wedge\{A, B\}$ and $\wedge\{A, B, C\}$, etc., and $A \equiv B$ for $(A \supset B) \wedge (B \supset A)$. We may abbreviate the parentheses (,) and use different brackets when they cause no confusions.

Let $p \in \mathcal{P}_0$. For a formula G in $\boldsymbol{L}_0$, we use the expression $G(p)$ to denote the specifications of all occurrences of p in G with the condition that G contains no other propositional variables. We say that a formula $G \in L_0$ with (3) is a *germinal form*:

$$G \text{ contains at least one logical connective or } \mathbf{B}_i(\cdot) \text{ for some } i \in N. \tag{3}$$

Let $\mathcal{G}$ be a *nonempty* set of germinal forms. This $\mathcal{G}$ is taken as granted in our formula-generation process. A candidate for $\mathcal{G}$ is the set of all germinal forms $\mathcal{G}^{All}$. This is a recursive set.

Now, let α be an ordinal with $0 < \alpha \leq \alpha^o$, and suppose that $\boldsymbol{L}_\beta$ is defined for all $\beta < \alpha$. We take a formula C in $\mathcal{P}_\alpha = \cup_{\beta<\alpha} \boldsymbol{L}_\beta$, which we call a *germ*. Then, an *α-permissible sequence,* $\langle C^\nu : \nu \geq 0 \rangle$, is generated from C and $G(p) \in \mathcal{G}$ by Rα0 and Rα1 :

Rα0: $C^0 = C$;

Rα1: $C^{\nu+1} = G(C^\nu)$ for all $\nu \geq 0$.

This generates recursively the sequence $\langle C^\nu : \nu \geq 0 \rangle$ from the initial condition $C^0 = C$ and by substituting C^ν for all occurrences of p in $G(p)$ for each $\nu \geq 0$.

For any α-permissible sequence $\langle C^\nu : \nu \geq 0 \rangle$, we allow the expression $\wedge \langle C^\nu : \nu \geq 0 \rangle$, which is the infinitary conjunction of all formulae C^ν's. We define $\mathcal{F}_\alpha = \{\wedge \langle C^\nu : \nu \geq 0 \rangle : \langle C^\nu : \nu \geq 0 \rangle$ is an α-permissible sequence$\}$. Note that $\mathcal{F}_\alpha$ is a countable set for any given set of germinal forms $\mathcal{G}$. When $\mathcal{G}$ is a recursive set, $\mathcal{F}_\alpha$ and L_α are recursively enumerable.

In addition to the sequences generated by Rα0 and Rα1, we also call the sequences generated by Iα2 *α-permissible sequences*. Thus, $\langle C^\nu : \nu \geq 0 \rangle$ is an α-permissible sequence if and only if $\wedge \langle C^\nu : \nu \geq 0 \rangle \in \boldsymbol{L}_\alpha$. We call $\wedge \langle C^\nu : \nu \geq 0 \rangle \in \boldsymbol{L}_\alpha$ simply an *α-infinite conjunction.*

We may denote $\langle C^\nu : \nu \geq 0 \rangle$ by Φ. We stipulate that $A \in \langle C^\nu : \nu \geq 0 \rangle$ iff $A \in \{C^\nu : \nu \geq 0\}$. When no confusion occurs, we use the same expression, $\wedge \Phi$, for a finite conjunction or an infinite conjunction. We also stipulate that the syntactical identity of $\langle C^\nu : \nu \geq 0 \rangle$ as an infinite sequence $\langle C^0, C^1, ... \rangle$. We write $\mathbf{B}_i(\Phi)$ for $\langle \mathbf{B}_i(C) : C \in \Phi \rangle$ if Φ is an infinite α-permissible sequence or $\{\mathbf{B}_i(C) : C \in \Phi\}$ if Φ is a finite set of formulae in $\boldsymbol{L}_\alpha$.

The subformulae of $A \in \boldsymbol{L}_\alpha$ are defined in the standard manner. Then, $\boldsymbol{L}_\alpha$ is subformula-closed.

Lemma 2.1. *Let $A \in \boldsymbol{L}_\alpha$. Any subformula of A belongs to $\boldsymbol{L}_\alpha$.*

Proof. Since $\mathcal{F}_0 = \emptyset$, **I**$\alpha$2 is vacuous; $\boldsymbol{L}_0$ is subformula-closed. Consider any $\alpha \leq \alpha^o$. Suppose that $\boldsymbol{L}_\beta$ is subformula closed for all $\beta < \alpha$. We prove by induction along **I**α1 - Iα2 that if $A \in \boldsymbol{L}_\alpha$ and C is an immediate subformula of A, then $C \in \boldsymbol{L}_\alpha$. Let $\langle C^\nu : \nu \geq 0 \rangle$ be defined by Rα0 - Rα1 with a germ $C \in \mathcal{P}_\alpha = \cup_{\beta<\alpha} \boldsymbol{L}_\beta$ and a germinal form $G \in \mathcal{G}$. Each C^ν is still in $\mathcal{P}_\alpha \subseteq \boldsymbol{L}_\alpha$.

Now, let A be defined by Iα1 or Iα2. In the case of Iα1, by definition, any subformula of A belongs to $\boldsymbol{L}_\alpha$. Consider the case of Iα2.(ii). Let $\langle D \supset C^\nu : \nu \geq 0\rangle$ be constructed by Iα2.(ii) from $D \in \boldsymbol{L}_\alpha$ and $\langle C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By the induction hypothesis, for each $\nu \geq 0$, we have $C^\nu \in \boldsymbol{L}_\alpha$; so, $D \supset C^\nu$ belongs to $\boldsymbol{L}_\alpha$ by Iα1. The cases of Iα2.(i) and Iα2.(iii) are similar.■

The set of formulae $\boldsymbol{L}_\alpha$ is increasing up to $\alpha = \omega$, but it becomes constant after $\alpha = \omega$. Thus, it suffices to consider the sets of formulae $\boldsymbol{L}_\alpha$ for $\alpha \leq \omega$.

Theorem 2.1. ***(Stopping at*** ω*): Let $\mathcal{G}$ be a fixed set of germinal forms. If $\alpha < \omega$, then $\boldsymbol{L}_\alpha \subsetneq \boldsymbol{L}_{\alpha+1}$; and if $\omega < \alpha \leq \alpha^o$, then $\boldsymbol{L}_\alpha = \boldsymbol{L}_\omega = \mathcal{P}_\omega$ $(= \cup_{\beta<\omega}\boldsymbol{L}_\beta)$.*

Proof. It holds that $\boldsymbol{L}_0 \subsetneq \boldsymbol{L}_1$. Let $1 \leq \alpha < \omega$. Suppose $\boldsymbol{L}_{\alpha-1} \subsetneq \boldsymbol{L}_\alpha$. Take an $A \in \boldsymbol{L}_\alpha - \boldsymbol{L}_{\alpha-1}$. Then, by (3), it follows from R$(\alpha+1)$1 that for an $(\alpha+1)$-recursive sequence $\langle C^\nu : \nu \geq 0\rangle$ generated from a germinal form (G^0, G^1) and germ A, the $(\alpha+1)$-infinitary conjunction $\wedge\langle C^\nu : \nu \geq 0\rangle$ in $\mathcal{F}_{\alpha+1}$ does not belong $\boldsymbol{L}_\alpha$. Hence, $\boldsymbol{L}_\alpha \subsetneq \boldsymbol{L}_{\alpha+1}$.

Consider the latter assertion. First, by Iω0-Iω2, $\mathcal{P}_\omega \subseteq \boldsymbol{L}_\omega \subseteq \boldsymbol{L}_\alpha$. No we show $\boldsymbol{L}_\omega \subseteq \mathcal{P}_\omega$. Take a germ C from $\mathcal{P}_\omega$, which belongs to $\boldsymbol{L}_\gamma$ for some $\gamma < \omega$. Since a germinal form G is a formula in $\boldsymbol{L}_0$, the conjunction of $\wedge\langle C^\nu : \nu \geq 0\rangle$ generated from C and G belongs to $\mathcal{F}_{\gamma+1}$. Hence, any formulae generated by Iω0- Iω2 belong to $\boldsymbol{L}_\alpha$ for some $\alpha < \omega$. Thus, $\boldsymbol{L}_\omega \subseteq \mathcal{P}_\omega = \cup_{\beta<\omega}\boldsymbol{L}_\beta$. Now, by induction over α up to α^o, we have $\mathcal{P}_\omega = \boldsymbol{L}_\omega = \boldsymbol{L}_\alpha$ for all α $(\omega \leq \alpha \leq \alpha^o)$.■

Here, we give two examples of germinal forms and infinitary formulae.

Example 2.1 (1) Positive introspection: Let $i \in N$ and $G_i^{PI} = \mathbf{B}_i(p) \in \mathcal{G}$. The sequence generated by Rα0-Rα1 from G_i^{PI} and $A' \in \mathcal{P}_\alpha$ is given as

$$\mathbf{B}_i^0(A') = A'; \text{and } \mathbf{B}_i^{\nu+1}(A') = \mathbf{B}_i(\mathbf{B}_i^\nu(A')) \text{ for } \nu \geq 0. \tag{4}$$

Then, $\langle \mathbf{B}_i^\nu(A') : \nu \geq 0\rangle$ is an α-permissible sequence, and we define $\mathbf{B}_i^\omega(A') := \wedge\langle \mathbf{B}_i^\nu(A') : \nu \geq 0\rangle$, which belongs to $\boldsymbol{L}_\alpha$ by Iα0. We consider the formula $\mathbf{B}_i^\omega\mathbf{B}_i(A)$ by letting $A' = \mathbf{B}_i(A)$. We will see in Section 4 that the formula $\mathbf{B}_i^\omega\mathbf{B}_i(A)$ is regarded as the extension of $\mathbf{B}_i(A)$ by requiring Axiom 4 in GL$(\boldsymbol{L}_\omega)$.

(2) Common knowledge: Let S be a nonempty set of $N = \{1, ..., n\}$. The *common knowledge* of a formula A among S is formulated with the germ A and the germinal form $G_S(p) = \wedge_{j\in S}\mathbf{B}_j(p))$ by

$$C_S^0(A) = A; \text{and } C_S^{\nu+1}(A) = \wedge_{i\in N}\mathbf{B}_j(C_S^\nu(A)) \text{ for all } \nu \geq 0. \tag{5}$$

Then, $C_S(A) := \wedge\langle C_S^\nu(A) : \nu \geq 0\rangle$. For notational clarity, we may denote $C_S^\nu(A)$ by $\mathbf{B}_S^\nu(A)$, and thus $\wedge_{i\in S}\mathbf{B}_i(C_S(A)) = \wedge\langle \mathbf{B}_S^\nu(A) : \nu \geq 0\rangle$.

The common belief of A among S using the common knowledge by $\wedge_{i\in S}\mathbf{B}_i(C_S(A))$ as $C_{BS}(A) := \wedge_{i\in S}\mathbf{B}_i(C_S(A))$. It is also defined by germ $\wedge_{i\in S}\mathbf{B}_i(A)$ and germinal form $\wedge_{i\in S}\mathbf{B}_i(p)$. In our logic, they are equivalent.

The depths of formulae in $\boldsymbol{L}_\alpha$ are relevant to the claim that our logics are "small" relative to the literature of infinitary logics. We introduce the *depth function* δ over $\boldsymbol{L}_\omega$, which assigns an ordinal number to each formula in $\boldsymbol{L}_\omega$. We define δ inductively as follows:

d0: $\delta(p) = 0$ for all propositional variables p;

d1: $\delta(\neg A) = \delta(A) + 1$, and $\delta(A \supset B) = \max(\delta(A), \delta(B)) + 1$;

d2: $\delta(\mathbf{B}_i(A)) = \delta(A) + 1$ for all $i \in N$;

d3: $\delta(\wedge\Phi) = \sup\{\delta(A) : A \in \Phi\} + 1$ for any $\wedge\Phi \in \boldsymbol{L}_\omega$.

In d3, Φ may be a finite set of formulae or an α-permissible sequence. For any set of formulae Γ, we define $\delta(\Gamma) = \sup\{\delta(A) : A \in \Gamma\}$. It follows from d0-d3 that for any $A \in \boldsymbol{L}_\omega$,

$$\delta(C) < \delta(A) \text{ for any proper subformula } C \text{ of } A. \tag{6}$$

Since $\boldsymbol{L}_0$ consists only of finitary formulae, we have $\delta(\boldsymbol{L}_0) = \sup\{\delta(A) : A \in \boldsymbol{L}_0\} = \omega$.

Consider the formula $\mathbf{B}_i^\omega(p) = \wedge\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$ in Example 2.1.(1). The transition from $\boldsymbol{L}_0$ to $\boldsymbol{L}_1$ generates the infinitary conjunction $\mathbf{B}_i^\omega(p) = \wedge\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$, and thus $\delta(\mathbf{B}_i^\omega(A))$ $= \omega + 1$. The depth of any formula D in $\boldsymbol{L}_1$ including $\mathbf{B}_i^\omega(A)$ takes the form $\omega + k$ for some finite k. In general, we have $\delta(D) < \omega + \omega = \omega 2$ and $\delta(\boldsymbol{L}_1) = \omega 2$. The following theorem generalizes this observation.

Theorem 2.2. ***(Depths of formulae).***
(1): *If $0 \leq \alpha < \omega$, then $\delta(A) < \omega(\alpha+1)$ for all $A \in \boldsymbol{L}_\alpha$; and $\delta(\boldsymbol{L}_\alpha) = \omega(\alpha+1)$.*
(2): *$\delta(A) < \omega^2$ for all $A \in \boldsymbol{L}_\omega$; and $\delta(\boldsymbol{L}_\omega) = \omega^2$.*

Proof. (1): For $\alpha = 0$, we have $\delta(A) < \omega$ for all $A \in \boldsymbol{L}_0$ and $\delta(\boldsymbol{L}_0) = \omega$. Now, let $1 \leq \alpha < \omega$, and suppose the induction hypothesis that $\delta(A) < \omega\alpha$ for all $A \in \boldsymbol{L}_{\alpha-1}$ and $\delta(\boldsymbol{L}_{\alpha-1}) = \omega\alpha$. Then, we prove the assertions for α. First, we show that $\delta(A) < \omega(\alpha+1)$ for all $A \in \boldsymbol{L}_\alpha$.

Let $\wedge\Phi \in \mathcal{F}_\alpha$. Since $\delta(A) < \omega\alpha$ for all $A \in \Phi$ by the induction hypothesis, we have $\delta(\wedge\Phi) \leq \omega\alpha + 1$ by d3. Hence, for any $A \in \mathcal{P}_\alpha \cup \mathcal{F}_\alpha$, we have $\delta(A) \leq \omega\alpha + 1$.

Now, consider Iα1. Suppose the other induction hypothesis that for any immediate subformula C of A generated by Iα1, $\delta(C) \leq \omega\alpha + k$ for some $k < \omega$. Then, by d1-d3, we have $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$.

Consider Iα2. The induction hypothesis is that $\delta(D) \leq \omega\alpha + k$ and $\delta(\wedge\Phi) \leq \omega\alpha + k$ for some $k < \omega$. Then, $\delta(D \supset C) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\wedge\{D \supset C : C \in \Phi\}) \leq ((\omega\alpha + k) + 1) + 1$. Also, $\delta(\mathbf{B}_i(C)) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\wedge\langle\delta(\mathbf{B}_i(C)) : C \in \Phi\rangle) \leq ((\omega\alpha + k) + 1) + 1$. Thus, for a formula A generated by Iα2, it still holds that $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$. By these two paragraphs and induction, it holds that $\delta(A) < \omega(\alpha+1)$ for all $A \in \boldsymbol{L}_\alpha$.

For $\delta(\boldsymbol{L}_\alpha) = \omega(\alpha+1)$, we show that for any $k < \omega$, there is a formula $C \in L_\alpha$ so that $\delta(C) \geq \omega\alpha + k$. This implies $\delta(L_\alpha) \geq \sup_k(\omega\alpha + k) = \omega\alpha + \omega = \omega(\alpha+1)$. By the conclusion of the previous paragraph, we have $\delta(L_\alpha) = \omega(\alpha+1)$. Now, the induction hypothesis is that there is a formula $A \in L_{\alpha-1}$ such that $\delta(A) \geq \omega(\alpha-1)$. This A as a germ and some germinal form G generates an α-permissible sequence $\langle C^\nu : \nu \geq 0\rangle$. By (3), it holds that for any $k < \omega$, there is a ν with $\delta(C^\nu) \geq \omega(\alpha-1) + k$. Hence, $\delta(\langle C^\nu : \nu \geq 0\rangle) = (\omega(\alpha-1) + \omega) + 1 = \omega\alpha + 1$. Then, using I$\alpha$1, for any $k < \omega$, we find a formula $C \in L_\alpha$ so that $\delta(C) \geq \omega\alpha + k$.

The first part of (2) follows (1), since $\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha$ by Theorem 2.1. The second part follows $\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha$ and (1); indeed, $\delta(\boldsymbol{L}_\omega) = \delta(\cup_{\alpha<\omega}\boldsymbol{L}_\alpha) = \sup\{\delta(\boldsymbol{L}_\alpha) : \alpha \geq 0\} = \sup\{\omega(\alpha+1) : \omega > \alpha \geq 0\} = \omega^2$.■

Our infinitary languages $\boldsymbol{L}_\alpha$ $(1 \leq \alpha \leq \omega)$ include infinitary conjunctions, but are not much larger than the finitary language $\boldsymbol{L}_0$. This fact can be seen in Table 2.1 and relies upon the restriction that infinitary conjunctions are constructed by Rα0-Rα1 and Iα2. Each $\boldsymbol{L}_\alpha$ $(\alpha < \omega)$ is closed only with Iα2, but not with respect to Rα0-Rα1, and $\boldsymbol{L}_\omega$ is closed with respect to these operations. These are not substantive restrictions as long as concrete infinitary concepts such as common knowledge are concerned.

Table 2.1: Depths and cardinalities

Rα	$\boldsymbol{L}_0$	$\subsetneq$	$\boldsymbol{L}_1$	$\subsetneq \cdot \quad \cdot \quad \cdot \subsetneq$	$\boldsymbol{L}_\omega$
#	$\aleph_0$		$\aleph_0$		$\aleph_0$
depth	ω	$<$	$\omega 2$	$<$	ω^2

Instead of **R**α0-**R**α1, Kaneko-Nagashima [11] assume:

KNα : $\mathcal{F}_\alpha = \{\wedge\Phi : \Phi$ is a countable subset of $\cup_{\beta<\alpha}\mathcal{L}_\beta\}$,

and Iα0-Iα1. We denote the set of formulae for step α by $\mathcal{L}_\alpha$ with $\mathcal{L}_0 = \boldsymbol{L}_0$. The set $\mathcal{L}_1$ is already uncountable. Also, $\mathcal{L}_\alpha$ does not stop at $\alpha = \omega$, e.g., $\cup_{\beta<\alpha}\mathcal{L}_\beta \subsetneq \mathcal{L}_\omega \subsetneq \mathcal{L}_{\omega+1}$ for all $\alpha \leq \omega$. In terms of depth δ, $\delta(\cup_{\alpha<\omega}\mathcal{L}_\alpha) = \omega^2$ but $\delta(\mathcal{L}_\omega) = \omega^2 + \omega$. This sequence $\mathcal{L}_\beta$ increases up to ω_1, where ω_1 is the first uncountable ordinal (here we assume $\alpha^o \geq \omega_1$). Tanaka-Ono [23] considered the smallest set, $\mathcal{L}^{TO}$, that is closed with respect to finitary operations on $\neg$, $\supset$, $\mathbf{B}_i(\cdot)$ and countable conjunctions:

TO: for any countable subset Φ of $\mathcal{L}^{TO}$, $\wedge\Phi$ belongs to $\mathcal{L}^{TO}$.

Then, it holds that $\mathcal{L}^{TO} = \cup_{\beta<\omega_1}\mathcal{L}_\beta$. This $\mathcal{L}^{TO}$ is the *smallest* infinitary language in the sense of Karp [10].

Remark 2.1 (Germinal forms of multi-variables): In the above development, we consider the case where a germinal form is given as one formula. For some infinitary concepts, we need their germinal forms to have multiple formulae. The main results in Section 3 can be extended to such a case. An example is an *epistemic infinitary regress*. Here, we illustrate this for case $n = 2$. A germinal form is given as $(\mathbf{B}_1(p_2), \mathbf{B}_2(p_1))$. Also, let $\mathbf{B}_1(A_1), \mathbf{B}_2(A_2) \in \mathcal{P}_\alpha = \cup_{\beta<\alpha}\boldsymbol{L}_\beta$. Then, the sequences $\langle \mathbf{Ir}_i^\nu[A_1, A_2] : \nu \geq 0\rangle$, $i = 1, 2$, are generated by Rα0-Rα1 as; for $i, j = 1, 2$ $(i \neq j)$,

$$\mathbf{Ir}_i^0[A_1, A_2] = \mathbf{B}_i(A_i); \text{ and } \mathbf{Ir}_i^{\nu+1}[A_1, A_2] = \mathbf{B}_i(\mathbf{Ir}_j^\nu[A_1, A_2]) \text{ for } \nu \geq 0, .$$

We write the conjunction $\mathbf{Ir}_i[A_1, A_2] := \wedge\langle \mathbf{Ir}_i^\nu[A_1, A_2] : \nu \geq 0\rangle$ for $i = 1, 2$.

Epistemic infinitary regress is a subjective concept in that each formula for i occurs in the scope of $\mathbf{B}_i(\cdot)$, and is an extension of common belief. If A_1 and A_2 are the same, then $\mathbf{Ir}_1[A_1, A_2] \wedge \mathbf{Ir}_2[A_1, A_2]$ is equivalent to the common belief of $A_1 = A_2$. If we take individuality and subjectivity more seriously than common knowledge, we would need this kind of multi-formulae germinal forms. These make treatments more complicated, but as long as the results given in the present paper are concerned, no essential amendments are required.

3 Epistemic Logics GL($\boldsymbol{L}_\alpha$) $(0 \leq \alpha \leq \omega)$

We formulate a Hilbert-style proof system and Kripke-semantics for epistemic logic GL($\boldsymbol{L}_\alpha$) = GL($\boldsymbol{L}_\alpha(\mathcal{G})$) ($0 \leq \alpha \leq \omega$ and a set of germinal forms $\mathcal{G}$). Then, we state the soundness-completeness theorem (Theorem 3.1), which will be proved in Section 5. We discuss the hierarchy of GL($\boldsymbol{L}_\alpha(\mathcal{G})$) with respect to both α and $\mathcal{G}$.

3.1 Hilbert-style proof theory

The base logic for epistemic logic GL($\boldsymbol{L}_\alpha$) (= GL($\boldsymbol{L}_\alpha(\mathcal{G})$) and $0 \leq \alpha \leq \omega$) is an infinitary classical logic defined by the following four axiom schemata and two inference rules: for all formulae $A, B, C, \wedge\Phi$ in $\boldsymbol{L}_\alpha$,

L1: $A \supset (B \supset A)$;

L2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

L3: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

L4: $\wedge\Phi \supset C$, where $C \in \Phi$;

Modus Ponens: $\dfrac{A \supset B \quad A}{B}$; and **$\wedge$-rule:** $\dfrac{\{A \supset C : C \in \Phi\}}{A \supset \wedge\Phi}$.

We add the following: for any $A, C, \wedge\Phi$ in $\boldsymbol{L}_\alpha$, and $i \in N$,

K: $\mathbf{B}_i(A \supset C) \supset (\mathbf{B}_i(A) \supset \mathbf{B}_i(C))$;

D: $\neg\mathbf{B}_i(\neg A \wedge A)$;

$\wedge$-Barcan: $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$;

Necessitation: $\dfrac{A}{\mathbf{B}_i(A)}$.

The above axiomatization is an infinitary version of epistemic logic KD^n with the $\wedge$-Barcan axiom. Indeed, infinitary aspects are included in L4, $\wedge$-rule and $\wedge$-Barcan, while the other axioms and inference rules do not directly operate on infinitary structures. The construction of $\boldsymbol{L}_\alpha$ guarantees the well-definedness of L4, $\wedge$-rule, and $\wedge$-Barcan: since $\wedge\Phi$ is from $\boldsymbol{L}_\alpha$, the instance $\wedge\Phi \supset C$ is also in $\boldsymbol{L}_\alpha$ for all $C \in \Phi$ by Lemma 2.1 and Iα1. The sequence $\langle A \supset C : C \in \Phi\rangle$ of upper formulae in $\wedge$-rule is α-permissible by Iα2.(i). Since $\mathbf{B}_i(\wedge\Phi) \in \boldsymbol{L}_\alpha$ by Iα1 and $\wedge\mathbf{B}_i(\Phi) \in \boldsymbol{L}_\alpha$ by Iα2.(ii), the formula $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$ of $\wedge$-Barcan is in $\boldsymbol{L}_\alpha$. Incidentally, if Φ is a *finite* set, $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$ is derived without using $\wedge$-Barcan.

Logic GL($\boldsymbol{L}_0$) is the same as the finitary KD^n. GL($\boldsymbol{L}_1$) extends it by allowing infinite conjunctions without nesting, GL($\boldsymbol{L}_\alpha$) extends it by allowing infinite conjunctions with nested depths up to α, and GL($\boldsymbol{L}_\omega$) allows arbitrary but finite nested depths.

A *proof* $P = \langle X, <; f\rangle$ in GL($\boldsymbol{L}_\alpha$) consists of a countable tree $\langle X, <\rangle$ and a function $f : X \to \boldsymbol{L}_\alpha$ with the following requirements:

(o): $\langle X, <\rangle$ has no infinite path from its root;[3]

(i): for each leaf x in $\langle X, <\rangle$, $f(x)$ is an instance of the axiom schemata;

(ii): for each node x in $\langle X, <\rangle$, $f(x)$ is a formula attached to x;

(iii): for each non-leaf x in $\langle X, <\rangle$,

$$\frac{\{f(y) : y \text{ is an immediate successor of } x\}}{f(x)}$$

is an instance of the inference rules, MP, $\wedge$-rule, or Nec.[4]

When A is attached to the root node of $P = \langle X, <; f\rangle$, we call P a *proof of* A. We say that A is *provable*, denoted by GL($\boldsymbol{L}_\alpha$) $\vdash A$, iff there is a proof of A in GL($\boldsymbol{L}_\alpha$). When the reference to GL($\boldsymbol{L}_\alpha$) is clear, we simply write $\vdash A$. For a (possibly infinite) set Γ of formulae, we write $\Gamma \vdash A$ iff $\vdash A$ or $\vdash \wedge\Phi \supset A$ for some finite subset Φ of Γ.

The following lemma states basic properties of the provability relation $\vdash$ in GL($\boldsymbol{L}_\alpha$). Since we adopt a particular axiomatization of classical logic, these need to be proved; (2) is a crucial step (Kaneko [9], Lemma 11.1). (3) is the converse of the $\wedge$-Barcan.

[4]We may formulate a proof in terms of a sequence ordered with ordinal numbers.

Lemma 3.1. *For any $A, B, C, \wedge\Phi \in \boldsymbol{L}_\alpha$, and $i \in N$,*
(1): $\vdash A \supset B$ *and* $\vdash B \supset C$ *imply* $\vdash A \supset C$;
(2): $\vdash [A \wedge B \supset C] \equiv A \supset (B \supset C)$; ***(3):*** $\vdash \mathbf{B}_i(\wedge\Phi) \supset \wedge\mathbf{B}_i(\Phi)$.

Remark 3.1. We can take the standard de Morgan method to define the disjunction formula as $\vee\Phi := \neg \wedge \{\neg A : A \in \Phi\}$ for a finite set of formulae Φ. For an α-permissible sequence Φ, this would work when we extend Iα2 to include $\langle \neg A : A \in \Phi \rangle$ for any $\wedge\Phi \in L_\alpha$.

Remark 3.2. The substitution-rule is regarded as basic (cf., Kleene [13], Chap.3). In our logic GL($\boldsymbol{L}_\alpha$), it holds that for any $A[p]$ and B in $\boldsymbol{L}_\alpha$,

$$\text{if} \vdash A[p] \text{ and } A[B] \in \boldsymbol{L}_\alpha, \text{ then } \vdash A[B], \tag{7}$$

where $A[p]$ is a formula in $\boldsymbol{L}_\alpha$ and $A[B]$ is the formula obtained from $A[p]$ by substituting B for all occurrences of p. However, the additional "$A[B] \in \boldsymbol{L}_\alpha$" holds for any α ($0 \leq \alpha \leq \omega$) if at least one of $A[p]$ and B is taken from L_0. This fact will be used later, e.g., in Lemma 4.2.

3.2 Kripke completeness and conservativity

A *Kripke frame* $\mathbb{K} = \langle W; R_1, ..., R_n \rangle$ is an $(n+1)$-tuple of a set of possible worlds and n accessibility relations over W, where W is an arbitrary nonempty set and R_i is a *serial* binary relation over W for each $i \in N$, i.e., for any $w \in W$, $(w, u) \in R_i$ for some $u \in W$. A *truth assignment* τ is a function from $W \times \mathcal{P}_0$ to $\{\top, \bot\}$. A pair $(\mathbb{K}, \tau)$ is a *Kripke model.*

The valuation $(\mathbb{K}, \tau, w) \models$ for $w \in W$ is defined over $\boldsymbol{L}_\alpha$ as follows: for any A, C, $\wedge\Phi \in \boldsymbol{L}_\alpha$,
V0: for any $p \in \mathcal{P}_0$, $(\mathbb{K}, w, \tau) \models p \Longleftrightarrow \tau(w, p) = \top$;
V1: $(\mathbb{K}, \tau, w) \models \neg A \Longleftrightarrow (\mathbb{K}, \tau, w) \not\models A$;
V2: $(\mathbb{K}, \tau, w) \models A \supset C \Longleftrightarrow (\mathbb{K}, \tau, w) \not\models A$ or $(\mathbb{K}, \tau, w) \models C$;
V3: $(\mathbb{K}, \tau, w) \models \wedge\Phi \Longleftrightarrow (\mathbb{K}, \tau, w) \models A$ for all $A \in \Phi$;
V4: $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(A) \Longleftrightarrow (\mathbb{K}, \tau, v) \models A$ for all v with wR_iv.

The well-definedness of the valuation $(K, \tau, w) \models$ depends on the subformula-closedness of L_α, which is guaranteed by Lemma 2.1. Since $\boldsymbol{L}_\alpha \subseteq \boldsymbol{L}_\omega$ ($\alpha \leq \omega$), the valuation $(\mathbb{K}, \tau, w) \models$ is uniform over $\boldsymbol{L}_\alpha$ for all $\alpha \leq \omega$; that is, it is defined over $\boldsymbol{L}_\omega$ and it can be restricted to $\boldsymbol{L}_\alpha$. For any $A \in \boldsymbol{L}_\alpha$, we write $\models A$ iff $(\mathbb{K}, \tau, w) \models A$ for all $\mathbb{K}$, $w \in W$ and τ.

We have the following soundness-completeness theorem; the proof of soundness is standard and mentioned below, and completeness will be proved in Section 5. The set of germinal forms $\mathcal{G}$ is still arbitrarily given, and we do not write it explicitly. Only when we treat $\mathcal{G}$ as parameters, we refer to it as GL($\boldsymbol{L}_\alpha(\mathcal{G})$).

Theorem 3.1. ***(Soundness and completeness for GL($\boldsymbol{L}_\alpha$))*** *Let α be an ordinal with $0 \leq \alpha \leq \omega$. For any $A \in \boldsymbol{L}_\alpha$, GL($\boldsymbol{L}_\alpha$) $\vdash A$ if and only if $\models A$.*

Soundness (the only-if part) implies the contradiction-freeness of logic GL($\boldsymbol{L}_\alpha$), which will be used in the proof of completeness.

Soundness is proved as follows: Let $P = \langle X, <; f \rangle$ be a proof of A in GL($\boldsymbol{L}_\alpha$). We prove by induction on the tree structure of P from its leaves that $\models C$ for each formula $C = f(x)$ attached to a node x of P. The proof is standard and is omitted.

Lemma 3.2. *(1): Let A be an instance of L1-L4 in $\boldsymbol{L}_\alpha$. Then $\models A$.*
(2): Let A be an instance of Axioms K, D, $\wedge$-Barcan in $\boldsymbol{L}_\alpha$. Then $\models A$.
(3): $\models$ satisfies inference rules MP, $\wedge$-rule, and Necessitation.

We only prove the truthfulness of the Barcan axiom. Let $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(\wedge\Phi)$, where Φ is an α-permissible sequence. Then, for any accessible $v \in W$ by R_i, it holds that $(\mathbb{K}, \tau, v) \models \wedge\Phi$, equivalently, $(\mathbb{K}, \tau, v) \models C$ for any $C \in \Phi$. This implies $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(C)$ for any $C \in \Phi$. Since $\wedge\mathbf{B}_i(\Phi)$ belongs to $\boldsymbol{L}_\alpha$ by Iα2.(ii), it follows from V3 that $(\mathbb{K}, \tau, w) \models \wedge\mathbf{B}_i(\Phi)$. Thus, $\mathbf{B}_i(\wedge\Phi) \supset \wedge\mathbf{B}_i(\Phi)$ is true.

To prove completeness, a difficulty with an infinitary logic is to show the existence of a maximal consistent set. For this purpose, Karp [10] assumes Axiom of Choice within her axiomatic system. However, we neither adopt Karp's method nor show the existence of a maximal consistent set. Instead, we adopt the Q-filter method due to Rasiowa-Sikorski [19] and the multi-modal extension given by Tanaka-Ono [23]. Here, a Q-filter plays the role of a maximal consistent set. The first two conditions in Iα2 correspond to the required conditions to utilize that method in the multi-modal extension. More details will come in Section 5.

We state a conservativity result as a corollary of Theorem 3.1. Recall that $\mathcal{G}^{All}$ is the set of all germinal forms.

Theorem 3.2. *(Conservativity) Let $\alpha \leq \omega$ and $\mathcal{G}$ any set of germinal forms. Then, for any $A \in \boldsymbol{L}_\alpha(\mathcal{G})$, $GL(\boldsymbol{L}_\alpha(\mathcal{G})) \vdash A$ if and only if $GL(\boldsymbol{L}_\omega(\mathcal{G}^{All})) \vdash A$.*

Proof. The *if* part is essential. Let $\mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G}^{All})) \vdash A$. Let $(\mathbb{K}, \tau)$ be any serial Kripke model, and w any world in $\mathbb{K}$. By Theorem 3.1, we have $(\mathbb{K}, \tau, w) \models A$. Because of subformula-closedness (Lemma 2.1) and the definition V0-V4 for $(\mathbb{K}, \tau, w) \models$, the statement $(\mathbb{K}, \tau, w) \models A$ is determined in $\boldsymbol{L}_\alpha(\mathcal{G})$. Since this holds for any $\mathbb{K}, \tau, w \in W$, we have $\mathrm{GL}(\boldsymbol{L}_\alpha(\mathcal{G})) \vdash A$ by Theorem 3.1.■

Table 3.1 Hierarchy of infinitary epistemic logics

$$\begin{array}{ccccccc}
\mathrm{GL}(\boldsymbol{L}_0) & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) & \rightrightarrows & \cdots & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G})) \\
 & & \downdownarrows & & & & \downdownarrows \\
 & & \cdots & & & & \cdots \\
 & & \downdownarrows & & & & \downdownarrows \\
 & & \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}^{All})) & \rightrightarrows & \cdots & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G}^{All}))
\end{array}$$

It follows from Theorem 3.2 that for any β ($\beta \leq \alpha \leq \omega$) and $\mathcal{G}' \subseteq \mathcal{G}$, $\mathrm{GL}(\boldsymbol{L}_\alpha(\mathcal{G}))$ is a conservative extension of $\mathrm{GL}(\boldsymbol{L}_\beta(\mathcal{G}'))$. Thus, we can describe the hierarchy of epistemic logics as in Table 3.1: Each row is a series of logics with the same $\mathcal{G}$, each column is a series with the same α, and $\rightrightarrows$ means that the latter is a conservative extension of the former. The weakest logic is $\mathrm{GL}(\boldsymbol{L}_0) = \mathrm{KD}^n$ and the strongest is $\mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G}^{All}))$.

Our statement in Theorem 3.2 uses fixed α and $\mathcal{G}$ but allow A to vary within $\boldsymbol{L}_\alpha(\mathcal{G})$. Alternatively, for a fixed $\mathcal{G}$, and for each fixed $A \in \boldsymbol{L}_\omega(\mathcal{G})$, we can find the smallest $\alpha_A < \omega$ such that $A \in \boldsymbol{L}_{\alpha_A}(\mathcal{G})$; then it follows from Theorem 3.2 that $\mathrm{GL}(\boldsymbol{L}_{\alpha_A}(\mathcal{G})) \vdash A \Longleftrightarrow \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G})) \vdash A$.

Another comment on Theorem 3.1 is: the completeness result remains if we add Axioms T, 4, and 5 either in combination or in isolation and if we add the corresponding conditions, reflexivity, transitivity, and euclidean on accessibility relation R_i ($i \in N$). Required modifications of the proof of the corresponding theorems will be stated in Remark 5.1.

We have adopted a KD-system, to evaluate the definability of each of those axiom in our GL($\boldsymbol{L}_\alpha$) along the above hierarchy, which will be given in Section 4. For this purpose, we give three meta-results; two are known (cf., Kaneko-Suzuki [12]) and the last one is new.

Lemma 3.3 is proved in GL($\boldsymbol{L}_0$) $=$ KD^n in [12], and holds in GL($\boldsymbol{L}_\alpha$) by Theorem 3.2. Recall the depth measure δ given in Section 2.[5]

Lemma 3.3. *(Depth lemma) Let A and C be two formulae in $\mathbf{L}_0$. Let $(i_1, ..., i_k)$ be a sequence of agents in N and $\delta(A) < k$. In GL($\mathbf{L}_\alpha$), if $\vdash A \supset \mathbf{B}_{i_1}...\mathbf{B}_{i_k}(C)$, then $\vdash \neg A$ or $\vdash C$.*

The second result is an extension of the *epistemic disjunction lemma* for KD^n. The following lemma is stated in GL(L_α), but can be proved in the same manner as in [12], i.e., by constructing a counter-model based upon Theorem 3.1. Recall Remark 3.1 about disjunction $\vee$.

Lemma 3.4. *(Epistemic Disjunction lemma) Let $A, C \in \mathbf{L}_\alpha$. In GL($\mathbf{L}_\alpha$), $\vdash \mathbf{B}_i(A) \vee \mathbf{B}_i(C)$ if and only if $\vdash \mathbf{B}_i(A)$ or $\vdash \mathbf{B}_i(C)$.*

The third result enables us to move forward/backward from the beliefs and their contents. This will be used in Section 4.1.

Lemma 3.5. *(Scope Lemma) Let $A, C \in \mathbf{L}_\alpha$. In GL($\mathbf{L}_\alpha$), $\vdash \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$ if and only if $\vdash A \supset C$.*

Proof. The *if* part is straightforward. We show the contrapositive of the *only-if* part. Suppose $\nvdash A \supset C$. By Theorem 3.1, there is a model $(\mathbb{K}, \tau)$ such that $(\mathbb{K}, \tau, w) \models A$ but $(\mathbb{K}, \tau, w) \nvDash C$ for some world $w \in W$. Now, we add a new world w^* to W so that $W^* = W \cup \{w^*\}$, $R_i^* = R_i \cup \{(w^*, w)\}$ and $R_j^* = R_i \cup \{(w^*, w^*)\}$ for all $j \neq i$. We extend τ to $\tau^* : W^* \times \mathcal{P}_0 \to \{\top, \bot\}$ so that $\tau^*(u, p) = \tau(u, p)$ for all $(u, p) \in W \times \mathcal{P}_0$ and $\tau^*(w^*, p)$ is arbitrary for all $p \in \mathcal{P}_0$. We have a new model $(\mathbb{K}^*, \tau^*)$. In this new model, all valuations are preserved from $(\mathbb{K}, \tau)$. Since agent i refers only to w at w^*, we have $(\mathbb{K}, \tau, w^*) \models \mathbf{B}_i(A)$ but $(\mathbb{K}, \tau, w^*) \nvDash \mathbf{B}_i(C)$. Hence, $(\mathbb{K}, \tau, w^*) \nvDash \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$. By Theorem 3.1, $\nvdash \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$.■

4 Applications

In the literature of epistemic logic, Axioms T, 4, and 5 are typically assumed. Although these axioms themselves are still finitary, Axioms 4 and 5 include some infinitary aspects. It is natural, then, to ask if they can be explicitly captured in our infinitary logics, and, if so, which ones in the hierarchy (1) can do this task. We take two steps for this question. First, we look for a formula enjoying each of these axioms in our infinitary extension GL($\boldsymbol{L}_\alpha$). Then, using such a formula, we consider the possibility of embedding a logic having such an axiom to GL($\boldsymbol{L}_\alpha$).[6] A similar consideration is given to the concept of common knowledge.

[5] In [12], the epistemic depth to count only the nested occurrences of $\mathbf{B}_i, i \in N$ is used for this lemma.

[6] Halpern *et al.* [5] consider a related issue. In the context of finitary logics, they consider two modalities, one called belief (KD45) and the other called knowledge (S5), and discuss whether the latter can be reduced to the former via various notions of definability. In contrast, our embedding results are about reducing one logic system (e.g., KD4) to another (e.g. GL($\boldsymbol{L}_\omega$)).

4.1 Evaluations of Axioms T, 4, and 5 in GL($\boldsymbol{L}_\alpha$)

We begin with the requirement that the target formula, denoted by $F_i(p)$, extends the belief operator $\mathbf{B}_i(\cdot)$ and enjoys the basic axioms K, D, and Necessitation. We fix one agent i throughout this subsection. We require a formula $F_i(p)$ in $\boldsymbol{L}_\alpha$ to satisfy the following: for any $A, C \in \boldsymbol{L}_\alpha$,

$$\begin{array}{rcl} \mathrm{F0}_i & : & F_i(A) \in \boldsymbol{L}_\alpha; \\ \mathrm{FE}_i & : & \vdash F_i(A) \supset \mathbf{B}_i(A); \\ \mathrm{FK}_i & : & \vdash F_i(A \supset C) \supset (F_i(A) \supset F_i(C)); \\ \mathrm{FN}_i & : & \vdash A \text{ implies } \vdash F_i(A), \end{array} \tag{8}$$

where $F_i(p)$ contains only propositional variable p and $\vdash$ is the provability relation in GL($\boldsymbol{L}_\alpha$). F0$_i$ means that $F_i(\cdot)$ is applied to any $A \in \boldsymbol{L}_\alpha$, and FE$_i$ that $F_i(\cdot)$ is an extension of the belief operator $\mathbf{B}_i(\cdot)$. FK$_i$ and FN$_i$ correspond to Axiom K and Necessitation. The corresponding requirement to Axiom D is implied by FE$_i$, i.e., $\vdash \neg F_i(A \wedge \neg A)$.

The above conditions are for a pair of $F_i(p)$ and $\boldsymbol{L}_\alpha$, since formulae A, C vary in $\boldsymbol{L}_\alpha$. Lemma 4.1 states that when $F_i(p) \in \boldsymbol{L}_\alpha$ satisfies F0$_i$, $F_i(p)$ is finitary or $\alpha = \omega$. In the first case, α is allowed to be arbitrary, and in the second, logic GL($\boldsymbol{L}_\omega$) is required.

Lemma 4.1. *If F0$_i$ holds for $F_i(p) \in \boldsymbol{L}_\alpha$, then $\delta(F_i(p)) < \omega$ or $\alpha = \omega$.*

Proof. Let $\delta(F_i(p)) \geq \omega$. Then, some infinitary conjunction $\wedge\Phi$ is included in $F_i(p)$ where Φ is generated by Rβ0 - Rβ1 for some $\beta \geq 1$. Since $F_i(p)$ contains only propositional variable p, so does $\wedge\Phi$. We can write $\Phi = \langle C^0(p), C^1(p), ...\rangle$. It holds by (3) that $\delta(C^\nu(F_i(p))) \geq \delta(F_i(p)) + \nu \geq \omega + \nu$ for all $\nu \geq 0$. Hence, $\delta(F_i(F_i(p))) \geq \delta(\wedge\langle C^0(F_i(p)), C^1(F_i(p)), ...\rangle) > \sup\{\omega + \nu : \nu \geq 0\} = \omega \cdot 2$. In general, we can prove by induction on $\beta \geq 1$ that $\delta(F_i^\beta(p)) > \omega \cdot \beta$ for all $\beta < \omega$. Using F0$_i$, $F_i^\beta(p) \in \boldsymbol{L}_\alpha$ for any $\beta < \omega$. Thus, $\omega^2 \leq \sup_{\beta<\omega} \delta(F_i^\beta(p)) \leq \delta(\boldsymbol{L}_\alpha)$. By Theorem 2.2, we have $\alpha = \omega$.■

Additional conditions below correspond to Axioms T, 4, and 5: for any $A \in \boldsymbol{L}_\alpha$,

$$\begin{array}{rcl} \mathrm{FT}_i & : & \vdash F_i(A) \supset A; \\ \mathrm{F4}_i & : & \vdash F_i(A) \supset F_i(F_i(A)); \\ \mathrm{F5}_i & : & \vdash \neg F_i(A) \supset F_i(\neg F_i(A)). \end{array}$$

We look for a formula $F_i(p)$ satisfying each of these in addition to F0$_i$ to FN$_i$. This question is analogous to Beth's explicit definability. The implicit definability is guaranteed by the soundness-completeness theorems for each choice of those properties (mentioned in Section 3.2).

In the case of Axiom T, we can observe that $\mathbf{B}_i(p) \wedge p$ satisfies F0$_i$, FE$_i$, and FT$_i$, and it is the deductively weakest among such formulae; we say that $F_i(p)$ is the *deductively weakest* among the formulae satisfying given conditions iff it satisfies them and for any $F'(p)$ among those formulae, $\vdash F_i'(A) \supset F_i(A)$ for any $A \in \boldsymbol{L}_\alpha$.

Theorem 4.1. *(Evaluation of Axiom T) Let $0 \leq \alpha \leq \omega$. In GL($\boldsymbol{L}_\alpha$), $\mathbf{B}_i(p) \wedge p$ is the deductively weakest among the formulae satisfying F0$_i$, FE$_i$, and FT$_i$.*

Proof. We can verify that $\mathbf{B}_i(p) \wedge p$ satisfies, F0$_i$, FE$_i$, and FT$_i$ in GL($\boldsymbol{L}_\alpha$). Let $F_i'(p)$ satisfy F0$_i$, FE$_i$, and FT$_i$. By FE$_i$, we have, $\vdash F_i'(A) \supset \mathbf{B}_i(A)$, and by FT$_i$, $\vdash F_i'(A) \supset A$. By $\wedge$-rule, we have $\vdash F_i'(A) \supset \mathbf{B}_i(A) \wedge A$, which holds for any $A \in \boldsymbol{L}_\alpha$. Thus, $\mathbf{B}_i(p) \wedge p$ is deductively weakest among $F_i(p)$ satisfying $\mathrm{F0}_i$, FE_i, and FT_i.■

This theorem holds for every α ($0 \leq \alpha \leq \omega$). Also, we can include FK_i and FN_i as required conditions in Theorem 4.1.

Now, we turn to Axiom 4. We prepare one lemma: we say that a *formula A is consistent* in $\mathrm{GL}(\mathbf{L}_\alpha)$ iff the set $\{A\}$ is consistent in $\mathrm{GL}(\mathbf{L}_\alpha)$. A formula A is not consistent if and only if $\vdash \neg A$.

Lemma 4.2. *Let $0 \leq \alpha \leq \omega$. Any $F_i(p)$ satisfying FN_i is consistent in $GL(L_\alpha)$.*

Proof. Suppose that $F_i(p)$ is not consistent in $\mathrm{GL}(\boldsymbol{L}_\alpha)$, which implies $\vdash \neg F_i(p)$. By the substitution-rule mentioned in Remark 3.2, it holds that $\vdash \neg F_i(p \supset p)$. On the other hand, by FN_i, $\vdash p \supset p$ implies $\vdash F_i(p \supset p)$. This is impossible because $\mathrm{GL}(\boldsymbol{L}_\alpha)$ is contradiction-free, as remarked just after Theorem 3.1.■

We assume that the germinal form $G_i^{PI} = \mathbf{B}_i(p)$ is in $\mathcal{G}$. Recall $\mathbf{B}_i^\omega(p) = \wedge\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$.

Theorem 4.2. *(Evaluation of Axiom 4) (1): Let $F_i(p) \in \boldsymbol{L}_\alpha$ satisfy $F0_i$, FE_i, FK_i,FN_i, and $F4_i$. Then $\alpha = \omega$ and $\vdash F_i(p) \supset \mathbf{B}_i^\omega \mathbf{B}_i(p)$.*

(2): $\mathbf{B}_i^\omega \mathbf{B}_i(p)$ is the deductively weakest among the formulae $F_i(p)$ satisfying $F0_i$, FE_i, FK_i, FN_i, and $F4_i$ in $GL(\boldsymbol{L}_\omega)$.

Proof. **(1)**: Let us prove $\mathrm{GL}(\boldsymbol{L}_\alpha) \vdash F_i(p) \supset \mathbf{B}_i^\nu \mathbf{B}_i(p)$ for all $\nu < \omega$. For $\nu = 0$, the claim is FE_i. Suppose $\vdash F_i(p) \supset \mathbf{B}_i^\nu \mathbf{B}_i(p)$. Then, by $\mathrm{F0}_i$, FN_i, and FK_i, we have $\vdash F_i(F_i(p)) \supset F_i(\mathbf{B}_i^\nu \mathbf{B}_i(p))$, and by $\mathrm{F4}_i$, we have $\mathrm{GL}(\boldsymbol{L}_\alpha) \vdash F_i(p) \supset \mathbf{B}_i^{\nu+1} \mathbf{B}_i(p)$.

Let $\delta(F_i(p)) < \omega$. Take a $\nu > \delta(F_i(p))$. By Lemma 3.3, we have $\vdash \neg F_i(p)$ or $\vdash p$ in $\mathrm{GL}(\boldsymbol{L}_0)$. The first is impossible since $F_i(p)$ is consistent in $\mathrm{GL}(\boldsymbol{L}_\alpha)$ by Lemma 4.2. The second is also impossible. Hence, $\delta(F_i(p)) \geq \omega$. By Lemma 4.1, $\alpha = \omega$. Using $\mathrm{F0}_i$, FE_i, FK_i, FN_i, $\mathrm{F4}_i$, we have $\mathrm{GL}(\boldsymbol{L}_\omega) \vdash F_i(p) \supset \mathbf{B}_i^\nu \mathbf{B}_i(p)$ for all $\nu < \omega$. Thus, $\mathrm{GL}(\boldsymbol{L}_\omega) \vdash F_i(p) \supset \mathbf{B}_i^\omega \mathbf{B}_i(p)$ by $\wedge$-rule.

(2): We can verify that $\mathrm{F0}_i$, FE_i, FK_i, FN_i, $\mathrm{F4}_i$ hold for $\mathbf{B}_i^\omega \mathbf{B}_i(p)$ in $\mathrm{GL}(\boldsymbol{L}_\omega)$. By (1), it is deductively weakest among $F_i(p)$ satisfying these requirements.■

In contrast to Theorem 4.1, Theorem 4.2 states that only in $\mathrm{GL}(\boldsymbol{L}_\omega)$, it is possible to have a formula having $\mathrm{F4}_i$ together with $\mathrm{F0}_i$, FE_i, FK_i, FN_i. For Axiom 5, the answer is entirely negative.

Theorem 4.3. *(Indefinability of Axiom 5 in $GL(\boldsymbol{L}_\alpha)$): There is no consistent formula $F_i(p)$ in $GL(\boldsymbol{L}_\alpha)$ $(0 \leq \alpha \leq \omega)$ such that it satisfies FE_i and $F5_i$.*

Proof. Suppose that there is some consistent formula $F_i(p)$ in $\mathrm{GL}(\boldsymbol{L}_\alpha)$ satisfying FE_i and $\mathrm{F5}_i$. Then, $\mathrm{F5}_i$ is equivalent to $\vdash F_i(p) \vee F_i(\neg F_i(p))$, which further implies, by FE_i, $\vdash \mathbf{B}_i(p) \vee \mathbf{B}_i(\neg F_i(p))$. By Lemma 3.4, we have $\vdash \mathbf{B}_i(p)$ or $\vdash \mathbf{B}_i(\neg F_i(p))$. By Lemma 3.5, we have $\vdash p$ or $\vdash \neg F_i(p)$. The former is impossible; and so is the latter because $F_i(p)$ is consistent in $\mathrm{GL}(\boldsymbol{L}_\alpha)$.■

This does not imply that Axiom 5 is impossible to be treated in our approach. As remarked in Section 3.2, we have the soundness-completeness theorem for the system including Axiom 5. Thus, Axiom 5 cannot be defined explicitly by a formula in $\mathrm{GL}(\boldsymbol{L}_\alpha)$, but it can still be defined by adding logical axioms.

4.2 Faithful embedding

In Section 4.1, we discussed the possibility of finding formulae in GL($\boldsymbol{L}_\alpha$) that satisfy the properties corresponding to Axioms T, 4, and 5. Another question is whether a logic with the respective axiom can be embedded into GL($\boldsymbol{L}_\alpha$). For T, the embedding result is available from $\boldsymbol{L}_\alpha$ to $\boldsymbol{L}_\alpha$ for any α in terms of language, for 4, it can only be from $\boldsymbol{L}_0$ to $\boldsymbol{L}_\omega$, but we have no embedding result for Axiom 5.

First, we consider the case of T and define a translation based on $F_i^T(p) = \mathbf{B}_i(p) \wedge p$. The F_i^T*-translator* $\psi^T : \boldsymbol{L}_\alpha \to \boldsymbol{L}_\alpha$ is inductively defined as follows: for all $A, C \in L_\alpha$ and $\wedge\Phi \in L_\alpha$,

E0: $\psi^T(p) = p$ if $p \in \mathcal{P}_0$; **E1**$_\alpha$: $\psi^T(\neg A) = \neg\psi^T(A)$;

E2$_\alpha$: $\psi^T(A \supset C) = \psi^T(A) \supset \psi^T(C)$; **E3**$_\alpha$: $\psi^T(\wedge\Phi) = \wedge\psi^T(\Phi)$;

E4$_\alpha$: $\psi^T(\mathbf{B}_i(A)) = F_i^T(\psi^T(A))$ and $\psi^T(\mathbf{B}_j(A)) = \mathbf{B}_j(\psi^T(A))$ for $j \neq i$.

Here, ψ^T is uniquely determined over L_0. When $\alpha \geq 1$, the definition ψ^T over $\boldsymbol{L}_\alpha$ needs one requirement on the set of germinal forms $\mathcal{G}$ to be closed under the translation ψ^T:

$$G \in \mathcal{G} \Longrightarrow \psi^T(G) \in \mathcal{G}. \tag{9}$$

The set of all germinal forms $\mathcal{G}^{All}$ satisfies (9). We have the following lemma, which is proved after Theorem 4.4.

Lemma 4.3. $\psi^T : \boldsymbol{L}_\omega \to \boldsymbol{L}_\omega$ *is uniquely defined by E1*$_\beta$ *to E4*$_\beta$ ($\beta \leq \omega$).

Now, we have the following theorem, where GL($\boldsymbol{L}_\alpha$)+T$_i$ denotes the logic GL($\boldsymbol{L}_\alpha$) plus Axiom T for $\mathbf{B}_i(\cdot)$. Then, the logic GL($\boldsymbol{L}_\alpha$)+T$_i$ is faithfully embedded into GL($\boldsymbol{L}_\alpha$) with ψ^T.

Theorem 4.4. *Let* $0 \leq \alpha \leq \omega$. *For any* $A \in \boldsymbol{L}_\alpha$, *GL*($\boldsymbol{L}_\alpha$)+*T*$_i$ $\vdash A$ *if and only if GL*($\boldsymbol{L}_\alpha$) $\vdash$ $\psi^T(A)$.

This theorem compares logic GL($\boldsymbol{L}_\alpha$)+T$_i$ with the fragment ψ^T(GL($\boldsymbol{L}_\alpha$)) obtained by the translator ψ^T. The *only-if* part is embedding, and the *if* part is faithfulness. It is more demanding than Theorem 4.1, since it makes a global comparison. It require well-definedness of the translator ψ^T; in particular, when a germinal form includes $B_i(\cdot)$. We need to show that ψ^T is defined coherently with the construction methods such as Rα0 - Rα1. The proof of Lemma 4.3 is essential.

Proof of Lemma 4.3: We show by induction over $\alpha \leq \omega$ that $\psi^T : \boldsymbol{L}_\alpha \to \boldsymbol{L}_\alpha$ is uniquely extended by E1$_\alpha$-E4$_\alpha$. E1$_0$-E4$_0$ uniquely define $\psi^T : \boldsymbol{L}_0 \to \boldsymbol{L}_0$. Suppose the induction hypothesis that $\psi^T : \mathcal{P}_\alpha$ $(= \cup_{\beta<\alpha}\boldsymbol{L}_\beta) \to \mathcal{P}_\alpha$ is uniquely defined. Then, we show that E1$_\alpha$-E4$_\alpha$ uniquely define $\psi^T : \boldsymbol{L}_\alpha \to \boldsymbol{L}_\alpha$. Thus, $\psi^T : \boldsymbol{L}_\omega \to \boldsymbol{L}_\omega$ is uniquely defined.

First, we prove that for any formula $G(p)$ in L_0 and $G'(p) = \psi^T(G(p))$,

$$\psi^T G(A) = G'(\psi^T A) \text{ for any } A \in \mathcal{P}_\alpha. \tag{10}$$

This is proved by induction on the length of formula G. If $G(p) = p$, then $\psi^T(G(A)) = \psi^T(A) = G'(\psi^T(A))$. We consider the induction step only for the case where the outermost connective of $G(p) \in L_0$ is $\mathbf{B}_i(\cdot)$, i.e., $G(p) = \mathbf{B}_i(D(p))$ for some $D(p)$ and the induction hypothesis that (10) holds for $D(p)$. Then, $G'(p) = \psi^T\mathbf{B}_i(D(p)) = \mathbf{B}_i(\psi^T D(p)) \wedge \psi^T D(p)$, and let $D'(p) = \psi^T D(p)$.

By the induction hypothesis, $\psi^T D(A) = D'(\psi(A))$. Thus, by $\mathrm{E4}_\alpha$, $\psi^T(G(A)) = \psi^T(\mathbf{B}_i D(A))$ $= \mathbf{B}_i(\psi^T D(A)) \wedge \psi^T D(A)$. Since $D'(p) = \psi^T D(p)$, the last formula is $\mathbf{B}_i(D'(\psi^T A)) \wedge D'(\psi^T A)$, which is $G'(\psi^T A)$.

Now, we show that ψ_T is well-defined over $\mathcal{F}_\alpha$. Suppose that $\wedge\langle C^\nu : \nu \geq 0\rangle$ is generated by by $\mathrm{R}\alpha0$-$\mathrm{R}\alpha1$ with a germ $A \in \mathcal{P}_\alpha$ and a germinal form $G(p)$. By (9), $G'(p) = \psi^T G(p) \in \mathcal{G}$, and let $A' = \psi^T(A) \in \mathcal{P}_\alpha$. Then, $\psi^T(\wedge\langle C^\nu : \nu \geq 0\rangle) = \wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$ by $\mathrm{E3}_\alpha$. Now, by (10), $\psi^T C^0 = \psi^T A = A'$ and $C'^{\nu+1} = \psi^T C^{\nu+1} = \psi^T G(C^\nu) = G'(\psi^T C^\nu) = G'(C'^\nu)$ for all $\nu \geq 0$. Hence, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$ is generated by germinal form G' and germ A'. Thus, $\psi^T(\wedge\langle C^\nu : \nu \geq 0\rangle) = \wedge\langle \psi^T C^\nu : \nu \geq 0\rangle \in \mathcal{F}_\alpha$.

Now, we extend ψ_T from $\mathcal{P}_\alpha \cup \mathcal{F}_\alpha$ to the entire L_α along $\mathrm{I}\alpha1$ - $\mathrm{I}\alpha2$. This is also by induction. The steps in $\mathrm{I}\alpha1$ are standard. Consider $\mathrm{I}\alpha2$. Let A, $\wedge\langle C^\nu : \nu \geq 0\rangle$, $\wedge\langle D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$, and assume the induction hypothesis that $\psi^T A$, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$, $\wedge\langle \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.

Consider $\mathrm{I}\alpha2.(i)$. Since $\psi^T A$ and $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$ are in L_α by the induction hypothesis, we have, using $\mathrm{I}\alpha2.(i)$, $\wedge\langle \psi^T A \supset \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. For all $\nu \geq 0$, since $A \supset C^\nu \in \boldsymbol{L}_\alpha$, it holds that $\psi^T(A \supset C^\nu) = \psi^T A \supset \psi^T C^\nu$ by $\mathrm{E2}_\alpha$. Hence, by $\mathrm{E3}_\alpha$, $\psi^T(\wedge\langle A \supset C^\nu : \nu \geq 0\rangle)$ $= \wedge\langle \psi^T(A \supset C^\nu) : \nu \geq 0\rangle = \wedge\langle \psi^T A \supset \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.

Consider $\mathrm{I}\alpha2.(ii)$: By the induction hypothesis, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle \in L_\alpha$, and by $\mathrm{I}\alpha2.(ii)$, $\wedge\langle \mathbf{B}_i(\psi^T C^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By $\mathrm{I}\alpha2.(iii)$, we have $\wedge\langle \mathbf{B}_i(\psi^T C^\nu) \wedge \psi^T C^\nu : \nu \geq 0\rangle \in L_\alpha$. This is written as $\wedge\langle \psi^T \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle \in L_\alpha$ by $\mathrm{E4}_\alpha$. Thus, $\psi^T(\wedge\langle \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle) = \wedge\langle \psi^T \mathbf{B}_i(C^\nu) :$ $\nu \geq 0\rangle = \wedge\langle \mathbf{B}_i(\psi^T C^\nu) \wedge \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. In the case of $j \neq i$, this proof becomes simpler.

Consider $\mathrm{I}\alpha2.(iii)$: By the induction hypothesis, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ and $\wedge\langle \psi^T D^\nu :$ $\nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By $\mathrm{I}\alpha2.(iii)$,we have $\wedge\langle \psi^T C^\nu \wedge \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By $\mathrm{E3}_\alpha$, this is written as $\wedge\langle \psi^T(C^\nu \wedge D^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. Hence, $\psi^T(\wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle) = \wedge\langle \psi^T(C^\nu \wedge D^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.■

Proof of Theorem 4.4. Take an arbitrary serial model $(\mathbb{K}, \tau)$ and let $\mathbb{K}^{ref}$ be the reflexive closure of $\mathbb{K}$ with respect to R_i. We show by induction on the length of A that

$$\text{for all } w \in W, \;\; (\mathbb{K}^{ref}, \tau, w) \models A \Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T(A). \tag{11}$$

This implies that $\mathbb{K}^{ref} \models A \Longleftrightarrow \mathbb{K} \models \psi^T(A)$. Since $\mathbb{K}$ is an arbitrary serial model, this equivalence implies, by the completeness theorem for $\mathrm{GL}(\boldsymbol{L}_\alpha)$ and $\mathrm{GL}(\boldsymbol{L}_\alpha)+\mathrm{T}_i$ (Theorem 3.1 and its variant) that $(\mathrm{GL}(\boldsymbol{L}_\alpha)+\mathrm{T}_i) \vdash A$ if and only if $\mathrm{GL}(\boldsymbol{L}_\alpha) \vdash \psi^T(A)$.

We can prove the assertion (11) by induction on the length of $A \in \boldsymbol{L}_\alpha$. We consider only the two cases: Case $A = \wedge\langle C^\nu : \nu \geq 0\rangle \in \mathcal{F}_\alpha$ and Case $A = \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$ generated by $\mathrm{I}\alpha2.(iii)$.

Consider Case $A = \wedge\langle C^\nu : \nu \geq 0\rangle \in \mathcal{F}_\alpha$. Then, $(\mathbb{K}^{ref}, \tau, w) \models \wedge\langle C^\nu : \nu \geq 0\rangle \Longleftrightarrow$ $(\mathbb{K}^{ref}, \tau, w) \models C^\nu$ for all $v \geq 0 \Longleftrightarrow (\mathbb{K}, \tau, v) \models \psi^T C^\nu$ for all $v \geq 0$ by the induction hypothesis. This is further equivalent to $(\mathbb{K}, \tau, w) \models \wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$, which is equivalent to $\Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T(\wedge\langle C^\nu : \nu \geq 0\rangle)$.

Consider Case $A = \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$. The induction hypothesis is: (11) holds for C^ν and D^ν for all $\nu \geq 0$. Now, $(\mathbb{K}^{ref}, \tau, w) \models \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle \Longleftrightarrow (\mathbb{K}, \tau, w) \models C^\nu \wedge D^\nu$ for all $\nu \geq 0 \Longleftrightarrow$ $(\mathbb{K}, \tau, w) \models \psi^T(C^\nu \wedge D^\nu)$ for all $\nu \geq 0$, which is equivalent to $(\mathbb{K}, \tau, w) \models \langle \psi^T(C^\nu \wedge D^\nu) : \nu \geq$ $0\rangle \Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$.■

In the case of Axiom 4, we use the translator ψ^4 based on $F_i^4(p) = \mathbf{B}_i^\omega \mathbf{B}_i(p)$; here, ψ^4 translates the formulae in $\boldsymbol{L}_0$ to $\boldsymbol{L}_\omega$. For this translation, we assume that $\mathbf{B}_i(p) \in \mathcal{G}$. The formal definition for $\psi^4 : \boldsymbol{L}_0 \longrightarrow \boldsymbol{L}_\omega$ is obtained by the same rules E0, $\mathrm{E1}_0$-$\mathrm{E3}_0$, but $\mathrm{E4}_0$ with $F_i^4(p) = \mathbf{B}_i^\omega \mathbf{B}_i(p)$ instead of $F_i^T(p)$. We have the following theorem.

Theorem 4.5. *For any $A \in \boldsymbol{L}_0$, KD^n+ $4_i \vdash A$ if and only if $GL(\boldsymbol{L}_\omega) \vdash \psi^4(A)$.*

Proof. Take an arbitrary Kripke model $(\mathbb{K},\tau)$ for KD^n, which is also a model for GL($\boldsymbol{L}_\omega$). We replace the accessibility relation R_i in $(\mathbb{K},\tau)$ by its transitive closure R_i^{tr}, and we denote the resulting Kripke model by $(\mathbb{K}^{tr},\tau)$. Then, KD$^n+4$ is Kripke complete with respect to those models $(\mathbb{K}^{tr},\tau)$.

We take an arbitrary Kripke model $(\mathbb{K},\tau)$ for KD^n. Then, we prove by induction on the length of $A\in\boldsymbol{L}_0$ that for any world $w\in W$, $(\mathbb{K}^{tr},\tau,w)\models A$ if and only if $(\mathbb{K},\tau,w)\models\psi^4(A)$.

Consider $A=\mathbf{B}_i(C)$. Let $(\mathbb{K}^{tr},\tau,w)\models\mathbf{B}_i(C)$. Then, $(\mathbb{K}^{tr},\tau,v)\models C$ for any $v\in R_i^{tr}(w)$. By the induction hypothesis, $(\mathbb{K},\tau,v)\models\psi_4(C)$ for any $v\in R_i^{tr}(w)$. Since R_i^{tr} is the transitive closure of R_i, it is equivalent to that $(\mathbb{K},\tau,v)\models\psi^4(C)$ for any v which is reachable by R_i. Thus, the latter means $(\mathbb{K},\tau,w)\models\mathbf{B}_i^\nu(\psi^4(C))$ for any $\nu\geq 1$. Thus, $(\mathbb{K},\tau,w)\models\mathbf{B}_i^\omega(\psi^4(C))$, i.e., $(\mathbb{K},\tau,w)\models\psi^4(\mathbf{B}_i(C))$. Tracing this proof, we have a proof of the converse.

For the cases of other connectives, the argument is similar.■

In contrast to Theorem 4.4, the above embedding result requires GL($\boldsymbol{L}_\alpha$) $=$ GL($\boldsymbol{L}_\omega$); indeed, the depth of the fragment $\psi^4(\boldsymbol{L}_0)$ is the same as that of $\boldsymbol{L}_\omega$, i.e., $\delta(\psi^4(\boldsymbol{L}_0))=\delta(\boldsymbol{L}_\omega)=\omega^2$. This is because the theorem embeds the entire KD^n+4_i. When we focus on a specific $A\in\boldsymbol{L}_0$, we can find the maximal iterations, α, of $\mathbf{B}_i(\cdot)$ inside A. Then, by Theorem 3.2 (conservativity), KD^n+ $4_i\vdash A$ is equivalent to GL($\boldsymbol{L}_\alpha$) $\vdash\psi^4(A)$.

4.3 Evaluation of common knowledge in GL($\boldsymbol{L}_\alpha$)

The concept of common knowledge can be formulated in a fixed-point extension of an finitary epistemic logic, often S5-type, (Halpern, *et al.* [2], Meyer-van der Hoek [16]). Here, we consider its KD^n variant, and show that this fixed-point logic is embedded to GL($\boldsymbol{L}_\alpha$).

Let $S\subseteq N$ with $S\neq\emptyset$. The finitary language $\boldsymbol{L}_0$ is extended by adding the unary operator symbol $\mathbf{C}_S(\cdot)$ to the basic symbols listed in Section 2.1, and use $L^{\mathbf{C}_S}$ to denote the extended language. A formula $\mathbf{C}_S(A)$ means common knowledge of A among the group of agents S.

The *common knowledge logic* CK($\boldsymbol{L}^{\mathbf{C}_S}$) is defined to be the extension of KD^n with the language $\boldsymbol{L}^{\mathbf{C}_S}$ by adding the following axiom scheme and an inference rule: for any A,$D\in L^{\mathbf{C}_S}$,

Axiom CKA: $\mathbf{C}_S(A)\supset[A\wedge\wedge_{i\in S}\mathbf{B}_i\mathbf{C}_S(A)]$;

Rule CKI: $\dfrac{D\supset[A\wedge\wedge_{i\in S}\mathbf{B}_i(D)]}{D\supset\mathbf{C}_S(A)}$.

The logical system is defined in the same way as in Section 3.1; the provability relation is denoted by $\vdash_C$. In this logic, it is shown by repeated use of CKA that $\vdash_C\mathbf{C}_S(A)\supset\mathbf{B}_S^\nu(A)$ for all $\nu\geq 0$, where $\mathbf{B}_S^\nu(A)$ is defined by $\mathbf{B}_S^0(A)=A$ and $\mathbf{B}_S^{\nu+1}(A)=\wedge_{i\in S}\mathbf{B}_i\mathbf{B}_S^\nu(A)$ for all $\nu\geq 0$. Thus, $\mathbf{C}_S(A)$ contains the common knowledge of A. Rule CKI means that if any D has the property described by CKA, then D contains $\mathbf{C}_S(A)$, i.e., $\mathbf{C}_S(A)$ is the deductively weakest among the formulae having the property.

The semantics for CK($\boldsymbol{L}^{\mathbf{C}_S}$) is defined by adding the following valuation step for $\mathbf{C}_S(\cdot)$:

$(M,w)\models\mathbf{C}_S(A)$ iff $(M,v)\models A$ for all $\mathbf{C}_S$-reachable v from w,

where v is *$\mathbf{C}_S$-reachable from w* iff there is a finite sequence $\langle w_0,...,w_m\rangle$ $(m\geq 0)$ such that $w_0=w$, $w_m=v$, and for all $k=0,...,m-1$, $(w_k,w_{k+1})\in R_i$ for some $i\in S$. Here, M is a serial model as in Section 3.2.

We have the completeness/soundness result for CK($\boldsymbol{L}^{\mathbf{C}_S}$), which is a variant of the well-known result (cf., [2], [16]).

Theorem 4.6. *For any $A \in L^{\mathbf{C}_S}$, A is valid if and only if CK($\boldsymbol{L}^{\mathbf{C}_S}$) $\vdash A$.*

In a similar manner to Sections 4.1, we look for a formula $F(p)$ in GL($\boldsymbol{L}_\alpha$) corresponding to $\mathbf{C}_S(p)$ in CK($\boldsymbol{L}^{\mathbf{C}_S}$). We reformulate CKA and CKI as the requirements for $F(p)$ in GL($\boldsymbol{L}_\alpha$): if $F(A)$ and then $D \in \boldsymbol{L}_\alpha$, then, in GL($\boldsymbol{L}_\alpha$),

$$\begin{array}{rcl} \mathrm{FCA}_\alpha & : & \vdash F(A) \supset A \wedge [\wedge_{i\in S}\mathbf{B}_i(F(A))]; \\ \mathrm{FCI}_\alpha & : & \text{if } \vdash D \supset A \wedge [\wedge_{i\in S}\mathbf{B}_i(D)], \text{ then} \vdash D \supset F(A), \end{array}$$

where $G_S(p) = \wedge_{i\in S}\mathbf{B}_i(p)$ is a germinal form in $\mathcal{G}$.

Theorem 4.7. *(Uniqueness) In GL($\boldsymbol{L}_\alpha$) ($1 \leq \alpha \leq \omega$), the common knowledge $F(p) = \wedge\{\mathbf{B}_S^\nu(p) : \nu \geq 0\}$ is a unique, up to the deductive equivalence, formula satisfying FCA$_\alpha$ and FCI$_\alpha$.*

Proof. It is easy to see from the germinal form $G_S(p) = \wedge_{j\in S}\mathbf{B}_j(p)$ that the formula $C_S(p) = \wedge\langle C_S^\nu(A) : \nu \geq 0\rangle$ given in (5) satisfies FCA$_\alpha$. Let $D \in \boldsymbol{L}_\alpha$ and suppose $\vdash D \supset A\wedge[\wedge_{i\in S}\mathbf{B}_i(D)]$. Using Nec and Axiom K for each $i \in S$, we have $\vdash D \supset C_S(A)$. Hence, the formula $C_S(p)$ satisfies both FCA$_\alpha$ and FCI$_\alpha$. Let $F'(p)$ be any formula in $\boldsymbol{L}_\alpha$ satisfying FCA$_\alpha$ and FCI$_\alpha$. Then, it follows from FCA$_\alpha$ for $F'(p)$ that $\vdash F'(A) \supset C_S(p)$. Since $C_S(p)$ satisfies FCA$_\alpha$, it follows from FCI$_\alpha$ for $F'(p)$ that $\vdash C_S(p) \supset F'(p)$. Thus, $\vdash F'(p) \equiv C_S(p)$.■

This theorem differs from Theorems 4.1 and 4.2 in that the unique determination of $F(p)$ is stated outside the systems. In fact, we can obtain parallel deductive uniqueness by adding additional requirements.

The faithful embedding of CK($\boldsymbol{L}^{\mathbf{C}_S}$) into GL($\boldsymbol{L}_\omega$) also holds: here we define the translator $\psi^{C_S} : \boldsymbol{L}^{\mathbf{C}_S} \to \boldsymbol{L}_\omega$ by E0 and E1$_0$ - E3$_0$, and

$$\begin{array}{rcl} \mathrm{E4}_0 & : & \psi^{C_S}(\mathbf{B}_i(A)) = \mathbf{B}_i(\psi^{C_S}(A)) \text{ for all } i \in N; \\ \mathrm{EC} & : & \psi^{C_S}(\mathbf{C}_S(A)) = C_S(\psi^{C_S}(A)). \end{array}$$

Then, we have the following theorem.

Theorem 4.8. *(Faithful embedding of CK($\boldsymbol{L}^{\mathbf{C}_S}$))* ***(1)*** *For any $A \in \boldsymbol{L}^{\mathbf{C}_S}$, CK($\boldsymbol{L}^{\mathbf{C}_S}$) $\vdash A$ if and only if GL($\boldsymbol{L}_\omega$) $\vdash \psi^{C_S}(A)$.*

(2) *For a fixed $A \in \boldsymbol{L}^{\mathbf{C}_S}$, there exists the smallest $\alpha_A < \omega$ such that CK($\boldsymbol{L}^{\mathbf{C}_S}$) $\vdash A$ if and only if GL($\boldsymbol{L}_\alpha$) $\vdash \psi^{C_S}(A)$.*

Part (1) of the theorem can be proved by observing that with the translation ψ^{C_S}, the Kripke semantics for CK($\boldsymbol{L}^{C_S}$) and for GL($\boldsymbol{L}_\omega$) are the same, and part (2) is a direct corollary of part (1) and Theorem 3.2. We omit the proof.

Remark 4.1 (Justified common knowledge): Sato [20] gave a logic including a concept related to common knowledge, which Artemov [1] called the *justified common knowledge logic* JL($\boldsymbol{L}^{\mathbf{J}}$). In this logic, the language $\boldsymbol{L}^{\mathbf{J}_S}$ is obtained from $\boldsymbol{L}_\omega^{\mathbf{C}_S}$ by replacing $\mathbf{C}_S(\cdot)$ by $\mathbf{J}_S(\cdot)$. Here, we consider the version in which the belief operators $\mathbf{B}_i(\cdot)$ obey KD^n, $\mathbf{J}_S(\cdot)$ obeys S4 axioms (including Nec), and

Interaction axiom: $\mathbf{J}_S(A) \supset \wedge_{i\in S}\mathbf{B}_i(A)$ for all $A \in \boldsymbol{L}^{\mathbf{J}_S}$.

The expression $\mathbf{J}_S(A)$ is interpreted as meaning that A is "obvious fact" in that it is known to all agents in S. Interaction Axiom connects $\mathbf{J}_S(A)$ to $\wedge_{i\in S}\mathbf{B}_i(A)$, but the converse is not guaranteed. In this logic, $\mathbf{J}_S(A)$ implies the common knowledge $C_S(A)$, i.e., $\vdash \mathbf{J}_S(A) \supset C_S(A)$.

Let us look for a formula $F(\cdot)$ representing $\mathbf{J}_S(\cdot)$ in GL($\boldsymbol{L}_\omega$). By requiring $F(\cdot)$ to have the conditions corresponding the S4 axiom and the Interaction axiom for $\mathbf{J}_S(\cdot)$, we have $\vdash F(A) \supset C_S(A)$. Also, $C_S(p)$ itself is a candidate for $F(p)$, but we have a different candidate for $F(p)$. Provided that the languages allow to talk about a group of agents T with $S \subsetneq T \subseteq N$, the common knowledge formula $C_T(p)$ for group T is also a candidate for $F(p)$; here Interaction axiom is still kept for S. Thus, the conditions do not determine $F(p)$ to be $C_S(S)$. We need to add one inference, similar to CKI, to choose the deductively weakest formula. Then, $\mathbf{J}_S(\cdot)$ is determined to be $C_S(A)$. In this case, the system JL($\boldsymbol{L}^{\mathbf{J}_S}$) together with this additional inference is faithfully embedded into GL($\boldsymbol{L}_\omega$).

5 Proof of the Completeness of GL($\boldsymbol{L}_\alpha$) by Q-filters

The standard method to construct a maximal consistent set is not available in our framework. Instead, we adopt the $\mathcal{Q}$-filter method to prove completeness of GL($\boldsymbol{L}_\alpha$). Since it is a long proof, we give a sketch here.

In Section 5.1, we provide a small summary of $\mathcal{Q}$-filters, which are extensions of prime filters in a Boolean algebra, and then we mention the lemmas due to Rasiowa-Sikorski [19] and Tanaka-Ono [23]. In Section 5.2, we define the Lindenbaum algebra based on GL($\boldsymbol{L}_\alpha$), and prepare for applications of the Rasiowa-Sikorski and Tanaka-Ono lemmas.

In Section 5.3, we construct a counter (Kripke) model based on $\mathcal{Q}$-filters in the Lindenbaum algebra, where the $\mathcal{Q}$-fliters are adopted as possible worlds. A key step is to show that A is true in a world w if and only if $[\![A]\!] \in w$, where $[\![A]\!]$ is the set of formulae deductively equivalent to A in GL($\boldsymbol{L}_\alpha$). This step requires Tanaka-Ono Lemma to deal with $\mathbf{B}_i(\cdot)$. Finally, we show that if $\nvdash A$, there is a $\mathcal{Q}$-filter w for which $[\![A]\!] \notin w$; the existence of such a $\mathcal{Q}$-filter w is guaranteed by the Rasiowa-Sikorski Lemma.

5.1 Boolean algebra and Q-filters

We give basic definition and relevant properties of a Boolean algebra (cf., Halmos [3] and Mendelson [15]), and state the Rasiowa-Sikorski Lemma and Tanaka-Ono Lemma.

Consider a Boolean algebra $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$. We define $a \leq b$ iff $a \sqcup b = b$. Then $\leq$ is a lattice ordering on $\mathbf{B}$ (that is, $a \sqcap b$ and $a \sqcup b$ are the *greatest lower bound* and *smallest upper bound* of a, b with respect to $\leq$). We say that a nonempty subset F of $\mathbf{B}$ is a *filter* iff **F1**(upward closed): $a \leq b$ and $a \in F$ imply $b \in F$; and **F2**($\sqcap$-closed): $a, b \in F$ implies $a \sqcap b \in F$. Also, we say that a filter F is *prime* iff **P1**(Non-triviality): $F \neq \mathbf{B}$; and **P2:**($\sqcup$-property): $a \sqcup b \in F$ implies $a \in F$ or $b \in F$. We have the following fact on a prime filter F:

$$a \in F \;\Leftrightarrow\; (-a) \notin F. \tag{12}$$

In the following, we write $a \to b$ for $-a \sqcup b = (-a) \sqcup b$. When F is a prime filter, $a \to b \in F$ if and only if $a \notin F$ or $b \in F$, since $(-a) \sqcup a = \mathbf{1} \in F$.

For any subset S of $\mathbf{B}$, the *greatest lower bound* of S in $(\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$ is denoted by $\sqcap S$, and the *least upper bound* of S is denoted by $\sqcup S$. Note that $\sqcap S$ and $\sqcup S$ may not exist, but if either exists, it is unique.

Now, we introduce a certain extension of a prime filter. Let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ be a pair of countable sets of nonempty subsets of $\mathbf{B}$ so that

(($\sqcap, \sqcup$)**-closed**): $\sqcap Q_1$ and $\sqcup Q_2$ exist for all $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$.

We say that a prime filter F is a *$\mathcal{Q}$-filter* iff

Q1: for any $Q_1 \in \mathcal{Q}_1$, $Q_1 \subseteq F$ implies $\sqcap Q_1 \in F$;

Q2: for any $Q_2 \in \mathcal{Q}_2$, $\sqcup Q_2 \in F$ implies $a \in Q_2$ for some $a \in F$.

These correspond to the conditions F2 and P2. The following is Rasiowa-Sikorski lemma (see also Tanaka-Ono [**23**]).

Lemma 5.1. ***(Rasiowa-Sikorski [19]).*** *Let $\mathbb{B}$ be a Boolean algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a pair of countable sets of nonempty subsets of $\mathbf{B}$ with $(\sqcap, \sqcup)$-closedness. For any $a, b \in \mathbf{B}$, if $a \not\leq b$, then there is a $\mathcal{Q}$-filter F such that $a \in F$ and $b \notin F$.*

For a given $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, we denote the set of all $\mathcal{Q}$-filters of $\mathbb{B}$ by $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$. The nonemptiness of $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ follows from Lemma 5.1 if $\mathbf{0} \neq \mathbf{1}$. The set $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ will be adopted for the set of all possible worlds in our construction of a Kripke model.

Since the logic GL($\boldsymbol{L}_\alpha$) has belief operators, Rasiowa-Sikorski lemma is not enough: We extend it, which is Tanaka-Ono lemma. We say that $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ is a *multi-modal algebra* iff

ma1: $(\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$ is a Boolean algebra;

ma2: for $i \in N$, $\Box_i$ is a unary operator on $\mathbf{B}$ satisfying the property that $\Box_i \mathbf{1} = \mathbf{1}$ and $\Box_i(a \sqcap b) = \Box_i a \sqcap \Box_i b$ for all $a, b \in \mathbf{B}$.

We define $\Box_i^{-1} F = \{x \in \mathbf{B} : \Box_i x \in F\}$ for any $F \subseteq \mathbf{B}$.

Let $\mathbb{B}$ be a multi-modal algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$ satisfying $(\sqcap, \sqcup)$-closedness. The following three conditions are crucial for the Tanaka-Ono Lemma: for all $i \in N$,

$q0$: for all $Q_1 \in \mathcal{Q}_1$, $\sqcap(\Box_i Q_1) := \sqcap\{\Box_i a : a \in Q_1\}$ exists and $\sqcap(\Box_i Q_1) = \Box_i(\sqcap Q_1)$;

$q1$: $\{\Box_i(a \to b) : b \in Q_1\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_1 \in \mathcal{Q}_1$;

$q2$: $\{\Box_i(b \to a) : b \in Q_2\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_2 \in \mathcal{Q}_2$.

Lemma 5.2. ***(Tanaka-Ono [23]).*** *Let $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ be a multi-modal algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$. Suppose that $\mathcal{Q}$ satisfies $(\sqcap, \sqcup)$-closedness, and the conditions $q0$, $q1$, and $q2$ for $i \in N$. Then, for any $i \in N$, $b \in \mathbf{B}$, and $F \in \mathbb{F}_{\mathcal{Q}}(\mathbb{B})$, if $\Box_i b \notin F$, there exists a $G \in \mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ such that $\Box_i^{-1} F \subseteq G$ and $b \notin G$.*

5.2 Lindenbaum algebra

For any $A, B \in \boldsymbol{L}_\alpha$, we define $A \sim B$ iff $\vdash (A \supset B) \wedge (B \supset A)$ in GL($\boldsymbol{L}_\alpha$). This relation $\sim$ is an equivalence relation over $\boldsymbol{L}_\alpha$. We take the quotient set $\boldsymbol{L}_\alpha / \sim$. For any $A \in \boldsymbol{L}_\alpha$, we denote, by $[\![A]\!]$, the equivalence class in $\boldsymbol{L}_\alpha / \sim$ including A.

In $\mathbf{B} := \boldsymbol{L}_\alpha / \sim$, we define elements $\mathbf{0}, \mathbf{1}$ and operations $\sqcap, \sqcup, -$, and $\Box_1, ..., \Box_n$ by

$\ell 1$: $\mathbf{0} = [\![\neg \mathbf{p}_0 \wedge \mathbf{p}_0]\!]$ and $\mathbf{1} = [\![\mathbf{p}_0 \supset \mathbf{p}_0]\!]$;
$\ell 2$: for any $A, B \in \boldsymbol{L}_\alpha$, $[\![A]\!] \sqcap [\![B]\!] = [\![A \wedge B]\!]$, $[\![A]\!] \sqcup [\![B]\!] = [\![\neg(\neg A \wedge \neg B)]\!]$, and $-[\![A]\!] = [\![\neg A]\!]$;
$\ell 3$: for any $A \in \boldsymbol{L}_\alpha$, $\Box_i [\![A]\!] = [\![\mathbf{B}_i(A)]\!]$ for $i \in N$.

Using these definitions, we have, for any $A, B \in \boldsymbol{L}_\alpha$,

$$[\![A]\!] \to [\![B]\!] = (-[\![A]\!]) \sqcup [\![B]\!] = [\![\neg A]\!] \sqcup [\![B]\!] = [\![\neg(\neg\neg A \wedge \neg B)]\!] = [\![A \supset B]\!]. \tag{13}$$

It follows from this that $\Box_i([\![A]\!] \to [\![B]\!]) = \Box_i([\![A \supset B]\!]) = [\![\mathbf{B}_i(A \supset B)]\!]$.

Lemma 5.3. $\mathbb{L} = (\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -, \Box_1, ..., \Box_n)$ *with* $\mathbf{B} = \boldsymbol{L}_\alpha / \sim$ *is a multi-modal algebra.*

Proof. We can show in the standard manner that $(\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -)$ with $\mathbf{B} = \boldsymbol{L}_\alpha / \sim$ is a Boolean algebra. It remains to show condition *ma*2. Let $i \in N$. Since $\vdash [(A \supset A) \supset \mathbf{B}_i(A \supset A)] \wedge [\mathbf{B}_i(A \supset A) \supset (A \supset A)]$, we have $\Box_i \mathbf{1} = \mathbf{1}$. Since $\vdash [\mathbf{B}_i(A \wedge C) \supset \mathbf{B}_i(A) \wedge \mathbf{B}_i(C)] \wedge [\mathbf{B}_i(A) \wedge \mathbf{B}_i(C) \supset \mathbf{B}_i(A \wedge C)]$, we have $\Box_i([\![A]\!] \sqcap [\![C]\!]) = \Box_i[\![A]\!] \sqcap \Box_i[\![C]\!]$.$\blacksquare$

In the following, we call the specific multi-modal algebra $\mathbb{L}$ given in Lemma 5.3 the *Lindenbaum algebra*. In the main step of the completeness theorem, we will use Lemmas 5.1 and 5.2. For this purpose, we should prove the following lemma.

Lemma 5.4. *For any* $\wedge\Phi \in \boldsymbol{L}_\alpha$ *and* $i \in N$,
(a)*:* $\sqcap\{[\![C]\!] : C \in \Phi\} = [\![\wedge\Phi]\!]$; *and* ***(b)****:* $\sqcap\{\Box_i[\![C]\!] : C \in \Phi\} = [\![\mathbf{B}_i(\wedge\Phi)]\!]$.

Proof. **(a)**: First, let us see that $[\![\wedge\Phi]\!]$ is a lower bound of $\{[\![C]\!] : C \in \Phi\}$. Since $\vdash \wedge\Phi \supset C$ for all $C \in \Phi$ by L4, we have $(-[\![\wedge\Phi]\!]) \sqcup [\![C]\!] = \mathbf{1}$. Let $C \in \Phi$. Then, we have

$$\begin{aligned} [\![\wedge\Phi]\!] &= [\![\wedge\Phi]\!] \sqcap \mathbf{1} = [\![\wedge\Phi]\!] \sqcap \langle(-[\![\wedge\Phi]\!]) \sqcup [\![C]\!]\rangle \\ &= \langle[\![\wedge\Phi]\!] \sqcap (-[\![\wedge\Phi]\!])\rangle \sqcup \langle[\![\wedge\Phi]\!] \sqcap [\![C]\!]\rangle = \mathbf{0} \sqcup \langle[\![\wedge\Phi]\!] \sqcap [\![C]\!]\rangle = [\![\wedge\Phi]\!] \sqcap [\![C]\!]. \end{aligned}$$

Hence, $[\![\wedge\Phi]\!] \leq [\![C]\!]$. Since C is arbitrary in Φ, $[\![\wedge\Phi]\!]$ is a lower bound of $\{[\![C]\!] : C \in \Phi\}$.

It remains to show that $[\![\wedge\Phi]\!]$ is the greatest lower bound of $\{[\![C]\!] : C \in \Phi\}$. Now, let $[\![D]\!]$ be a lower bound of $\{[\![C]\!] : C \in \Phi\}$. This means $[\![D]\!] \leq [\![C]\!]$, i.e., $[\![D]\!] \sqcup [\![C]\!] = [\![C]\!]$, for any $C \in \Phi$. Let $C \in \Phi$. Then $(-[\![D]\!]) \sqcup [\![C]\!] = (-[\![D]\!]) \sqcup ([\![D]\!] \sqcup [\![C]\!]) = (-[\![D]\!] \sqcup [\![D]\!]) \sqcup [\![C]\!] = \mathbf{1} \sqcup [\![C]\!] = \mathbf{1}$. This implies $\vdash D \supset C$. Since C is arbitrary in Φ, we have, by $\wedge$-rule, we have $\vdash D \supset \wedge\Phi$. This means that $[\![\wedge\Phi]\!]$ is greater than or equal to $[\![D]\!]$ in $\mathbb{L}$. Thus, $[\![\wedge\Phi]\!]$ is the greatest lower bound of $\{[\![C]\!] : C \in \Phi\}$.
(b): Since $\vdash \mathbf{B}_i(\wedge\Phi) \supset \mathbf{B}_i(A)$ for all $A \in \Phi$, and since $\{\Box_i[\![A]\!] : A \in \Phi\} = \{[\![\mathbf{B}_i(A)]\!] : A \in \Phi\}$, $[\![\mathbf{B}_i(\wedge\Phi)]\!]$ is a lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$. Now, let $[\![D]\!]$ be a lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$. Using the same argument as in (a), we have $\vdash D \supset \mathbf{B}_i(A)$ for all $A \in \Phi$. Thus, $\vdash D \supset \wedge\mathbf{B}_i(\Phi)$ by $\wedge$-rule. By $\wedge$-Barcan, we have $\vdash D \supset \mathbf{B}_i(\wedge\Phi)$. This means that $[\![\mathbf{B}_i(\wedge\Phi)]\!]$ is the greatest lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$.$\blacksquare$

Now we define a pair $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ as follows:

$$\mathcal{Q}_1 = \{\{[\![A]\!] : A \in \Phi\} : \wedge\Phi \in \boldsymbol{L}_\alpha\} \text{ and } \mathcal{Q}_2 = \emptyset. \tag{14}$$

Then, $\mathcal{Q}_1$ is a countable set. According to Iβ0-Iβ2 ($\beta \leq \alpha$) of Section 2.1, the sets in $\mathcal{Q}_1$ are finite or countably infinite. Then, we have the following lemma.

Lemma 5.5. *(1): $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies $(\sqcap, \sqcup)$-closedness.*

(2): $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies the conditions $q0$, $q1$, $q2$.

Proof. Since $\mathcal{Q}_2 = \emptyset$, the $(\sqcap, \sqcup)$-closedness for $\sqcup$ and $q2$ are vacuous.

(1): Consider a $Q \in \mathcal{Q}_1$. This Q is written as $\{[\![A]\!] : A \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$. Since $\sqcap Q = [\![\wedge\Phi]\!]$ by Lemma 5.4.(a), $\sqcap Q$ belongs to $\mathbf{B} = \boldsymbol{L}_\alpha/\sim$.

(2)$(q0)$: We show that for any $Q \in \mathcal{Q}_1$, $\sqcap(\Box_i Q) := \sqcap\{\Box_i a : a \in Q\}$ exists and $\sqcap(\Box_i Q) = \Box_i(\sqcap Q)$. Since $Q \in \mathcal{Q}_1$, $\{\Box_i a : a \in Q\}$ is expressed as $\{[\![\mathbf{B}_i(A)]\!] : A \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$. By I$\beta$1-I$\beta$2, $\wedge\Phi \in \boldsymbol{L}_\alpha$ implies $\wedge\mathbf{B}_i(\Phi) \in \boldsymbol{L}_\alpha$. Then, by Lemma 5.4.(b) and $\vdash \wedge\mathbf{B}_i(\Phi) \equiv \mathbf{B}_i(\wedge\Phi)$,

$$\sqcap(\Box_i Q) = [\![\wedge\langle\mathbf{B}_i(A) : A \in \Phi\}]\!] = [\![\mathbf{B}_i(\wedge\Phi)]\!] = \Box_i[\![\wedge\Phi]\!] = \Box_i \sqcap Q.$$

$(q1)$: Let $Q \in \mathcal{Q}_1$ and $a \in \mathbf{B}$. We show $\{\Box_i(a \to b) : b \in Q\} \in \mathcal{Q}_1$. Since $a = [\![A]\!]$ for some $A \in \boldsymbol{L}_\alpha$ and Q is also expressed as $\{[\![B]\!] : B \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$, we have, by (13),

$$\{\Box_i(a \to b) : b \in Q\} = \{[\![\mathbf{B}_i(A \supset B)]\!] : B \in \Phi\}. \tag{15}$$

Since $\wedge\langle A \supset B : B \in \Phi\rangle \in \boldsymbol{L}_\alpha$ by Iα2.(i), we have $\wedge\langle\mathbf{B}_i(A \supset B) : B \in \Phi\rangle \in \boldsymbol{L}_\alpha$ by Iα2.(ii). Let $\Phi' = \langle\mathbf{B}_i(A \supset B) : B \in \Phi\rangle$. Then, since $\wedge\Phi' \in \boldsymbol{L}_\alpha$, we have, by (15), $\{\Box_i(a \to b) : b \in Q\} = \{[\![\mathbf{B}_i(A \supset B)]\!] : B \in \Phi\} \in \mathcal{Q}_1$.■

5.3 Construction of a counter model

Recall that $\mathbb{L} = (\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -, \Box_1, ..., \Box_n)$ with $\mathbf{B} = \boldsymbol{L}_\alpha/\sim$ is the Lindenbaum algebra given in Lemma 5.3. Also, let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ be given by (14). Now, we define a Kripke frame $\mathbb{K} = (W; R_1, ..., R_n)$ and an assignment τ as follows:

(i): $W = \mathbb{F}_\mathcal{Q}(\mathbb{L})$, where $\mathbb{F}_\mathcal{Q}(\mathbb{L})$ is the set of all $\mathcal{Q}$-filters for $\mathbb{L}$;

(ii): for all $i \in N$, wR_iu if and only if $\Box_i^{-1}w \subseteq u$;

(iii): for any $w \in W$ and any propositional variable p, $\tau(w, p) = \top$ if and only if $[\![p]\!] \in w$.

The nonemptiness of $\mathbb{F}_\mathcal{Q}(\mathbb{L})$ follows from Lemma 5.1 and the contradiction-freeness of GL$(\boldsymbol{L}_\alpha)$ noted in the end of Section 3.2.

We will prove that $\mathbb{M} = (\mathbb{K}, \tau) = (W; R_1, ..., R_n, \tau)$ is a Kripke model.

Lemma 5.6. *R_i is serial for each $i \in N$.*

Proof. Let $w \in W$. Consider $\Box_i[\![\neg\mathbf{p}_0 \wedge \mathbf{p}_0]\!] = [\![\mathbf{B}_i(\neg\mathbf{p}_0 \wedge \mathbf{p}_0)]\!]$ by $\ell1$. By Axiom D, we have $\Box_i[\![\neg\mathbf{p}_0 \wedge \mathbf{p}_0]\!] = [\![\neg\mathbf{p}_0 \wedge \mathbf{p}_0]\!] = \mathbf{0}$. Since w is a prime filter, we have $\mathbf{0} \notin w$. By Lemma 5.2 (Tanaka-Ono Lemma), we have $u \in \mathbb{F}_\mathcal{Q}(\mathbb{L})$ such that $\Box_i^{-1}w \subseteq u$, i.e., wR_iu, and $\mathbf{0} \notin u$.■

The following lemma is central to the completeness theorem.

Lemma 5.7. *For any $A \in \boldsymbol{L}_\alpha$ and $w \in W$, $(\mathbb{K}, \tau, w) \models A$ if and only if $[\![A]\!] \in w$.*

Proof. We prove the assertion by induction along the definition Iβ0-Iβ2 ($\beta \leq \alpha$) of formulae. Consider a propositional variable p. Then $(\mathbb{K}, \tau, w) \models p \Leftrightarrow \tau(w, p) = \top \Leftrightarrow [\![p]\!] \in w$.

Now, consider a non-propositional formula A in $\boldsymbol{L}_\beta$. Suppose that A is generated by Iβ1. Here, the induction hypothesis is simply that that the assertion holds for any proper subformulae of A. The case $\wedge$ is applied to an infinitary conjunctive formula generated directly Rβ0-Rβ1.

$(\supset)$: Let $(\mathbb{K},\tau,w) \models A \supset B$. Then $(\mathbb{K},\tau,w) \not\models A$ or $(\mathbb{K},\tau,w) \models B$. By the induction hypothesis, we have $[\![A]\!] \notin w$ or $[\![B]\!] \in w$. Since $[\![\neg A]\!] \in w$ or $[\![B]\!] \in w$, and since $[\![\neg A]\!] \leq [\![A \supset B]\!]$ and $[\![B]\!] \leq [\![A \supset B]\!]$, we have $[\![A \supset B]\!] \in w$.

Let $[\![A \supset B]\!] \in w$. Then $[\![\neg A \vee B]\!] = [\![\neg A]\!] \sqcup [\![B]\!] \in w$. Since w is a prime filter, we have $[\![\neg A]\!] \in w$ or $[\![B]\!] \in w$. Hence $[\![A]\!] \notin w$ or $[\![B]\!] \in w$. By the induction hypothesis, we have $(\mathbb{K},\tau,w) \not\models A$ or $(\mathbb{K},\tau,w) \models B$. Thus, $(\mathbb{K},\tau,w) \models A \supset B$.

$(\neg)$: The proof is similar to that for $(\supset)$ and is omitted.

$(\mathbf{B}_i)$: Let $(\mathbb{K},\tau,w) \models \mathbf{B}_i(A)$. Then $(\mathbb{K},\tau,u) \models A$ for any u with $(w,u) \in R_i$. By the induction hypothesis, $[\![A]\!] \in u$ for any u with $(w,u) \in R_i$. Now, on the contrary, suppose that $\Box_i[\![A]\!] \notin w$. Then, by Lemma 5.2 (Tanaka-Ono Lemma), there is a $u \in \mathbb{F}_{\mathcal{Q}}(\mathbb{L})$ such that $\Box_i^{-1}w \subseteq u$ and $[\![A]\!] \notin u$. This is a contradiction. Hence, $[\![\mathbf{B}_i(A)]\!] = \Box_i[\![A]\!] \in w$.

Let $[\![\mathbf{B}_i(A)]\!] = \Box_i[\![A]\!] \in w$. Then $[\![A]\!] \in u$ for all u with $\Box_i^{-1}w \subseteq u$. By the induction hypothesis, we have $(\mathbb{K},\tau,u) \models A$ for all u with $(w,u) \in R_i$. Hence, $(\mathbb{K},\tau,u) \models \mathbf{B}_i(A)$.

$(\wedge)$: Let $\wedge\Phi$ be a finite conjunctive formula generated by Iβ1, or an infinite conjunctive formula generated by Rβ0-Rβ1. In the latter case, any $A \in \Phi$ belongs to $\cup_{\gamma<\beta}\boldsymbol{L}_\gamma$. In either case, the induction hypothesis can be that the assertion holds for any $A \in \Phi$. In these case, we have the following proof.

Let $(\mathbb{K},\tau,w) \models \wedge\Phi$. Then $(\mathbb{K},\tau,w) \models A$ for all $A \in \Phi$. By the induction hypothesis, we have $[\![A]\!] \in w$ for all $A \in \Phi$. Then $\sqcap\{[\![A]\!] : A \in \Phi\}$ exists by Lemma 5.5.(1), and it belongs to w by Q1. Hence, $[\![\wedge\Phi]\!] = \sqcap\{[\![A]\!] : A \in \Phi\} \in w$.

Let $[\![\wedge\Phi]\!] \in w$. Then $[\![\wedge\Phi]\!] \leq [\![A]\!]$ for all $A \in \Phi$. Since w is a filter, we have $[\![A]\!] \in w$ for all $A \in \Phi$ by F1. Hence $(\mathbb{K},\tau,w) \models A$ for all $A \in \Phi$ by the induction hypothesis, which implies $(\mathbb{K},\tau,w) \models \wedge\Phi$.

Now, we consider the cases of Iβ2.(i), Iβ2.(ii), and Iβ2.(iii). Suppose that $\wedge\Phi = \wedge\langle D \supset C_\nu : \nu \geq 0\rangle$, $\wedge\Phi = \wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle$, or $\wedge\Phi = \wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle$ be generated by Iβ2.(i), Iβ2.(ii), or Iβ2.(iii) from D, $\wedge\langle C_\nu : \nu \geq 0\rangle$, and $\wedge\langle D_\nu : \nu \geq 0\rangle$. Here, the induction hypothesis is that the assertion holds form D, $\wedge\langle C_\nu : \nu \geq 0\rangle$, and $\wedge\langle D_\nu : \nu \geq 0\rangle$.

Let $(\mathbb{K},\tau,w) \models \wedge\langle D \supset C_\nu : \nu \geq 0\rangle$. Then $(\mathbb{K},\tau,w) \models D \supset C_\nu$, i.e., $(\mathbb{K},\tau,w) \not\models D$ or $(\mathbb{K},\tau,w) \models C_\nu$, for all $\nu \geq 0$. The latter part implies $(\mathbb{K},\tau,w) \models \wedge\langle C_\nu : \nu \geq 0\rangle$. By the induction hypothesis, we have $[\![D]\!] \notin w$ or $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$. Since w is a prime filter, we have $[\![D]\!] \rightarrow [\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$, which implies $[\![D \supset \wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$ by (13). Since $\vdash (D \supset \wedge\langle C_\nu : \nu \geq 0\rangle) \equiv \wedge\langle D \supset C_\nu : \nu \geq 0\rangle$, we have $[\![\wedge\langle D \supset C_\nu : \nu \geq 0\rangle]\!] \in w$. The converse can be obtained by tracing back this argument.

Let $(\mathbb{K},\tau,w) \models \wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle$. This implies $(\mathbb{K},\tau,w) \models \mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)$. Let u be any world with $(w,u) \in R_i$. Then, $(\mathbb{K},\tau,u) \models \wedge\langle C_\nu : \nu \geq 0\rangle$. By the induction hypothesis, we have $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in u$. Now, on the contrary, suppose that $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \notin w$. Then, by Lemma 5.2 (Tanaka-Ono Lemma), there is a $u^o \in \mathbb{F}_{\mathcal{Q}}(\mathbb{L})$ such that $\Box_i^{-1}w \subseteq u^o$ and $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \notin u^o$. Since $(w,u^o) \in R_i$ by the definition of R_i, this is a contradiction. Hence, $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$. Thus, $[\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle)]\!] = [\![\mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)]\!] = \Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$, using $\wedge$-Barcan.

Conversely, let $[\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle)]\!] \in w$. Then, $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] = [\![\mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)]\!] = [\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle]\!] \in w$ using $\wedge$-Barcan. Let $u \in W$ be an arbitrary world with $\Box_i^{-1}w \subseteq u$. Then, $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in u$. By the induction hypothesis, we have $(\mathbb{K},\tau,u) \models \wedge\langle C_\nu : \nu \geq 0\rangle$. Since u is arbitrary with $(w,u) \in R_i$, we have $(\mathbb{K},\tau,u) \models \mathbf{B}_i(C_\nu)$ for all $\nu \geq 0$. Hence, $(\mathbb{K},\tau,u) \models \wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle$.

Let $(\mathbb{K}, \tau, w) \models \wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle$. Then $(\mathbb{K}, \tau, w) \models C_\nu \wedge D_\nu$, i.e., $(\mathbb{K}, \tau, w) \models C_\nu$ and $(\mathbb{K}, \tau, w) \models D_\nu$ for all $\nu \geq 0$. This implies $(\mathbb{K}, \tau, w) \models \wedge\langle C_\nu : \nu \geq 0\rangle$ and $(\mathbb{K}, \tau, w) \models \wedge\langle D_\nu : \nu \geq 0\rangle$. By the induction hypothesis, we have $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$ and $[\![\wedge\langle D_\nu : \nu \geq 0\rangle]\!] \in w$. Since w is a filter, we have $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \sqcap [\![\wedge\langle D_\nu : \nu \geq 0\rangle]\!] \in w_\nu$. Since $\vdash [\langle C_\nu : \nu \geq 0\rangle \wedge \langle D_\nu : \nu \geq 0\rangle] \equiv \wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle$, we have $[\![\wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle]\!] \in w$. The converse can be obtained by tracing back this argument.$\blacksquare$

The final step of completeness is to show that for any $A \in \boldsymbol{L}_\alpha$, if $\nvdash A$, then $(\mathbb{K}, \tau, w) \nvDash A$ for some world $w \in W$ and some assignment τ. Indeed, suppose $\nvdash A$. This means that $[\![A]\!] \neq \mathbf{1}$, which implies $[\![A]\!] \ngeq \mathbf{1}$. Applying Lemma 5.1 (Rasiowa-Sikorski lemma) to $[\![A]\!]$ and $\mathbf{1}$, there is a $\mathcal{Q}$-filter F such that $\mathbf{1} \in F$ and $[\![A]\!] \notin F$. Denote F by w. Then, by Lemma 5.7, we have $(\mathbb{K}, \tau, w) \nvDash A$.$\blacksquare$

Remark 5.1. Lemma 5.6 can be extended to other epistemic axioms, T, 4, or 5, and the corresponding conditions, reflexivity, transitivity, or euclidean for R_i. Here, we check that transitivity is derived Axiom 4: Let $\Box_i^{-1} w \subseteq u$ and $\Box_i^{-1} u \subseteq v$ and $[\![A]\!] \in \Box_i^{-1} w$. Then, $[\![\mathbf{B}_i(A)]\!] \in w$. Since $\vdash \mathbf{B}_i(A) \supset \mathbf{B}_i\mathbf{B}_i(A)$, we have $[\![\mathbf{B}_i(A)]\!] \leq [\![\mathbf{B}_i\mathbf{B}_i(A)]\!]$. Since w is a filter, we have $[\![\mathbf{B}_i\mathbf{B}_i(A)]\!] \in w$; so $[\![\mathbf{B}_i(A)]\!] \in \Box_i^{-1} w$. Hence, $[\![\mathbf{B}_i(A)]\!] \in u$, i.e., $[\![A]\!] \in \Box_i^{-1} u$. Repeating this argument, we have $[\![A]\!] \in \Box_i^{-1} v$.

Euclidean is derived by Axiom 5: Recall the euclidean condition that wR_iu and wR_iv imply uR_iv. Suppose $\Box_i^{-1} w \subseteq u$ and $\Box_i^{-1} w \subseteq v$. Suppose that for some A, $[\![A]\!] \in \Box_i^{-1} u$ but $[\![A]\!] \notin v$. Since $\Box_i^{-1} w \subseteq v$, we have $[\![\mathbf{B}_i(A)]\!] \notin w$. Thus, $[\![\neg\mathbf{B}_i(A)]\!] \in w$. By Axiom 5, $[\![\mathbf{B}_i(\neg\mathbf{B}_i(A))]\!] \in w$. Thus, $[\![\neg\mathbf{B}_i(A))]\!] \in \Box_i^{-1} w \subseteq u$, which is a contradiction to $[\![A]\!] \in \Box_i^{-1} u$. Hence, $\Box_i^{-1} u \subseteq v$.

6 Conclusions

We have developed a series of small infinitary epistemic logics. This series may be regarded as lying between the fixed-point logic approach and the traditional infinitary logic approach. Our approach, like the traditional infinitary logic approach, allows for an explicit formulation of infinitary concepts such as common knowledge, but, at the same time, like the fixed point approach, we can control infinitary expressions by imposing specific germinal forms and limited number of layers of infinitary expressions. These advantages allow us to evaluate the required depths of a specific epistemic concept, and deepens our understanding of the two approaches.

Besides the fixed-point logics discussed in the present paper, there is a large literature on the fixed-point approach and the modal μ-calculus (cf., Venema [24], Jäger *et al.* [7], Jäger-Studer [8]). Section 4.3 connect some particular fixed-point logics to our small infinitary logics. In particular, our embedding result of the common knowledge logic (Theorem 4.8) is indicative to a more general relationship between these two approaches, but this remains to be an open problem.

Finally, our approach can be applied to game theory. An important question in game theory is to understand implications of common beliefs of payoff maximization to each individual agent's decision making. One solution concept based on this idea is *rationalizability* (cf. Osborne-Rubinstein [18]). We can formalize this solution concept as a decision criterion and characterize infinite regress of beliefs of the criterion. The full epistemic characterization of this concept can be done within GL($\boldsymbol{L}_2$) but not in GL($\boldsymbol{L}_1$). The conservativity result implies that it is without loss of generality to focus on epistemic logic GL($\boldsymbol{L}_2$) for this game theoretical application, which is a small part in the series in (1).

7 Appendix: Application to Rationalizability in Game Theory

The application of GL($\boldsymbol{L}_\alpha$) here has purposes. From the viewpoint of game theory, it gives a clear representation of a solution concept called the *rationalizable actions (strategies)* (cf., Myerson[17] and Osborne-Rubinstein [18]). The original formulation is given in the standard game theory language, and in the recent game theory literature, it is formulated in terms of common probabilistic beliefs. Our logic GL($\boldsymbol{L}_\alpha$) allows for an explicit treatment of it in terms of epistemic infinite regress, rather than the common beliefs, and enables us to evaluate the epistemic depths the theory requires. From the viewpoint of our logic framework, it gives a concrete example of a discourse requiring only GL($\boldsymbol{L}_\alpha$) exactly with $\alpha = 2$, which differs from the depths required by various concepts discussed in this paper.

A 2-person game is given as $G = (\{1,2\}, S_1, S_2, g_1, g_2)$, where 1 and 2 are agents, S_i is a finite nonempty set of *available actions*, and $g_i : S_1 \times S_2 \to R$ is the *payoff function* of agent $i = 1, 2$. In our logic, we represent the payoff functions g_1 and g_2 in terms of preference relations. In this game, the two agents simultaneously choose their actions, and hence have to infer each's choice from the available information. In fact, for the characterization of rationalizable actions, we need only the language without detailed information about g_1 and g_2. For the existence (playability) of rationalizable actions, we need some information about g_1 and g_2. We will return to this point after stating the main theorems.

We use the following atomic propositions as propositional variables:

preference symbols $\mathrm{Pr}_i(s_1, s_2 : t_1, t_2)$ for $i = 1, 2$ and $(s_1, s_2), (t_1, t_2) \in S_1 \times S_2$;

decision symbols $\mathrm{I}_i(s_i)$ for $s_i \in S_i$ and $i = 1, 2$;

auxiliary propositional variables $p(s_i)$ for $s_i \in S_i$ and $i = 1, 2$.

The atomic proposition $\mathrm{Pr}_i(s_1, s_2 : t_1, t_2)$ is interpreted as "agent i weakly prefers (s_1, s_2) to (t_1, t_2)". We write this as $\mathrm{Pr}_i(s_i; s_j : t_i; t_j)$, where $\{i, j\} = \{1, 2\}$. In the following, we denote the other agent by j. The expression $\mathrm{I}_i(s_i)$ is interpreted as "s_i is a *possible* final decision for agent i"; in the following theory, $\mathrm{I}_i(s_i)$ is determined based on primitive preferences, $\mathrm{Pr}_i(s_1, s_2 : t_1, t_2)$. The auxiliary propositional variables $p(s_i)$ are used for simplicity.

Let $i, j = 1, 2$. We introduce the formula $\mathrm{Bst}_i(s_i; t_j)$: for $s_i \in S_i$ and $t_j \in S_j$,

$$\mathrm{Bst}_i(s_i; t_j) := \wedge\{\mathrm{Pr}_i(s_i; t_j : s_i'; t_j) : s_i' \in S_i\}. \tag{16}$$

This means that s_i is a best action when agent j chooses t_j.

We use the concept of an epistemic infinite regress $\mathbf{Ir}_i[A_1, A_2] = \wedge\langle \mathbf{Ir}_i^\nu[A_1, A_2] : \nu \geq 0\rangle$ for $i = 1, 2$ stated in Remark 2.1. We denote the germinal form for this by $G^{IR} = (\mathbf{B}_1(p_1), \mathbf{B}_2(p_2))$. To describe "rationalizable strategies", we need the other germinal form $G^R = [(G_1^R(s_1), G_2^R(s_2)) : (s_1, s_2) \in S]$ given by

$$G_i^R(s_i) = \bigvee_{t_j \in S_j} \langle \mathbf{B}_j(p(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle \text{ for } s_i \in S_i \text{ and } i = 1, 2. \tag{17}$$

Recall Remark 3.1: $\vee\Phi$ is defined to be $\neg \wedge \{\neg A : A \in \Phi\}$ for a finite nonempty set of formulae Φ in $\boldsymbol{L}_\alpha$. In this appendix, the language $\boldsymbol{L}_\alpha$ is defined based on the germinal forms $\mathcal{G}^{IR+R} = \{G^{IR}, G^R\}$. As stated in Remark 2.1, the Soundness-Completeness Theorem (Theorem 3.1) holds for GL($\boldsymbol{L}_\alpha$).

Since $G^R = [(G_1^R(s_1), G_2^R(s_2)) : (s_1, s_2) \in S]$ consists of $|S_1 \times S_2|$-number of formulae, it would be better to repeat Steps Rα0 and Rα1. They generate the set of formulae $\{\mathrm{Rat}_i^\nu(s_i) : \nu \geq 0,$

$s_i \in S_i$, and $i = 1, 2\}$:

$$\begin{aligned} \mathrm{R}\alpha 0 &: \mathrm{Rat}_i^0(s_i) = \vee_{t_j \in S_j} \mathrm{Bst}_i(s_i; t_j) \text{ for } s_i \in S_i \text{ and } i = 1, 2; \\ \mathrm{R}\alpha 1 &: \mathrm{Rat}_i^{\nu+1}(s_i) = \underset{t_j \in S_j}{\vee} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle \text{ for } s_i \in S_i \text{ and } i = 1, 2. \end{aligned} \tag{18}$$

For each $s_i \in S_i$ and $i = 1, 2$, we have the sequence $\langle \mathrm{Rat}_i^\nu(s_i) : \nu \geq 0\rangle$, and we denote its conjunction $\wedge\langle \mathrm{Rat}_i^\nu(s_i) : \nu \geq 0\rangle$ by $\mathrm{Rat}_i(s_i)$. An action $s_i \in S_i$ is called a *rationalizable action (strategy)* iff $\mathrm{Rat}_i^\nu(s_i)$ holds for all $\nu \geq 0$.[7] Each formula in (18) is in $\boldsymbol{L}_0$, but their conjunction $\wedge\langle \mathrm{Rat}_i^\nu(s_i) : \nu \geq 0\rangle$ belongs to $\boldsymbol{L}_1(\mathcal{G}^R)$.

The formula $\mathrm{Rat}_i(s_i)$ appears in the mind of agent i; thus, it occurs in the scope $\mathbf{B}_i(\cdot)$. However, to avoid complications, we often consider $\mathrm{Rat}_i(s_i)$ without the scope $\mathbf{B}_i(\cdot)$.

Here, we consider $\mathrm{Rat}_i(s_i)$ as a concept derived from a more basic description of agent i's (*ex ante*) decision making. We adopt the following criterion of how agent i makes a decision:

$$\boldsymbol{D}_i^R : \wedge_{s_i \in S_i}(\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle).$$

Thus, this requires agent i to find some predicted decision t_j for j and i's decision s_i to be a best response to t_j. Here, we adopt $\boldsymbol{D}_i^R$ for the case of "rationalizable action"; but we have different criteria, for example, "Nash strategies", which will be discussed in a different place.

Agent i uses the criterion D_i^R to infer his final decision. Hence, D_i^R occurs in the mind of agent i, which is expressed as $\mathbf{B}_i(D_i^R)$. Since t_j occurs in $\mathbf{B}_j(\mathrm{I}_j(t_j))$ in D_i^R, this requires i to predict j's decision. This prediction is inferred by the imaginary agent j following D_j^R in the mind of agent i. The source for this inference is formally expressed as $\mathbf{B}_i\mathbf{B}_j(D_j^R)$. Again, this includes $\mathbf{B}_i(\mathrm{I}_i(t_i))$ in D_j^R, and by the parallel argument to the above, $\mathbf{B}_i\mathbf{B}_j\mathbf{B}_i(D_i^R)$ is required. Unless we stop this argument at some finite level, we would have an infinite regress:

$$\mathbf{B}_i(D_i^R) \to \mathbf{B}_i\mathbf{B}_j(D_j^R) \to \mathbf{B}_i\mathbf{B}_j\mathbf{B}_i(D_i^R) \to ... \tag{19}$$

Each formula is in $\boldsymbol{L}_0$, but their conjunction $\mathbf{Ir}_i[\boldsymbol{D}^R] := \mathbf{Ir}_i[D_1^R, D_2^R]$ belongs to $\boldsymbol{L}_1(\mathcal{G}^{IR+R})$. This motivates the term "epistemic infinite regress".

The expression "infinite regress" often indicates a difficulty in finding an intended meaning to be described. However, we regard the formula $\mathbf{Ir}_i[\boldsymbol{D}^R]$ as a system of equations with unknowns $\mathrm{I}_1(s_1)$ and $\mathrm{I}_2(s_2)$, and would like to find these unknown formulae. First, we show that any solution to $\mathbf{Ir}_i[\boldsymbol{D}^R]$ is logically stronger than or equal to $\mathrm{Rat}_i(s_i)$. This theorem will be proved in the end of this appendix.

Theorem 7.1. ***(Necessity)**. Let $s_i \in S_i, s_j \in S_j$ and $i, j = 1, 2$ $(i \neq j)$. Then,*

***(1):** $GL(\boldsymbol{L}_1(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset [\mathbf{B}_i(I_i(s_i)) \supset \mathbf{B}_i(Rat_i(s_i))]$;*

***(2):** $GL(\boldsymbol{L}_2(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset \mathbf{Ir}_i[I_i(s_i) \supset Rat_i(s_i); I_j(s_j) \supset Rat_j(s_j)]$.*

The first states that any individual decision determined by $\mathbf{Ir}_i[\boldsymbol{D}^R]$ is a rationalizable action. The second state that these implications for $i = 1, 2$ form also an infinite regress. For the first, the epistemic logic $\mathrm{GL}(\boldsymbol{L}_\alpha(\mathcal{G}^{IR+R}))$ of depth $\alpha = 1$ is sufficient, but for the second, $\alpha = 2$ is required since $\mathrm{Rat}_i(s_i)$ occurs in $\mathbf{Ir}_i[\cdot;\cdot]$.

Now, consider the converse of the above theorem. As stated, we regard $\boldsymbol{D}^R = [D_1^R, D_2^R]$ as a system of equations with unknowns $\{I_i(s_i)\}_{t_i \in S_i}, i = 1, 2$. Candidates for those unknowns are

[7] In the game theory literature, this (eliminating all $\mathbf{B}_i$ and $\mathbf{B}_j$) is called a *point-rationalizable strategy* in that it takes no probabilistic beliefs into account.

formulae validating $\boldsymbol{D}^R = [D_1^R, D_2^R]$ when we plug them to the unknowns. As candidates, we consider only $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ where $\mathcal{R}_i = \{\mathrm{Rat}_i(s_i)\}_{s \in S_i}, i = 1, 2$. We use $\boldsymbol{D}^R(\mathcal{R}) = [D_1^R(\mathcal{R}), D_1^R(\mathcal{R})]$ to denote the formulae obtained from $\boldsymbol{D}^R = [D_1^R, D_2^R]$ by plugging $\mathcal{R}_i = \{\mathrm{Rat}_i(t_i)\}_{t_i \in S_i}$ to $\{\mathrm{I}_i(t_i)\}_{t_i \in S_i}$ for $i = 1, 2$. When $\boldsymbol{D}^R(\mathcal{R})$ holds, these $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ are regarded as solutions for $\boldsymbol{D}^R = [D_1^R, D_2^R]$. This argument is formulated as:

$\mathbf{WD}_i^R$: $D_i^R(\mathcal{R}) \supset \wedge_{t_i \in S_i}(\mathrm{Rat}_i(t_i) \supset \mathrm{I}_i(t_i))$.

We write $\mathbf{WD}^R = (\mathbf{WD}_1^R, \mathbf{WD}_2^R)$.

When this is assumed, we have the following theorem, which will be proved below.

Theorem 7.2. ***(Full Characterization).*** *Let* $(s_1, s_2) \in S_1 \times S_2$ *and* $i = 1, 2$. *Then,*

(1): $GL(\boldsymbol{L}_2(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\mathbf{WD}^R] \supset \mathbf{Ir}_i[Rat_1(s_1) \supset I_1(s_1), Rat_2(s_2) \supset I_2(s_2)]$;

(2): $GL(\boldsymbol{L}_2(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\mathbf{D}^R] \wedge \mathbf{Ir}_i[\mathbf{WD}^R] \supset \mathbf{Ir}_i[Rat_1(s_1) \equiv I_1(s_1), Rat_2(s_2) \equiv I_2(s_2)]$.

The second is the full characterization of $\mathrm{I}_1(s_1)$ and $\mathrm{I}_2(s_2)$, which follows from Theorem 7.1.(2) and Theorem 7.2.(1). Thus, a full epistemic characterization of a rationalizable action is done in $\mathrm{GL}(\boldsymbol{L}_2(\mathcal{G}^{IR+R}))$.

In addition to the above full characterization, we can talk about the game theoretical content when we assume the epistemic infinite regress of the payoff structures. That is, for $i = 1, 2$, we denote the following set of formulae by Γ_i :

$$\{\mathrm{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) \geq g_i(t_1, t_2)\} \cup \{\neg \mathrm{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) < g_i(t_1, t_2)\}.$$

Then, we assume the infinite regress $\mathbf{Ir}_i[\mathbf{\Gamma}] = \mathbf{Ir}_i[\wedge\Gamma_1, \wedge\Gamma_2]$. Then, we have

$$\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathrm{B}_i(\mathrm{Rat}_i(s_i)) \Longleftrightarrow s_i \text{ is a rationalizable action;} \tag{20}$$

$$\mathrm{GL}(\boldsymbol{L}_2(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i[\mathbf{D}^R] \wedge \mathbf{Ir}_i[\mathbf{WD}^R(\mathcal{R})] \wedge \mathbf{Ir}_i[\mathbf{\Gamma}] \supset I_i(s_i) \text{ for some } s_i \in S_i. \tag{21}$$

The right-hand side of (20) is stated in $G = (\{1, 2\}, S_1, S_2, g_1, g_2)$, and the existence of a rationalizable action is known. Then, we have (21) by Theorem 7.2.(1).

Now, let us prove the above two theorems. Fist, we mention Lemma 7.1, in which $\mathrm{GL}(\boldsymbol{L}_2)$ is required for (3), but $\mathrm{GL}(\boldsymbol{L}_1)$ is enough for the others as far as content formulae in $\boldsymbol{L}_1$.

Lemma 7.1. ***(1):*** $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \equiv \mathbf{B}_i(\mathbf{Ir}_i^o[\boldsymbol{A}])$, *where* $\mathbf{Ir}_i^o[\boldsymbol{A}] := A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}]$;

(2): *if* $\vdash A_j$ *for* $j = 1, 2$, *then* $\vdash \mathbf{Ir}_i[\boldsymbol{A}]$; ***(3):*** $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i[\mathbf{Ir}_i^o[\boldsymbol{A}]; \mathbf{Ir}_j^o[\boldsymbol{A}]]$;

(4): $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i(A_1, A_2) \supset \mathbf{Ir}_i(C_1, C_2)$, *equivalently,*
$\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \supset [\mathbf{Ir}_i(A_1, A_2) \supset \mathbf{Ir}_i(C_1, C_2)]$;

(5): $\vdash \mathbf{Ir}_i[A_1, A_2] \wedge \mathbf{Ir}_i[C_1, C_2] \equiv \mathbf{Ir}_i[A_1 \wedge C_1, A_2 \wedge C_2]$.

Proof. We prove (1), (3), and (4).
(1): Recall $\mathbf{Ir}_i[\boldsymbol{A}] = \wedge\langle \mathbf{Ir}_i^\nu[\boldsymbol{A}] : \nu \geq 0\rangle$, where $\mathbf{Ir}_i^0[\boldsymbol{A}] = \mathbf{B}_i(A_i)$ and $\mathbf{Ir}_i^{\nu+1}[\boldsymbol{A}] = \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j^\nu[\boldsymbol{A}])$ for all $\nu \geq 0$. Hence, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j^\nu[\boldsymbol{A}])$ for all $\nu \geq 0$; so $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \wedge\langle \mathbf{B}_i(\mathbf{Ir}_j^\nu[\boldsymbol{A}]) : \nu \geq 0\rangle$. By $\wedge$-Barcan, we have $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(\wedge\langle \mathbf{Ir}_j^\nu[\boldsymbol{A}] : \nu \geq 0\rangle)$. Thus, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}])$. The converse is similar.
(3): By (1), $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(\mathbf{Ir}_i^o[\boldsymbol{A}])$ for $i = 1, 2$. Suppose that $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i^\nu[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]]$ for $i = 1, 2$. Since $\vdash \mathbf{B}_i(\mathbf{Ir}_j[\boldsymbol{A}]) \supset \mathbf{B}_i(\mathbf{Ir}_j^\nu[\mathbf{Ir}_i^o[\boldsymbol{A}], \mathbf{Ir}_i^o[\boldsymbol{A}]])$ by Nec and K, and since $\vdash \mathbf{Ir}_i[\boldsymbol{A}]) \supset$

$\mathbf{B}_i(A_i)$ by (1), we have $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i^{\nu+1}[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_1^o[\boldsymbol{A}]]$. Hence, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]]$.
(4): It suffices to show that $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i(A_1, A_2) \supset \mathbf{Ir}_i(C_1, C_2)$. It is proved by induction over ν that $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i[A_1, A_2] \supset \mathbf{Ir}_i^\nu[C_1, C_2]$ for all $\nu \geq 0$. By $\wedge$-rule, we have the result.■

Using the epistemic content $\mathbf{Ir}_i^o[\boldsymbol{A}] := A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}]$ of $\mathbf{Ir}_i[\boldsymbol{A}]$, we have the following lemma.

Lemma 7.2. *$GL(\boldsymbol{L}_1(\mathcal{G}^{IR+R})) \vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [I_i(s_i) \supset Rat_i^\nu(s_i)]$ for all $\nu \geq 0$ and $i = 1, 2$.*

Proof. We show it by induction on ν. Since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset D_i^R$ and $\vdash D_i^R \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j} \mathrm{Bst}_i(s_i; t_j)]$, we have the assertion for $\nu = 0$. Suppose the assertion for ν. Then, $\vdash \mathbf{Ir}_j[\boldsymbol{D}^R] \supset [\mathbf{B}_j(\mathrm{I}_j(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j) \supset \mathbf{B}_j(\mathrm{Rat}_j^\nu(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j)]$. Hence, we have $\vdash \mathbf{Ir}_j[\boldsymbol{D}^R] \supset [\vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle \supset \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle]$. Since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset \mathbf{Ir}_j[\boldsymbol{D}^R]$, we have $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle \supset \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle]$. Also, since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle]$, we have $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle]$. Thus, $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \mathrm{Rat}_i^{\nu+1}(s_i)]$. Hence, we have the assertion for $\nu + 1$.■

Proof of Theorem 7.1.(1). This is obtained by Lemma 7.2.
(2): Lemma 7.2 implies $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset (\mathrm{I}_i(s_i) \supset \mathrm{Rat}_i(s_i))$ for $i = 1, 2$. By Lemma 7.1.(2), we have $\vdash \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R] \supset (\mathrm{I}_1(s_1) \supset \mathrm{Rat}_1(s_1)), \mathbf{Ir}_2^o[\boldsymbol{D}^R] \supset (\mathrm{I}_2(s_2) \supset \mathrm{Rat}_2(s_1))]$. Using Lemma 7.1.(4), we have $\vdash \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R], \mathbf{Ir}_2^o[\boldsymbol{D}^R]] \supset \mathbf{Ir}_i[\mathrm{I}_1(s_1) \supset \mathrm{Rat}_1(s_1), \mathrm{I}_2(s_2) \supset \mathrm{Rat}_2(s_1)]$. Since $\vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R], \mathbf{Ir}_2^o[\boldsymbol{D}^R]]$ by Lemma 7.1.(3), we have the assertion.■

Now, we turn to the proof of Theorem 7.2. We show: for $i = 1, 2$, $\vdash D_i^R(\mathcal{R})$, i.e.,

$$\vdash \mathrm{Rat}_i(s_i) \supset \vee_{t_j \in S_j} \langle \mathrm{B}_j(\mathrm{Rat}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle \text{ for } s_i \in S_i. \tag{22}$$

Then, it follows from Lemma 7.1.(2) that $\vdash \mathbf{Ir}_i[D_1^R(\mathcal{R}), D_2^R(\mathcal{R})]$. By Lemma 7.1.(4),

$$\vdash \mathbf{Ir}_i[D_1^R(\mathcal{R}), D_2^R(\mathcal{R})] \wedge \mathbf{Ir}_i[\mathbf{WD}_1^R, \mathbf{WD}_2^R] \supset \mathbf{Ir}_i[\mathrm{Rat}_1(s_1) \supset \mathrm{I}_1(s_1), \mathrm{Rat}_1(s_1) \supset \mathrm{I}_2(s_1)].$$

Hence, $\vdash \mathbf{Ir}_i[\mathbf{WD}_1^R, \mathbf{WD}_2^R] \supset \mathbf{Ir}_i[\mathrm{Rat}_1(s_1) \supset \mathrm{I}_1(s_1), \mathrm{Rat}_1(s_1) \supset \mathrm{I}_2(s_1)]$, which is Theorem 7.2.(1). Combining this with Theorem 7.1.(2) by Lemma 7.1.(5), we have the second assertion of Theorem 7.2.

The first step for (22) is the following lemma.

Lemma 7.3. *(Monotonicity) For all $\nu \geq 0$, $s_i \in S_i$, and $i = 1, 2$, $\vdash Rat_i^{\nu+1}(s_i) \supset Rat_i^\nu(s_i)$, which is proved in $GL(L_0)$.*

Proof. We prove the assertion by induction over $\nu \geq 0$. Recall $\mathrm{Rat}_i^0(s_i) = \vee_{t_j \in S_j} \mathrm{Bst}_i(s_i; t_j)$. Since $\mathrm{Rat}_i^1(s_i) = \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^0(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle$, we have $\vdash \mathrm{Rat}_i^1(s_i) \supset \vee_{t_j \in S_j} \mathrm{Best}(s_i; t_j)$, i.e., $\vdash \mathrm{Rat}_i^1(s_i) \supset \mathrm{Rat}_i^0(s_i)$. Suppose that $\vdash \mathrm{Rat}_i^{\nu+1}(s_i) \supset \mathrm{Rat}_i^\nu(s_i)$ for $i = 1, 2$. This implies $\vdash \mathbf{B}_j(\mathrm{Rat}_j^{\nu+1}(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j) \supset \mathbf{B}_j(\mathrm{Rat}_j^\nu(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j)$, and then $\vdash \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu+1}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle \supset \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j) \rangle$, i.e., $\vdash \mathrm{Rat}_i^{\nu+2}(s_i) \supset \mathrm{Rat}_i^{\nu+1}(s_i)$.■

Now, we prove (22). The proof of part (1) is based on the soundness/completeness (Theorem 3.1); the finiteness of S_i, and also Lemma 7.3 are used. In the following lemma, we write $\wedge_\nu$ for $\wedge_{0 \leq \nu < \omega}$.

Lemma 7.4. *$GL(L_1(\mathcal{G}^R)) \vdash \mathrm{Rat}_i(s_i) \supset \vee_{t_j \in S_j} [\mathbf{B}_j(\mathrm{Rat}_j(t_j)) \wedge Bst_i(s_i; t_j)]$.*

Proof. First, we recall $\wedge_\nu \mathrm{Rat}_i^\nu(s_i) = \wedge_\nu \vee_{t_j \in S_j}[\mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)]$. We prove $\vdash \wedge_\nu \mathrm{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\wedge_\nu \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle \wedge \mathrm{Bst}(s_i; t_j)]$. By rule I$\alpha$2.(*ii*) with $\alpha = 1$, $\wedge_\nu \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle$ is permissible conjunction. Since $\vdash \wedge_\nu \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle \equiv \mathbf{B}_j(\wedge_\nu \mathrm{Rat}_j^\nu(t_j))$, we have $\vdash \wedge_\nu \mathrm{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\mathbf{B}_j(\wedge_\nu \mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}(s_i; t_j)]$, which is the assertion.

Let $\mathcal{M} = (\mathcal{F}, \tau)$ be a serial Kripke model, and w any possible world in W. Suppose $(\mathcal{M}, w) \models \wedge_\nu \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle$. Then, $(\mathcal{M}, w) \models \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle$ for any $\nu \geq 0$. Let

$$T_j^\nu = \{t_j \in S_j : (\mathcal{M}, w) \models \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\} \text{ for } \nu \geq 0.$$

Since $(\mathcal{M}, w) \models \vee_{t_j \in S_j} \langle \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle$, we have $T_j^\nu \neq \emptyset$ for all ν. Since $(\mathcal{M}, w) \models \mathrm{Rat}_j^{\nu+1}(s_j) \supset \mathrm{Rat}_j^\nu(s_j)$ by Lemma 7.3 and Soundness, we have $T_j^\nu \supseteq T_j^{\nu+1}$ for all $\nu \geq 0$. Since S_j is a finite set, there is some ν_0 such that T_j^ν is constant for all $\nu \geq \nu_0$. Hence, we find an $s_j \in \cap_\nu T_j^\nu$, which implies $(\mathcal{M}, w) \models \langle \wedge_\nu \mathbf{B}_j(\mathrm{Rat}_j^\nu(s_j))\rangle \wedge \mathrm{Bst}_i(s_i; s_j)$. Thus, $(\mathcal{M}, w) \models \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle \wedge \mathrm{Bst}_i(s_i; t_j)]$. Thus, $(\mathcal{M}, w) \models \wedge_\nu \mathrm{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle \wedge \mathrm{Bst}_i(s_i; t_j)]$. Since $\mathcal{F}, \tau, w \in W$ are all arbitrary, we have $\vdash \wedge_\nu \mathrm{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\mathrm{Rat}_j^\nu(t_j))\rangle \wedge \mathrm{Bst}_i(s_i; t_j)]$ by completeness.■

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