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Approximate Quasi-Linearity for Large Incomes

Mamoru Kaneko

Waseda INstitute of Political EConomy
Waseda University
Tokyo,Japan

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Mamoru Kaneko†

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Abstract

Quasi-linear utility functions are widely used in economics and game theory as a convenient assumption on agents. Quasi-linearity ignores income effects on individual economic activities: Indeed, It is captured by the condition of no-income effects. However, we can observe many social/economic problems, such as in housing markets, where income effects are non-negligible relative to agents' economic activities. In this paper, we consider how quasi-linearity holds for large incomes for agents. We give an axiomatic approach to this problem, and study its implications and applications to some economic and game theoretic problems.

Key words: Quasi-linear Utility Function, Approximate Quasi-Linearity, Cauchy Condition, Normality, Cooperative Game, Lindahl-ratio Equilibrium

1. Introduction

Quasi-linear utility functions are widely used in economics and game theory. This assumption simplifies developments of theories a lot; for example, in the theory of TU cooperative games, Pareto optimality for a given coalition of players can be expressed by a one-dimensional value of the maximum total surplus, while in the theory without the assumption, Pareto optimality should be described by a set of feasible utility vectors for the coalition. In the cost-benefit analysis, similarly, the total surplus (minus the total cost) from a policy is used as the criterion to recommend it or not.

Quasi-linearity ignores income effects in individual economic activities. Indeed, it is captured by the no-income effect condition: a simple axiomatization of quasi-linearity is found in Aumann [1] and Kaneko [7] (see also Kaneko-Wooders [10], Mas-Collel *et al.*[16]). Thus, quasi-linearity is an approximation of the structure of a utility function

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†Faculty of Political Science and Economics, Waseda University, Tokyo 169-8050, Japan (mkanekoepi@waseda.jp)

in a domain of large incomes. However, income effects are typically observed when expenditures for economic activities in question are non-negligible relative to incomes; typical examples are individual behavior of purchase of a house, automobile etc. Hence, it is desirable to study quasi-linearity from the domain allowing negligible income effects. As far as the present author knows, only Miyake [12], [13], and [14] studied quasi-linearity from the domain including income effects. In this paper, we study how much the case of income effects and case of no income effects are reconciled: indeed, we give an axiomatic approach to this problem and study its implications.

Miyake [12] studies the above problem in the classical economics context where the consumption set is given as the nonnegative orthant R^2_+ of R^2 and the utility function U is given as a function of C^2. Under the normality condition on income effects as well as quasi-concavity, he gave various conditions to guarantee the result that the utility function U is approximated by a quasi-linear function for large incomes.[1] In [13], he studied the behavior of the demand function for large incomes under similar conditions.[2] Miyake [14] is closer to our study, specifically, in Section 4.

Here, our treatment is more direct to approximate quasi-linearity than in [12]. We start with the characterization of quasi-linearity. Let $\succsim$ be a given preference relation over $X \times R_+$, where X is an arbitrary set of alternatives in question and R_+ is the set of nonnegative real numbers, interpreted as a consumption level measured by a composite commodity (Marshall's money, see Hicks [5], Chap.III, and [6], Chap.5). In addition to certain basic conditions on $\succsim$, when we add the *no-income-effects* condition – C4^{NI} in Section 2, we have a quasi-linear utility function $u : X \to R$ so that for all $(x, c), (x', c') \in X \times R_+$,

$$(x, c) \succsim (x', c') \iff u(x) + c \geq u(x') + c'. \tag{1.1}$$

Our main theorem (Theorem 3.1) replaces the no-income-effect condition C3^{NI} by a weaker condition, C4 – a Cauchy property, given in Section 3, and states that a utility function $U : X \times R_+ \to R$ representing $\succsim$ is approximated by a quasi-linear utility function u in the sense that for all $(x, c) \in X \times R_+$, there is a c_0 such that

$$|U(x, c) - (u(x) + c)| < \varepsilon \text{ for all } c \geq c_0. \tag{1.2}$$

Both functions U and u are derived from $\succsim$ with the basic conditions. Condition (1.2) itself was first mentioned in Miyake [12]. We will show that under other basic conditions, our condition C4 is equivalent to (1.2).

Condition (1.2) means that $u(x) + c$ approximates $U(x, c)$ for large c. The essential part of this is that $u(x) < \infty$ is independent upon c. An implication of Theorem 3.1

[1] He used the term "asymptotic quasi-linearity". We use "approximate quasi-linearity" to emphasize approximation of a utility function including income effects by a quasi-linear utility function.

[2] There are a number of articles studying the behavior of the individual demand function with "small" income effects. See [13] for this literature.

is that (1.2) does not necessarily hold for a function $U(x, c)$ familiar to us. We should consider the meaning of "utility" from a given x for large c. The implication is justified by assuming that x is tradable in society, though we exclude familiar mathematical functions from the candidates of approximate quasi-linearity. This will be discussed in the end of Section 3.1.

To study quasi-linearity as well as the implication mentioned above more clearly, we give another set of sufficient conditions for (1.2) in terms of normality, which is a weakening of the no-income effect condition C4^{NI}. This will be given Section 4.

Since economic theory and/or game theory with quasi-linear utility functions are well investigated, it is convenient to have connections from these cases to the large finite cases. Specifically, we ask the question of how we can convert results obtained with quasi-linear utility functions to cases with large finite incomes. In this paper, we apply our theorem to the theory of cooperative (TU) game theory, and to the theory of Lindahl-ratio equilibrium in a public goods economy.

Diagram 1.1 gives a schematic explanation of these applications. We start with a base model E and its quasi-linear approximation E_q, which is the arrow $\longrightarrow$ in Diagram 1.1. Then, some results are obtained in E_q, described in the right-bottom corner; then they are converted to the left-bottom corner. By our result (1.2), the converted results hold approximately in E_q.

Base model E	$\longrightarrow$ large incomes	TU-model E_q
$\Uparrow$ approximately		$\Downarrow$ analysis
Results	$\longleftarrow$ conversion	Results

Diagram 1.1

This paper is written as follows: Section 2 reviews the characterization of a quasi-linear utility function in terms of a preference relation by Kaneko [7]. In Section 3, we give a characterization for a preference relation to be approximately represented by a quasi-linear utility function, and we consider its application to the theory of TU cooperative games. Section 4 gives another axiomatization in terms of normality, and an application to the theory of Lindahl-ratio equilibrium. Section 5 extends the result in Section 3 to expected utility theory. Section 6 concludes this paper with a few remarks.

2. Quasi-linear Utility Function

A *preference relation* $\succsim$ is a binary relation over $X \times R_+$. An expression $(x, c) \succsim (x', c')$ means that (x, c) is weakly preferred to (x', c'). First, we assume Condition C0:

C0 (Complete preordering): $\succsim$ is complete and transitive over $X \times R_+$.

Under C0, we define the *strict part* $\succ$ and the *indifference part* $\sim$ as follows: $(x, c) \succ$

$(x', c') \Longleftrightarrow$ not $(x', c') \succsim (x, c)$; and $(x, c) \sim (x', c') \Longleftrightarrow (x, c) \succsim (x', c')$ and $(x', c') \succsim (x, c)$.

We assume the following three basic conditions. Some results holds without some of them, but we do not discuss such details.

C1 (Monotonicity): for any $x \in X$, if $c > c'$, then $(x, c) \succ (x, c')$;

C2 (Monetary substitutability): if $(x, c) \succ (x', c')$, then there is an $\alpha > 0$ such that $(x, c) \sim (x', c' + \alpha)$.

C3 (Fixed reference): There is an $x_0 \in X$ such that $(x, 0) \succsim (x_0, 0)$ for all $x \in X$.

Condition C1 is coherent with the interpretation of R_+ in terms of the composite commodity. Condition C2 may be interpreted as meaning that the economic activities behind the composite commodity R_+ is rich enough to substitute for a transition from x' to x. Condition C3 means that x_o is the worst alternative in X with zero consumption. This is guaranteed by C0 when X is a finite set.[3]

Quasi-linearity can be captured by adding Condition C4^{NI};

C4^{NI} (No income effects): if $(x, c) \sim (x', c')$ and $\delta > 0$, then $(x, c + \delta) \sim (x', c' + \delta)$.

This was given in the case of the domain $X \times R$, instead of $X \times R_+$, in Kaneko [7] (cf., also Kaneko-Wooders [10]), where ξ of $(x, \xi) \in X \times R$ means the increment or decrement from the normalized initial consumption level 0. In the domain $X \times R_+$, $c \in R_+$ is a nonnormalized consumption level, and we can impose an explicit income constraint.

Proposition 2.1 (Quasi-linearity). A preference relation $\succsim$ on $X \times R_+$ satisfies Conditions C0 to C3, and C4^{NI} if and only if there is a function $u : X \to R$ such that $u(x) \geq u(x_0)$ for all $x \in X$ and (1.1) holds for all $(x, c), (x', c') \in X \times R_+$.

Proof. The *only-if* part is essential. Then, let $(x, c) \in X \times R_+$. Since $(x, 0) \succsim (x_0, 0)$ by C3, we have a unique $\alpha_x \geq 0$ by C1 and C2 so that $(x, 0) \sim (x_0, \alpha_x)$. Then, by C4^{NI}, $(x, c) \sim (x_0, \alpha_x + c)$. Define $u : X \to R$ by $u(x) = \alpha_x$ for all $x \in X$.

Now, let $(x, c), (x', c') \in X \times R_+$. Then, by the above definition of α_x and B1,

$$\begin{aligned} (x, c) &\succsim (x', c') \Longleftrightarrow (x_0, \alpha_x + c) \sim (x, c) \succsim (x', c') \sim (x_0, \alpha_{x'} + c') \\ &\Longleftrightarrow \alpha_x + c \geq \alpha_{x'} + c' \Longleftrightarrow u(x) + c \geq u(x') + c'. \end{aligned}$$

■

[3] When X is an infinite set with some topology, under C0, a sufficient condition for C3 is: for any $y \in Y$, $\{(x, 0) \in X \times R : (y, 0) \succsim (x, 0)\}$ is a compact set in $X \times R$. This is proved by using the finite intersection property.

3. Approximate Quasi-Linearity

We give a condition for a preference relation $\succsim$ to be approximated by a quasi-linear utility function as idealization. This approximate representation theorem is given as Theorem 3.1. Also, we give an application to the theory of TU cooperative games.

3.1. Condition for approximate quasi-linearity

Consider the problem of when Condition $C4^{NI}$ holds approximately for large incomes. This is answered by relaxing $C4^{NI}$ in the following way:

C4 (Approximate monetary Substitutes): Let $x, x' \in X$. For any $\varepsilon > 0$, there is a $c_0 \geq 0$ such that for any $c, c' \geq c_0$ and $\alpha, \alpha' \geq 0$, if $(x, c) \sim (x', c + \alpha)$ and $(x, c') \sim (x', c' + \alpha')$, then $|\alpha - \alpha'| < \varepsilon$.

The additional α, α' are compensations for the transitions from x to x' with consumptions c, c', and C4 requires these to be close for large c and c'. This is a kind of the Cauchy property of a sequence $\{a_\nu\}$ (cf., Royden-Fitzpatrick [17], Section 1.5); it will be shown that α, α' converge as c becomes large. Condition C4 is an weakening of $C4^{NI}$ under C1; when $C4^{NI}$ is assumed, the conclusion of C4 becomes $|\alpha - \alpha'| = 0$.

It is informative to state the following separately from the main theorem.

Lemma 3.1. Suppose that $\succsim$ satisfies C0 to C3. Then there is a real-valued function $\delta : X \times R_+ \to R$ such that

$$(x, c) \sim (x_0, \delta(x, c) + c) \text{ for all } (x, c) \in X \times R_+; \tag{3.1}$$

$$\text{for any } (x, c), (x', c') \in X \times R_+,\ (x, c) \succsim (x', c') \Longleftrightarrow \delta(x, c) + c \geq \delta(x', c') + c'. \tag{3.2}$$

Proof. Consider any $(x, c) \in X \times R_+$. Then, $(x, c) \succsim (x_0, 0)$ by C0, C1, and C3. Thus, there is a unique $\delta(x, c) + c$ by C1 and C2 such that $(x_0, \delta(x, c) + c) \sim (x, c)$. Here, $\delta(x, c)$ may be negative but $\delta(x, c) + c$ is nonnegative. Thus, we have the function $\delta(\cdot, \cdot) : X \times R_+ \to R$ satisfying (3.1). We show (3.2). Let $(x, c), (x', c') \in X \times R_+$. Then, by C0 and C1, $(x_0, \delta(x, c) + c) \sim (x, c) \succsim (x', c') \sim (x_0, \delta(x', c') + c') \Longleftrightarrow \delta(x, c) + c \geq \delta(x', c') + c'$.■

Define $U : X \times R_+ \to R$ by

$$U(x, c) = \delta(x, c) + c \text{ for all } (x, c) \in X \times R_+. \tag{3.3}$$

Then, C4 implies that this U is approximated by a quasi-linear function.

Theorem 3.1 (Approximate Quasi-Linearity). Let $\succsim$ be a preference relation on $X \times R_+$ satisfying C0 to C3, and U the function given by (3.3). Then, $\succsim$ satisfies C4 if and only if for each $x \in X$,

$$-\infty < u(x) = \lim_{c \to +\infty} \delta(x, c) < \infty, \tag{3.4}$$

$$\begin{aligned} \text{for any } \varepsilon &> 0, \text{ there is some } c_0 \geq 0 \text{ such that} \\ \text{for any } c &\geq c_0, \ |U(x, c) - (u(x) + c)| < \varepsilon. \end{aligned} \tag{3.5}$$

Proof. (If): Suppose that the function u given by (3.4) satisfies (3.5). We show that C4 holds for $\succsim$. Let $x, x' \in X$. Let $\varepsilon > 0$ be given. Then, by (3.5), there is some $c_0 \geq 0$ such that for any $d, d' \geq c_0$, $|U(x, d) - (u(x) + d)| < \frac{\varepsilon}{4}$ and $|U(x', d') - (u(x') + d')| < \frac{\varepsilon}{4}$. Also, these function values such as $u(x) + d$ can be all positive. Now, let and $c, c' \geq c_o$, and suppose $U(x, c) = U(x', c + \alpha)$ and $U(x, c') = U(x', c' + \alpha')$. Using the second with $d' = c$, we have

$$\begin{aligned} \left|(u(x') + c + \alpha) - (u(x) + c)\right| &< \left|U(x', c + \alpha) + \tfrac{\varepsilon}{4} - (u(x) + c)\right| \\ &\leq |U(x, c) - (u(x) + c)| + \tfrac{\varepsilon}{4} \leq \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \tfrac{\varepsilon}{2}, \end{aligned} \tag{3.6}$$

where the equality is obtained since $U(x, c) = U(x', c + \alpha)$ and the inequality is from $|U(x, d) - (u(x) + d)| < \frac{\varepsilon}{4}$. Replacing c, α by c', α', we have $|(u(x') + c' + \alpha') - (u(x) + c')| < \frac{\varepsilon}{2}$. Now, we evaluate $|\alpha - \alpha'|$:

$$\begin{aligned} \left|\alpha - \alpha'\right| &= \left|[(u(x') + c + \alpha) - (u(x') + c))] - [(u(x) + c' + \alpha') - (u(x) + c'))]\right| \\ &= \left|[(u(x') + c + \alpha) - (u(x) + c))] - [(u(x') + c' + \alpha') - (u(x) + c'))]\right| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The first equality is a tautological formula, and the second is obtained from the first by exchanging the second $u(x')$ with the first $u(x)$. The last inequality follows from (3.6) with c, α and c', α'.

(Only-if): Let $\delta(x, c)$ be the function given by (3.1). We show that for each fixed $x \in X$, there is a $u(x) \in R$ with (3.4). Consider the sequence $\{\delta(x, \nu)\} = \{\delta(x, \nu) : \nu = 1, ...\}$. C4 states that for any $\varepsilon > 0$, there is a ν_0 such that for any $\nu, \nu' \geq \nu_0$, $|\delta(x, \nu) - \delta(x, \nu')| < \varepsilon$. This means that $\{\delta(x, \nu)\}$ is a Cauchy sequence. Hence, it converges to some real number, which is denoted by $u(x)$. This is (3.4). Now, each $\delta(x, \nu)$ in $\{\delta(x, \nu)\}$ is defined for a natural number $\nu \geq 1$. However, we prove $\lim_{c \to +\infty} \delta(x, c) = u(x)$. Let ε be an arbitrary positive number. Then there is a ν_0 such that for any $\nu \geq \nu_0$, $|\delta(x, \nu) - u(x)| < \varepsilon/2$. By C4, there is a c_0 such that for any $\nu \geq c_0$ and $c \geq c_0$, $|\delta(x, \nu) - \delta(x, c)| < \varepsilon/2$. Now, let $\nu_1 = \max(\nu_0, c_0)$. Then, for any $c \geq \nu_1$, we have

$|\delta(x,c) - u(x)| \leq |\delta(x,c) - \delta(x,\nu_1)| + |\delta(x,\nu_1) - u(x)| < \varepsilon$. Thus, $\lim_{c\to+\infty} \delta(x,c) = u(x)$. This is (3.4).

By Lemma 3.1 and (3.3), the function $U(x,c) = \delta(x,c) + c$ represents $\precsim$ in the sense of (3.1). Then, since $\lim_{c\to\infty} \delta(x,c) = u(x)$, we have $|U(x,c) - (u(x)+c)| = |(\delta(x,c)+c) - (u(x)+c)| = |\delta(x,c) - u(x))| \to 0$ as $c \to \infty$. This is (3.5). ■

Approximate quasi-linearity is first studied in Miyake [12]. He aimed to study the Marshallian demand theory: he starts with the domain $X \times R_+ = R_+ \times R_+$, and assumes that a utility function U of C^2 (twice continuously differentiable in the interior of $R_+ \times R_+$) is given. Also, U is assumed to be quasi-concave and satisfies normality (formulated in terms of first and second partial derivatives) in $R_+ \times R_+$. Then, he gave some other conditions to guarantee approximate quasi-linearity in the sense (3.5), and various results on the limit demand function (see also Miyake [13]).

Quasi-linearity is usually justified in the case where the expenditures for choices from X are small relative to an income. It would be natural to ask whether this interpretation is applied to an arbitrarily given utility function. Consider $U(x,c) = u(x) + \sqrt{c}$, which satisfies the law of diminishing marginal utility for c and differentiability at any c (i.e., it is locally approximated at c by a linear function). Thus, this function appears to be a candidate for approximate quasi-linearity. In fact, this observation is incorrect. To see it, consider the following necessary condition of (3.5): for each $x \in X$,

$$\{\delta(x,c) : c \in R_+\} \text{ is bounded,} \tag{3.7}$$

since $\delta(x,c)$ converges to $u(x)$ as $c \to \infty$ by (3.4) of Theorem 3.1. Thus, the compensation for x from x_o is bounded even if c is very large.

Indeed, $U(x,c) = u(x) + \sqrt{c}$ (and its monotone transformation) for $(x,c) \in X \times R_+$ violates (3.7). Choose an x with $u(x) > u(x_o)$ and let $h = u(x) - u(x_0)$. Then, $u(x) + \sqrt{c} = u(x_0) + \sqrt{c + \delta(x,c)}$, i.e., $h = \sqrt{c+\delta(x,c)} - \sqrt{c}$; so $\delta(x,c) = (h+\sqrt{c})^2 - c = h^2 + 2h\sqrt{c}$. Hence, $\delta(x,c) \to +\infty$ as $c \to +\infty$; (3.7) is violated. A positive example is: $U(x,c) = (1 - \frac{1}{1+c})u(x) + c$. In this case, (3.5) holds, and (3.7) holds. Nevertheless, a lot of familiar examples are excluded from approximate quasi-linearity.

The exclusion may give raise to some inconvenience. This could be avoided by allowing nondifferentiablity (some kink) with respect to c; up to some c_o, a familiar functional form is adopted but beyond c_o, some other functional form compatible with approximate quasi-linearity can be adopted. Hence, the above inconvenience is not a serious problem.[4]

[4] Kaneko-Ito [11] conducted an equilibrium-econometric analysis to study how utility functions have "significant income effects", adopting the utility functions of the form $U(x,c) = u(x) + c^{\alpha}$ $(0 < \alpha < 1)$. It was shown that this α is bounded away from 1 using rental housing market data in Tokyo. Since incomes of households are distributed in some interval, (3.5) is not contradictory to this empirical study.

Since boundedness (3.7) is a key for consideration of approximate quasi-linearity, we study it in a formal manner. We represent boundedness in terms of the preference relation $\succsim$. This will be used in another characterization of approximate quasi-linearity in Section 4.

C5 (Boundedness for compensations): For any $x \in X$, there is an $m > 0$ such that $(x_o, c+m) \succsim (x, c)$ for any $c \in R_+$.

That is, there is a compensation m for x_o from x independent of consumption level c. Under C0 to C3, this is equivalent to boundedness of $\delta(x, \cdot)$ with fixed $x \in X$.

Lemma 3.2. Suppose that $\succsim$ satisfies C0 to C3. Then, $\succsim$ satisfies C5 if and only if (3.7) holds for $\delta(x, \cdot)$ for each $x \in X$.

Proof. *If*: By (3.7), there is an $m \in R_+$ such that $m > \delta(x, c)$ for all $c \in R_+$. By (3.1), we have $(x_o, c+\delta(x, c)) \sim (x, c)$ for any c. By C1, we have $(x_o, c+m) \succsim (x, c)$ for any c. *Only-if*: By (3.1), $(x_o, c + \delta(x, c)) \sim (x, c)$ for any $c \in R_+$. However, $(x_o, c + m) \succsim (x_o, c + \delta(x, c)) \sim (x, c)$ for any $c \in R_+$. By C1, we have $m \geq \delta(x, c))$ for any $c \in R_+$.$\blacksquare$

We now raise the question of whether or not approximate quasi-linearity is an appropriate concept from the viewpoint of economics. Our theory formulates "large income" simply as "c tends to ∞". Mathematically, there are two possibilities: (A) $\delta(x, c)$ is in a bounded region, and (B) it goes to ∞. A subtlety is in the interpretation of "large incomes". To have a meaningful interpretation, we should consider how much richness is hidden behind the compound commodity c and/or richness of X. The two mathematical possibilities are examined from the socio-economic point of view.

When income gets larger for a person, his scope of consumption (economic behavior in general) gets larger. Suppose that there is an alternative y, hidden behind the composite commodity c or in X, similar to x in the sense that he can switch from x to y. When this is applied to any person in a similar economic situation, a value of each of x or y is more or less determined in the socioeconomic situation. In this interpretation, $\delta(x, c)$ is not very different from the social/market value. Here, possibility (A) is justified, and approximate quasi-linearity is applied.

In possibility (B), alternative x is unique and has no substitution for the person either behind the composite commodity or in X; x may be indispensable for him and its value may be unbounded when $c \to \infty$. In this case, approximate quasi-linearity does not hold.

3.2. An application to the theory of TU cooperative games

Here, we consider an application of Theorem 3.1 to the theory of TU cooperative games (cf., Osborne-Rubinstein [18], Chap.13, Maschler *et al.* [15], Chap.16). This is one example for Diagram 1.1.

We denote the set of players by $N = \{1, ..., n\}$. For each nonempty subset $S \subseteq N$, X_S is given as a *finite* nonempty set of social alternatives controlled by S, and $C_S : X_S \to R$ be a cost function. It can be assumed that $X_S \cap X_{S'} = \emptyset$ if $S \neq S'$. The value $C_S(x)$ for each $x \in X_S$ is allocated among the members in S. Let $X^i = \cup_{i \in S \subseteq N} X_S$. Each player $i \in N$ has a preference relation $\succsim_i$ over the set $X^i \times R_+$, and has an initial income $I_i \geq 0$. Here, $(x, c_i) \in X_S \times R_+$ means that an alternative x for S is chosen, and player i's consumption is c_i after paying his cost assignment. This economy is expressed as $E = (\{C_S\}_{S \subseteq N}, \{\succsim_i\}_{i \in N}, \{I_i\}_{i \in N})$, which we call the *base economy*. The above formulation includes market games (cf., Maschler *et al.* [15]), voting games (cf., Kaneko-Wooders [10]), and some others.

Under C0-C4 for the preference relations $\succsim_i$ for each $i \in N$, we have two functions $u_i : X^i \to R$ and $U_i : X^i \times R_+ \to R$ satisfying (3.3) and (3.5). The TU-economy is given as $E_q = (\{C_S\}_{S \subseteq N}, \{u_i\}_{i \in N}, \{I_i\}_{i \in N})$. In E_q, we can define the characteristic function game (TU game) (N, v) by, for all $S \subseteq N$,

$$v(S) = \max_{x \in X_S} \left(\sum_{i \in S} u_i(x) - C_S(x) \right). \tag{3.8}$$

Here, this maximization is well-defined for each S, since each X_S is a finite set. This is the maximum total surplus obtained by S.

We ask the question of how (N, v) is related to the base economy E. The aim of (N, v) is to consider a distribution of the total surplus for each S expressed by v. Such a distribution is described by an imputation: A vector $\alpha_S = \{\alpha_i\}_{i \in S}$ is called an *S-imputation* iff $\Sigma_{i \in S}\ \alpha_i = v(S)$ and $\alpha_i \geq v(\{i\})$ for all $i \in S$. We denote the set of all S-imputations in (N, v) by $I_S(N, v)$. Then, the question is what the set $I_S(N, v)$ is in the base economy E, in particular, what corresponds to α_i in $\alpha_S \in I_S(N, v)$.

Let $\alpha_S = \{\alpha_i\}_{i \in S} \in I_S(N, v)$ and x_S^* a solution for (3.8). The cost assignment for player $i \in S$ is given as $\gamma_i(\alpha_i) := u_i(x_S^*) - \alpha_i$. Indeed, $\alpha_i = u_i(x_S^*) - \gamma_i(\alpha_i)$ is the net surplus for player i. We can construct an S-allocation in the base economy E :

$$\psi(\alpha_S) = (x_S^*, \{I_i - \gamma_i(\alpha_i)\}_{i \in S}). \tag{3.9}$$

When I_i is large enough for each $i \in S$, this $\psi(\alpha_S)$ satisfies the budget constraints for all $i \in S$. In the TU-economy E_q, the utility level for player i is given as

$$u_i(x_S^*) + I_i - \gamma_i(\alpha_i) = u_i(x_S^*) + I_i - (u_i(x_S^*) - \alpha_i) = I_i + \alpha_i. \tag{3.10}$$

That is, the surplus α_i obtained from cooperation of S in (N, v) is the increment of utility from the initial I_i. If the initial state is normalized as 0, the utility level is exactly α_i.

We look at (3.10) in the base economy E. We fix an S. The question is how the cost allocation $\{\gamma_i(\alpha_i)\}_{i \in S}$ is interpreted in E. Recall that under C0 to C4 for the preference relations $\succsim_i$ for each $i \in S$, we have two functions $u_i : X^i \to R$ and $U_i : X^i \times R_+ \to R$ satisfying (3.3) and (3.5).

Theorem 3.2 (Approximation by a TU-game): For any $\varepsilon > 0$, there is an $I^* \geq 0$ such that for all $I_i \geq I^*$ $(i \in S)$ and for all $\alpha_S = \{\alpha_i\}_{i \in S} \in I_S(N, v)$, [5]

$$I_i \geq \gamma_i(\alpha_i) \text{ for all } i \in S; \tag{3.11}$$

$$|U_i(x_S^*, I_i - \gamma_i(\alpha_i)) - (I_i + \alpha_i)| < \varepsilon \text{ for all } i \in S, \tag{3.12}$$

Proof. First, we fix a player $i \in S$. The set $\{\gamma_i(\alpha_i) : \alpha_S \in I_S(N, v)\}$ is bounded above. Let I_i^0 be an income level greater than the maximum of this set. Hence, for all $I_i \geq I_i^0$, we have (3.11) for i.

Consider (3.12) for i. Applying Theorem 3.1 to i, we have some c_i^* such that for any $c_i \geq c_i^*$, $|U_i(x_S^*, c_i) - (u_i(x_S^*) + c_i)| < \varepsilon$. Since $I_i^1 - \gamma_i(\alpha_i) = I_i^1 - (u_i(x_S^*) - \alpha_i)$ and $\alpha_i \geq v(\{i\})$ for all $\alpha_S \in I_S(N, v)$, we can take an I_i^1 so that $I_i^1 - (u_i(x_S^*) - \alpha_i) \geq c_i^*$ for all $\alpha_S \in I_S(N, v)$. Then, we have, for all $I_i \geq I_i^1$,

$$\begin{aligned} & |U_i(x_S^*, I_i - (u_i(x_S^*) - \alpha_i)) - (I_i + \alpha_i)| \\ = \; & |U_i(x_S^*, I_i - (u_i(x_S^*) - \alpha_i)) - (u_i(x_S^*) + I_i - (u_i(x_S^*) - \alpha_i))| < \varepsilon \end{aligned}$$

for all $\alpha_S \in I_S(N, v)$. We take $I^* = \max\{I_i^0, I_i^1 : i \in S\}$. Then, for this I^*, (3.11) and (3.12) hold for all $i \in S$. ■

In Theorem 3.2, we focus on a particular coalition S. The theorem can be extended to the existence of I^* uniformly for all $S \subseteq N$. Once this is obtained, we can apply it to a solution theory for (N, v). For example, the core of (N, v) can be translated into the approximate core in the base economy E. Thus, the theory of TU cooperative games is viewed as an ideal approximation of the theory without the TU assumption for large incomes. This is the conversion described in the bottom of Diagram 1.1.

4. Characterization in terms of Normality

We have shown that under C0 to C3, Condition C4 is exactly equivalent to approximate quasi-linearity. However, it may be more convenient to have a sufficient condition for approximate quasi-linearity to study applications in economics. Here, we weaken Condition C4^{NI} (No income effects) in a different manner from C4; it is normality, which together with C5 (Boundedness) implies C4. Then, we will apply this result to the theory of Lindahl-ratio equilibrium in a public good economy, which is another example of conversion suggested in Diagram 1.

[5] The set $I_S(N, v)$ is nonempty under some additional condition, e.g., $v(S) \geq \Sigma_{i \in S} v(\{i\})$.

4.1. Normality and approximate quasi-linearity

We relax Condition C4^{NI} as follows:

C4^{NM} (Normality): Let $(x, c), (x', c') \in X \times R_+$ and $\alpha \geq 0$. Then, if $(x, c) \sim (x', c')$ and $c \leq c'$, then $(x, c + \alpha) \succsim (x', c' + \alpha)$.

That is, an additional income α gives more satisfaction if consumption c is less.[6] It is the normality condition, meaning that if an income is increased, a better choice x would be more demanded than the other x'.[7]

Recall that $\delta : X \times R_+ \to R$ is the function given in Lemma 3.1. Condition C4^{NM} is closely related to the monotonicity of $\delta(x, c)$ with respect to $c \in R_+$.

Lemma 4.1 (Monotonicity): Suppose C0 to C3 and C4^{NM} for $\succsim$. Let $x, x' \in X$.

(1): If $(x, 0) \succsim (x', 0)$, then $(x, c) \succsim (x', c)$ for all $c \geq 0$.

(2): $\delta(x, c) \geq 0$ for all $c \in R_+$, and $\delta(x, c)$ is a (weakly) increasing function of c.

(3): For each $c \in R_+$, if $(x, 0) \succsim (x', 0)$, then $\delta(x, c) \geq \delta(x', c)$.

Proof. **(1)**: Since $(x, 0) \succsim (x', 0)$, we have $(x, 0) \sim (x', \alpha)$ for some $\alpha \geq 0$ by C1. Hence, we have $(x, 0 + c) \succsim (x', \alpha + c)$ by C3^{NM}. By C1, we have $(x, c) \succsim (x', c)$.

(2): By definition, $(x, c) \sim (x_o, \delta(x, c) + c)$ for any $c \in R_+$. Since $(x, c) \succsim (x_o, c)$ by (1), it holds by C1 that $\delta(x, c) \geq 0$. Now, let $\varepsilon \geq 0$. Then since $(x, c) \sim (x_o, \delta(x, c) + c)$, we have, by C4^{NM}, $(x, c+\varepsilon) \succsim (x_o, \delta(x, c)+c+\varepsilon)$. Since $(x, c+\varepsilon) \sim (x_o, \delta(x, c+\varepsilon)+c+\varepsilon)$, we have $(x_o, \delta(x, c + \varepsilon) + c + \varepsilon) \succsim (x_o, \delta(x, c) + c + \varepsilon)$ by C0. This and C1 imply $\delta(x, c + \varepsilon) \geq \delta(x, c)$.

(3): Let $(x, 0) \succsim (x', 0)$. By (3.1) and (1), $(x_o, c + \delta(x, c)) \sim (x, c) \succsim (x', c) \sim (x_o, c + \delta(x', c))$. By C0 and C1, we have $\delta(x, c) \geq \delta(x', c)$.■

Under C5 in addition to C0 to C3, the function $\delta(x, c)$ is bounded for each $x \in X$, and under C4^{NM}, it is increasing with respect to $c \in R_+$. Hence, for each $x \in X$, $\delta(x, c)$ converges to some real number, denoted by $u(x)$, as $c \to \infty$. This is (1) of Theorem 4.1. Then, C4 is derived.

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[6] The strict version of this condition is used in Kaneko [9] (also, see Kaneko-Wooders [10]).

[7] When we assume $X = R_+$, we can show that the demand for the commodity in $X = R_+$ is weakly monotonic with an income under the assumption that $\succsim$ is weakly monotonic with respect to $x \in X = R_+$. Let $p > 0$. Suppose that $(x, I - px) \succsim (x', I - px')$ and $x > x'$. Then, by C1 and C2, $(x, I - px) \sim (x', I - px' + \alpha)$ for some $\alpha \geq 0$. Let $I' > I$. Then, since $I - px < I - px' + \alpha$, we have $(x, I' - px) \succsim (x', I' - px' + \alpha) \succsim (x', I' - px')$ by C3^{NM}. This means that the quantity demanded is weakly increasing when an income increases.

Theorem 4.1 (Characterization by normality). Suppose C0 to C3, C4^{NM}, and C5 on $\succsim$.

(1): For each $x \in X$, $\lim_{c\to+\infty} \delta(x,c) = u(x) < +\infty$ and $\delta(x,c) \le u(x)$ for all $c \ge 0$.

(2): C4 holds for $\succsim$.

Proof. We show (2). By Theorem 3.1, it suffices to show (3.5). Since $\delta(x,c)$ converges to $u(x)$, for any $\varepsilon > 0$, there is a $c_o \in R_+$ such that $|\delta(x,c) - u(x)| < \varepsilon$ for any $c \ge c_o$. This is written as $|(\delta(x,c)+c) - (u(x)+c)| < \varepsilon$ for any $c \ge c_o$, which is (3.5).■

Miyake [13] gave a result (Theorem 2 in p.561) corresponding to Theorem 4.1 in a similar framework. We have focused to approximate quasi-linearity, and our Theorem 4.1 is more direct than his. He studied the behavior of "willingness-to-pay" and willingness-to-accept", and he gave many results on the behavior of these concepts.

Now, let us see the convex structure on a preference relation $\precsim$ and the derived function $u : X \to R$. Suppose that X has a convex structure (not necessarily $X = R_+$). Then, $\precsim$ is said to be *convex* iff $\{(x',c') \in X \times R_+ : (x,c) \precsim (x',c')\}$ is a convex set for any $(x,c) \in X \times R_+$. This is equivalent to that the utility function U given by (3.3) is quasi-concave. In this case, we have the further result that the quasi-linear function u approximating U is concave. Additionally, we give a monotonicity result on u.

Lemma 4.2. Suppose C0 to C3, C4^{NM}, and C5 on $\succsim$. Let $u : X \to R$ be the function given by Theorem 3.1.

(1) (Convexity): If $\precsim$ is convex, then u is a concave function.

(2) (Monotonicity): If $(x,0) \succsim (x',0)$, then $u(x) \ge u(x')$.

Proof. (1): Let $x, x' \in X$ and $c \in R_+$. Suppose $(x',c) \precsim (x,c)$. Then, by C1, C2, we have a unique c' such that $(x',c') \sim (x,c)$. It implies $\delta(x',c') + c' = \delta(x,c) + c$. Here, $c' \ge c$ by Lemma 4.1.(1), and we denote $c' = c'(c)$.

Let $\lambda \in (0,1)$. Then, by convexity for $\succsim$, we have $(\lambda x + (1-\lambda)x', \lambda c + (1-\lambda)c')$ $\succsim (x,c) \sim (x',c')$. Thus, $\delta(\lambda x + (1-\lambda)x', \lambda c + (1-\lambda)c')$ $+(\lambda c + (1-\lambda)c') \ge \lambda(\delta(x,c) + c) + (1-\lambda)(\delta(x',c') + c')$, which implies

$$\delta(\lambda x + (1-\lambda)x', \lambda c + (1-\lambda)c') \ge \lambda\delta(x,c) + (1-\lambda)\delta(x',c'). \tag{4.1}$$

This holds for any c with $c' = c'(c)$. When $c \to \infty$, $c'(c) \to \infty$. Since $\lim_{c\to+\infty} \delta(x,c)$ $= u(x)$ and $\lim_{c'\to+\infty} \delta(x',c') = u(x')$, we have, by (4.1), $u(\lambda x + (1-\lambda)x') \ge u\delta(x) + (1-\lambda)u(x')$.

(2): Let $(x,0) \succsim (x',0)$. By Lemma 4.1.(3), $\delta(x,c) \ge \delta(x',c)$. Then, we have $u(x) = \lim_{c\to\infty} \delta(x,c) \ge \lim_{c\to\infty} \delta(x',c) = u(x')$.■

Thus, convexity and monotonicity for $\succsim$ are preserved as concavity in the limit function u. On the other hand, continuity of $\succsim$ is not preserved in u (cf., Debreu [2] for continuity of $\succsim$). For example, the preference relation $\succsim$ defined by $U(x,c) = (1 - (\frac{1}{x+1})^c) + c$

for all $(x,c) \in R^2_+$ is continuous, and $\delta(x,c)$ is given by $1-(\frac{1}{x+1})^c$. Here, $u(x) = \lim_{c\to\infty}\delta(x,c)$ is given as $u(0)=0$ and $u(x)=1$ for $x>0$; this u is discontinuous.[8]

4.2. Lindahl-ratio equilibrium for a public goods economy

Here, we apply the above result on approximate quasi-linearity to the theory of Lindahl-ratio equilibrium (simply, a ratio equilibrium) in a public goods economy (cf., Kaneko [8] and van den Nouweland, *et al.* [19]). We consider a public good economy and its cost allocation problem. Let $X = R_+$. A cost function $C : X \to R_+$ is given as a convex and strictly monotone function over X with $C(0)=0$. Each player $i \in N$ has a preference relation $\succsim_i$ over $X \times R_+$ and an income $I_i \geq 0$. We call $E = (C; \{\succsim_i\}_{i\in N}, \{I_i\}_{i\in N})$ the *base (public good) economy*. We assume that each $\succsim_i$ satisfies C0-C3, C4^{NM}, and C5, and that it is convex and weakly monotone over $X \times R_+$.

We say that $r = (r_1, ..., r_n)$ is a *ratio vector* iff $\Sigma_{i\in N}\ r_i = 1$. A pair $(x^*, r) = (x^*, (r_1, ..., r_n))$ of an $x^* \in R_+$ and an ratio vector $(r_1, ..., r_n)$ is called a *ratio equilibrium* in the base economy E iff for all $i \in N$,

$$r_iC(x^*) \leq I_i; \tag{4.2}$$

$$(x^*, I_i - r_iC(x^*)) \succsim_i (x_i, I_i - r_iC(x_i)) \text{ for all } x_i \geq 0 \text{ with } r_iC(x_i) \leq I_i. \tag{4.3}$$

That is, with an appropriate choice of a ratio vector for cost-sharing, every player agrees on the same choice x^*. Kaneko [8] proved the existence of a ratio equilibrium under the continuity assumptions on the preference relations, $\precsim_i, i \in N$, and the above assumption on the cost function C. Our present question is how the result for the quasi-linear case is converted to this economy.

Now, for each $i \in N$, we have $u_i : X = R_+ \to R$ with $\lim_{c\to+\infty}\delta_i(x,c) = u_i(x)$ for each $x \in X$. The QL-(public good) economy is given as $E_q = (C; \{u_i\}_{i\in N})$. In the QL-economy E_q, a pair $(x^*, r) = (x^*, (r_1, ..., r_n))$ of an $x^* \in R_+$ and an ratio vector $(r_1, ..., r_n)$ is called a *ratio equilibrium in* E_q iff

$$u_i(x^*) - r_iC(x^*) \geq u_i(x_i) - r_iC(x_i) \text{ for all } x_i \in X. \tag{4.4}$$

Here, no budget constraints are assumed.

A ratio equilibrium $(x^*, r) = (x^*, (r_1, ..., r_n))$ in E_q with $x^* > 0$ can be regarded as an approximate ratio equilibrium for large incomes in E. We have the following theorem under C0-C3, C4^{NM}, C5 over each $\succsim_i$.

[8] It holds that $u_i(x) + c$ is continuous if and only if $u_i(x)$ is continuous, and that it is quasi-concave if and only if $u_i(x)$ is concave (cf., Kaneko [7]).

Theorem 4.2 (Conversion of a ratio equilibrium from E_q to E). Let $(x^*, r) = (x^*, (r_1, ..., r_n))$ be a ratio equilibrium in E_q with $x^* > 0$. Then, for any $\varepsilon > 0$, there is an I^* such that for any $I_i \geq I^*$ and $i \in N$,

$$I_i \geq r_i C(x^*); \tag{4.5}$$

$$U_i(x^*, I_i - r_i C(x^*)) + \varepsilon \geq U_i(x, I_i - r_i C(x_i)) \text{ for all } x_i \in R_+ \text{ with } I_i \geq r_i C(x_i). \tag{4.6}$$

Proof. Consider any $i \in N$. We choose I_i^0 so that (4.5) holds for I_i^0. Consider (4.6). Let $x_i \in R_+$ and $I_i \geq I_i^0$ with $I_i \geq r_i C(x_i)$. Then, $u_i(x^*) - r_i C(x^*) \geq u_i(x_i) - r_i C(x_i)$ by (4.4), and $u_i(x_i) \geq \delta(x_i, I_i - r_i C(x_i))$ by Lemma 4.1. By these, we have

$$\begin{array}{rcl} u_i(x_i^*) + I_i - r_i C(x_i^*) & \geq & u_i(x_i) + I_i - r_i C(x_i) \\ & \geq & \delta(x_i, I_i - r_i C(x_i)) + I_i - r_i C(x_i). \end{array}$$

Now, by Theorem 3.1, we can choose a large $I_i^1 \geq I_i^0$ so that if $I_i \geq I_i^1$, we have $U_i(x_i^*, I_i - r_i C(x_i^*)) + \varepsilon > u_i(x^*) + I_i - r_i C(x^*)$. Hence, by this (4.4), we have $U_i(x_i^*, I_i - r_i C(x_i^*)) + \varepsilon > u_i(x^*) + I_i - r_i C(x^*) > \delta(x_i, I_i - r_i C(x_i)) + I_i - r_i C(x_i) = U_i(x_i, I_i - r_i C(x_i))$.

Let $I^* = \max\{I_i^1 : i \in N\}$. we have (4.5) and (4.6) for any $I_i \geq I^*$ and $i \in N$. ■

We finalize this section with a remark on the existence of a ratio equilibrium. In addition to the above assumptions, we assume also that each $\precsim_i$ is convex. Then, by Lemma 4.2, $u_i : R_+ \longrightarrow R$ is a concave and monotone function by Lemma 4.2, and then, $u_i(x)$ is continuous over $(0, +\infty)$.

The existence of a ratio equilibrium is closely related to the maximization of the total surplus:

$$\max_{x \in X} \left(\sum_{i \in N} u_i(x) - C(x) \right). \tag{4.7}$$

The existence of an inner optimal solution $x^* > 0$ is obtained by the continuity of this objective function over $(0, +\infty)$ under the condtion that $\Sigma_{i \in N}\ u_i(x) - C(x) > \Sigma_{i \in N}\ u_i(0) - C(0)$ for some $x > 0$ and $\Sigma_{i \in N}\ u_i(x') - C(x') < 0$ for some $x' > 0$. Here, we assume that (4.7) has an inner optimal solution $x^* > 0$.

Lemma 4.3. Let $x^* > 0$ be a solution for (4.7). Then, there is a ratio vector $r = (r_1, ..., r_n)$ such that (r, x^*) is a ratio equilibrium in the economy E_q.

Proof. Let $g(x) = \Sigma_{i \in N}\ u_i(x) - C(x)$. Then, g is a concave function, we have the left and right derivatives g'^- and g'^+ at each $x \in (0, \infty)$. Then,

$$g'^+(x^*) = \Sigma_{i \in N} u_i'^+(x^*) - C'^+(x^*) \leq 0 \leq g'^-(x^*) = \Sigma_{i \in N} u_i'^-(x^*) - C'^-(x^*). \tag{4.8}$$

Now, for $\theta \in [0, 1]$, let $\alpha_i(\theta) = \theta u_i'^+(x^*) + (1 - \theta) u_i'^-(x^*)$ for all $i \in N$. Since $C(x)$ is increasing and concave, it holds by (4.8) that $\alpha_i(\theta) \geq 0$ for all $i \in N$ and $\theta \in [0, 1]$.

Then, since $\Sigma_{i\in N}\ u_i'^{+}(x^*) \leq C'^{+}(x^*) \leq C'^{-}(x^*) \leq \Sigma_{i\in N}\ u_i'^{+}(x^*)$, we find a $\theta^* \in [0,1]$ so that $C'^{+}(x^*) \leq \Sigma_{i\in N}\alpha_i(\theta^*) \leq C'^{-}(x^*)$. Let $r_i = \alpha_i(\theta^*)/\Sigma_{j\in N}\alpha_j(\theta^*)$ for all $i \in N$. Then, we have

$$\begin{aligned} u_i'^{+}(x^*) - r_i C'^{+}(x^*) &= u_i'^{+}(x^*) - (\theta^* u_i'^{+}(x^*) + (1-\theta^*)u_i'^{-}(x^*)) \leq 0 \\ &\leq u_i'^{-}(x^*) - (\theta^* u_i'^{+}(x^*) + (1-\theta^*)u_i'^{-}(x^*)) = u_i'^{-}(x^*) - r_i C'^{-}(x^*). \end{aligned}$$

Hence, each player i maximize $u_i(x) - r_iC(x)$ at $x = x^*$. ■

5. Extension to Expected Utility

Quasi-linear utility functions are also used in the environment with risks, in game theory and economics. In this case, the characterization of quasi-linearity should be extended to include expected utility theory, or *vice versa*. This was discussed in Kaneko-Wooders [10]. Here, we will discuss the extension of Theorem 3.1 taking expected utility theory into account. The extension is a concatenation of our theory and expected utility theory.

Let $m_F(X \times R_+) := \{f : X \times R_+ \to [0,1] : \Sigma_{(x,c)\in T}\ f(x,c) = 1$ for some finite subset T of $X \times R_+\}$, i.e., the set of all probability distributions with finite supports over $X \times R_+$. Regarding $m_F(X \times R_+)$ as a subset of the linear space of all real-valued functions endowed with the standard sum and scalar (real) multiplication, $m_F(X \times R_+)$ is a convex set, i.e., if $f, g \in m_F(X \times R_+)$ and $\lambda \in [0,1]$, then the convex combination (mixture) $\lambda f * (1-\lambda)g$ belongs to $m_F(X \times R_+)$. Let $\succsim^*$ be a binary relation over $m_F(X \times R_+)$. We can regard $X \times R_+$ as a subset of $m_F(X \times R_+)$ by the identity mapping.

We assume the following:

Condition E0 (Complete preordering): $\succsim^*$ is a complete and transitive relation on $m_F(X \times R_+)$;

Condition E1 (Substitute): If $f \succ^* g \succ^* h$, then $\lambda f * (1-\lambda)h \sim^* g$ for some $\lambda \in [0,1]$;

Condition E2 (Independence): For any $f, g, h \in m_F(X \times R_+)$ and $\lambda \in (0,1)$,

  (1): $f \succ^* g$ implies $\lambda f * (1-\lambda)h \succ^* \lambda g * (1-\lambda)h$;

  (2): $f \sim^* g$ implies $\lambda f * (1-\lambda)h \sim^* \lambda g * (1-\lambda)h$.

It is known (cf., Herstein-Milnor [4], Fishburn [3], Kaneko-Wooders [10]) that these three conditions are enough to derive a utility function $U^* : m_F(X \times R_+) \to R$ representing $\succsim^*$ and satisfying $U^*(\lambda f * (1-\lambda)g) = \lambda U^*(f) + (1-\lambda)U^*(g)$ for all $f, g, h \in m_F(X \times R_+)$ and $\lambda \in [0,1]$.

Restricting the preference relation $\succsim^*$ to $X \times R_+$, we have the preference relation over $\succsim$ on $X \times R_+$, which satisfies Condition C0. Conditions E1-E2 require nothing about $\succsim$ over the base set $X \times R_+$.We can assume C1-C4 on $\succsim$. We denote the restriction of U^* to the base set $X \times R_+$ by U.

Theorem 5.1 (Expected utility theory version). Suppose that a preference relation $\succsim^*$ over $m_F(X \times R_+)$ satisfies E0-E2, and that the derived preference $\succsim$ on $X \times R_+$ satisfies C1-C4.

(1): There is a utility function $U^* : m_F(X \times R_+) \to R$ such that

$$U^*(f) = \Sigma_{(x,c)\in T_f}\ f(x,c)U(x,c) \text{ for each } f \in m_F(X \times R_+), \tag{5.1}$$

where T_f is a finite support of $f \in m_F(X \times R_+)$.

(2): There is a (strictly) monotone $f : R \to R$ such that

$$U(x,c) = f(\delta(x,c) + c) \text{ for all } (x,c) \in X \times R_+. \tag{5.2}$$

(3): There is a function $u : X \to R$ such that (3.5) holds for each $x \in X$.

Proof. (1) is known from expected utility theory.

(2): It is shown in Lemma 3.1 that over the domain $X \times R_+$, the relation $\succsim$ is represented by the function $\delta(x,c) + c$. This implies that if $\delta(x,c) + c = \delta(x',c') + c'$, then $U(x,c) = U(x',c')$. Hence, we can define a function $f : \{\delta(x,c) + c : (x,c) \in X \times R_+\} \to R$ by $f(\delta(x,c)+c) = U(x,c)$ for all $(x,c) \in X \times R_+$. This f is monotone, and can be extended to R.

(3): This is simply Theorem 3.1. ■

We have still the difference that Theorem 5.1.(3) is stated in terms of $\delta(x,c) + c$ rather than U. Expected utility theory is cardinal, while the theory in Section 3 is ordinal. Hence, it may be informative to connect (3) with (2) directly. This connection is made to assume the risk neutrality:

E3: (Risk Neutrality): $\frac{1}{2}(x,c) * \frac{1}{2}(x,c') \sim^* (x, \frac{1}{2}c + \frac{1}{2}c')$.

This is a condition on interactions between our theory and expected utility theory. Then, the function f can be linear, in particular, we can assume

$$U^*(x,c) = \delta(x,c) + c \text{ for all } (x,c) \in X \times R_+. \tag{5.3}$$

In sum, we obtain the approximately quasi-linear function by adding E3 in the extended theory. Of course, if we assume risk aversion (lover), we have f to be a concave (convex) function.

6. Conclusions

We gave characterizations of a preference relation $\precsim$ to be approximately represented by a quasi-linear utility function for large incomes. The main condition is C4, which is a weakening of the no-income effect condition C4^{NI}. It guarantees the limit function $u(x) = \lim_{c\to+\infty} \delta(x, c)$, which is a representation of opportunity monetary equivalence from the origin x_o to alternative x. Mathematically speaking, C4 excludes a lot of candidates. In the end of Section 3.1, we have argued that the existence of the limit function $u(x)$ is justified in the case where the composite commodity behind c is rich enough or the alternatives in X are rich enough. This interpretation may be formulated in a general manner, and we evaluate our argument rigorously. These remain open.

We gave two applications of our result to the theories of TU-cooperative games and of Lindahl-ratio equilibrium for a public goods economy. We discussed the conversions of results in the theories of TU cooperative games into the base economy for large incomes, and the conversion of Lindahl-ratio equilibrium in a public goods economy also into the base economy. These are taken in the direction from the limit cases to the base economies, and we did not analyze the other direction from the base economies to the limit. More general studies of those conversions in both directions remain to be an open problem.

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