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Monotonic redistribution: Integrating performance-based allocation and equity

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Abstract

We show that a proportional taxation redistribution rule regarding heterogeneity among members of a society is implied by four axioms: efficiency, total wealth monotonicity, contribution monotonicity and additivity. Our rule is an extension of Casajus (2015)'s redistribution rule in which each member of a society is identified only with his/her income.

Keywords: redistribution, proportional taxation, monotonicity, weighted division.
JEL Classification: D63, H20.

1 Introduction

Consider a society consisting of many households. The configuration should vary from one household to another. We now consider two households as examples. Suppose that household A consists of three people, a schoolchild and his parents, and household B consists of two people, a man and a woman. Assume that the total annual income of household A is the same as that of B. Our question is how we should treat them as *equals* under a taxation and redistribution process. To answer the question, we explore a redistribution rule which takes into accounts both contributions (*i.e.*, income) and heterogeneity among households in the simple setup proposed by Casajus (2015).

In this paper, we make an axiomatic case for proportional taxation regarding heterogeneity among members of a society. To achieve this purpose, we first show that a redistribution rule satisfies *efficiency* (E), *total monotonicity for no-contributors* (TMN), *contribution monotonicity* (CM) and *additivity* (AD) if and only if the rule is a proportional taxation rule with the weighted redistribution based on heterogeneity among households. The property (E) requires that the total amount of a society's wealth (*i.e.*, the sum of contributions) is redistributed without any loss. The property (TMN) describes that if the total amount of wealth does not decrease, then the reward for a member whose contributions are zero

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also does not decrease. Contribution monotonicity (CM) requires that if a member's contributions do not decrease with keeping the society's total wealth unchanged, then the rewards for the member should not decrease. The axiom (AD) is a requirement for the consistency of the rule, which is described in Section 2. Our result shows that these properties can be a rationale for the redistribution based on heterogeneity among members.

Our results contribute to the recent exploration of the allocation rule integrating contribution-based and equity-based systems. Casajus (2015) introduces three plausible properties for redistribution, *efficiency* (E), *monotonicity* (M) and *symmetry* (S), and shows that his redistribution rule is the only rule satisfying these three properties. In words, his rule is described as follows: first, each member pays tax in proportion to his contributions and, second, the sum of tax in the society is completely equally redistributed to every member. That is, Casajus (2015) presumes that heterogeneity among agents is generated only from the difference of incomes. In contrast, we consider the other type of heterogeneity which is not captured by the difference of incomes. As we will see later, symmetry takes an important role to reflect this difference.

The rest of this paper is organized as follows. In Section 2, we introduce the model and redistribution rules. In Section 3, we conclude the paper. Some proofs are relegated to the Appendix.

2 The Model and Redistribution Rules

We consider the model of a society studied by Casajus (2015). Let $N = \{1, \cdots, n\}$ be the set of members of the society. We assume $n \geq 2$. Each member $i \in N$ is identified with his individual contribution $x_i \in \mathbb{R}$. Let $x = (x_1, ..., x_n) \in \mathbb{R}^N$. We allow for negative contributions. The society's total wealth is the sum of individual contributions $\sum_{i \in N} x_i$. A redistribution rule for the society is a function $f : \mathbb{R}^N \to \mathbb{R}^N$. For each $i \in N$ and $x \in \mathbb{R}^N$, the value $f_i(x)$ means the reward which the member i receives after redistribution.

2.1 Casajus (2015)'s Approach: The Equal Division

Casajus (2015) postulates the following three axioms, which reflect the plausible properties every acceptable redistribution rule should satisfy.

Axiom 1 (Efficiency (E))**.** For all $x \in \mathbb{R}^N$, we have $\sum_{i \in N} f_i(x) = \sum_{i \in N} x_i$.

By efficiency, Casajus (2015) assumes that (i) the sum of rewards should not be greater than the society's wealth to be distributed, and (ii) the society does not dispose of any part of its wealth without distributing it.

Axiom 2 (Symmetry (S))**.** For all $x \in \mathbb{R}^N$ and $i, j \in N$ such that $x_i = x_j$, we have $f_i(x) = f_j(x)$.

One can understand symmetry as a requirement of fairness. In this model, only contributions distinguish a member from another. Symmetry states that two members who contribute the same should receive the same.

Axiom 3 (Monotonicity (M))**.** For all $x, y \in \mathbb{R}^N$ and $i \in N$ such that $\sum_{l \in N} x_l \geq \sum_{l \in N} y_l$ and $x_i \geq y_i$, we have $f_i(x) \geq f_i(y)$.

This property requires that if both the total wealth and the contributions of a particular member do not decrease, then the reward for this member also does not decrease.

The following theorem shows that a redistribution rule satisfying these three properties is only the rule redistributing the society's wealth by dividing equally.[*1]

Theorem 2.1 (Casajus (2015))**.** Let $n \neq 2$. A redistribution rule $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (E), (S) and (M) if and only if there exists some $\tau \in [0,1]$ such that $f_i(x) = (1-\tau) \cdot x_i + \tau \cdot \sum_{l \in N} \frac{x_l}{n}$ for all $x \in \mathbb{R}^N$ and $i \in N$.

2.2 Our Approach: The Weighted Division

Our objective is to offer another redistribution rule which takes heterogeneity into accounts by extending Casajus (2015)'s rule. To describe the heterogeneity, we introduce a weight $w = (w_i)_{i \in N} \in \mathbb{R}^N_+$ satisfying $\sum_{i \in N} w_i = 1$. Let $\mathcal{W}$ denote the set of weights. Now, we begin by replacing his monotonicity with the following two weaker monotonicity axioms.

Axiom 4 (Total Monotonicity for No-contributors (TMN))**.** For all $x, y \in \mathbb{R}^N$ and each agent $i \in N$ such that $x_i = y_i = 0$, if $\sum_{l \in N} x_l \geq \sum_{l \in N} y_l$, then $f_i(x) \geq f_i(y)$.

Axiom 5 (Contribution Monotonicity (CM))**.** For all $x, y \in \mathbb{R}^N$ with $\sum_{l \in N} x_l = \sum_{l \in N} y_l$ and each agent $i \in N$, if $x_i \geq y_i$, then $f_i(x) \geq f_i(y)$.

Each of these monotonicity axioms is weaker than monotonicity. Therefore, we have [(M) $\Rightarrow$ (TMN) and (CM)]. In general, the opposite direction is not true.

Axiom 6 (Additivity (AD))**.** For all $x, y \in \mathbb{R}^N$, $f(x) + f(y) = f(x+y)$

Although additivity is rather technical, we can interpret it as a consistency of the rule. To see this, we consider a profile $x \in \mathbb{R}^N$ and suppose that x consists of two finer profiles $y, z \in \mathbb{R}^N$ satisfying $x = y + z$ (such as contributions for part-time jobs y and full-time jobs z). Additivity suggests that the sum of the redistribution results, $f(y) + f(z)$ should be equal to that of the aggregated profile $f(x)$. In other words, additivity keeps redistribution of x coherent against any intentional partition into y and z. Casajus (2015)'s rule also meets additivity.

By these axioms together with efficiency, we propose a following redistribution rule.

Theorem 2.2. A redistribution rule $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM) and (AD) if and only if there exists some $\tau \in [0,1]$ and $w \in \mathcal{W}$ such that $f_i(x) = (1-\tau) \cdot x_i + \tau \cdot w_i \cdot \sum_{l \in N} x_l$ for all $x \in \mathbb{R}^N$ and $i \in N$.

[*1] Casajus (2016) and Yokote and Casajus (2017) provide another axiomatization for the redistribution of Casajus (2015)'s rule.

Proof. See Appendix A. ☐

The two monotonicity axioms and the lack of symmetry axiom differentiate our rule from that of Casajus (2015). In order to reflect heterogeneity among members, we drop symmetry and, instead of symmetry, adopt additivity. As a result, the difference is reflected in how to divide the total wealth $\tau \sum_{l \in N} x_l$. We employ the weighted division $\tau w_i \sum_{l \in N} x_l$ in stead of the equal division $\tau \frac{1}{n} \sum_{l \in N} x_l$. We consider w_i as the member i's relative *priority*, and $w = (w_1, ..., w_n)$ as a priority profile.

Note that a priority profile $w = (w_1, ..., w_n)$ is endogenously determined without depending on x and τ. To see this, let $N = \{1, 2\}$. An example of our rule is of the form $f_1(x) = (1-\tau) \cdot x_1 + \tau \cdot \frac{1}{3} \cdot \sum_{l \in N} x_l$ and $f_2(x) = (1-\tau) \cdot x_2 + \tau \cdot \frac{2}{3} \cdot \sum_{l \in N} x_l$. An example of redistribution rules in which w depends on x is as follows.

Example 2.3. Fix $\varepsilon \in \mathbb{R}$ ($\varepsilon \neq 0$) and define the rule $f : \mathbb{R}^N \to \mathbb{R}^N$ by

$$f_i(x) = (1-\tau) \cdot x_i + \tau \cdot \frac{(x_i + \varepsilon)}{\sum_{l \in N}(x_l + \varepsilon)} \cdot \sum_{l \in N} x_l.$$

Clearly, the weight of this function depends on x. The functions of this form are generally ruled out by additivity.[*2]

We evaluate this result in descriptive and normative ways. From the descriptive point of view, our rule can be seen as a class (or a set) of many particular redistribution rules depending on τ and w. Each particular rule depicts an actual redistribution rule. For example, Casajus (2015)'s plausible rule is the specific form of our general rule with setting $w = (\frac{1}{n}, ..., \frac{1}{n})$. Moreover, even the *dominator* rule, *i.e.*, $f_{i^*}(x) = \sum_{j \in N} x_j$ and $f_i(x) = 0$ for every $i \neq i^*$, also can be covered by our rule with the setting $\tau = 1$ and $w_{i^*} = 1$. Namely, the axioms above are needed to be satisfied by a rule to be a redistribution rule that we collect taxes with the fixed rate τ and redistribute it based on the weight w which is independent of τ and contributions x. The axiomatization result does not suggest that we justify these redistribution rules (including the dominator rule) but that our axioms above rule out many artificial redistribution rules as one can see in Appendix B. Next, from a normative point of view, we distinguish symmetry from the other properties. Symmetry, which represents *fairness* in this model, is a subjective value judgment. In other words, as mentioned in Section 1, "what is fair" depends on the situation In contrast to symmetry, the other properties (additivity, monotonicity and efficiency) are more associated with the *structure* of redistribution. Our result can be thought of as a characterization based on the proprieties that the structure of a redistribution rule should satisfy.

We should also mention a technical result. Considering the characterization proposed by Casajus (2015) (see Theorem 2.1), one may conjecture that our result can be implied only by (E) and (M). This conjecture is not true because of Example 2.3. However, the following results clarify the axiomatic structures of our rule and Casajus (2015)'s.

[*2] To consider a redistribution rule with an *exogenous* profile of priorities, we need the different framework which contains w as one of its primitives. Abe and Nakada (2016) discuss this problem in the framework of TU-cooperative games.

Lemma 2.4. The following two statements are true:

(i) (TMN), (CM) and (AD) $\Rightarrow$ (M),

(ii) (M) $\Rightarrow$ (TMN) and (CM).

Proof. See Appendix C. □

This result shows that for every additive redistribution rules, [(TMN) and (CM)] is logically equal to (M). Therefore, we obtain another characterization.

Theorem 2.5. A redistribution rule $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (E), (M) and (AD) if and only if there exists some $\tau \in [0, 1]$ and $w \in \mathcal{W}$ such that $f_i(x) = (1 - \tau) \cdot x_i + \tau \cdot w_i \cdot \sum_{l \in N} x_l$.

Proof. This follows from Lemma 2.4 and Theorem 2.2. □

This set of axioms is obtained by replacing (S) in Theorem 2.1 with (AD). Axiom (AD) is needed for w to be independent from τ and x. Note that the three axioms are mutually independent. Example 2.3 is the rule satisfying (E) and (M) but not (AD). See also Appendix B.

3 Concluding Remarks

We consider a redistribution rule which takes into accounts both contributions and heterogeneity among members in a society. Under the simple framework, we give an axiomatic rationale for a proportional taxation rule with the weighted redistribution.

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Appendix A. Proof of Theorem 2.2

Proof. It is clear that the rule satisfies all the axioms. We suppose that a rule $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (E), (TMN), (CM) and (AD).

For any $c \in \mathbb{R}$, let us define $X^c = \{x \in \mathbb{R}^N \,|\, \sum_{l \in N} x_l = c\}$.

Claim 1: For each $i \in N$, there exist functions $\phi_i : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(x) = \phi_i(x_i, \sum_{l \in N} x_l) + \alpha_i(\sum_{l \in N} x_l)$.

We first take any $c \in \mathbb{R}$. For any $i \in N$ and any $x, \bar{x} \in X^c$ with $x_i = \bar{x}_i$, we have

$$f_i(x) \overset{\text{(CM)}}{=} f_i(\bar{x}) \; =: \alpha_i(c, x_i). \tag{A.1}$$

Specifically, we denote

$$\alpha_i(c) := \alpha_i(c, 0). \tag{A.2}$$

Moreover, for any $i \in N$ and $x, y \in X^c$, we define

$$\begin{aligned} f_i(x) - f_i(y) &\overset{(A.1)}{=} \alpha_i(c, x_i) - \alpha_i(c, y_i) \\ &=: \phi_i(x_i, y_i, c). \end{aligned} \tag{A.3}$$

Hence, for any $i \in N$ and any $x, y \in X^c$ with $y_i = 0$, we have

$$\phi_i(x_i, y_i, c) \overset{(A.3)}{=} f_i(x) - f_i(y) \overset{(A.1),(A.2)}{=} f_i(x) - \alpha_i(c). \tag{A.4}$$

Note that $f_i(x) - \alpha_i(c)$ does not depend on y. Now, for any $i \in N$ and $x \in X^c$, let

$$\phi_i(x_i, c) := f_i(x) - \alpha_i(c). \tag{A.5}$$

Then, for any $i \in N$ and $x \in \mathbb{R}^N$, we obtain

$$f_i(x) \overset{(A.5)}{=} \phi_i(x_i, \sum_{l \in N} x_l) + \alpha_i(\sum_{l \in N} x_l). \tag{A.6}$$

This completes Claim 1.

Before moving to Claim 2, we show that $\alpha_i(c)$ is nondecreasing with respect to c, *i.e.*, $\alpha_i(c) \geq \alpha_i(c')$ for $c \geq c'$. To see this, we first confirm that for any $c \in \mathbb{R}$,

$$\phi_i(0, c) \overset{(A.5),(A.4)}{=} \phi_i(0, 0, c) \overset{(A.3),(A.1)}{=} \alpha_i(c, 0) - \alpha_i(c, 0) = 0. \tag{A.7}$$

Next, for any $c, c' \in \mathbb{R}$ with $c \geq c'$,

$$\begin{aligned} \alpha_i(c) - \alpha_i(c') &= [0 + \alpha_i(c)] - [0 + \alpha_i(c')] \\ &\overset{(A.7)}{=} [\phi_i(0, c) + \alpha_i(c)] - [\phi_i(0, c') + \alpha_i(c')] \\ &\overset{(A.6),(TMN)}{\geq} 0. \end{aligned}$$

Similarly, for any $i \in N$, $\phi_i(\cdot, \sum_{l \in N} x_l)$ is nondecreasing with respect to the first argument with keeping the second argument unchanged, because of (CM).

Claim 2: For each $i \in N$, $x \in \mathbb{R}^N$ and $c \in \mathbb{R}$, $\phi_i(cx_i, c\sum_{l \in N} x_l) = c\tau_i x_i$ for some $\tau_i \in \mathbb{R}_+$ and $\alpha_i(c\sum_{l \in N} x_l) = c\alpha_i(\sum_{l \in N} x_l)$.

For any $k \in N$, let $z^k \in \mathbb{R}^N$ be the vector such that $z^k_k = 1$ and $z^k_j = 0$ for any $j \neq k$. In view of Claim 1, we have

$$f_i(z^k) \overset{(A.7)}{=} \begin{cases} \phi_i(1, 1) + \alpha_i(1) & \text{if } \; i = k, \\ \alpha_i(1) & \text{otherwise.} \end{cases}$$

By (AD), for any $q \in \mathbb{Q}$, we have $f_i(qz^k) = qf_i(z^k)$. Hence, for any $i \neq k$, we have $\alpha_i(q) = q\alpha_i(1)$; and for k, $\phi_k(q, q) + \alpha_k(q) = q\phi_k(1, 1) + q\alpha_k(1)$. Since, for any $j \in N$, $\alpha_j(\cdot)$ is monotonically nondecreasing, *i.e.*, $\alpha_j(q) \geq \alpha_j(q')$ for $q \geq q'$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, we obtain, for all $c \in \mathbb{R}$,

$$\alpha_i(c) = c\alpha_i(1). \tag{A.8}$$

Thus, for any $x, y \in \mathbb{R}^N$, we have

$$\begin{aligned}
&\phi_i(x_i + y_i, \sum_{l\in N}(x_l + y_l)) + \alpha_i(\sum_{l\in N}(x_l + y_l)) \\
&\overset{(A.1)}{=} f_i(x + y) \overset{(AD)}{=} f_i(x) + f_i(y) \\
&\overset{(A.1)}{=} \left[\phi_i(x_i, \sum_{l\in N} x_l) + \alpha_i(\sum_{l\in N} x_l)\right] + \left[\phi_i(y_i, \sum_{l\in N} y_l) + \alpha_i(\sum_{l\in N} y_l)\right] \\
&\overset{(A.8)}{=} \left[\phi_i(x_i, \sum_{l\in N} x_l) + \phi_i(y_i, \sum_{l\in N} y_l)\right] + \alpha_i(\sum_{l\in N}(x_l + y_l)),
\end{aligned}$$

which implies that

$$\phi_i(x_i + y_i, \sum_{l\in N}(x_l + y_l)) = \phi_i(x_i, \sum_{l\in N} x_l) + \phi_i(y_i, \sum_{l\in N} y_l). \tag{A.9}$$

By this, for any $x \in \mathbb{R}^N$, we have

$$\phi_i(x_i, \sum_{l\in N} x_l) \overset{(A.9)}{=} \phi_i(x_i, x_i) + \phi_i(0, \sum_{l\neq i} x_l) \overset{(A.7)}{=} \phi_i(x_i, x_i) =: \phi_i(x_i).$$

Recall that ϕ_i is monotonically nondecreasing with respect to the first argument. Then, there exists nonnegative constant $\tau_i \geq 0$ such that

$$\phi_i(x_i, \sum_{l\in N} x_l) = \tau_i \cdot x_i.^{*3} \tag{A.10}$$

This completes Claim 2.

Claim 3: There exists $\tau \in [0, 1]$ and $w \in \mathcal{W}$ such that for any $x \in \mathbb{R}^N$, $\alpha_i(\sum_{l\in N} x_l) = (1-\tau)w_i \sum_{l\in N} x_l$.

Similar to Claim 2, from the fact that $\alpha_i(\cdot)$ is monotonically nondecreasing and additive, it follows that there exists a nonnegative constant $\eta_i \geq 0$ such that for any $c \in \mathbb{R}$,

$$\alpha_i(c) = c\eta_i. \tag{A.11}$$

For any $k \in N$, let $z^k \in \mathbb{R}^N$ be the vector such that $z^k_k = 1$ and $z^k_j = 0$ for any $j \neq k$. We have

$$1 \overset{(E)}{=} \sum_{j\in N} f_j(z^k) = f_k(z^k) + \sum_{j\neq k} f_j(z^k) \overset{(A.1)}{=} \phi_k(1, 1) + \alpha_k(1) + \sum_{j\neq k} \alpha_j(1) \overset{(A.11)}{=} \tau_k + \sum_{j\in N} \eta_j.$$

Hence, $\sum_{j\in N} \eta_j = 1 - \tau_k$ for any $k \in N$, which implies that $\tau_k = \tau_{k'} =: \tau$ for any $k, k' \in N$. Now, we define $w_i := \frac{1}{1-\tau}\eta_i$ (therefore, $w_i \geq 0$ because of $\eta_i \geq 0$) for every $i \in N$. Then, for any $x \in \mathbb{R}^N$, we have

$$\alpha_i(\sum_{l\in N} x_l) \overset{(A.11)}{=} \eta_i \sum_{l\in N} x_l = (1 - \tau)w_i \sum_{l\in N} x_l.$$

Moreover, from $\sum_{j\in N} \eta_j = 1 - \tau$ and $\eta_j \geq 0$ for all $j \in N$, it follows that $\tau \leq 1$, which completes Claim 3.

By Claim 1–3, we complete the proof. □

*3 If an additive function $f : \mathbb{R} \to \mathbb{R}$ is monotonic on an interval of positive length, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = cx$. If f is monotonically nondecreasing, then $c \geq 0$. See Chapter 2 of Jung (2011) .

Appendix B. Independence of Axioms

Example B.1. Consider the function $f_i^{\mathrm{E}}(x) = 0$ for any $i \in N$ and any $x \in \mathbb{R}^N$. This function satisfies all axioms except (E).

Example B.2. Consider the following function: for any $i \in N$ and any $x \in \mathbb{R}^N$,

$$f_i^{\mathrm{TMN}}(x) = 2x_i - \frac{1}{n}\sum_{j \in N} x_j.$$

This function satisfies all axioms except for (TMN).

Example B.3. Consider the following function: for any $i \in N$ and any $x \in \mathbb{R}^N$,

$$f_i^{\mathrm{CM}}(x) = \frac{2}{n}\sum_{j \in N} x_j - x_i.$$

This function satisfies all axioms except for (CM).

Example B.4. Consider the following function: for any $i \in N$ and any $x \in \mathbb{R}^N$,

$$f_i^{\mathrm{AD}}(x) = \begin{cases} x_i & \text{if } \sum_{j \in N} x_j < 1, \\ \frac{1}{n}\sum_{j \in N} x_j & \text{if } \sum_{j \in N} x_j \geq 1. \end{cases}$$

This function satisfies all axioms except for (AD).

Appendix C. Proof of Lemma 2.4

Proof. The second statement is clear. We now show the first statement. Let f be a redistribution rule satisfying (TMN), (CM) and (AD). For any $i \in N$ and any $x, y \in \mathbb{R}^N$ with $\sum_{l \in N} x_l \geq \sum_{l \in N} y_l$ and $x_i \geq y_i$, it is enough to consider the following cases:

Case 1: $\sum_{l \in N} x_l = \sum_{l \in N} y_l$ and $x_i \geq y_i$,
Case 2: $\sum_{l \in N} x_l > \sum_{l \in N} y_l$ and $x_i \geq y_i$,

In Case 1, it is straightforward form (CM) that $f_i(x) \geq f_i(y)$.

In Case 2, fix $i \in N$ and let $j \in N$ with $j \neq i$. Define $x'_j := \sum_{l \in N} x_l - \sum_{l \in N} y_l > 0$ and $x'_k := 0$ for every $k \neq j$. Moreover, we define $x^* := x - x'$. We have $\sum_{l \in N} x^*_l = \sum_{l \in N} y_l$ and $x^*_i = x_i \geq y_i$. By (CM), we obtain

$$f_i(x^*) \overset{\text{(CM)}}{\geq} f_i(y). \tag{C.1}$$

From (AD), it follows that $f(\mathbf{0}) = \mathbf{0}$.[*4] Hence, we have

$$f_i(x') \overset{\text{(TMN)}}{\geq} f_i(\mathbf{0}) \overset{\text{(AD)}}{=} 0. \tag{C.2}$$

[*4] Let f satisfy (AD). Assume that there exists $i \in N$ such that $f_i(\mathbf{0}) \neq 0$. Define $a := f_i(\mathbf{0}) \neq 0$. We have

$$f_i(\mathbf{0}) + f_i(\mathbf{0}) = 2a \neq a = f_i(\mathbf{0}) = f_i(\mathbf{0} + \mathbf{0}).$$

This contradicts to (AD).

Thus, we obtain

$$f_i(x) \overset{(\mathrm{AD})}{=} f_i(x^*) + f_i(x') \overset{(\mathrm{C.1})(\mathrm{C.2})}{\geq} f_i(y).$$

□

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