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Abstract

Our objective is to analyze the relationship between the Shapley value and the core from the perspective of the potential of a game. To this end, we introduce a new concept, *generalized HM-potential*, which is a generalization of the potential function defined by Hart and Mas-colell (1989). We show that the Shapley value lies in the core if and only if the maximum of the generalized HM-potential of a game is less than a cutoff value. Moreover, we show that this is equivalent to the minimum of the generalized HM-potential of a game being greater than another, different cutoff value. We also provide a geometric characterization of the class of games in which the Shapley value lies in the core, which also shows the relationship with convex games and average convex games as a corollary. Our results suggest a new approach to utilizing the potential function in cooperative game theory.

Keywords: Shapley value; Core; Potential; Cooperative game
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1 Introduction

One of the objectives in cooperative game theory is to explore desirable solutions. The Shapley value (Shapley, 1953b) and the core should be the most well-known solution concepts. The Shapley value is a single-valued solution, which assigns payoffs to each player based on his/her contributions to cooperative coalitions. Since the seminal study of Shapley (1953b), many researchers have sought to clarify the properties behind the Shapley value.[*1] The Shapley value not only has normative properties but also a variety of applications and strategic foundations.[*2] The core is a set-valued solution, which is a set of payoff allocations from which no players attempt to deviate by forming a group. Its axiomatic properties and strategic foundations are also well-studied.[*3] Because of its generality, the core is used not only in cooperative game theory but also in a wide range of economic theories.

The Shapley value sometimes lies in the core and sometimes does not. If it is in the core, the Shapley value is stable in terms of coalitional deviations and is an attractive core selection. This fact raises the traditional question: What is the condition for the Shapley value to lie in the core? One of the most eminent conditions is *convexity*, introduced by Shapley (1971). He shows that if a game is convex, the Shapley value lies in the core. Inarra and Usategui (1993) and Izawa and Takahashi (1998) show that a condition, called *average convexity*, is also a sufficient condition.[*4] In addition

[*1] See Young (1985), Casajus (2011, 2014), and Casajus and Yokote (2017) and references therein.

[*2] Shapley and Shubik (1954) apply the Shapley value to evaluate the distribution of power among the members of a committee system. Hart and Moore (1990) use the Shapley value as each agent's payoff to analyze the incomplete contract model. Gul (1989), Pérez-Castrillo and Wettstein (2001) and McQuillin and Sugden (2016) provide implementation procedures for obtaining the Shapely value as the sub-game perfect equilibrium outcome of the game.

[*3] Consistency properties play a central role in axiomatic characterizations of the core. Davis and Maschler (1965), Moulin (1985), Peleg (1986) and Tadenuma (1992) introduce different types of consistencies and axiomatize the core. Abe (2017) axiomatically characterizes the core for games with externalities. Perry and Reny (1994) offer a non-cooperative game through which a core element is implemented.

[*4] Average convexity is also analyzed by Sprumont (1990). He calls it *quasi-convexity* in his work. However, his approach is totally different from those of Inarra and Usategui (1993) and Izawa and Takahashi (1998). He defines the Shapley value with respect to every subset of the grand coalition and considers an allocation scheme for all possible coalitions. He uses quasi-convex games for an allocation scheme to be population monotonic.

to the sufficient conditions above, they provide some necessary and sufficient conditions. Although these necessary and sufficient conditions are important steps toward understanding the relationship between the Shapley value and the core, considering their technical expression, it might be difficult to derive applicable insights from these conditions.*5 Our objective is to provide new necessary and sufficient conditions for the Shapley value to be in the core.

To this end, we introduce a generalized potential function, which is a generalization of the potential function introduced by Hart and Mas-Colell (1989), and provide the conditions for the optimization of the generalized potential function. The potential function is a function that assigns a real number to each game.*6 Hart and Mas-Colell (1989) show that the potential function uniquely exists and that the marginal contributions of each player to the potential function coincide with the Shapley value. Moreover, they show that the potential function is represented as a linear transformation of each game: The potential function is given as a weighted sum of each coalition's collective payoff with the given weight profile. By weakening the restrictions of weights offered by Hart and Mas-Colell (1989), we offer certain classes of weights. Then, our generalized potential function is defined over the set of all games and the set (or class) of weight profiles. Therefore, once we fix a game, the generalized potential function is a function over the set of weight profiles.

We show that the Shapley value lies in the core if and only if the maximum of the generalized potential function in a particular class of weight profiles is less than a threshold value. We also present its dual characterization, that is, the Shapley value lies in the core if and only if the minimum of the generalized potential function in another set of weight profiles is greater than another threshold value. Moreover, this approach is straightforwardly applied to any linear solution.*7 We show that,

*5 We discuss their conditions in Section 5.

*6 We follow Morris and Ui (2005)'s terminology for the generalized potential. They introduce the concept in non-cooperative games and analyze the robustness of equilibria to incomplete information.

*7 Various linear solutions are intensively studied to complement the Shapley value: weighted Shapley values (Shapley, 1953a; Chun, 1988, 1991; Kalai and Samet, 1987; Nowak and Radzik, 1995; Yokote, 2015), egalitarian Shapley values and their generalization (Joosten, 1996; Casajus and Huettner, 2013, 2014; van den Brink, Funaki and Ju, 2013; Abe and Nakada, 2017; Yokote, Kongo and Funaki, 2017), and the CIS value (Driessen and Funaki, 1991). See also Yokote and Funaki (2017) for other solutions.

for any linear solution, there is a corresponding set of weight profiles such that the linear solution lies in the core if and only if the minimum of the generalized potential function in the set of weights is greater than zero.

Our result fills the gap between non-cooperative and cooperative game theory regarding the usage of the potential function. Monderer and Shapley (1996) introduce the potential function in non-cooperative game theory, which is a function on the action space that aggregates players' preferences over action profiles.[*8] The potential function in non-cooperative game theory is practically employed because its maximizer designates a Nash equilibrium. This result implies that optimizing the potential function gives an equilibrium without considering the complicated fixed point argument. Moreover, some subsequent studies suggest a relationship between the stability of equilibria and the optimizers of the potential function.[*9] In contrast to non-cooperative games, the potential function is rarely used to find solutions and stability in cooperative games. We show that the potential function can play a similar role in cooperative games as in non-cooperative games: Optimizing our generalized potential function provides us with information about the solution and stability, namely, the Shapley value and the core.

We also provide a geometric characterization of the class of games in which the Shapley value lies in the core.[*10] As mentioned above, the condition for the Shapley value to lie in the core is characterized by minimizing the generalized potential function with respect to weight profiles in a certain set of weights. Since the function is linear with respect to weight profiles, the minimization above is equivalent to minimizing the generalized potential function with respect to weights in a cone generated by the set of weight profiles. As elaborated below, we show that this is, moreover, equivalent to a game being in the dual cone constructed by the set of underlying weight

[*8] For the relationship between the potential function in cooperative games and that in non-cooperative games, see Ui (2000) and Nakada (2017).

[*9] Hofbauer and Sorger (1999, 2002) show that the maximizer of the potential function has stability properties in the dynamics process. Ui (2001) shows that the unique maximizer of the potential function is robust to incomplete information.

[*10] Shapley (1971) provides the geometric characterization of the core in convex games. Marinacci and Montrucchio (2004) provide a similar characterization in terms of the Choquet integral with respect to the underlying game. In contrast, we characterize the class of games: a geometrical structure of the class of games for which the Shapley value lies in the core.

profiles. This geometric characterization uncovers the new reason that convexity and average convexity allow the Shapley value to be in the core and provides an intuition for the mutual relationship among the classes, which we discuss in Section 5.

The rest of this paper is organized as follows. Section 2 provides basic concepts. In Section 3, we introduce the definition of the generalized potential function. Our main results are offered in Section 4. We compare our results with existing results in Section 5. Section 6 is the conclusion of this paper. Some proofs are relegated to the Appendix.

2 Preliminaries

2.1 TU-games

Let $N = \{1, \cdots, n\}$ be the set of players and a function $v : 2^n \to \mathbb{R}$ with $v(\emptyset) = 0$ denote a characteristic function. A *coalition* of players is a nonempty subset of the player set: $S \subseteq N$. We denote the cardinality of coalition S by $|S|$. We use n to denote $|N|$. A cooperative game with transferable utility (a TU-game) is (N, v). We fix the player set N throughout this paper and typically use v instead of (N, v) to denote a game. Let $\mathcal{G}_N$ be the set of all TU-games with the player set N.

For each nonempty $T \subseteq N$, a unanimity game $u_T \in \mathcal{G}_N$ is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Shapley (1953a) shows that any game $v \in \mathcal{G}_N$ can be represented as a unique linear combination of unanimity games: For any game $v \in \mathcal{G}_N$, there are unique values λ_T^v, $\emptyset \neq T \subseteq N$ such that

$$v(S) = \sum_{\emptyset \neq T \subseteq N} \lambda_T^v u_T(S) = \sum_{\emptyset \neq R \subseteq S} \lambda_R^v, \tag{2.1}$$

where $\lambda_T^v = \sum_{\emptyset \neq R \subseteq T} (-1)^{|T|-|R|} v(R)$. For simplicity, we omit v and write λ_T instead of λ_T^v when there is no ambiguity from doing so. We use λ to denote the vector $(\lambda_T)_{\emptyset \neq T \subseteq N} \in \mathbb{R}^{2^n - 1}$.

2.2 Shapley value and Core

Let σ be a permutation of N. For any game v, player i's marginal contribution in σ is

$$mc_{i,\sigma} = v(\rho_i^\sigma \cup \{i\}) - v(\rho_i^\sigma), \tag{2.2}$$

where ρ_i^σ is the set of predecessors of player i in σ. Let Π be the set of all permutations. The Shapley value $Sh(v)$ is given as follows: For each $i \in N$,

$$Sh_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} mc_{i,\sigma}. \tag{2.3}$$

In view of the linearity of the Shapley value, $Sh(v)$ satisfies

$$Sh_i(u_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

From (2.1), it follows that $Sh_i(v) = \sum_{T \subseteq N, i \in T} \lambda_T/|T|$, where $\lambda_T/|T|$ is called *Harsanyi's dividend* to the members of T.

The core $C(v)$ is the set of allocations given by

$$C(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{j \in N} x_j \leq v(N) \text{ and } \sum_{j \in S} x_j \geq v(S) \text{ for all } S \subseteq N \right\}. \tag{2.4}$$

We say that a vector $\gamma \in \mathbb{R}^{2^n-2}$ is a balancing weight if for any $i \in N$, it satisfies $\sum_{S:\ i \in S \subset N} \gamma_S = 1$. Bondareva (1963) and Shapley (1967) study the class of games in which the core is nonempty in terms of balancing weights.

Theorem 1 (Bondareva, 1963; Shapley, 1967)**.** For each $v \in \mathcal{G}_N$, $C(v) \neq \emptyset$ if and only if, for any balancing weight $\gamma \in \mathbb{R}^{2^n-2}$, we have

$$\sum_{\emptyset \neq S \subseteq N} \gamma_S v(S) \leq v(N).$$

A game is said to be balanced if it has a nonempty core. Let $\mathcal{G}_N^C = \{v \in \mathcal{G}_N | C(v) \neq \emptyset\}$ be the set of balanced games.

3 Generalized potentials

Hart and Mas-colell (1989) introduce the potential function. For each $w \in \mathbb{R}^n_+$, a function $P^w : \mathcal{G}_N \to \mathbb{R}$ is said to be a *weighted HM-potential* if it satisfies

$$\textstyle\sum_{i\in N} w_i DP_i^w(v) = v(N)$$

where $DP_i^w(v)$ is the marginal contribution of a player i, which is defined as

$$DP_i^w(v) = P^w(v) - P^w(v^{-i}).$$

Note that $v^{-i} \in \mathcal{G}_N$ is the game given by $v^{-i}(S) = v(S \cap (N \setminus i))$ for any nonempty $S \subseteq N$. Hart and Mas-colell (1989) show that the potential P^w is uniquely given by

$$P^w(v) = \sum_{T\subseteq N} \Big(\frac{1}{\sum_{i\in T} w_i}\Big)\lambda_T.$$

In particular, if $w = (1, \cdots, 1)$, we say that P^w is the *HM-potential.* Let P denote the HM-potential and $DP_i(v) = P(v) - P(v^{-i})$. They show that $DP_i(v) = Sh_i(v)$ for any game v and any player $i \in N$.

We now introduce the following generalized potential function, which plays a central role in this paper.

Definition 1. A function $V : \mathcal{G}_N \times \mathbb{R}^{2^n-1}_+ \to \mathbb{R}$ is a *generalized HM-potential* if for any game $v \in \mathcal{G}_N$ and vector $w \in \mathbb{R}^{2^n-1}_+$,

$$V(v, w) = w \cdot \lambda$$

Note that w_T does not necessarily require that $w_T = \frac{1}{\sum_{i\in T} w_i}$. If w satisfies $w_T = \frac{1}{\sum_{i\in T} w'_i}$ for all $\emptyset \neq T \subseteq N$ and some $w' = (w'_i)_{i\in N}$, then this function is a weighted potential for $w' = (w'_i)_{i\in N}$.

4 Results

4.1 Maximization of the generalized potential

For any nonempty $S \subseteq N$, we define

$$L_S = \sum_{j \in S} \sum_{\sigma \in \Pi} mc_{j,\sigma}. \tag{4.1}$$

Defining $L_i := L_{\{i\}}$, we have

$$L_S = \sum_{j \in S} L_j. \tag{4.2}$$

In other words, L_S is the sum of each member's total marginal contribution $L_i = \sum_{\sigma \in \Pi} mc_{i,\sigma}$. From (2.3), it follows that

$$Sh_i(v) = \frac{1}{n!} L_i. \tag{4.3}$$

Let L be a vector given by L $= (L_S)_{\emptyset \neq S \subseteq N}$.

Lemma 1. Suppose $v(N) > 0$. The Shapley value lies in the core if and only if for any $\gamma = (\gamma_S)_{\emptyset \neq S \subset N} \in \mathbb{R}_+^{2^n - 1}$ with $\sum_{\emptyset \neq S \subseteq N} \gamma_S L_S \leq 1$, we have

$$\sum_{\emptyset \neq S \subseteq N} \gamma_S v(S) \leq \frac{1}{n!}.$$

Proof. See Appendix A. □

Let $\mathcal{W}^{SC1}(v) = \{w \in \mathbb{R}_+^{2^n - 1} | w_T = \sum_{T \subseteq S} \gamma_S$ and $\sum_{S \subseteq N} \gamma_S L_S \leq 1\}$. The set $\mathcal{W}^{SC1}$ allows us to write Lemma 1 as follows.

Lemma 2. Suppose that $v(N) > 0$. The Shapley value lies in the core if and only if

$$\max_{w \in \mathcal{W}^{SC1}(v)} V(v, w) \leq \frac{1}{n!}.$$

We say that two games v, v' are *strategically equivalent* if there exist $a \in \mathbb{R}_{++}$ and $b = (b_i)_{i \in N} \in \mathbb{R}^n$ such that $v'(S) = av(S) + \sum_{i \in S} b_i$ for all $S \subseteq N$. For a given v, strategic equivalence generates an equivalent class in $\mathcal{G}_N$. We denote by $[v]$ the set of all strategically equivalent games to v. For any game v, it is clear that there exists a game $v' \in [v]$ such that $v'(N) > 0$. Moreover, since the Shapley value and the core satisfy covariance, we have $Sh(v) \in C(v) \Leftrightarrow Sh(v') \in C(v')$.[*11] Therefore, by Lemma 2, we obtain the following result.

Proposition 1. For any v, the Shapley value lies in the core if and only if there exists $v' \in [v]$ such that $v'(N) > 0$ and

$$\max_{w \in \mathcal{W}^{SC1}(v')} V(v', w) \leq \frac{1}{n!}.$$

4.2 Minimization of the generalized potential

For any $\emptyset \neq T, S \subseteq N$, let $q_T(S) = \sum_{j \in T \cap S} \frac{1}{|S|} = \frac{|T \cap S|}{|S|}$. Each $q_T(S)$ is the proportion of members of T in S. To consider how the members in coalition T additionally make a gain by cooperating with the members in S, we define

$$\psi_T(S) = \begin{cases} 0 & \text{if } \ q_T(S) = 1, \\ q_T(S) & \text{if } \ q_T(S) < 1. \end{cases} \tag{4.4}$$

Hence, for any $S \subseteq N$, $\psi_T(S)$ is also given as $\sum_{j \in T} \psi_{\{j\}}(S)$ as long as $\sum_{j \in T} \psi_{\{j\}}(S) < 1$; otherwise (namely, if the summation is 1), $\psi_T(S)$ is zero. Note that for any nonempty $S, T \subseteq N$,

$$q_T(S) = 1 \iff S \subseteq T.$$

Moreover, as for $\psi_T(S)$,

$$\psi_T(S) = 0 \iff S \subseteq T \text{ or } S \cap T = \emptyset. \tag{4.5}$$

For each $T \subseteq N$, we define $w^T \in \mathbb{R}^{2^n - 1}$ as

$$w^T(S) = \psi_T(S) \text{ for each } S \subseteq N. \tag{4.6}$$

Let $\mathcal{W}^{SC2} = \{w \in \mathbb{R}^{2^n-1} | w^T \text{ for some } \emptyset \neq T \subseteq N\}$. Then, we provide the following characterization result.

Proposition 2. The Shapley value lies in the core if and only if

$$\min_{w \in \mathcal{W}^{SC2}} V(v, w) \geq 0.$$

[*11] A solution $f : \mathcal{G}_N \rightrightarrows \mathbb{R}^n$ satisfies covariance if for any $v \in \mathcal{G}_N$ and any $x \in f(v)$, $ax + b \in f(v')$, where $a \in \mathbb{R}_{++}$, $b = (b_i)_{i \in N} \in \mathbb{R}^n$, and $v'(S) = av(S) + \sum_{i \in S} b_i$ for all $S \subseteq N$.

Proof. See Appendix A. $\square$

Summarizing Propositions 1 and 2, we obtain the following dual characterizations.

Theorem 2. For any game v, the following statements are equivalent.

(i) The Shapley value lies in the core.

(ii) There exists $v' \in [v]$ such that $v'(N) > 0$ and $\max_{w \in \mathcal{W}^{SC1}(v')} V(v', w) \leq \frac{1}{n!}$.

(iii) $\min_{w \in \mathcal{W}^{SC2}} V(v, w) \geq 0$.

In contrast to $\mathcal{W}^{SC1}$, $\mathcal{W}^{SC2}$ does not depend on game v. This result suggests that even the calculations of the Shapley value and the core are no longer needed to check $Sh(v) \in C(v)$.

While Proposition 2 is described in terms of λ, which depends on v, it can be translated into the expression using v instead of λ. Let $U \in \mathbb{R}^{2^n-1} \times \mathbb{R}^{2^n-1}$ be the matrix whose column vector is each unanimity game u_T. For any game v, in view of (2.1), we can write $v = U\lambda$. Then, for any $w \in \mathbb{R}_+^{2^n-1}$, we have $V(w, v) = w \cdot \lambda = \underbrace{(wU^{-1})}_{\tilde{w}} \cdot v = \tilde{w} \cdot v$. Let $\tilde{\mathcal{W}}^{SC2} = \{\tilde{w} | \tilde{w} = wU^{-1}, w \in \mathcal{W}^{SC2}\}$. Since $\mathcal{W}^{SC2}$ and U^{-1} do not depend on v, $\tilde{\mathcal{W}}^{SC2}$ does not depend on v, either. In the following example, we demonstrate how to apply our result. Tables 1-4 in Example 1 show $\mathcal{W}^{SC2}, U, U^{-1}$, and $\tilde{\mathcal{W}}^{SC2}$ for three players.

Example 1. Suppose that $N = \{1, 2, 3\}$. As mentioned in Section 2, λ is uniquely determined for v. The inequalities of Proposition 2 are as follows.

$$T = \{1\};\ \frac{1}{2}\lambda_{\{1,2\}} + \frac{1}{2}\lambda_{\{1,3\}} + \frac{1}{3}\lambda_{\{1,2,3\}} \geq 0,$$

$$T = \{1, 2\};\ \frac{1}{2}\lambda_{\{1,3\}} + \frac{1}{2}\lambda_{\{2,3\}} + \frac{2}{3}\lambda_{\{1,2,3\}} \geq 0.$$

If $T = \{1, 2, 3\}$, it is always satisfied with $0 \geq 0$ (see the bottom row of Table 1).

Similarly, we can write the conditions in terms of v.

$$
\begin{aligned}
T=\{1\};\ & -\frac{2}{3}v(\{1\})-\frac{1}{6}v(\{2\})-\frac{1}{6}v(\{3\}) \\
& +\frac{1}{6}v(\{1,2\})+\frac{1}{6}v(\{1,3\})-\frac{1}{3}v(\{2,3\})+\frac{1}{3}v(\{1,2,3\})\geq 0, \\
T=\{1,2\};\ & \frac{1}{6}v(\{1\})+\frac{1}{6}v(\{2\})-\frac{1}{3}v(\{3\}) \\
& -\frac{2}{3}v(\{1,2\})-\frac{1}{6}v(\{1,3\})-\frac{1}{6}v(\{2,3\})+\frac{2}{3}v(\{1,2,3\})\geq 0.
\end{aligned}
$$

For $T=\{1,2,3\}$, it similarly holds with $0\geq 0$. We have only to verify such inequalities for every $\emptyset\neq T\subseteq N$ to determine whether the Shapley value lies in the core.

$T\backslash S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$
$\{2\}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$
$\{3\}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
$\{1,2\}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$
$\{1,3\}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$
$\{2,3\}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{2}{3}$
$\{1,2,3\}$	0	0	0	0	0	0	0

Table 1 $\mathcal{W}^{SC2}$.

$S\backslash T$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	1	0	0	0	0	0	0
$\{2\}$	0	1	0	0	0	0	0
$\{3\}$	0	0	1	0	0	0	0
$\{1,2\}$	1	1	0	1	0	0	0
$\{1,3\}$	1	0	1	0	1	0	0
$\{2,3\}$	0	1	1	0	0	1	0
$\{1,2,3\}$	1	1	1	1	1	1	1

Table 2 U.

$S\backslash T$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	1	0	0	0	0	0	0
$\{2\}$	0	1	0	0	0	0	0
$\{3\}$	0	0	1	0	0	0	0
$\{1,2\}$	-1	-1	0	1	0	0	0
$\{1,3\}$	-1	0	-1	0	1	0	0
$\{2,3\}$	0	-1	-1	0	0	1	0
$\{1,2,3\}$	1	1	1	-1	-1	-1	1

Table 3 U^{-1}.

$T\backslash S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{3}$
$\{2\}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$
$\{3\}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
$\{1,2\}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{3}$
$\{1,3\}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{1}{6}$	$\frac{2}{3}$
$\{2,3\}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{3}$
$\{1,2,3\}$	0	0	0	0	0	0	0

Table 4 $\tilde{\mathcal{W}}^{SC2} = \mathcal{W}^{SC2}U^{-1}$.

4.3 Geometric characterization

We provide a geometric characterization of $\mathcal{G}_N^{SC}$, which is useful to compare the results in Sections 4 and 5. We begin with a brief introduction on cones. Let X be a nonempty subset of $\mathbb{R}^l$. We denote by $coni(X) = \{z | z = \sum_{x \in X} t_x x, t_x \geq 0\}$ the conical hull of X, namely, the minimal convex cone containing X. Moreover, let $coni(X)^* = \{p \in \mathbb{R}^l | p \cdot x \geq 0, x \in coni(X)\}$ be the dual cone of $coni(X)$, and let $coni(X)^\circ = \{p \in \mathbb{R}^l | p \cdot x \leq 0, x \in coni(X)\}$ be the polar cone of $coni(X)$. Now, it follows from Proposition 2 and the linearity of V that

$$\begin{aligned}\min_{w \in \mathcal{W}^{SC2}} V(v, w) \geq 0 &\Leftrightarrow \forall w \in coni(\mathcal{W}^{SC2}), w \cdot \lambda \geq 0 \\ &\Leftrightarrow \forall \tilde{w} \in coni(\tilde{\mathcal{W}}^{SC2}), \tilde{w} \cdot v \geq 0 \\ &\Leftrightarrow v \in coni(\tilde{\mathcal{W}}^{SC2})^*.\end{aligned}$$

Let $\mathcal{G}_N^{SC} = \{v \in \mathcal{G}_N | Sh(v) \in C(v)\}$ be the set of games for which the Shapley value lies in the core. Then, by the above discussion, we provide the following geometric characterization of $\mathcal{G}_N^{SC}$.

Theorem 3. $\mathcal{G}_N^{SC} = coni(\tilde{\mathcal{W}}^{SC2})^*$.

Our result can also be applied for the class of balanced games $\mathcal{G}_N^C$. As introduced in Section 2, a balancing weight is $\gamma = (\gamma_S)_{\emptyset \neq S \subseteq N} \in \mathbb{R}^{2^n - 1}$ satisfying $\sum_{S:\ i \in S \subseteq N} \gamma_S = 1$ for any $i \in N$. In view of the redundancy of γ_N and $v(N)$, for a given balancing weight γ, we define $\gamma'_S = \gamma_S$ for every nonempty $S \subsetneq N$ and $\gamma'_N = -1$. Let Λ be the set of γ', formally, $\Lambda = \{\gamma' \in \mathbb{R}_+^{2^n - 2} \times \{-1\} | \sum_{S: i \in S \subsetneq N} \gamma'_S = 1\ \forall i \in N\}$. In view of Theorem 1, for any game $v \in \mathcal{G}_N, C(v) \neq \emptyset$ if and only if

$$\max_{\gamma \in \Lambda} \gamma \cdot v \leq 0.$$

Note that the set Λ is a convex polytope (see, for example, Peleg and Sudhölter (2007)). Hence, Λ is the convex hull of its extreme points.[*12] Let $\hat{\Lambda}$ denote the set of extreme points of Λ. Theorem 1 is written as follows.

[*12] The explicit description of $\hat{\Lambda}$ in general n-player games is not known. This is because it is generally difficult to construct extreme points of a convex polyhedron.

Theorem 4. For any game $v \in \mathcal{G}_N, C(v) \neq \emptyset$ if and only if

$$\max_{\gamma \in \hat{\Lambda}} \gamma \cdot v \leq 0.$$

Note that $\hat{\Lambda}$ is independent of v. In the same manner as $\mathcal{W}^{SC2}$, we have $\max_{\gamma \in \hat{\Lambda}} \gamma \cdot v \leq 0$ if and only if $\max_{\tilde{\gamma} \in coni(\hat{\Lambda})} \tilde{\gamma} \cdot v \leq 0$. The latter condition is equivalent to $v \in coni(\Gamma)^{\circ}$. Hence, we obtain the following direct connection between the balancing weights and balanced games.

Theorem 5. $\mathcal{G}_N^C = coni(\hat{\Lambda})^0$.

For any class of games, we can find a corresponding conical expression as long as we obtain a class of weights representing the class of games through the generalized potential. In Section 5, we investigate other important classes.

5 Relationship with existing results

5.1 Convex and average convex games

Convex games and average convex games are known as the classes of games in which the Shapley value lies in the core. In this subsection, we first show that some systems of weights define the two classes of games and, then, explore their relationship through the weights.

We begin with convex games. A game v is convex if for any $i, j \in N$ $(i \neq j)$, any $S \subseteq N \setminus \{i, j\}$ and $T = S \cup \{j\}$,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Shapley (1971) shows that if a game v is convex, then $Sh(v) \in C(v)$. To revisit convexity through the generalized potential, we offer the following expression for convexity.

Lemma 3. The following statements are equivalent.

(i): A game v is convex.

(ii): For any $S \subseteq N$ with $|S| = 2$ and any T with $S \subseteq T \subseteq N$,

$$\sum_{R: S \subseteq R \subseteq T} \lambda_R \geq 0.$$

Proof. See Appendix B. $\square$

One can verify Lemma 3 through the following example. Let $N = \{1, 2, 3\}$. For player 1's marginal contributions, we have

$$\begin{aligned}
v(\{1,2,3\}) - v(\{2,3\}) \geq v(\{1,2\}) - v(\{2\}) &\iff \lambda_{\{1,2,3\}} + \lambda_{\{1,3\}} \geq 0, \\
v(\{1,2,3\}) - v(\{2,3\}) \geq v(\{1,3\}) - v(\{3\}) &\iff \lambda_{\{1,2,3\}} + \lambda_{\{1,2\}} \geq 0, \\
v(\{1,2\}) - v(\{2\}) \geq v(\{1\}) &\iff \lambda_{\{1,2\}} \geq 0, \\
v(\{1,3\}) - v(\{3\}) \geq v(\{1\}) &\iff \lambda_{\{1,3\}} \geq 0.
\end{aligned}$$

The inequalities for players 2 and 3 also hold in the same manner.

Now, we provide the system of weights for convex games. Let $\alpha \subseteq N$ be a coalition such that $|\alpha| = 2$. For a given α, we define $\beta \subseteq N$ as $\alpha \subseteq \beta$. For given α and β, we define $w^{\alpha,\beta} \in \mathbb{R}^{2^n-1}$ as follows: For any $S \subseteq N$,

$$w_{\alpha,\beta}(S) = \begin{cases} 1 & \text{if } \alpha \subseteq S \subseteq \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Let $\mathcal{W}^{SC3} = \{w_{\alpha,\beta} \in \mathbb{R}^{2^n-1} |\ \alpha, \beta \subseteq N,\ |\alpha| = 2,\ \alpha \subseteq \beta\}$. Note that $\mathcal{W}^{SC3}$ is independent of v. Hence, we can write Lemma 3 as follows.

Proposition 3. A game v is convex if and only if

$$\min_{w \in \mathcal{W}^{SC3}} V(v, w) \geq 0.$$

Proof. We have $\min_{w \in \mathcal{W}^{SC3}} V(v, w) = \min_{\alpha,\beta:\ |\alpha|=2,\ \alpha\subseteq\beta} w_{\alpha,\beta} \cdot \lambda$. This entails (ii) of Lemma 3. Note that, in view of (5.1), $w_{\alpha,\beta} \cdot \lambda = \sum_{\alpha \subseteq R \subseteq \beta} \lambda_R$. $\square$

Let $\mathcal{G}_N^{CONV}$ be the set of convex games. Proposition 3 implies the following result in the same manner as Theorem 3.

Theorem 6. $\mathcal{G}_N^{CONV} = coni(\tilde{\mathcal{W}}^{SC3})^*$.

Next, we consider average convex games. A game is average convex if for any nonempty $S, T \subseteq N$ with $S \subseteq T$,

$$\sum_{j \in S} (v(T) - v(T \setminus \{j\})) \geq \sum_{j \in S} (v(S) - v(S \setminus \{j\})).$$

Note that convexity implies average convexity.[*13] Inarra and Usategui (1993) show that if a game v is average convex, then $Sh(v) \in C(v)$.[*14] Similar to convex games, the following relationship holds.

Lemma 4. The following statements are equivalent.

(i): A game v is average convex.

(ii): For any $S, T \subseteq N$ with $S \subseteq T \subseteq N$,

$$\sum_{\substack{R \subseteq T \\ R\cap S \neq \emptyset, R\cap(T\setminus S)\neq\emptyset}} |R \cap S| \cdot \lambda_R \geq 0.$$

Proof. See Appendix B. □

We also provide the system of weights for average convex games. For given α and β, we define $\omega^{\alpha,\beta} \in \mathbb{R}^{2^n-1}$ as follows: For any $S \subseteq N$,

$$\omega_{\alpha,\beta}(S) = \begin{cases} |S \cap \alpha| & \text{if } S \subseteq \beta, S \cap \alpha \neq \emptyset, \text{ and } S \cap (\beta \setminus \alpha) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Let $\mathcal{W}^{SC4} = \{\omega_{\alpha,\beta} \in \mathbb{R}^{2^n-1} | \ \alpha, \beta \subseteq N, \ \alpha \subseteq \beta\}$. Note that $\mathcal{W}^{SC4}$ is independent of v. The following proposition shows that $\mathcal{W}^{SC4}$ defines average convexity.

Proposition 4. A game v is average convex if and only if

$$\min_{\omega \in \mathcal{W}^{SC4}} V(v, \omega) \geq 0.$$

[*13] For the counter examples of the opposite direction, see examples in Inarra and Usategui (1993) and Izawa and Takahashi (1998).

[*14] Inarra and Usategui (1993) also offer the necessary and sufficient condition as follows: $v \in \mathcal{G}_N^{SC}$ if and only if for any $T \subseteq N$,

$$\sum_{\emptyset \neq S \subseteq N} \frac{(n-s)!(s-1)!}{n!} h_T(S)(v(S) - v(S \setminus T) - v(S \cap T)) \geq 0,$$

where

$$h_T(S) = \begin{cases} |S| \cdot (\frac{|S \cap T|}{|S|} - \frac{|T \setminus S|}{|N \setminus S|}) & \text{if } S \subseteq N, S \neq N, \\ |T| & \text{if } S = N. \end{cases}$$

Izawa and Takahashi (1998) also show that $v \in \mathcal{G}_N^{SC}$ if and only if for any $T \subseteq N$,

$$\sum_{S \subset N} \sum_{i \in S \cap T} \frac{(n-s)!(s-1)!}{n!} (v^i(S) - v^i(S \cap T)) \geq 0.$$

Our result can be seen as another interpretation of this result.

Proof. We have $\min_{w\in\mathcal{W}^{SC4}} V(v,w) = \min_{\alpha,\beta,\ \alpha\subseteq\beta} w_{\alpha,\beta}\cdot\lambda$. This entails (ii) of Lemma 4. Note that, in view of (5.2), $w_{\alpha,\beta}\cdot\lambda = \sum_{\substack{R\subseteq\beta \\ R\cap\alpha\neq\emptyset,\, R\cap(\beta\setminus\alpha)\neq\emptyset}} |R\cap\alpha|\lambda_R$. $\square$

Let $\mathcal{G}_N^{AVCONV}$ be the set of convex games. Proposition 4 implies the following result.

Theorem 7. $\mathcal{G}_N^{AVCONV} = coni(\tilde{\mathcal{W}}^{SC4})^*$.

Now, we analyze the relationship among $\mathcal{W}^{SC2}, \mathcal{W}^{SC3}$, and $\mathcal{W}^{SC4}$. First, we offer the following relationship between $\mathcal{W}^{SC3}$ and $\mathcal{W}^{SC4}$.

Lemma 5. For each $\omega\in\mathcal{W}^{SC4}$, it can be represented as sum of weights in $\mathcal{W}^{SC3}$.

Proof. See Appendix B. $\square$

Second, we show that each weight $w\in W^{SC2}$ is decomposed into weights in $\mathcal{W}^{SC4}$.

Lemma 6. For any nonempty subset $T\subseteq N$ and $w_T\in W^{SC2}$, we have

$$w_T = \sum_{\emptyset\neq\alpha\subseteq T}\ \sum_{\substack{\gamma\subseteq N\setminus T \\ \beta:=\alpha\cup\gamma}} \kappa_{T,\alpha,\beta}\,\omega_{\alpha,\beta},$$

where $w_{\alpha,\beta}\in\mathcal{W}^{SC4}$ for any α,β and

$$\begin{aligned}\kappa_{T,\alpha,\beta} &= \sum_{\beta\subseteq R\subseteq N} (-1)^{|R|-|\beta|}\frac{1}{|R|} \\ &= \frac{1}{n\cdot({}_{n-1}C_{m-1})} \geq 0,\end{aligned}$$

where ${}_nC_r$ means a combination $\begin{pmatrix} n \\ r\end{pmatrix} = \frac{n!}{r!(n-r)!}$.

Proof. See Appendix B. $\square$

From Propositions 2 and 4 and Lemma 6, the fact that the Shapley value lies in the core in average convex games is written as follows. For any nonempty $T\subseteq N$,

$$\begin{aligned}V(v,w_T) = w_T\cdot\lambda &= \left(\sum_{\alpha\in F^T}\ \sum_{\alpha\subseteq\beta\subseteq N}\kappa_{T,\alpha,\beta}\,\omega_{\alpha,\beta}\right)\cdot\lambda \\ &= \sum_{\alpha\in F^T}\ \sum_{\alpha\subseteq\beta\subseteq N}\kappa_{T,\alpha,\beta}\left(\omega_{\alpha,\beta}\cdot\lambda\right) \\ &\geq 0.\end{aligned}$$

The last inequality holds because of average convexity (Proposition 4).

By Lemma 5, we know that, for each $\omega \in \mathcal{W}^{SC4}$, there are nonnegative coefficients $\{\mu_{\omega,w}\}_{w\in\mathcal{W}^{SC3}}$ such that $\omega = \sum_{w\in\mathcal{W}^{SC3}} \mu_{\omega,w} w$. Combining this result with Lemma 6, we can say that, for each $w_T \in W^{SC2}$,

$$\begin{aligned}
w_T &= \sum_{\emptyset\neq\alpha'\subseteq T} \sum_{\substack{\gamma'\subseteq N\setminus T \\ \beta':=\alpha'\cup\gamma'}} \kappa_{T,\alpha',\beta'}\omega_{\alpha',\beta'} \\
&= \sum_{\emptyset\neq\alpha'\subseteq T} \sum_{\substack{\gamma'\subseteq N\setminus T \\ \beta':=\alpha'\cup\gamma'}} \kappa_{T,\alpha',\beta'}\big(\sum_{w\in\mathcal{W}^{SC3}} \mu_{\omega_{\alpha',\beta'},w} w\big) \\
&= \sum_{w\in\mathcal{W}^{SC3}} \underbrace{\big(\sum_{\emptyset\neq\alpha'\subseteq T} \sum_{\substack{\gamma'\subseteq N\setminus T \\ \beta':=\alpha'\cup\gamma'}} \kappa_{T,\alpha',\beta'}\mu_{\omega_{\alpha',\beta'},w}\big)}_{\eta_{T,\alpha,\beta}} w \\
&= \sum_{\alpha\subseteq\beta\subseteq N} \eta_{T,\alpha,\beta} w_{\alpha,\beta}
\end{aligned}$$

where $w_{\alpha,\beta} \in \mathcal{W}^{SC3}$ for each α, β with $|\alpha| = 2$. By construction, $\eta_{T,\alpha,\beta} \geq 0$. Therefore, we can decompose $w \in W^{SC2}$ into weights in $\mathcal{W}^{SC3}$ in a similar manner as Lemmas 5 and 6.

Lemma 7. For any nonempty subset $T \subseteq N$ and $w_T \in W^{SC2}$, we have

$$w_T = \sum_{\alpha\subseteq\beta\subseteq N} \eta_{T,\alpha,\beta} w_{\alpha,\beta},$$

for some $\eta_{T,\alpha,\beta} \geq 0$ and $w_{\alpha,\beta} \in \mathcal{W}^{SC3}$ for each α, β with $|\alpha| = 2$.

From Propositions 2 and 3 and Lemma 7, one can verify that the Shapley value lies in the core in convex games as follows: For any nonempty subset $T \subseteq N$,

$$\begin{aligned}
V(v, w_T) = w_T \cdot \lambda &= \left(\sum_{\alpha\in F^T} \sum_{\alpha\subseteq\beta\subseteq N} \eta_{T,\alpha,\beta} w_{\alpha,\beta} \right) \cdot \lambda \\
&= \sum_{\alpha\in F^T} \sum_{\alpha\subseteq\beta\subseteq N} \eta_{T,\alpha,\beta} \left(w_{\alpha,\beta} \cdot \lambda \right) \\
&\geq 0.
\end{aligned}$$

The last inequality holds because of convexity (Proposition 3).

Finally, summarizing Lemmas 5, 6, and 7, we obtain the following geometric relationship among $\mathcal{W}^{SC2}, \mathcal{W}^{SC3}$, and $\mathcal{W}^{SC4}$.

Theorem 8. $coni(\mathcal{W}^{SC2}) \subsetneq coni(\mathcal{W}^{SC4}) \subsetneq coni(\mathcal{W}^{SC3})$.

Theorem 8 provides a new explanation for why the classes of convex games and average convex games are larger than those of games for which the Shapley value lies in the core, as

$$\underbrace{coni(\tilde{\mathcal{W}}^{SC3})^*}_{=\mathcal{G}_N^{CONV}} \subsetneq \underbrace{coni(\tilde{\mathcal{W}}^{SC4})^*}_{=\mathcal{G}_N^{AVCONV}} \subsetneq \underbrace{coni(\tilde{\mathcal{W}}^{SC2})^*}_{=\mathcal{G}_N^{SC}}.$$

5.2 Commander games

Yokote, Funaki, and Kamijo (2017) provide a necessary and sufficient condition for the Shapley value to be in the core by considering another basis, which is called *commander games*. For each nonempty $T \subseteq N$, a commander game $\bar{u}_T \in \mathcal{G}_N$ is defined as

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |T \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Yokote, Funaki, and Kamijo (2016) show that $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ is a basis of $\mathcal{G}_N$, that is, each game v is represented as $v = \sum_{\emptyset \neq T \subseteq N} d_T \bar{u}_T$, where $d = (d_T)_{\emptyset \neq T \subseteq N}$ is the coefficient for the corresponding $\bar{u}_T$. Moreover, they show that, for any $i \in N$, $d_{\{i\}} = Sh_i(v)$, that is, the coefficient corresponding to each singleton set coincides with the Shapley value. By this expression, we can write

$$\begin{aligned} v(S) &= \sum_{\emptyset \neq T \subseteq N} d_T \bar{u}_T(S) \\ &= \sum_{i \in S} Sh_i(v) + \sum_{\emptyset \neq T \subseteq N; |T| \geq 2, |T \cap S| = 1} d_T. \end{aligned} \tag{5.3}$$

Let $v^{Sh} \in [v]$ be a strategically equivalent game such that $v^{Sh}(S) = v(S) - \sum_{i \in S} Sh_i(v)$. From the above discussion, they provide the following result.

Proposition 5 (Yokote, Funaki and Kamijo, 2017)**.** $Sh(v) \in C(v)$ if and only if $\mathbf{0} \in C(v^{Sh})$.

By (5.3), this condition is equivalent to stating that, for any nonempty $T \subseteq N$,

$$\sum_{\emptyset \neq S \subseteq N; |S| \geq 2, |S \cap T| = 1} d_S \leq 0. \tag{5.4}$$

We clarify the relationship between Proposition 2 and Proposition 5. Let $E \in \mathbb{R}^{2^n-1} \times \mathbb{R}^{2^n-1}$ be the matrix that corresponds to (5.3), meaning that $v - Sh = Ed \leq \mathbf{0}$. Since both unanimity games $\{u_T\}_{\emptyset \neq T \subseteq N}$ and commander games $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ are the basis of $\mathcal{G}_N$, for any game v,

$$v = U\lambda = \bar{U}d,$$

where $\bar{U} \in \mathbb{R}^{2^n-1} \times \mathbb{R}^{2^n-1}$ is the matrix whose column vector is a commander game $\bar{u}_T$. Therefore, $d = (\bar{U}^{-1}U)\lambda$, meaning that $Ed = E(\bar{U}^{-1}U)\lambda$. Let $\mathcal{W}^{SC5}$ be the set of row vectors of $E(\bar{U}^{-1}U)$. By this expression, condition (5.4) is equivalent to

$$\max_{w \in \mathcal{W}^{SC5}} V(v, w) \leq 0. \tag{5.5}$$

Let $\tilde{\mathcal{W}}^{SC5}$ be the set of row vectors of $E(\bar{U}^{-1}U)U^{-1} = E\bar{U}^{-1}$. Then, by the same logic as in Theorem 3, we have

$$\mathcal{G}_N^{SC} = coni(\tilde{\mathcal{W}}^{SC5})^0.$$

To see the relationship between $\mathcal{W}^{SC5}$ and $\mathcal{W}^{SC2}$, we regard $\mathcal{W}^{SC2}, \mathcal{W}^{SC5} \in \mathbb{R}^{2^n-1} \times \mathbb{R}^{2^n-1}$ as the matrix, each of whose row vectors corresponds to each $w \in \mathcal{W}^{SC2}$ and $w \in \mathcal{W}^{SC5}$. For any $v \in \mathcal{G}_N$, remember that $\frac{1}{n!}\mathrm{L} = (\sum_{i \in S} Sh_i(v))_{\emptyset \neq S \subseteq N}$. Then, for any $v \in \mathcal{G}_N$,

$$\begin{aligned} -\mathcal{W}^{SC2}\lambda = U\lambda - \frac{1}{n!}\mathrm{L} &= \bar{U}d - \frac{1}{n!}\mathrm{L} \\ &= Ed \\ &= E(\bar{U}^{-1}U)\lambda \\ &= \mathcal{W}^{SC5}\lambda. \end{aligned}$$

where the third equality holds because of the definition of E. Therefore, we have $\mathcal{W}^{SC5} = -\mathcal{W}^{SC2}$. In this sense, condition (5.5), which is equivalent to Proposition 5, is the mirror image of Proposition 2. The following Tables 5-8 show the description of $\bar{U}, \bar{U}^{-1}, E$ and $\mathcal{W}^{SC5}$ for three players.

$S \backslash T$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	1	0	0	1	1	0	1
$\{2\}$	0	1	0	1	0	1	1
$\{3\}$	0	0	1	0	1	1	1
$\{1,2\}$	1	1	0	0	1	1	0
$\{1,3\}$	1	0	1	1	0	1	0
$\{2,3\}$	0	1	1	1	1	0	0
$\{1,2,3\}$	1	1	1	0	0	0	0

Table 5 $\bar{U}$.

$S \backslash T$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{3}$
$\{2\}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$
$\{3\}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
$\{1,2\}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
$\{1,3\}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$
$\{2,3\}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{3}$
$\{1,2,3\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$

Table 6 $\bar{U}^{-1}$.

$T\backslash S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	0	0	0	1	1	0	1
$\{2\}$	0	0	0	1	0	1	1
$\{3\}$	0	0	0	0	1	1	1
$\{1,2\}$	0	0	0	0	1	1	0
$\{1,3\}$	0	0	0	1	0	1	0
$\{2,3\}$	0	0	0	1	1	0	0
$\{1,2,3\}$	0	0	0	0	0	0	0

Table 7 E.

$T\backslash S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1\}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$
$\{2\}$	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{3}$
$\{3\}$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$
$\{1,2\}$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$
$\{1,3\}$	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{2}{3}$
$\{2,3\}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{2}{3}$
$\{1,2,3\}$	0	0	0	0	0	0	0

Table 8 $\mathcal{W}^{SC5} = E\bar{U}^{-1}U$.

6 Concluding remarks

In this paper, we introduce the generalized potential function and offer the necessary and sufficient conditions for the Shapley value to be in the core by optimizing the generalized potential. The traditional potential function only characterizes the Shapley value in cooperative games, while it is practically utilized to find Nash equilibria in non-cooperative games. One of our contributions lies in the novelty that we suggest a new approach to put “a real number assigned to a game” into practice in cooperative game theory through generalizing the traditional potential function. We conclude the paper with the following two other possibilities for generalization.

6.1 Other linear solutions

Throughout the previous sections, we considered the relationship between the Shapley value and the core. However, as briefly mentioned in Section 1, our approach applies to any linear solution instead of the Shapley value. A solution $f : \mathcal{G}_N \to \mathbb{R}^n$ is *linear* if for any $c, c' \in \mathbb{R}$ and $v, v' \in \mathcal{G}_N$, $f(cv + c'v') = cf(v) + c'f(v')$. If a solution f is linear, in view of (2.1), $f(v) = \sum_{\emptyset \neq T \subseteq N} \lambda_T f(u_T)$. Then, for any nonempty subset $T \subseteq N$, we define $w_T^f \in \mathbb{R}^{2^n-1}$ as

$$w_T^f(S) = \begin{cases} \sum_{i \in T} f_i(u_S) & \text{if } S \nsubseteq T, \\ \sum_{i \in T} f_i(u_S) - 1 & \text{if } S \subseteq T. \end{cases}$$

Let $\mathcal{W}^f$ be the set of all w_T^f. Note that this construction corresponds to that of (4.6). We have $f(v) \in C(v)$ if and only if

$$\min_{w \in \mathcal{W}^f} V(v, w) \geq 0$$

in a similar manner as Proposition 2.

6.2 Coincidence of linear solutions

Our approach is also valid to identify the coincidence of different linear solutions. As noted above, for any liner solution f, it holds that $f_i(v) = \sum_{T \subseteq N} \lambda_T f_i(u_T)$ for each $i \in N$. For each $i \in N$, let us define $\theta_i^{f,f'} \in \mathbb{R}^{2^n-1}$ as $\theta_i^{f,f'}(T) = f_i(u_T) - f_i'(u_T)$ for each $T \subseteq N$. Then, let $\Theta^{f,f'} = \{\theta_i^{f,f'} \in \mathbb{R}^{2^n-1} |$ for some $i \in N\}$. Similar to the above discussion, two linear solutions f and f' coincide if and only if for any $\theta_i \in \Theta^{f,f'}$,

$$V(v, \theta_i) = 0. \tag{6.1}$$

We slightly abuse notation and regard $\Theta^f \in \mathbb{R}^n \times \mathbb{R}^{2^n-1}$ as the matrix whose row vector is $\theta_i^{f,f'}$. Then, this is equivalent to

$$v \in coni(\tilde{\Theta}^{f,f'})^\perp$$

where $coni(\tilde{\Theta}^{f,f'})^\perp$ is an orthogonal complement of $coni(\tilde{\Theta}^{f,f'})$, which implies that v is included in the null space of $\tilde{\Theta}^{f,f'}$. Yokote, Funaki and Kamijo (2017) discuss the conditions for the Shapley value to coincide with the CIS value and the ENSC value, based on the covariance of these solutions and commander games. Their results can be obtained by condition (6.1) in the same manner as in Section 5.2. Moreover, our result has an advantage: It applies to any linear solution. The application to linear solutions is not necessarily induced from the covariance of solutions and commander games. As demonstrated above, we believe that the generalized potential is a useful tool to study the relationship among various solutions of TU-games.

A. Omitted Proofs in Section 4

Proof of Lemma 1. We assume $Sh(v) \in C(v)$ and consider the following linear programing problem:

$$\min_{x \in \mathbb{R}_+} x \text{ subject to } x\mathrm{L} \geq v, \tag{A.1}$$

where v is a vector $(v(S))_{\emptyset \neq S \subseteq N}$.[*15] From the assumption $Sh(v) \in C(v)$, it follows that $\sum_{j \in S} Sh_j(v) \geq v(S)$ for any $S \subseteq N$, which implies $\frac{1}{n!} L_S \geq v(S)$ by (4.2) and (4.3). Hence, (A.1) is feasible (*e.g.*, $x = \frac{1}{n!}$). The corresponding dual problem is given as follows:

$$\max_{\gamma \in \mathbb{R}_+^{2^n - 1}} \gamma \cdot v \text{ subject to } \gamma \cdot \mathrm{L} \leq 1. \tag{A.2}$$

This problem is feasible for any v ($e.g., \gamma = \mathbf{0}$). By the duality theorem, the two optimal values coincide, say z. Hence, for any γ with $\sum_{\emptyset \neq S \subseteq N} \gamma_S L_S \leq 1$, we have

$$\sum_{\emptyset \neq S \subseteq N} \gamma_S v(S) \leq z.$$

Moreover, by substituting $\frac{1}{n!}$ for x in the minimization problem, we obtain $z \leq \frac{1}{n!}$. Hence,

$$\sum_{\emptyset \neq S \subseteq N} \gamma_S v(S) \leq \frac{1}{n!}. \tag{A.3}$$

Next, on the contrary, we assume that (A.3) holds for any γ with $\sum_{\emptyset \neq S \subseteq N} \gamma_S L_S \leq 1$. The maximization problem (A.2) is feasible (*e.g.*, $\gamma = \mathbf{0}$), and its maximum, say z, should be less than or equal to $\frac{1}{n!}$. By the assumption, for any γ with $\gamma \cdot \mathrm{L} \leq 1$,

[*15] $x \geq y$ if $x_i \geq y_i$ for each i.

$\gamma \cdot v$ has the upper bound $\frac{1}{n!}$, which implies that the minimization problem (A.1) is feasible.[*16] By the duality theorem, its minimum coincides with $z \le \frac{1}{n!}$.

Hence, z is a feasible solution of (A.1) and satisfies

$$zL_N \ge v(N). \tag{A.4}$$

Moreover, we have

$$L_N = \sum_{j\in N}\sum_{\sigma\in\Pi} mc_{j,\sigma} = \sum_{\sigma\in\Pi} v(N) = n!\cdot v(N).$$

From $v(N) > 0$, it follows that $z \ge \frac{1}{n!}$. Hence, $z = \frac{1}{n!}$. Substituting $\frac{1}{n!}$ for x in the constraints of (A.1), we obtain $Sh(v) \in C(v)$. □

Proof of Proposition 2. Fix $T \subseteq N$ and let $w^T \in \mathcal{W}^{SC2}$. For any v, we have

$$\begin{aligned}
V(v, w^T) &= \sum_{S\subseteq N} w_S^T \lambda_S = \sum_{S\subseteq N} \psi_{T,S}\lambda_S \\
&= \sum_{S:S\subseteq T} \psi_{T,S}\lambda_S + \sum_{S:S\not\subseteq T} \psi_{T,S}\lambda_S \\
&\overset{(4.4),\ (4.5)}{=} \left(\sum_{S:S\subseteq T} q_{T,S}\lambda_S - \sum_{S:S\subseteq T} \lambda_S\right) + \sum_{S:S\not\subseteq T} q_{T,S}\lambda_S \\
&= \sum_{S\subseteq N} q_{T,S}\lambda_S - \sum_{S:S\subseteq T} \lambda_S \\
&= \sum_{S\subseteq N} \left(\sum_{j\in T\cap S} \frac{1}{|S|}\right)\lambda_S - \sum_{S:S\subseteq T} \lambda_S \\
&= \sum_{j\in T}\sum_{S:j\in S} \frac{1}{|S|}\lambda_S - \sum_{S:S\subseteq T} \lambda_S \\
&= \left(\sum_{j\in T} Sh_j(v)\right) - v(T).
\end{aligned}$$

Hence, $Sh(v) \in C(v)$ if and only if $V(v, w^T) \ge 0$ for all $w^T \in \mathcal{W}^{SC2}$. This completes the proof. □

[*16] Since the maximization problem is feasible and has the upper bound, it has an optimal solution. Since the maximization problem has an optimal solution, the corresponding minimization problem is feasible.

B. Omitted Proofs in Section 5

Proof of Lemma 3. Let $i, j \in N$ $(i \neq j)$ and $S \subseteq N \setminus \{i, j\}$. Define $T = S \cup \{j\}$. We have

$$\begin{aligned}
v(T \cup \{i\}) - v(T) - [v(S \cup \{i\}) - v(S)] &= \sum_{R \subseteq T \cup \{i\}} \lambda_R - \sum_{R \subseteq T} \lambda_R - \left[\sum_{R \subseteq S \cup \{i\}} \lambda_R - \sum_{R \subseteq S} \lambda_R \right] \\
&= \sum_{\substack{R \subseteq T \cup \{i\} \\ i \in R}} \lambda_R - \left[\sum_{\substack{R \subseteq S \cup \{i\} \\ i \in R}} \lambda_R \right] \\
&= \sum_{\substack{R \subseteq S \cup \{i,j\} \\ i \in R}} \lambda_R - \sum_{\substack{R \subseteq S \cup \{i\} \\ i \in R}} \lambda_R \\
&= \sum_{\substack{R \subseteq S \cup \{i,j\} \\ i \in R,\ j \in R}} \lambda_R.
\end{aligned}$$

$(i) \Rightarrow (ii)$: Setting $S' := \{i, j\}$ and $T' := S \cup \{i, j\}$ competes the proof.

$(ii) \Rightarrow (i)$: We have $\sum_{\substack{R \subseteq S \cup \{j,i\} \\ i \in R,\ j \in R}} \lambda_R \geq 0$ and obtain convexity. $\square$

Proof of Lemma 4. For any nonempty $S, T \subseteq N$ with $S \subseteq T$, we have

$$\begin{aligned}
\sum_{j \in S} (v(T) - v(T \setminus \{j\})) - \sum_{j \in S} (v(S) - v(S \setminus \{j\})) &= \sum_{j \in S} \left(\sum_{R : j \in R \subseteq T} \lambda_R - \sum_{R : j \in R \subseteq S} \lambda_R \right) \\
&= \sum_{j \in S} \sum_{\substack{R : j \in R \\ R \not\subseteq S,\ R \subseteq T}} \lambda_R \\
&= \sum_{\substack{R \subseteq T \\ R \not\subseteq S,\ R \not\subseteq T \setminus S}} |R \cap S| \cdot \lambda_R.
\end{aligned}$$

This completes the proof. $\square$

Proof of Lemma 5. Fix $\alpha, \beta \subseteq N$ with $\alpha \subseteq \beta$. Let us enumerate $\beta \setminus \alpha = \{k_1, \cdots, k_{|\beta \setminus \alpha|}\}$ and set $k_0 = \emptyset$. Then, we want to show that

$$\omega_{\alpha,\beta} = \sum_{j \in \alpha} \Big(\sum_{i=1}^{|\beta \setminus \alpha|} w_{\{j,k_i\},\beta \setminus \{k_0, \cdots, k_{i-1}\}} \Big)$$

where $w_{\{j,k_i\},T\setminus\{k_0,\cdots,k_{i-1}\}} \in \mathcal{W}^{SC3}$. It suffices to consider $S \subseteq \beta$. First, we consider $S \subseteq N$ such that $S \cap \alpha = \emptyset$ or $S \cap (\beta \setminus \alpha) = \emptyset$. Then, by (5.1) and (5.2), $\omega_{\alpha,\beta}(S) = 0 = \sum_{j\in\alpha}(\sum_{i=1}^{|\beta\setminus\alpha|} w_{\{j,k_i\},\beta\setminus\{k_0,\cdots,k_{i-1}\}})(S)$. Second, we consider $S \subseteq N$ such that $S \cap \alpha \neq \emptyset$ and $S \cap (\beta \setminus \alpha) \neq \emptyset$. Let $S = \{j_1, \cdots, j_s\} \cup \{k_{i_1}, \cdots, k_{i_s}\}$ where each element is enumerated and $\{j_1, \cdots, j_s\} \subseteq \alpha$, $\{k_{i_1}, \cdots, k_{i_s}\} \subseteq \beta \setminus \alpha$. Then, we have

$$\begin{aligned}\sum_{j\in\alpha}(\sum_{i=1}^{|\beta\setminus\alpha|} w_{\{j,k_i\},\beta\setminus\{k_0,\cdots,k_{i-1}\}})(S) &= \sum_{j\in S\cap\alpha}(\sum_{i=1}^{|\beta\setminus\alpha|} w_{\{j,k_i\},\beta\setminus\{k_0,\cdots,k_{i-1}\}})(S) \\ &= \sum_{j\in S\cap\alpha} w_{\{j,k_{i_1}\},\beta\setminus\{k_0,\cdots,k_{i_1-1}\}}(S) \\ &= |S\cap\alpha| \\ &= \omega_{\alpha,\beta}.\end{aligned}$$

This completes the proof. □

Proof of Lemma 6. Decomposition. Fix $\emptyset \neq T \subseteq N$. It suffices to consider $S \subseteq \beta$. First, we consider $S \subseteq N$ such that $T \cap S = \emptyset$ or $S \setminus T = \emptyset$. We have $\omega_{\alpha,\beta}(S) = 0$ for any $\alpha \subseteq T$ and any $\beta := \alpha \cup \gamma$ with $\gamma \subseteq N \setminus T$ by (5.2), which results in

$$\sum_{\emptyset\neq\alpha\subseteq T}\sum_{\substack{\gamma\subseteq N\setminus T\\ \beta:=\alpha\cup\gamma}} \kappa_{\alpha,\beta}\omega_{\alpha,\beta}(S) = \sum_{\emptyset\neq\alpha\subseteq T}\sum_{\substack{\gamma\subseteq N\setminus T\\ \beta:=\alpha\cup\gamma}} \kappa_{\alpha,\beta}\cdot 0 = 0.$$

In view of (4.6) and (4.5), $w_T(S) = 0$. Hence the equality holds.

Next, we consider $S \subseteq N$ such that $T \cap S \neq \emptyset$ and $S \setminus T \neq \emptyset$. We have

$$\begin{aligned}\sum_{\emptyset\neq\alpha\subseteq T}\sum_{\substack{\gamma\subseteq N\setminus T\\ \beta:=\alpha\cup\gamma}} \kappa_{\alpha,\beta}\omega_{\alpha,\beta}(S) &= \sum_{\emptyset\neq\alpha\subseteq T}\sum_{\substack{\gamma\subseteq N\setminus T\\ \beta:=\alpha\cup\gamma}} \left(\sum_{\beta\subseteq R\subseteq N}(-1)^{|R|-|\beta|}\frac{1}{|R|}\right)\omega_{\alpha,\beta}(S) \\ &\overset{(5.2)}{=} \sum_{S\subseteq\beta\subseteq N}\sum_{\beta\subseteq R\subseteq N}(-1)^{|R|-|\beta|}\frac{1}{|R|}|S\cap(\beta\cap T)| \\ &= \sum_{S\subseteq\beta\subseteq N}\sum_{\beta\subseteq R\subseteq N}(-1)^{|R|-|\beta|}\frac{1}{|R|}|S\cap T| \\ &= \sum_{S\subseteq R\subseteq N}\sum_{S\subseteq\beta\subseteq R}(-1)^{|R|-|\beta|}\frac{1}{|R|}|S\cap T| \\ &= |S\cap T|\sum_{S\subseteq R\subseteq N}\frac{1}{|R|}\sum_{S\subseteq\beta\subseteq R}(-1)^{|R|-|\beta|}.\end{aligned}$$

For the first summation, we divide $S \subseteq R \subseteq N$ into two cases: $S = R$ and $S \subsetneq R \subseteq N$. For any R with $S \subsetneq R \subseteq N$, we have $\sum_{S\subseteq\beta\subseteq R}(-1)^{|R|-|\beta|} = 0$ by the binomial theorem.[*17] If $S = R$, then $\sum_{S\subseteq\beta\subseteq R}(-1)^{|R|-|\beta|} = (-1)^0 = 1$. Hence, we obtain

$$|S \cap T| \sum_{S\subseteq R\subseteq N} \frac{1}{|R|} \sum_{S\subseteq\beta\subseteq R} (-1)^{|R|-|\beta|} = |S \cap T| \cdot \frac{1}{|S|} \cdot 1 = \frac{|S \cap T|}{|S|}.$$

By (4.6) and (4.5), we obtain $w_T(S) = \frac{|S\cap T|}{|S|}$ if S satisfies $T \cap S \neq \emptyset$ and $S \setminus T \neq \emptyset$.

Nonnegativity. Now, we show that the coefficients are nonnegative:

$$\sum_{\beta\subseteq R\subseteq N} (-1)^{|R|-|\beta|} \frac{1}{|R|} = \frac{1}{n \cdot ({}_{n-1}C_{m-1})} \geq 0.$$

Note that $\sum_{\beta\subseteq R\subseteq N}(-1)^{|R|-|\beta|}\frac{1}{|R|} = \sum_{k=0}^{n-m}(-1)^k \cdot \frac{{}_{n-m}C_k}{m+k}$ where $m = |\beta|$. Hence, we obtain

$$\begin{aligned}
\sum_{k=0}^{n-m}(-1)^k \cdot \frac{{}_{n-m}C_k}{m+k} &= \frac{1}{m}\Big(\sum_{k=0}^{n-m}(-1)^k \cdot {}_{n-m}C_k \cdot \frac{1}{1+\frac{k}{m}}\Big) \\
&= \frac{1}{m}\Big(\sum_{k=0}^{n-m}(-1)^k \cdot {}_{n-m}C_k \cdot \int_0^1 y^{\frac{k}{m}}\, dy\Big) \\
&= \frac{1}{m}\Big(\int_0^1 \sum_{k=0}^{n-m}(-1)^k \cdot {}_{n-m}C_k \cdot y^{\frac{k}{m}}\, dy\Big) \\
&= \frac{1}{m}\Big(\int_0^1 \sum_{k=0}^{n-m} {}_{n-m}C_k \cdot (-y^{\frac{1}{m}})^k dy\Big) \\
&= \frac{1}{m}\int_0^1 (1 - y^{\frac{1}{m}})^{n-m} dy.
\end{aligned}$$

*17 We have $\sum_{b'=0}^{r-s}\binom{r-s}{b'}(-1)^{r-s-b'} = \sum_{b'=0}^{r-s}\binom{r-s}{b'}(-1)^{r-s-b'}(1)^{b'} = ((-1)+(1))^{r-s} = 0$.

Let $I(m,n) = \int_0^1 (1-y^{\frac{1}{m}})^{n-m} dy$. By integral by parts, we have

$$\begin{aligned}
I(m,n) &= \int_0^1 (1-y^{\frac{1}{m}})^{n-m} dy \\
&= \int_0^1 y'(1-y^{\frac{1}{m}})^{n-m} dy \\
&= [y(1-y^{\frac{1}{m}})]_0^1 - \int_0^1 (n-m)y(1-y^{\frac{1}{m}})^{n-m-1}(-\frac{1}{m}y^{\frac{1-m}{m}})dy \\
&= \frac{n-m}{m}\int_0^1 y^{\frac{1}{m}}(1-y^{\frac{1}{m}})^{n-m-1}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{n-m}{m}I(m,n-1) - I(m,n) &= \frac{n-m}{m}I(m,n) \\
\Leftrightarrow I(m,n) &= \frac{n-m}{n}I(m,n-1).
\end{aligned}$$

Since

$$\begin{aligned}
I(m,n) &= \frac{n-m}{n}I(m,n-1), \\
I(m,n-1) &= \frac{n-m-1}{n-1}I(m,n-2), \\
&\vdots \\
I(m,m+1) &= \frac{1}{m+1}I(m,m),
\end{aligned}$$

and $I(m,m) = 1$, we obtain

$$I(m,n) = \frac{(n-m)\cdot(n-m-1)\cdots 2\cdot 1}{n\cdot(n-1)\cdots(m+1)} = \frac{(n-m)!m!}{n!}.$$

Thus, we have

$$\sum_{k=0}^{n-m}(-1)^k \cdot \frac{{}_{n-m}C_k}{m+k} = \frac{1}{m}I(m,n) = \frac{(n-m)!(m-1)!}{n!} = \frac{1}{n\cdot {}_{n-1}C_{m-1}}.$$

This completes the proof. □

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