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Shuige Liu†

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Abstract

We introduce *I(influence)-structure* of a game which describes that for each player, who influence his payoff, and study its relation with players' behavior in social context. We first give a necessary and sufficient condition for the existence of pure-strategy Nash equilibrium (NE) in terms of I-structure. This result has some intriguing applications, while it is too stringent to be applied directly to social games. We then use it as an approximation. We introduce *ε-I-structure* which describes only salient influencers of each player and define *ε-approximation* of the original game. We show that each NE of an ε-approximation is an approximated NE of the original game, and connect ε-I-structure with those approximated NE's. Since ε-I-structure can be interpreted by players' bounded cognitive ability, these results relate behavior pattern in a social game and subjective individual worlds.

Keywords: *influence structure, existence of Nash equilibrium, ε-Nash equilibrium.*

1. Introduction

A basic assumption in game theory is that each player has certain preferences among the outcomes, while he may not have a full control of them (cf. Luce and Raiffa [11], p.1). That is, a player's payoff is influenced of the choices by his own as well as some others'. In this paper, we formulate this influencer-influencee structure in a game, called *I(influence)-structure*, and study its relation with players' behavior pattern in social context. I-structure describes that for each player, who influence his payoff. Since a player only needs to consider those influencers' choices when making a decision, I-structure should be relevant to the stable behavior pattern in a game, e.g., pure-strategy Nash equilibrium (NE). Our first result is a condition for the existence of NE. This is illustrated in the following example.

Example 1.1 (Location game with three stores of different sizes) Players 1, 2, and 3 consider the locations for their new stores in a town. Each $i = 1, 2, 3$ has two strategies: to locate in the front of the train station ($\mathbf{s}_{i1}$), or in the residential area ($\mathbf{s}_{i2}$). Player 1 is a department store; for him, no matter what others chose, the front of train station is more profitable. Player 2's is a middle-sized super market, and his payoff is influenced by player 1 as well as he himself. For him, a location different from player 1's generates a higher payoff. Player 3's is a small

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convenient store, and his payoff is influenced by the choices of both players 1, 2, and himself: when players 1 and 2 locate at the same place, the other location is more profitable; when they choose differently, following player 1 is better. This situation is described as follows:

Table 1.1

1\2 ($\mathbf{s}_{31}$)	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	40,5,0	40,10,2
$\mathbf{s}_{12}$	20,10,1	20,5,3

Table 1.2

1\2 ($\mathbf{s}_{32}$)	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	40,5,3	40,10,1
$\mathbf{s}_{12}$	20,10,2	20,5,0

The I-structure of this game is illustrated in Figure 1.1. Here, an arrow indicates the direction of an influence. For example, the arrow from 1 to 2 means that player 1 has influence on player 2. Especially, an arrow around a player means he influences himself, or he is *reflexive*.

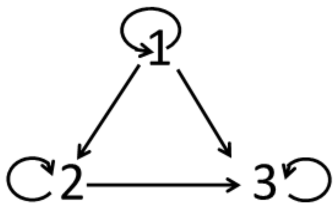

Figure 1.1: The I-structure in Example 1.1.

Does this game have a NE? By looking at Tables 1.1 and 1.2, we can see $(\mathbf{s}_{11}, \mathbf{s}_{22}, \mathbf{s}_{31})$ is a NE. A faster approach is to look at its I-structure. First, since player 1 is only influenced by himself, he has some dominant strategy s_1^*. Second, since player 2 is influenced by player 1 and himself, he has some best response s_2^* against s_1^*. Finally, player 3 has some best response s_3^* to (s_1^*, s_2^*). It can be seen that (s_1^*, s_2^*, s_3^*) is a NE for this game.

Further, it should be noted that the latter approach can be applied to any game having that I-structure, that is, NE exists for any game with I-structure as that in Figure 1.1.

Example 1.1 suggests some relation between I-structure and existence of NE. We show in Theorem 3.1 that, for an I-structure, each game corresponding to it has a NE if and only if it does not contain any *reflexive cycle*. Here a reflexive cycle is a set of more than one reflexive players among whom the influence relations form a cycle. We require that a reflexive cycle consists of more than one player, that is, a reflexive player does not form a reflexive cycle. By this definition, the I-structure in Figure 1.1 has no reflexive cycle, and each game corresponding to it has a NE. On the other hand, an I-structure with a reflexive cycle has some corresponding game without NE.

This result connects NE with a group of games sharing some structure rather than with one game having specific payoff functions. The if-part gives a sufficient condition for the existence of NE in terms of I-structure. The only if-part is weaker, stating that when some reflexive cycle exists, there is some game with that I-structure having no NE. Theorem 3.1 can be seen as a non-cooperative counterpart of Theorem 2.7 in Kaneko and Wooders [8], which connects the nonemptiness of the core with the structure of the basic coalitions of an cooperative game.

The concept of I-structure may reminds some readers of games on networks (cf. Jackson and Zenou [4]). They have the same mathematical form but different perspecives. A networks is given exogenously, and some rule assigns it with a specific game. Studies on networks focus on the relation between the rule and NE's. In contrast, our approach takes an I-structures as endogenous of a game. An I-structure corresponds to many games. We study the relation between the configuration of an I-structure and NE's in games corresponding to it.

I-structure seems related to potential games (cf. Monderer and Shapley [12]) in two senses. First, the concept "influence" is defined by the change of payoffs by someone's unilateral change of strategies, which looks similar to the potential function. Actually, it can be proved that if π contains no reflexive cycle, then each game with π is a generalized ordinal potential game (see Liu [10] for the proof). Second, it may be wondered whether ordinal potentiality and I-structure without reflexive cycle are related to each other since both guarantee the existence of NE. We have had examples in Liu [10] showing that they are logically independent.

Theorem 3.1 has some intriguing applications. In Section 3.2, we show that it provides an algorithm for enumerating NE's in some games and gives some insight into best-response dynamics. However, our main interest is using I-structure to investigate the individuals' behavior pattern in a society as a game. Here we meet a difficulty. Theorem 3.1 states that the existence of a stable behavior pattern is guaranteed only if the I-structure is hierachical. In other words, even a pair of reflexive and mutual influenced individuals, which is ubiquitous, may expose the whole society to the risk of having no NE. Hence, the insight provided by Theorem 3.1 seems limited.

To overcome this difficulty, we relax the requirement for I-structure and define the *ε-I-structure* of a game. I-structure requires that for each player, every player having influence on him should be considered no matter that influence is salient or subtle. In ε-I-structure only those having salient influence are considered. Based on it, we define an *ε-approximation* of the original game. Theorem 4.1 shows that each NE of an ε-approximation is a ε-NE (i.e., an approximated NE) of the original game. Theorem 4.2 connects ε-I-structure and existence of ε-NE in the original game. Since an ε-I-structure is simpler than the I-structure, it is more probably of having no reflexive cycles, and the ε-approximation based on it has a NE. Hence, Theorems 4.1 and 4.2 imply that even if a game has no NE, it may have an approximated one by ignoring some subtle influence structures among players.

This idea is illustrated in the following example, which is described in an indicative manner.

Example 1.2 (Location game with three main stores and many small ones) We add 100 players, player 4,...,103, to Example 1.1, which are all small stalls, each having subtle influence on himself and others. By Theorem 3.1, this game may have no NE. On the other hand, by ignoring those small influences, in its ε-I-structure remains only the hierachical structure among players 1, 2 and 3 as in Figure 1.1 and 100 solitude points. An ε-approximation based on this ε-I-structure has some NE, which is an approximated NE of the original game.

In game theory, ε-NE has long been used to describe players' bounded rationalities in repeated games (cf. Radner [14])[1]. Since any unilateral change from an ε-NE generates a profit less than ε, a player with bounded rationality may not bother to do it. Here, we use it in a different sense: an ε-NE results from each player's bounded cognitive ability, i.e., he fails to perceive or ignores those whose influence on him is subtle.

In this sense, ε-I-structure provides an approach to study a social game with players of bounded cognitive ability. A social game may have a large number of players and a complicated objective I-structure, i,e., each player may be influenced by many players in varying degress, salient or small. On the other hand, a player with bounded cognitive ability may consider only those who have salient influences on him. This situation is captured by ε-I-structure. Theorems 4.1 and 4.2 connect these two structures, showing that even a behavior pattern is not stable

[1] It is called *ε-equilibrium* in literature (cf. Rubinstein [16]). Here we call it ε-NE since we want to emphasize its conceptual similarity with NE and differentiate from another ε-equilibrium in the literature of market equilibrium theory (cf. Starr [17]).

in the objective situation, it is approximatedly stable from the view points of individuals with bounded cognitive ability.

This paper is organized as follows. Section 2 gives basic definitions and some preliminary results. Section 3 presents the necessary and sufficient condition in terms of I-structure for the existence of NE, and discusses briefly some applications of this result. Section 4 defines ε-I-structure and ε-approximation of a game, and study their relations with the ε-NE in the original game. Section 5 gives a summary and discusses some remaining problems.

2. Basic Definitions

Let $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a finite strategic form game, where N is the finite set of players, S_i is the nonempty set of strategies and $u_i : \Pi_{j\in N} S_j \to \mathbb{R}$ is the payoff function for player $i \in N$. Let $I \subseteq N$. We write S_I for $\Pi_{i\in I} S_i$ and S_{-I} for $\Pi_{j\in N-I} S_j$. For each $s \in S$, we write s_I for $(s_i)_{i\in I}$ and s_{-I} for $(s_j)_{j\in N-I}$, and we denote s by $(s_I; s_{-I})$. For simplicity, when $I = N$, we use s instead of s_N; when $I = \{i\}$, we write s_i and s_{-i} for $s_{\{i\}}$ and $s_{N-\{i\}}$, respectively. These abbreviations are also applied to S_N, $S_{\{i\}}$, and $S_{N-\{i\}}$.

An *I(influence)-structure* of G is a function $\pi : N \to 2^N$ satisfying that for each $i \in N$ and $s_{\pi(i)} \in S_{\pi(i)}$,

$$u_i(s_{\pi(i)}; s_{-\pi(i)}) = u_i(s_{\pi(i)}; s'_{-\pi(i)}) \text{ for all } s_{-\pi(i)}, s'_{-\pi(i)} \in S_{-\pi(i)}. \tag{2.1}$$

That is, player i's payoff is influenced only by the choices of players in $\pi(i)$. When $\pi(i) = \emptyset$, we stipulate that $(s_{\pi(i)}; s_{-\pi(i)}) = s_{-\pi(i)} = s$ for each $s \in S$, and (2.1) becomes $u_i(s) = u_i(s')$ for all $s, s' \in S$. $\pi(i) = \emptyset$ means that i is not influenced by any players, that is, his payoff is constant. Mathematically, an I-structure is a directed graph.

A game G may have multiple I-structures. A trivial I-structure is $\pi(i) = N$ for each $i \in N$. There is the smallest I-strcuture. To see this, for two I-structures π and π', we define $\pi \cap \pi'$ by $(\pi \cap \pi')(i) = \pi(i) \cap \pi'(i)$ for each $i \in N$. Then we can see that $\pi \cap \pi'$ is also an I-structure of G. This implies the existence of the smallest I-structure. It can be seen that π is the smallest I-structure of G if and only if for each $i \in N$, $\pi(i) = \{j \in N : u_i(s_j; s_{-j}) \neq u_i(s'_j; s_{-j})$ for some $s_j, s'_j \in S_j$ and $s_{-j} \in S_{-j}\}$, that is, $\pi(i)$ is the set of players who substantively influence i's payoff through some unilateral change of strategies.

Though the smallest I-structure is more efficient, in this paper our discussion is not constrained to it. The reason is that we want to take the viewpoint of each player, for whom calculating the precise boundary of substantive influencers is demanding on cognitive ability. To capture the viewpoint of a player, let π as an I-structure of G. Since for each $i \in N$, u_i does not depend on $S_{-\pi(i)}$ by (2.1), the payoff function for each player can be restricted to $S_{\pi(i)}$. We define $u_i^\pi : S_{\pi(i)} \to \mathbb{R}$ as

$$u_i^\pi(s_{\pi(i)}) = u_i(s_{\pi(i)}; s_{-\pi(i)}) \text{ for each } s_{\pi(i)} \in S_{\pi(i)} \text{ and } s_{-\pi(i)} \in S_{-\pi(i)}. \tag{2.2}$$

When $\pi(i) = \emptyset$, we stipulate u_i^π to be a function with empty domain. For each $i \in N$, $(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)$ can be seen as his *individual world*. We define $\Gamma = \{(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)\}_{i\in N}$ to be the *collection of individual worlds*.

G and Γ can be regarded as equivalent in two senses. First, there is a one-to-one correspondence between them. Formally speaking, for a directed graph $\pi : N \to 2^N$ and S_i for each $i \in N$, we use $\mathbf{G}(\pi, \{S_i\}_{i\in N})$ and $\mathbf{\Gamma}(\pi, \{S_i\}_{i\in N})$ to denote the sets of games $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ with π as its I-structure and of $\Gamma = \{(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)\}_{i\in N}$, respectively.[2] Let $\phi : \mathbf{G}(\pi, \{S_i\}_{i\in N}) \to \mathbf{\Gamma}(\pi, \{S_i\}_{i\in N})$ be the function satisfying (2.2). By construction,

[2] It can be shown that for each $\pi : N \to 2^N$ and $\{S_i\}_{i\in N}$, $\mathbf{G}(\pi, \{S_i\}_{i\in N}) \neq \emptyset$ since a game with $u_i(s) = 0$ for all $i \in N$ and $s \in S$ belongs to $\mathbf{G}(\pi, \{S_i\}_{i\in N})$. Further, we can show that when $|S_i| \geq 2$ for each $i \in N$, there exists $G \in \mathbf{G}(\pi, \{S_i\}_{i\in N})$ with π as its smallest I-structure. For a proof and detailed disucssion, see Liu [10].

ϕ is almost bijective, that is, when $\pi(i) \neq \emptyset$ for each $i \in N$, ϕ is bijective; when $\pi(i) = \emptyset$ for some $i \in N$, ϕ is surjective, and all G's corresponding to the same Γ can be seen as equivalent with respect to the indifference among choices of each i with $\pi(i) = \emptyset$.

Second, G and Γ have the same Nash equilibrium. We say that $s^* \in S$ is a *pure-strategy Nash equilibrium* (*NE*) of $\Gamma = (N, \pi, \{S_i\}_{i \in N}, \{u_i^\pi\}_{i \in N})$ iff for each $i \in N$ with $i \in \pi(i)$, $u_i^\pi(s_i^*; s_{\pi(i)-\{i\}}^*) \geq u_i^\pi(s_i; s_{\pi(i)-\{i\}}^*)$ for all $s_i \in S_i$. Let $G \in \mathbf{G}(\pi, \{S_i\}_{i \in N})$ and $\Gamma = \phi(G)$. It can be seen that $s \in S$ is a NE for G if and only if it is a NE for $\phi(G)$.

Although G and Γ are equivalent, in the following we focus on Γ since Γ has two advantages. First, the domain of the utility function for each player in Γ is smaller, which makes it more manipulable in mathematics. Second, since Γ is the collection of individual worlds, it provides a subjective viewpoint different from the objective one of G on the whole social situation. In Section 3, the first advantage is emphasized, and all results hold there if we replace Γ by G. In Section 4, the difference between the interpretations of G and Γ will be evident.

3. Influence Structure and Nash Equilibrium

3.1. I-Structure and Existence of NE

Let $\pi : N \to 2^N$ be a directed graph. A player i is called *reflexive* iff $i \in \pi(i)$. A *cycle* in $\pi : N \to 2^N$ is a finite sequence $i_0, ..., i_k$ $(k > 0)$ in N satisfying the following two conditions:

C0: $i_t \in \pi(i_{t+1})$ and $i_t \neq i_{t+1}$ for each $t = 0, ..., k$;

C1: $i_k \in \pi(i_0)$.

Since this definition does not allow one-player cycle, a cycle and reflexivity are distinguished. A cycle $i_0, ..., i_k$ is called *reflexive* iff each i_t is reflexive for $t = 0, ..., k$.

The following theorem gives a necessary and sufficient condition on π for the existence of NE's in all games with an I-structure π.

Theorem 3.1 (I-Structure and Existence of NE). Let $\pi : N \to 2^N$ and $|S_i| \geq 2$ for all $i \in N$. Then each $\Gamma \in \mathbf{\Gamma}(\pi, \{S_i\}_{i \in N})$ has a NE if and only if π contains no reflexive cycle.

Figure 3.1 gives three I-structures π_1, π_2 and π_3. π_1 has no cycle. In π_2, $(1, 2, 3)$ is a cycle but not reflexive since player 3 is not reflexive. π_3 has a reflexive cycle $(1, 2, 3)$. By Theorem 3.1, any game with π_1 or π_2 as its I-structure has a NE, while some Γ with π_3 has no NE.

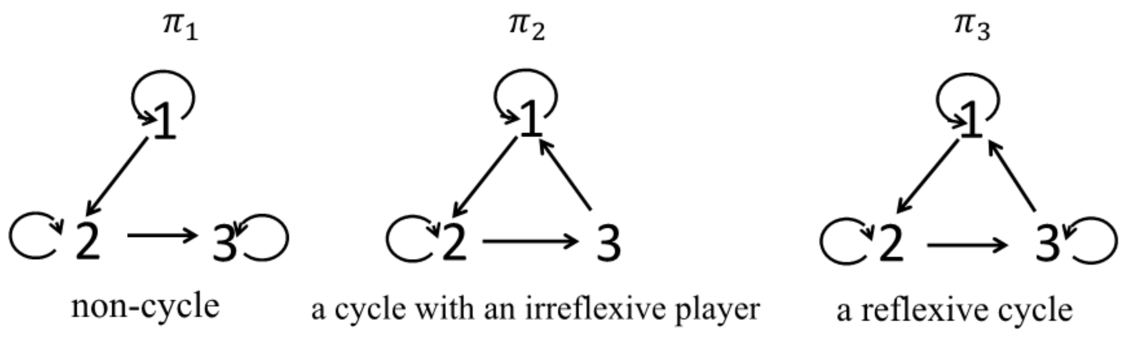

Figure 3.1: Three I-structures

First, we show that each game with π_1 has a NE. Let $\Gamma \in \mathbf{\Gamma}(\pi_1, \{S_i\}_{i \in N})$. As in Example 1.1, player 1 has a dominant strategy s_1^*, player 2 is has a best response s_2^* to s_1^*, and player 3 has some best response s_3^* to (s_1^*, s_2^*). (s_1^*, s_2^*, s_3^*) is a NE for Γ.

Here, we have stratified N and constructed a NE along the strata: We start from some player not influenced by anyone else and choose a dominant strategy of him. Then we choose a best response for each player influenced by him, and so on. Finally we obtain a NE. This approach can be generalized and applied to all π without cycle. Given $\pi : N \to 2^N$, for each $N' \subseteq N$, we define $\pi_{N'} : N' \to 2^{N'}$ by letting $\pi_{N'}(i) = \pi(i) \cap N'$ for each $i \in N'$. In graph theory we have the following lemma (cf. Harary [3], p.200):

Lemma 3.1. If π has no cycle, then the following two statements hold:

(a) $B(\pi) := \{i \in N : \pi(i) \subseteq \{i\}\} \neq \emptyset$;

(b) $\pi_{N'}$ has no cycle for each $N' \subseteq N$.

For π without cycle, Lemma 3.1 provides an approach to stratify N and construct a NE inductively along that stratification: let $N_1 = B(\pi)$, $N_2 = B(\pi_{N-N_1})$, $N_3 = B(\pi_{N-N_1 \cup N_2})$, etc.[3] Since π has no cycle, by Lemma 3.1(a), $N_1 \neq \emptyset$. By (b), π_{N-N_1} has no cycle, and $N_2 \neq \emptyset$ if $N - N_1 \neq \emptyset$, etc. Since N is finite, finally every player will be included in some N_k. Since each player in N_1 has either dominant strategies (i.e., $\pi(i) = \{i\}$) or a constant payoff (i.e., $\pi(i) = \emptyset$), and a player in each stratification $k > 1$ is influenced only those in the previous stratifications and (perhaps) himself, we can choose a best response for each $i \in N - N_1$ inductively. In this manner, we construct a NE.

This does not work for π_2. Since $(1, 2, 3)$ is a cycle, $B(\pi_2) = \emptyset$. Nevertheless, since that cycle is not reflexive, we can start from the irreflexive player 3. Since 3's choice does not affect his own payoff, let s_3^* be an arbitrary strategy of him. Since $\pi_{\{1,2\}}$ has no cycle, and we can stratify $\{1, 2\}$: First we take player 1. Since he is only influenced by himself and player 3, he has a a best response s_1^* to s_3^*. Then let s_2^* be a best response for player 2 to s_1^*. We can see that (s_1^*, s_2^*, s_3^*) is an NE.

In general, for π in which each cycle is not reflexive, we can first take away all irreflexive players. Then the remaining sub-I-structure has no cycle, and we can stratify it. Since each irreflexive player has a constant payoff, and each player in one stratum is influenced by only himself and some players in previous strata, we can construct a NE inductively along the strata. This is the basic idea in our proof of the If-part of Theorem 3.1.

Proof of the If-part of Theorem 3.1. Let $\pi : N \to 2^N$ having no reflexive cycle. We define N_t ($t \in \mathbb{N}$) by induction as follows:

N0: Let $N_0 = \{i \in N : i \notin \pi(i)\}$;

N1: Suppose that N_t has been defined for all $t \leq k$ for some $k \geq 0$. Then we let $N_{k+1} = B(\pi_{N - \cup_{t \leq k} N_t})$.

N_0 is the set of all irreflexive players. Since π has no reflexive cycle, each cycle (if any) has some player in N_0, and π_{N-N_0} has no cycle. By Lemma 3.1, if $N - N_0 \neq \emptyset$, then $N_1 \neq \emptyset$; if $N - N_0 \cup N_1 \neq \emptyset$, $N_2 \neq \emptyset$, etc. This process will stop at some ℓ, and each player will be included in a unique N_k ($0 \leq k \leq \ell$). In this manner, we have stratified N into $N_0, ..., N_\ell$.

Let $\Gamma = \{(\pi(i), \{S_j\}_{j \in \pi(i)}, u_i^\pi)\}_{i \in N}$. We construct $s^* \in S$ along $N_0, ..., N_\ell$ as follows:

S0: For each $i \in N_0$, let s_i^* be an arbitrary strategy in S_i;

S1: Suppose that, for some $k \leq \ell$, we have defined s_j^* for all $j \in \cup_{t \leq k} N_k$. That is, for each $i \in N_{k+1}$ and each $j \in \pi(i) - \{i\}$, s_j^* has been defined. Then for each $i \in N_{k+1}$, let s_i^* be a best response to $s_{\pi(i)-\{i\}}^*$.[4]

For each $i \in N_0$, since $i \notin \pi(i)$, his unilateral change of strategies does not alter his payoff.

[3] Here, for each $N' \subseteq N$, $B(\pi_{N'}) = \{i \in N' : \pi_{N'}(i) \subseteq \{i\}\}$.

[4] It is possible that for some $i \in N_1$, $\pi(i) = \{i\}$. Then s_i^* is just a dominant strategy for i.

For $i \in N_k$ ($0 < k \leq \ell$), since s_i^* is a best response to $s^*_{\pi(i)-\{i\}}$, $u_i^\pi(s_i^*; s_{-i}^*) \geq u_i^\pi(s_i; s_{-i}^*)$ for all $s_i \in S_i$. Therefore, s^* is a NE for Γ. //

When there is some reflexive cycle in π, we cannot stratify N. Consider π_3 in Figure 3.1. Since each player is influenced by himself as well as someone else, we cannot start the stratification N as we did before. Theorem 3.1 states that, given S_i with $|S_i| \geq 2$ for each $i \in N$, some $\Gamma \in \mathbf{\Gamma}(\pi, \{S_i\}_{i\in N})$ has no NE. For simplicity, let $S_i = \{a, b\}$ for $i = 1, 2, 3$. Consider $\Gamma = \{(\pi_3(i), \{S_j\}_{j\in\pi_3(i)}, u_i^{\pi_3})\}_{i\in N}$ as follows:

Table 3.1. $u_1^{\pi_3}$

$1\backslash 3$	a	b
a	0	1
b	1	0

Table 3.2. $u_2^{\pi_3}$

$2\backslash 1$	a	b
a	1	0
b	0	1

Table 3.3. $u_3^{\pi_3}$

$3\backslash 2$	a	b
a	1	0
b	0	1

Γ is a Matching-Pennies style game: player 1 gets a higher payoff when his choice is different from his influencer's (i.e., player 3's), while other players get a higher payoff when their choices coincide with their influencers'. It can be seen that Γ has no NE, since for any $s \in S$, if $s_1 = s_3$, then player 1 can deviate; if $s_1 \neq s_3$, then at least one in players 2 and 3 can deviate.

For more complicated π and/or larger S_i ($i \in N$), still we can construct a Matching-Pennies style game for players in a minimal reflexive cycle[5]. This is the basic idea in our proof of the only-if part of Theorem 3.1.

Proof of the Only-if part of Theorem 3.1. We show its contrapositive, that is, for π with some reflexive cycle and S_i with $|S_i| \geq 2$ for each $i \in N$, we construct some $\Gamma \in \mathbf{\Gamma}(\pi, \{S_i\}_{i\in N})$ such that Γ has no NE.

Let $i_0, ..., i_k$ be a minimal reflexive cycle in π. For simplicity, we denote $\mathbf{s}_{i1}$ by a and $\mathbf{s}_{i2}$ by b for each $i = i_0, ..., i_k$. We define $u_i^\pi : S_{\pi(i)} \to \mathbb{R}$ for each $i \in N$ by

G1. For $i = i_0$, since $i_0, i_k \in \pi(i_0)$, for each $s_{\pi(i_0)} \in S_{\pi(i_0)}$, let

$$u_{i_0}^\pi(s_{\pi(i_0)}) = \begin{cases} 1 & \text{if } (s_{i_0}, s_{i_k}) = (a,b) \text{ or } (b,a) \\ 0 & \text{if } (s_{i_0}, s_{i_k}) = (a,a) \text{ or } (b,b) \\ -1 & \text{if } (s_{i_0}, s_{i_k}) \in \{a,b\} \times (S_{i_k} - \{a,b\}) \cup \\ & (S_{i_0} - \{a,b\}) \times \{a,b\} \\ -2 & \text{otherwise} \end{cases} \tag{3.1}$$

G2. For $i = i_t$, $t = 1, ..., k$, since $\{i_{t-1}, i_t\} \in \pi(i_t)$, for each $s_{\pi(i_t)} \in S_{\pi(i_t)}$, let

$$u_{i_t}^\pi(s_{\pi(i_t)}) = \begin{cases} 1 & \text{if } (s_{i_t}, s_{i_{t-1}}) = (a,a) \text{ or } (b,b) \\ 0 & \text{if } (s_{i_t}, s_{i_{t-1}}) = (a,b) \text{ or } (b,a) \\ -1 & \text{if } (s_{i_t}, s_{i_{t-1}}) \in \{a,b\} \times (S_{i_{t-1}} - \{a,b\}) \cup \\ & (S_{i_t} - \{a,b\}) \times \{a,b\} \\ -2 & \text{otherwise} \end{cases} \tag{3.2}$$

G3. For $i \in N - \{i_0, ..., i_k\}$, let $u_i^\pi(s_{\pi(i)}) = 0$ for each $s_{\pi(i)} \in S_{\pi(i)}$,

G1 and **G2** are illustrated in Figure 3.2. It can be seen that for each $i = i_0, ..., i_k$, (1) any $s_i \in S_i - \{a, b\}$ is dominated by a; (2) on the $\{a, b\}$-block, player i_0 gets a higher payoff when his choice is different from his influencer's (i.e., player i_k's), while other players get a higher payoff when their choices coincide with their influencers'.

Let $\Gamma = \{(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)\}_{i\in N}$. We show that Γ has no NE. Let $s \in S$. We consider the following four cases for (s_{i_0}, s_{i_k}), which is illustrated in Figure 3.3:

[5] For two cycles c, c', c' is said to be a *subcycle* of c iff each element of c' is also an element of c. A reflexive cycle is said to be *minimal* iff it has no proper reflexive subcycle.

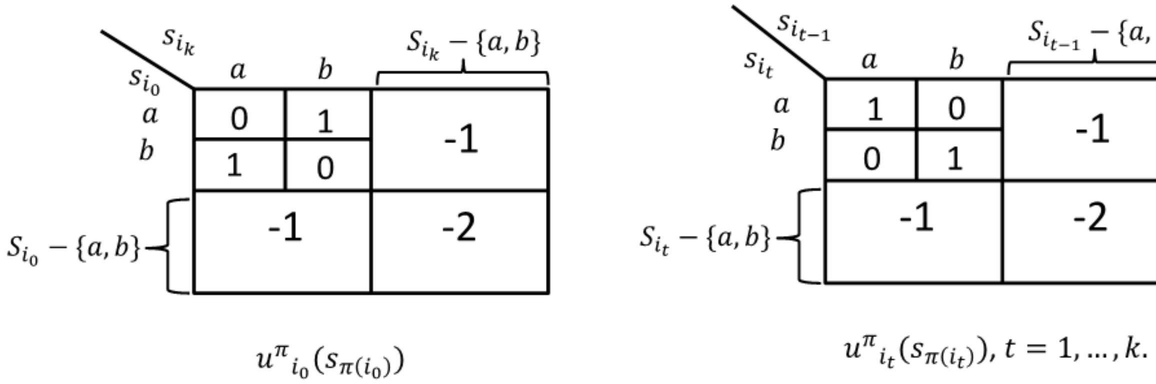

Figure 3.1: Figure 3.2: $u^{\pi}_{i_0}$ and $u^{\pi}_{i_t}$ $(t = 1, ..., k)$

Case 1. $(s_{i_0}, s_{i_k}) \in (S_{i_0} - \{a, b\}) \times S_{i_k}$. Since s_{i_0} is dominated by a, s is not a NE.

Case 2. $(s_{i_0}, s_{i_k}) \in \{a, b\} \times (S_{i_k} - \{a, b\})$. Since s_{i_k} is dominated by a, s is not a NE.

Case 3. $(s_{i_0}, s_{i_k}) = (a, a)$ or (b, b). Since player 1 can improve his payoff by choosing b if $(s_{i_0}, s_{i_k}) = (a, a)$ and a when $(s_{i_0}, s_{i_k}) = (b, b)$, s is not a NE.

Case 4. $(s_{i_0}, s_{i_k}) = (a, b)$ or (b, a). For $(s_{i_0}, s_{i_k}) = (a, b)$, consider the set $A = \{t \in \{1, ..., k\} : s_{i_t} \neq a\}$. Since $s_{i_k} \neq a$, $k \in A$ and $A \neq \emptyset$. Let t^* be the smallest element of A, that is, $s_{i_{t^*-1}} = a$, and $s_{i_{t^*}} \neq a$. Then by (3.2), i_{t^*} can deviate to a to improve his payoff from 0 to 1. When $(s_{i_0}, s_{i_k}) = (b, a)$, similarly we can find some player who can improve his payoff by deviating to b. Therefore, s is not a NE.

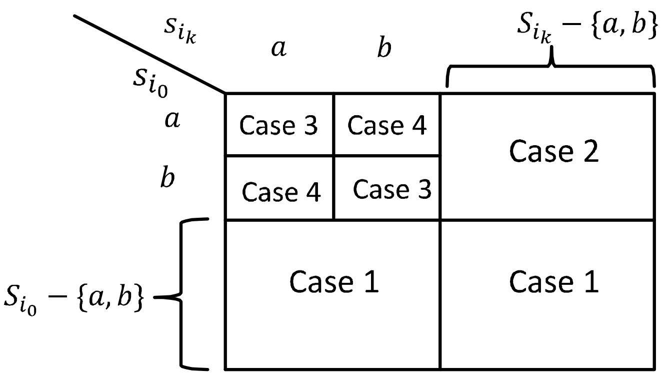

Figure 3.3: Four cases of (s_{i_0}, s_{i_k}).

Since cases 1-4 exhaust all possibilities for (s_{i_0}, s_{i_k}), we have shown that Γ has no NE. //

3.2. Some Applications of Theorem 3.1

Theorem 3.1 has some intriguing applications. In this subsection we show two of them. One is an algorithm for enumerating NE's in some games, the other is a condition for the convergence in best-response dynamics.

Enumeration of NE's. In general, enumerating NE's in a game is difficult (cf. Papadimitious [13]). Theorem 3.1 provides an algorithm for enumerating NE's in a game whose I-structure has no reflexive cycle. Let Γ have such an I-sturcture. It can be see that each NE s^* in Γ has the following structure: for each $i \in N_0$, s_i^* is an arbitrary strategy in S_i; for each $i \in N_1$, s_i^* is a best response to those s_j^*, $j \in \pi(i) - \{i\}$, and so on. Hence, to enumerate NE's in Γ, we need only to consider for each $s_{N_0} \in S_{N_0}$, how many best response $s_{N_1} \in S_{N_1}$ to it exist, and for each (s_{N_0}, s_{N_1}) where s_{N_1} is a best-response vector to s_{N_0}, how many best best response $s_{N_2} \in S_{N_2}$ exist. Continuing this process, finally we get the number of NE's in Γ.

However, this algorithm brings out another problem: Given an arbitrary game G, how to determine its I-structure? To do this we need to compare differences between payoffs by some unilateral change of strategies for each player, which requires a tremendous amount of calculation even when the game's size is modest (cf. Liu [10]). Enumerating NE's seems still difficult.

Convergence in Best-response Dynamics. Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a finite strategic form game.[6] A *best response* (*BS*) *mapping* for $i \in N$ is a function $b_i : S \to S_i$ satisfying that for each $s \in S$, $b_i(s)$ is a BS to s_{-i} if s_i is not a BS to s_{-i} and $b_i(s) = s_i$ if s_i is a BS to s_{-i}. BS mapping may not be unique. Let $\{b_i\}_{i \in N}$ be BS mappings in G, and $\mathcal{N} = (N^1, ..., N^k)$, where $N^t \subseteq N$ for each $t = 1, ..., k$ and $N = \cup_{t=1}^k N^t$. A *BS sequence in G* is a sequence $s^0, s^1, ...$ in S such that (1) s^0 is an arbitrary element in S; (2) for each $t > 0$ and $i \in N$,

$$s_i^t = \begin{cases} s_i^{t-1} & \text{if } i \notin N^{t \bmod k} \\ b_i(s^{t-1}) & \text{if } i \in N^{t \bmod k} \end{cases}$$

A BS sequence is said to be *convergent* iff there is some $M \in \mathbb{N}$ such that for each $M' \geq M$, $s^{M'} = s^M$. It can be seen that if a BS sequence is convergent, it converges to a NE.

Whether a BS sequence is convergent depends on G, $\{b_i\}_{i \in N}$, $\mathcal{N}$, and s^0. A well-known result in algorithmic game theory is that when $\mathcal{N}$ is a permutation of $\{\{i\}\}_{i \in N}$, i.e., at each step only one player can deviate, then if G is a ordinal potential game, for any $\{b_i\}_{i \in N}$ and s^0 the BS sequence converges (cf. Roughgarden [15], p.217). This result does not hold for every $\mathcal{N}$. We call G *globally BS convergent* iff for any $\{b_i\}_{i \in N}$, $\mathcal{N}$, and s^0, the BS sequence convergent. From Theorem 3.1, we have the following condition for global BS convergence.

Proposition 3.1. Let $\pi : N \to 2^N$ and $|S_i| \geq 2$ for all $i \in N$. Then each $G \in \mathbf{G}(\pi, \{S_i\}_{i \in N})$ is globally BS convergent if and only if π contains no reflexive cycle.

The proof of this statement is almost the same with that of Theorem 3.1. For the if part, for any $\mathcal{N}$, $\{b_i\}_{i \in N}$, and $s \in S$, $b_i(s) = s_i$ for each $i \in N_0$, and we can always wait until all players in N_1 choose their best responses, and so on. Only the convergence speed varies among different $\mathcal{N}$. It can be seen that $\mathcal{N} = \{N_0, N_1, ...\}$ leads to the fastest convergence. The only-if part holds since we can always find a game without NE when π has some reflexive cycle.

4. ε-I-Structure, ε-Approximation and Approximated NE

Our purpose is to apply I-structure to study the influence relation in a social game and its effect on players' behavior pattern. Theorem 3.1 states that for the existence of a stable behavior pattern, i.e., a NE, is guaranteed only if the I-structure is hierachical, i.e., either it has no cycle or has a irreflexive person in each cycle, both of which seems unrealistic in a social situation. It is sensible to assume that in most social situations, people influence each other (i.e., cycles exist) and each individual influences his own payoff (i.e., reflexive). Hence, the application of Theorem 3.1 seems quite limited.

[6]Here we return to strategic form game for a while since this is easier to be related to literatures of potential games and algorithmetic game theory.

To solve this problem, in this section we use I-structure as an approximation rather than a precise description of the situation. We relax that requirement in I-structure that each influencer should be contained in $\pi(i)$, and define an ε-I-structure of a game where those whose influence on i is subtle are excluded. Based on it, we define an ε-approximation of the original game, and show that the NE of the ε-approximation is a ε-NE of the original game. Finally, as a parallel to Theorem 3.1, we connect ε-I-structure with the existence of ε-NE of the original game.

Let $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ and $\varepsilon \geq 0$. $\pi : N \to 2^N$ is called an *ε-I-structure* of G iff for each $i \in N$ and $s_{\pi(i)} \in S_{\pi(i)}$,

$$|u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i(s_{\pi(i)}; s'_{-\pi(i)})| \leq \varepsilon \text{ for all } s_{-\pi(i)}, s'_{-\pi(i)} \in S_{-\pi(i)}. \quad (4.1)$$

When $\varepsilon = 0$, (4.1) coincides with (2.1). ε-I-structure extends I-structure by allowing exclusion of players having subtle influence (less than ε) on i. This can be interpreted from players' bounded cognitive ability, that is, each player fails to or ignores those whose influence on him is small. The following Lemma shows that π is an ε-I-structure of G if and only if each player i has an approximated payoff function on $\pi(i)$.

Lemma 4.1. π is an ε-I-structure of G if and only if for each $i \in N$, there is $u_i^\pi : S_{\pi(i)} \to \mathbb{R}$ satisfying

$$|u_i^\pi(s_{\pi(i)}) - u_i(s)| \leq \frac{\varepsilon}{2} \text{ for all } s \in S. \quad (4.2)$$

Proof. **(Only-if part)** For each $i \in N$, we define $u_i^\pi : S_{\pi(i)} \to \mathbb{R}$ by

$$u_i^\pi(s_{\pi(i)}) = \frac{1}{2}[\max_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)}) + \min_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)})] \text{ for each } s_{\pi(i)} \in S_{\pi(i)}.$$

Since π satisfies (4.1), for each $s \in S$,

$$\begin{aligned} u_i(s) - u_i^\pi(s_{\pi(i)}) &\leq \max_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i^\pi(s_{\pi(i)}) \\ &= \frac{1}{2}[\max_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)}) - \min_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)})] \leq \frac{\varepsilon}{2}; \end{aligned}$$

and

$$\begin{aligned} u_i(s) - u_i^\pi(s_{\pi(i)}) &\geq \min_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i^\pi(s_{\pi(i)}) \\ &= \frac{1}{2}[\min_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)}) - \max_{s_{-\pi(i)}\in S_{-\pi(i)}} u_i(s_{\pi(i)}; s_{-\pi(i)})] \geq -\frac{\varepsilon}{2}. \end{aligned}$$

That is, $|u_i(s) - u_i^\pi(s_{\pi(i)})| \leq \frac{\varepsilon}{2}$. Hence (4.2) is satisfied.

(If part) Suppose that for each $i \in N$, there is $u_i^\pi : S_{\pi(i)} \to \mathbb{R}$ satisfying satisfying (4.2). Let $i \in N$ and $s_{\pi(i)} \in S_{\pi(i)}$, $s_{-\pi(i)}, s'_{-\pi(i)} \in S_{-\pi(i)}$. Then

$$\begin{aligned} |u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i(s_{\pi(i)}; s'_{-\pi(i)})| &= |u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i^\pi(s_{\pi(i)}) + u_i^\pi(s_{\pi(i)}) - u_i(s_{\pi(i)}; s'_{-\pi(i)})| \\ &\leq |u_i(s_{\pi(i)}; s_{-\pi(i)}) - u_i^\pi(s_{\pi(i)})| + |u_i(s_{\pi(i)}; s'_{-\pi(i)}) - u_i^\pi(s_{\pi(i)})| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, π is an ε-I-structure of G. //

Lemma 4.1 can be explained from two viewpoints. The only-if part is from an outsider's (a researcher's) viewpoint, stating that given an ε-I-structure, some approximated payoff function can be constructed for each player. The if part gives substantial meaning to an ε-I-structure from the viewpoint of players. In contrast to the objective u_i, a player has an subjective u_i^π built on ignorence of subtle influences. It is probable when G represents a situation with many players and complicated objective I-structure while each individual has only bounded cognitive ability. In this sense, $\Gamma = \{(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)\}_{i\in N}$ is a collection of subjective individual worlds approximating the objective world G. We call Γ an ε-*approxiamtion* of G.

This seems like the approximation model approach in the literature of bounded rationality in game theory (Rubinstein [16]). However, as pointed out by Kline [9], approximation model approach takes bounded rationality only as a numerical approximation of the full rationality; it does not explore the structural difference between them. Some literature tried to fill this gap by giving "bounded rationality" some substantial content. For example, algorithmetic theory interprets ε from bounded computational ability of players (e.g., Kalai [5]). Here, we interpret it from players' bounded cognitive ability.

The following theorem shows that the NE of an ε-approximation is an approximated NE for the original game.

Theorem 4.1 (ε-Approximation and ε-NE). Let $\Gamma = \{(\pi(i), \{S_j\}_{j\in\pi(i)}, u_i^\pi)\}_{i\in N}$ be an ε-approximation of G. If s^* is a NE for Γ, then it is an ε-NE for G.

Here, an ε-*NE* for G is a strategy profile $s \in S$ satisfying that for each $i \in N$, $u_i(s_i; s_{-i}) + \varepsilon \geq u_i(s_i'; s_{-i})$ for all $s_i' \in S_i$.

Proof of Theorem 4.1. Let s^* be a NE in Γ. We show that s^* is an ε-NE in G, that is, for each $i \in N$, $u_i(s_i^*; s_{-i}^*) + \varepsilon \geq u_i(s_i; s_{-i}^*)$ for all $s_i \in S_i$. Let $i \in N$. We consider the following cases:

(1) $i \in \pi(i)$. Then for each $s \in S$, $s_{\pi(i)} = (s_i; s_{\pi(i)-i})$. It follows from (**??**) that

$$-\frac{\varepsilon}{2} \leq u_i(s_i^*; s_{-i}^*) - u_i^\pi(s_i^*; s_{\pi(i)-i}^*) \leq \frac{\varepsilon}{2}, \tag{4.3}$$

and for each $s_i \in S_i$,

$$-\frac{\varepsilon}{2} \leq u_i^\pi(s_i; s_{\pi(i)-i}^*) - u_i(s_i; s_{-i}^*) \leq \frac{\varepsilon}{2}. \tag{4.4}$$

Combine (4.3) and (4.4), we have

$$-\varepsilon \leq u_i(s_i^*; s_{-i}^*) - u_i(s_i; s_{-i}^*) + [u_i^\pi(s_i; s_{\pi(i)-i}^*) - u_i^\pi(s_i^*; s_{\pi(i)-i}^*)] \leq \varepsilon. \tag{4.5}$$

Since s^* is a a NE in Γ, $u_i^\pi(s_i; s_{\pi(i)-i}^*) - u_i^\pi(s_i^*; s_{\pi(i)-i}^*) \leq 0$. Then it follows from (4.5) that $-\varepsilon \leq u_i(s_i^*; s_{-i}^*) - u_i(s_i; s_{-i}^*) + [u_i^\pi(s_i; s_{\pi(i)-i}^*) - u_i^\pi(s_i^*; s_{\pi(i)-i}^*)] \leq u_i(s_i^*; s_{-i}^*) - u_i(s_i; s_{-i}^*)$, that is, $u_i(s_i^*; s_{-i}^*) + \varepsilon \geq u_i(s_i; s_{-i}^*)$.

(2) $i \notin \pi(i)$.Then for each $s_i \in S_i$, $(s_i^*; s_{-i}^*)|_{\pi(i)} = (s_i; s_{-i}^*)|_{\pi(i)} = s_{\pi(i)}^*$. Hence

$$\begin{aligned} |u_i(s_i^*; s_{-i}^*) - u_i(s_i; s_{-i}^*)| &= |u_i(s_i^*; s_{-i}^*) - u_i^\pi(s_{\pi(i)}^*) + u_i^\pi(s_{\pi(i)}^*) - u_i(s_i; s_{-i}^*)| \\ &\leq |u_i(s_i^*; s_{-i}^*) - u_i^\pi(s_{\pi(i)}^*)| + |u_i^\pi(s_{\pi(i)}^*) - u_i(s_i; s_{-i}^*)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, we have $-\varepsilon \leq u_i(s_i^*; s_{-i}^*) - u_i(s_i; s_{-i}^*)$, that is, $u_i(s_i^*; s_{-i}^*) + \varepsilon \geq u(s_i; s_{-i}^*)$. Here we have shown that s^* is an ε-NE in G. //

An ε-I-structure is intended to smaller than an I-structure, and therefore is more probable to contain no reflexive cycle, that is, an ε-approximation based on it is more probable to have some NE. Theorem 4.1 states that this NE is an approximated one, i.e., an approximated stable behavior pattern, in the original game.

Using Lemma 4.1 and Theorem 4.1, we can connect ε-influence structure with ε-NE of G. This can be seen as a parallel to Theorem 3.1. Given π and S_i with $|S_i| \geq 2$ for each $i \in N$, we use $\mathbf{G}_\varepsilon(\pi, \{S_i\}_{i\in N})$ to denote the set of all games with π as their ε-I-structure. Then

Theorem 4.2 (ε-I-structure and ε-NE). Each $G \in \mathbf{G}_\varepsilon(\pi, \{S_i\}_{i\in N})$ has an ε-NE if and only if π contains no reflexive cycle.

Proof. The only-if part can be proved in a similar manner as for the only-if part of Theorem 3.1. Here we only show the if-part. Let $G \in \mathbf{G}_\varepsilon(\pi, \{S_i\}_{i\in N})$. Since π is an ε-I-structure of G, it follows from Lemma 4.1 that there is some $\Gamma \in \mathbf{\Gamma}(\pi, \{S_i\}_{i\in N})$ which is an ε-approximation of G. Since π contains no reflexive cycle, it follows from Theorem 3.1 that Γ has a NE s^*. By Lemma 4.1, s^* is an ε-NE of G. //

Theorem 4.1 and 4.2 connect objective social situation with subjective individual worlds, showing that even if some behavior pattern is not stable objectively, it is approximatedly stable from the viewpoint of individuals with bounded cognitive ability.

5. Conclusions

In this paper, we introduced the concept of I-structure, discussed its relation with NE, and used it to approximate a game. In Theorem 3.1, we showed that given an I-structure, all games corresponding to it has a NE if and only if it contains no reflexive cycle. Then, we relaxed I-structure and defined ε-I-structure, based on which we define the ε-approximation of the original game. In Theorem 4.1 we show that each NE of an ε-approximation is a ε-NE of the original game; In Theorem 4.2 we show the relation between an ε-I-structure and existence of ε-NE in all games with it as a parallel Theorem 3.1.

Some important problems have not been touched upon yet. In our defintion of ε-approximation, we allowed each player to focus on a part of players. However, for a player with bounded cognitive ability, it is probable that he has his subjective criterion not only on players but also on his strategy set. To formulate such situation, one possibility is to introduce a pair $\langle S'_i, r_i\rangle$ for each $i \in N$, where S'_i is a set of *representative strategies* of i, and $r_i : S_i \to S'_i$ is a surjection called the *representing function*. The intended meaning is that each player combines some detailed strategies and considers them as a whole, that is, each $s'_i \in S'_i$ is a representative of all strategies in $r_i^{-1}(s_i)$. Based on this, we can define an *extended ε-approximation* of G as $\Gamma^e = \{(\pi(i), \{S'_i\}_{i\in N}, \{r_i\}_{i\in N}, u_i^\pi)\}_{i\in N}$ satisfying that for each $i \in N$,

$$|u_i^\pi((r_j(s_j))_{j\in\pi(i)}) - u_i(s)| \leq \frac{\varepsilon}{2} \text{ for each } s \in S.$$

It can be shown that each NE in Γ^e is a ε-NE of G. Some studies in the literature on the simplified strategies and computational complexity (cf. Chen [1]) can be applied to study the structure of $\langle S'_i, r_i\rangle$ and its relation with the NE in G. The relation between π and $\langle S'_i, r_i\rangle$ and their effect on NE is still an open problem.

Further, it is possible to connect I-structure with epistemic depth of interpersonal inference (cf. Kaneko and Suzuki [7]) and cognitive hierachy theory (cf. Cramer et al. [2]). In Example 1.1, since player 1 is only influenced by himself, he needs only to consider his own dominant strategies, that is, the epistemic depth for him to make a decision is 1. For player 2 that is 2 since he has to consider the choices of himself and player 1, and for player 3 that is 3 since he has to consider himself, player 2, and player 1 in the mind of player 2. In general, it can be seen that when the I-structure has no reflexive cycle, each player's epistemic depth of interpersonal inference is finite, while a reflexive cycle may lead to infinitite ones. Also, ε-approximation can be seen from this viewpoint: by focusing on some salient influencers, the epistemic depth of each player may become shallower. In this direction, a detailed study is needed.

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