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Abstract

The concept of "compromise" often plays an important role to determine how to divide a resource. In cooperative game theory, some solutions based on compromise among players have been proposed. In this paper, we discuss the relationship among the compromise solutions. We define the *both-edges weak convexity* and show that the Weber set is a subset of the *compromise set* if and only if the game is both-edges weakly convex. In addition, we offer some conditions for the coincidence among point-valued solutions based on compromise.

Keywords: Cooperative game; Compromise set; τ-value; CIS-value; ENSC-value
JEL Classification: C71

1 Introduction

One of the main purposes of the cooperative game theory is to provide the theoretical framework for problems of distribution. In fact, the cooperative game theory has been proposing a number of distinctive solutions. For example, the *core* is describing the payoff distributions from which every group of players does not deviate, and, moreover, the *Shapley value* is the solution which provides each player with the payoff depending on her *marginal contribution*. In particular, some solutions are dedicated to the concept of compromise among players. Tijs (1981) and Tijs and Lipperts (1982) respectively proposed the τ*-value* and the *compromise set* which is also referred to as "*core cover*". In order to clarify the concept of compromise, they defined the two types of bounds, namely, the *utopia demand* and the *minimal right*. The utopia demand illustrates the idea that a player has the upper bound she can claim to obtain at most, while the minimal right describes the lower bound she is entitled to receive at least.

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In most cases, it is not feasible to provide all players with their utopia demands even if they form the most efficient coalition. On the other hand, their minimal rights do not satisfy the efficiency. The compromise solutions indicate what distribution to be chosen between the upper bound and the lower bound.

Furthermore, van den Brink and Funaki (2009) analyzed some solutions based on the distinctive type of compromise concept, namely, *CIS-value* and *ENSC-value*. The CIS-value guarantees one's payoff from the individual coalition as her lower bound and then equally divides the surplus. The ENSC-value, first, proposes the utopia demand and then indicates a feasible distribution reflecting the utopia demand. They described these solutions as distributions equally sharing the surplus among all players. Nevertheless, we interpret these distributions as the results of equal amount of compromise.

In contrast to the solutions defined by the normative properties, such as the core and the Shapley value, the compromise solutions have not been analyzed enough. It is the aim of this paper to clarify the relationship among the compromise solutions.

This paper is organized as follows. In Section 2, we introduce some basic definitions. In particular, we offer some solutions and important results describing the relationship among the solutions. Section 3 is devoted to analyze the compromise set in both-edges weakly convex games defined in this section. We consider some point-valued compromise solutions in Section 4. The τ-value, the CIS-value and the ENSC-value are classified into this section.

2 Preliminaries

A transferable utility cooperative game (TU-game) is a pair (N, v), where $N = \{1, ..., n\}$ is a finite set of players and $v : 2^N \to R$ is a function assigning to every coalition $S \subseteq N$ a real number $v(S)$ such that $v(\{\emptyset\}) = 0$. The set of all TU-games with player set N is denoted by Γ^N. For notational simplicity, we refer to a game as v instead of (N, v) and denote, for example, $v(\{12\})$ by $v(12)$.

2.1 Definitions

First, we introduce two contrastive vectors which have been studied by Tijs (1981).

Definition 2.1 (Tijs (1981)). Let $v \in \Gamma^N$. The *utopia demand* of player $i \in N$ is given by

$$M_i(v) = v(N) - v(N \setminus i).$$

The *utopia vector* $M(v) \in R^N$ consists of the utopia demand of every player, *i.e*,

$$M(v) = (M_1(v), ..., M_n(v)).$$

The utopia demand of player i describes the maximum amount player i can receive in the grand coalition N, since if player i claims her payoff more than $M_i(v)$ then the other players $N \setminus i$ can exclude i and protect their total payoff $v(N \setminus i)$, or they suffer their total payoff less than $v(N \setminus i)$.

Definition 2.2 (Tijs (1981))**.** Let $v \in \Gamma^N$. The *minimal right* of player $i \in N$ is defined by

$$m_i(v) = \max_{S \subseteq N : i \in S} \left[v(S) - \sum_{j \in S \setminus i} M_j(v) \right].$$

The *minimal right vector* $m(v) \in R^N$ consists of the minimal right of every player, *i.e*,

$$m(v) = (m_1(v), ..., m_n(v)).$$

We can see the minimal right of player i as the maximum payoff the player i can achieve by giving their utopia demand to the rest of players in the coalition.

By using these two vectors, Tijs and Lipperts (1982) introduced a set-valued solution.

Definition 2.3 (Tijs and Lipperts (1982))**.** The *compromise set* of game $v \in \Gamma^N$ is defined by

$$CC(v) = \left\{ x \in R^N \;\middle|\; \sum_{j \in N} x_j = v(N),\ m(v) \leq x \leq M(v) \right\}.$$

The compromise set has been originally introduced as "core cover" by Tijs and Lipperts (1982). In this paper, we refer to this concept as compromise set. The compromise set of a game is the set of all efficient payoff vectors between the utopia vector and the minimal right vector, which illustrates a set of compromised payoff between their maximum and minimum. A game is said to be *compromise admissible* if the compromise set is non-empty. We denote by CA^N the set of all compromise admissible games with the player set N.

We define the imputation set $I(v)$ as a set of efficient payoff vectors satisfying the individual rationality, *i.e*,

$$I(v) = \left\{ x \in R^N \;\middle|\; \sum_{j \in N} x_j = v(N),\ x_j \geq v(j) \ \ for\ all\ j \in N \right\}.$$

Moreover, we define the *core* of a game v as a set of payoff vectors from which any coalition does not deviate, formally,

$$Core(v) = \left\{ x \in I(v) \;\middle|\; \sum_{j \in S} x_j \geq v(S) \ \ for\ all\ S \subseteq N \right\}.$$

Tijs and Lipperts (1982) proved that any payoff vector in the core belongs to the compromise set and that the compromise set is a subset of the imputation set in general, that is, for any $v \in \Gamma^N$ it holds that

$$Core(v) \subseteq CC(v) \subseteq I(v).$$

In addition, Quant, Borm, Reijnierse and van Velzen (2005) showed the compromise set coincides with the core if and only if for any $S \subseteq N$,

$$v(S) \leq \max\left\{\sum_{j\in S} m_j(v),\ v(N) - \sum_{j\in N\setminus S} M_j(v)\right\}.$$

They called a game v satisfying $Core(v) = CC(v)$ *compromise stable.*

2.2 Weber Set and Compromise Set

The concept of Weber set was introduced by Weber (1988). The extreme points of the Weber set are described as *marginal vectors*. An order of player set N is denoted by a bijective function $\sigma : N \to N$. Here $\sigma(k)$ refers to the k-th player in the order σ. Let $\Pi(N)$ denotes the set of all orders of N. The marginal vector $z^\sigma(v)$ is defined by

$$z^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), ..., \sigma(k)\}) - v(\{\sigma(1), ..., \sigma(k-1)\}) \quad for\ k = 1, ..., n.$$

Then, the Weber set of a game v is equal to the convex hull of marginal vectors. That is,

$$W(v) = conv\{z^\sigma(v)\,|\sigma \in \Pi(N)\,\}.$$

Weber (1988) showed that for any $v \in \Gamma^N$, $Core(v) \subseteq W(v)$. Furthermore, Shapley (1971) and Ichiishi (1981) proved that $Core(v) = W(v)$ if and only if the game v is *convex*, where we say a game v is *convex* if $v(T) - v(T \setminus i) \leq v(S) - v(S \setminus i)$ for any player $i \in N$ and for any two coalitions, S, T $(T \subseteq S)$ such that $i \in S$ and $i \in T$.

In addition, Quant, Borm, Reijnierse and van Velzen (2005) characterized a significant class of games by these solution concepts. They proved that a game is convex and compromise stable if and only if the game is strategically equivalent to a bankruptcy game. This fact implies that the core, the compromise set and the Weber set coincide if and only if the game is strategically equivalent to a bankruptcy game because of the definitions of the convex game and the compromise stable game.

3 Compromise Set and Both-edges Weakly Convex Game

In the previous section, we introduced some important relationships among the solution concepts. In this section, we examine some new relationships of the compromise set with other well known solution concepts and offer a new class of games, *both-edges weakly convex* games.

3.1 Both-edges Weakly Convex Games

Definition 3.1. Let $v \in \Gamma^N$. A game v is said to be *both-edges weakly convex* if for any $i \in N$ and for any $S \subseteq N$ with $i \in S$,

$$v(i) \leq v(S) - v(S \setminus i) \leq v(N) - v(N \setminus i).$$

Clearly, any convex game is both-edges weakly convex. In the both-edges weakly convex games, the following Lemma 3.2 and Lemma 3.3 hold.

Lemma 3.2. If a game v is both-edges weakly convex then for all $i \in N$, $v(i) = m_i(v)$.

Proof. First, we show that $v(i) = m_i(v)$ for all $i \in N$ if and only if for any $i \in N$ and for any $S \subseteq N$ with $i \in S$,

$$v(S) - \sum_{j \in S \setminus i} M_j(v) \leq v(i). \tag{3.1}$$

If $v(i) = m_i(v)$ for all $i \in N$ then we have

$$v(i) = m_i(v) = \max_{S \subseteq N : i \in S} \left[v(S) - \sum_{j \in S \setminus i} M_j(v) \right].$$

Hence, we obtain (3.1). If the inequality (3.1) holds, we can take an arbitrary player i as S and obtain

$$v(S) - \sum_{j \in S \setminus i} M_j(v) = v(i) - \sum_{j \in i \setminus i} M_j(v) = v(i).$$

Thus we can conclude that $v(i) = m_i(v)$.

Second, we show that if a game v is both-edges weakly convex then it satisfies (3.1). Let i_0 be any player in N. Let $S = \{i_0, ..., i_k\}$, where $0 \leq k \leq n$. It holds that,

$$\begin{aligned} v(S) - v(i_0) &= v(S) - v(S \setminus i_k) \\ &\quad + v(S \setminus i_k) - v((S \setminus i_k) \setminus i_{k-1}) \\ &\quad \vdots \\ &\quad + v(S \setminus \{i_k, ..., i_2\}) - v(i_0). \end{aligned} \tag{3.2}$$

In view of the both-edges weak convexity, for each line of (3.2), inequalities $v(S) - v(S \setminus i_k) \leq M_{i_k}(v)$, $v(S \setminus i_k) - v((S \setminus i_k) \setminus i_{k-1}) \leq M_{i_{k-1}}(v)$,..., and $v(S \setminus \{i_k, ..., i_2\}) - v(i_0) \leq M_{i_1}(v)$ hold. Hence, for any $i_0 \in N$, we obtain $v(S) - v(i_0) \leq \sum_{j \in S \setminus i_0} M_j(v)$ which implies (3.1). $\square$

Lemma 3.3. Let $v \in \Gamma^N$. The following two sentences are equivalent:

(i) $W(v) \subseteq CC(v)$,

(ii) for any permutation $\sigma \in \Pi(N)$, $m(v) \leq z^\sigma \leq M(v)$.

Proof. First, we show (i)⇒(ii). In view of the definition of the Waber set, we have $conv\{z^\sigma(v) \mid \sigma \in \Pi(N)\} \subseteq CC(v)$. Hence, for any permutation $\sigma \in \Pi(N)$, $z^\sigma \in CC(v)$. Hence, the definition of the compromise set implies (ii).

Second, we prove (ii)⇒(i). For any $x \in W(v)$, it holds that $x = \sum_{\sigma\in\Pi(N)} \alpha_\sigma z^\sigma$ for some weight vector $(\alpha_\sigma)_{\sigma\in\Pi(N)}$ satisfying $\sum_{\sigma\in\Pi(N)} \alpha_\sigma = 1$. Hence, by (ii), it follows that $m(v) \leq x \leq M(v)$, which implies $x \in CC(v)$. □

In view of Lemma 3.2 and Lemma 3.3, we can obtain the following result.

Theorem 3.4. A game $v \in \Gamma^N$ is both-edges weakly convex if and only if $W(v) \subseteq CC(v)$.

Proof. Suppose $W(v) \subseteq CC(v)$. By Lamma 3.3, for any permutation $\sigma \in \Pi(N)$, we have $m(v) \leq z^\sigma \leq M(v)$. Hence, for every $j = 1, 2, ..., n$,

$$m_{\sigma(j)}(v) \leq z^\sigma_{\sigma(j)}(v) \leq M_{\sigma(j)}(v). \tag{3.3}$$

In the case of $j = 1$, it holds that

$$m_{\sigma(1)}(v) \leq z^\sigma_{\sigma(1)}(v) = v(\sigma(1)) - v(\emptyset) = v(\sigma(1)).$$

Since this inequality holds for all $\sigma \in \Pi(N)$, we obtain $m_i(v) \leq v(i)$ for any $i \in N$. Hence, we can conclude that $m_i(v) = v(i)$ for any $i \in N$ and, in view of (3.3) and the definition of $M(v)$, that for any $\sigma \in \Pi(N)$

$$v(\sigma(j)) \leq v(\{\sigma(j), \sigma(j-1), ..., \sigma(1)\}) - v(\{\sigma(j-1), ..., \sigma(1)\}) \leq v(N) - v(N \setminus \sigma(j)).$$

Thus, for all $i \in N$ and $S \subseteq N$ with $i \in S$, we have

$$v(i) \leq v(S) - v(S \setminus i) \leq v(N) - v(N \setminus i).$$

On the contrary, we suppose a game v is both-edges weakly convex. Then, for any $\sigma \in \Pi(N)$, it follows that

$$v(\sigma(j)) \leq v(\{\sigma(j), \sigma(j-1), ..., \sigma(1)\}) - v(\{\sigma(j-1), ..., \sigma(1)\}) \leq v(N) - v(N \setminus \sigma(j)).$$

By Lemma 3.2, we have

$$m_{\sigma(j)}(v) \leq z^\sigma_{\sigma(j)} \leq M_{\sigma(j)}(v)$$

and obtain $W(v) \subseteq CC(v)$. □

3.2 Compromise Set and Stable Set in Convex Games

We define the *stable set*, V, as a set of payoff vectors satisfying,

$$V \cap Dom(V) = \emptyset \;\; and$$
$$V \cup Dom(V) = I(v),$$

where let $Dom(V)$ denotes a set of imputations dominated by V, formally,

$$Dom(V) = \left\{ x \in I(v) \,\middle|\, \exists (S, y) \in 2^N \times V such\ that\ y_j > x_j \ \forall j \in S \ and\ v(S) \geq \sum_{j \in S} y_j \right\}.$$

We can see the stable set as a set of payoff vectors that are internally stable (represented by the upper condition) and externally stable (by the lower condition).

The class of *zero-normalized games* E is given by

$$E = \left\{ v \in \Gamma^N \,\middle|\, v(S) + \sum_{i \in N \setminus S} v(i) \leq v(N) \ \forall S \subset N \right\}.$$

Rafels and Tijs (1997) proved the following theorem.

Theorem 3.5 (Rafels and Tijs (1997))**.** If $v \in E$, then,

$$C(v) \cup Dom(I(v) \cap W(v)) = I(v).$$

Theorem 3.5 describes that the set of imputations can be divided into some set-valued solutions.

Corollary 3.6. If a game v is convex, then the compromise set satisfies the external stability, *i.e.*,

$$CC(v) \cup Dom(CC(v)) = I(v).$$

Proof. Suppose that a game v is convex. Then, it is easy to see that v is both-edges weakly convex and $v \in E$. In view of Theorem 3.3 and Theorem 3.5, we have

$$I(v) = C(v) \cup Dom(I(v) \cap W(v)) \subset CC(v) \cup Dom(CC(v)).$$

Thus, $CC(v)$ is satisfying the external stability. □

Corollary 3.7. The compromise set coincide with the stable set in games strategically equivalent to bankruptcy games.

The compromise set describes result of compromise between the minimal payoff, $m(v)$, every player is entitled to receive, and the maximum payoff, $M(v)$, every player can claim at most.

On the other hand, the stable set illustrates the stability in the sense of domination, namely, the external stability and the internal stability. Hence, we can conclude Corollary 3.7 shows that the stability based on compromise meets the stability of domination in bankruptcy games.

4 The Relationship among Compromise Values

In this section, we analyze the compromise solutions assigning a single vector to each game. First we introduce τ-*value* as the most basic compromise value, which has been studied by Tijs (1981).

Definition 4.1 (Tijs (1981))**.** For any $v \in CA^N$, τ-*value* is given by

$$\tau(v) = m(v) + k(M(v) - m(v)),$$

where

$$k = \frac{v(N) - \sum_{j \in N} m_j(v)}{\sum_{j \in N} M_j(v) - \sum_{j \in N} m_j(v)}$$

if $\sum_{j \in N} M_j(v) - \sum_{j \in N} m_j(v) \neq 0$, otherwise $k = 0$.

The τ-value is the single payoff vector satisfying the efficiency on the line between $m(v)$ and $M(v)$. It is clear that the τ-value is defined by the two points, $m(v)$ and $M(v)$, which does not necessarily satisfy the efficiency. We can interpret the weight k as a degree of compromise in this game. In addition, we can see not only the τ-value also the *CIS-value* and the *ENSC-value* as the members of the compromise solutions.

Definition 4.2. Let $v \in \Gamma^N$. For any $i \in N$, the *CIS-value* is defined by

$$CIS_i(v) = v(i) + \frac{1}{n}\left[v(N) - \sum_{j \in N} v(j)\right].$$

Definition 4.3. Let $v \in \Gamma^N$. For any $i \in N$, the *ENSC-value* is defined by

$$ECSC_i(v) = v(N) - v(N \setminus i) + \frac{1}{n}\left[v(N) - \sum_{j \in N} (v(N) - v(N \setminus j))\right].$$

According to van den Brink and Funaki (2009), these two values can be described as distributions equally sharing the surplus among players. The CIS-value allocates $v(i)$ to every player i and then equally divide the surplus $v(N) - \sum_{j \in N} v(j)$ into every player. On the other hand, the ENSC-value, first, assume their utopia point, that is $M(v)$, and then equally makes all player give up the amount between the feasible amount $v(N)$ and the utopia point $M(v)$.

In contrast to the τ-value, the CIS-value employs $v(i)$ as its lower bound, while the ENSC-value adopts the utopia point as its upper bound.

In general, the three types of compromise solutions, namely, the τ-value, the CIS-value and the ENSC-value, indicate respectively different distributions. In order to understand the difference among these three concepts, we can use our both-edges weakly convex class. The following proposition depicts the fact.

Proposition 4.4. Let a game v is both-edges weakly convex. It holds that for any $i \in N$,

$$\tau_i(v) = CIS_i(v) + \frac{v(N) - \sum_{j\in N} v(j)}{\sum_{j\in N} \lambda_j}(ENSC_i(v) - CIS_i(v)),$$

where $\lambda_i = v(N) - v(N \setminus i) - v(i)$ for all $i \in N$.

Proof. The proof is straightforward. If v is both-edges weakly convex, by Lemma 3.2, we have

$$\begin{aligned}
\tau_i(v) &= v(i) + \frac{v(N) - v(N\setminus i) - v(i)}{\sum_{j\in N}[v(N) - v(N\setminus j) - v(j)]}\left[v(N) - \sum_{j\in N} v(j)\right] \\
&= v(i) + \frac{\lambda_i}{\sum_{j\in N}\lambda_j}\left[v(N) - \sum_{j\in N} v(j)\right] \\
&= \frac{\sum_{j\in N}\lambda_j - (v(N) - \sum_{j\in N} v(j))}{\sum_{j\in N}\lambda_j} CIS_i(v) + \frac{v(N) - \sum_{j\in N} v(j)}{\sum_{j\in N}\lambda_j} ENSC_i(v) \\
&= CIS_i(v) + \frac{v(N) - \sum_{j\in N} v(j)}{\sum_{j\in N}\lambda_j}(ENSC_i(v) - CIS_i(v)).
\end{aligned}$$

□

Proposition 4.4 shows that the τ-value can be a convex combination of the CIS-value and the ENSC-value. Since we can regard λ_i as an amount of compromise which is admissible to player i, the weight can be seen as the ratio of two factors: the difference between the feasible amount $v(N)$ and the total amount of $v(i)$ and the total amount of admissible compromise.

In addition, the following proposition clarifies the direct relationship between ENSC and τ-value.

Proposition 4.5. For any $v \in \Gamma^N$, $\tau(v) = ENSC(v)$ if and only if

$$v(N) = \sum_{j\in N} M_j(v) \;\; \textit{or} \;\; \textit{for any } i, j \in N,\; M_i(v) - m_i(v) = M_j(v) - m_j(v).$$

Proof. We can prove this fact arithmetically. By the definitions of these solutions, for every

$i \in N$, we have

$$\tau_i(v) - ENSC_i(v) = m_i(v) + \frac{v(N) - \sum_{j\in N} m_j(v)}{\sum_{j\in N} M_j(v) - \sum_{j\in N} m_j(v)}(M(v) - m(v))$$
$$-M_i(v) + \frac{1}{n}\left(\sum_{j\in N} M_j(v) - v(N)\right)$$
$$= \left(v(N) - \sum_{j\in N} M_j(v)\right)\left(\frac{M_i(v) - m_i(v)}{\sum_{j\in N} M_j(v) - \sum_{j\in N} m_j(v)} - \frac{1}{n}\right).$$

□

Note that the relationship between the CIS-value and the τ-value is not parallel to Proposition 4.5. As Proposition 4.4 shows, the coincidence of lower bounds, $v(i) = m_i(v)$, does not imply the coincidence of the τ-value with the CIS-value.

5 Conclusion

In this paper, we provided a theoretical relationship among solutions based on the concept of compromise. We offered the class of both-edges weakly convex games and the set-valued compromise solutions in Section 3. We showed that the compromise set becomes a subset of the Weber set if and only if the game satisfies the both-edges weak convexity. Section 4 was devoted to the point-valued compromise solutions. We examined that the τ-value lies on the line between the CIS-value and the ENSC-value in the both-edges weakly convex games. Moreover, it was also shown that the τ-value coincides with the ENSC-value if and only if the utopia demand is feasible for all players or the amount of admissible compromise, $M_i(v) - m_i(v)$, is the same among all players.

Through this paper, we restricted our attention to the games in *coalition function form* (CFF). By introducing the concept of partition of the player set, we can extend our discussion to games with externalities. The games with externalities are often referred to as *partition function form* (PFF) games. While the worth of a coalition depends only on the coalition in CFF games, it depends also on the partition of the player set. In general, solution concepts are affected by the externalities across coalitions. However, some solutions consisting of $v(N \setminus i)$ and $v(N)$ are not affected by them. For example, the ENSC-value meets this property. On the other hand, the CIS-value and the τ-value do not satisfy it. This approach has been introduced by de Clippel and Serrano (2008) as the "externality-free" property. They proved that a variant of the Shapley value is characterized by some axioms free from externalities. We can apply the externality-free property to determine the lower and upper bounds for the compromise in PFF games. This interesting topic is left for future work.

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