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Abstract

This paper provides testable implications of decision making under evolving attention, using a revealed preference approach. There are various bounded rationality models that analyze static decision making under limited consideration, models with a dynamic component are yet to be cultivated. Our model inherits the main idea of the limited attention model, while we add an assumption that agents' attention evolve over time. We allow that agents do not pay attention to every alternative in the constraint set, while we assume that the alternatives she does pay attention to depend on her past choices/consideration. Characterization results are given for three types of evolution of attention.

KEYWORDS: Revealed preference; Limited consideration; Limited attention; Consideration set; Bounded rationality

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1 Introduction

Revealed preference theory has had a great effect on various fields of economics. In consumer theory, in particular, the reveled preference approach originated by Samuelson (1938) has been studied extensively. The basic idea of this approach is that we can infer that the agent prefers alternative x to y whenever x is chosen when y is available. This argument seems plausible, but it implicitly assumes that the agent is aware of all the alternatives in the constraint set that she faces. It may be in reality that the agent prefers y, but choose x because she overlooks y.

Papers such as Masatlioglu et al. (2012) and Lleras et al. (2017) relax the 'full-attention' assumption and assume that the agent may only be aware of a subset of the constraint set (*limited consideration*). For example, a consumer seeking to buy a new PC may not have information about all the affordable PCs in the market, or an agent using a search engine may only pay attention to the search results displayed on the first page. Masatlioglu et al. (2012) present a general model of choice under limited attention in a static choice setting. It is assumed that the *consideration set*, the set of alternatives that attracts attention, does not change when removing an ignored alternative. This is known as the *attention filter* property.

In this paper, we introduce and characterize a model that inherits the main idea from Masatlioglu et al. (2012), and add to it a dynamic component. That is, we consider a dynamic model where the agent makes decisions under limited attention. As in Masatlioglu et al. (2012), we require that the consideration set does not change when removing an ignored alternative. In addition to this, since we are considering a dynamic model, we require the that the agent's attention evolves over time. We study three cases of how the agent's consideration sets evolve, and provide characterizations of the choice behavior under each variation of evolving attention. In addition to characterization of the model, we show how the preference, attention, and inattention of the agent can be inferred from the observed choices.

Since we assume that the agent makes decisions in multiple periods, the observed choice dataset is richer than that in Masatlioglu et al. (2012). We assume that a choice function is observed for each time period. This can be interpreted as hypothetical choices that the agent made from each constraint set. Suppose for example that the agent answers a survey of what she would choose. In addition to this, we assume that there is a *realized* constraint set at each time period that the agent faces in reality. With this richer dataset, we will be able to infer the agent's preference more sharply compared to the model in Masatlioglu et al. (2012).

Other related models are rational shortlisting models (Manzini and Mariotti, 2007; Au and Kawai, 2011), categorize-then-choose models (Manzini and Mariotti, 2012), and rationalization models (Cherepanov et al., 2010). They all have some components in common: an agent makes a multiple step decision, where she first eliminates some alternatives based on some first-step binary relation(s), then chooses her preference maximizing alternative from a list of alternatives that survive elimination in the previous period(s). Papers such as Tyson (2013), de Clippel and Rozen (2014), and Inoue and Shirai (2016) also deal with limited consideration models. These papers give revealed preference tests for limited consideration models based on a partial data set.

This paper is organized as follows. In section 2 we introduce the notations and basic concepts of the model. In sections 3, 4, and 5, we study three variations of how the agent's attention evolves. In each section, we present a characterization of the model, and then show how inference of preference, attention, and inattention is possible under each variation of evolving attention.

2 The model

Let X be a finite set of alternatives, and denote by Ω the collection of nonempty subsets of X, i.e. $\Omega \equiv 2^X \backslash \{\varnothing\}$. We consider X as the set of alternatives that may be available for the agent to choose, and $B \in \Omega$ as a constraint set from which the agent chooses an alternative. Assume that we observe dataset $\mathcal{O} = \{(c_t, B_t)\}_{t \in \mathcal{T}}$ of a single agent, where $\mathcal{T} = \{1, \ldots, T\}$. That is, we observe the choice function $c_t : \Omega \to X$ for every time period $t = 1, \ldots, T$. For any constraint set $B \in \Omega$ and time period t, $c_t(B)$ is considered as a hypothetical choice that the agent made at time t, perhaps obtained by a survey that asked her what she would choose from constraint set B. We observe this for every constraint set B for every time t. In addition to this, we observe a constraint set $B_t \in \Omega$, which is the constraint set that the agent actually faces at time t. $c_t(B_t)$ is the choice that the agent actually makes at time t when facing the constraint set B_t.

In this model, we allow that the agent has limited attention. That is, when facing a constraint set B, we allow that the agent does not pay attention to all the alternatives in B, and chooses her preference-maximizing choice over only a subset of B that she pays attention to. We can formulate this by defining a map on Ω, which we call the *consideration set mapping*. For each period t, define the consideration set mapping by $\Gamma_t : \Omega \to \Omega$ such that $\Gamma_t(B) \subseteq B$ for each $B \in \Omega$. Fixing time t, we focus on the case where the consideration set mapping Γ_t is an *attention filter*, which is the property required in the static limited attention model in Masatlioglu et al. (2012). When the consideration set mapping is an attention filter, removing an ignored alternative does not change the consideration set, i.e. $x \notin \Gamma_t(B)$ implies $\Gamma_t(B) = \Gamma_t(B \backslash \{x\})$.

The feature of our model is that the agent makes decisions sequentially over time. In addition to the requirement that the consideration set mapping Γ_t is an attention filter for fixed t, we cast restrictions on how the agent's attention evolves over time. Thus, we study below several cases of how the collection of consideration set mappings $(\Gamma_t)_{t \in \mathcal{T}} \equiv (\Gamma_1, \dots, \Gamma_T)$ evolves over time.

In the first case that we consider, the actual choices that the agent made in the previous period will be added to what she paid attention to in the past. That is, for any constraint set $B \in \Omega$ and time $t \in \mathcal{T}$, the constraint set will evolve in the following way: $\Gamma_t(B) \supseteq \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$. We call this evolution of attention *type-I evolution*. Note that in this case, fixing some constraint set B, the agent's consideration sets grow over time in terms of set inclusion, i.e. $\Gamma_t(B) \supseteq \Gamma_{t-1}(B)$.

The second case that we consider is a model where the consideration set consists exactly of the actual choices that the agent chose previously plus alternatives that she paid attention to in the past. That is, for any constraint set $B \in \Omega$ and time $t \in \mathcal{T}$, the constraint set will evolve in the following way: $\Gamma_t(B) = \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$. We call this evolution of attention *type-II evolution*. This model is a special case of the type-I evolution model, since whenever an agent has type-II evolution of attention, her attention trivially follows a type-I evolution. Note that type-I evolution allows the agent's consideration sets to grow in addition to the alternatives that we require to attract attention, while type-II evolution does not allow this.

The final case that we consider is a model where we allow the agent's consideration sets to evolve in any way under the restriction that she pays attention to the alternatives that she actually chose in the past. The agent's consideration set mapping evolves as $\Gamma_t(B) \supseteq \{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$, which we call *type-III evolution* of attention. In contrast to type-I and type-II evolution of attention, the consideration sets do not grow under type-III evolution.

We study each case in detail in the following sections. Below we formally define the concept of rationalizability of the observed dataset $\mathcal{O}$ under the evolving attention assumptions.

DEFINITION 1. The observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t \in \mathcal{T}}$ is *rationalizable under limited attention with type-i evolution of attention* for i $\in$ {I, II, III}, if there exist consideration set mappings for each time t, $(\Gamma_t)_{t \in \mathcal{T}}$, and an irreflexive, complete, and transitive binary relation $\succ$ on X, where:

- consideration set mapping $\Gamma_t : \Omega \to \Omega$ for all $t \in \mathcal{T}$ is an attention filter, i.e. $\Gamma_t(B) = \Gamma_t(B \backslash \{x\})$ whenever $x \notin \Gamma_t(B)$ for all $B \in \Omega$.
- $c_t(B)$ is the $\succ$-maximal alternative in $\Gamma_t(B)$; i.e. $c_t(B) \succ x$ for all $x \in \Gamma_t(B) \backslash \{c_t(B)\}$.
- consideration set mappings $(\Gamma_t)_{t \in \mathcal{T}}$ follow a type-i evolution of attention.

For notational simplicity, when the observed dataset is rationalizable under limited attention with type-I evolution of attention, we say that the dataset is *LA-rationalizable with type-I evolution* or *LA(I)-rationalizable.* The cases for type-II and type-III evolution of attention will be analogously referred to.

Note that while we assume that the agent's consideration set mappings evolves over time, we assume that she has an underlying preference that does not change over time.

Below we introduce some core concepts of decision making under limited attention with evolving attention.

DEFINITION 2. Let the observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t \in \mathcal{T}}$ be LA-rationalizable with type-i evolution of attention, for i $\in$ {I, II, III}. Then we say that;

- *it is robust to infer that an alternative x is preferred to y* if for all consideration set mappings and preference $((\Gamma_t)_{t \in \mathcal{T}}, \succ)$ that LA(i)-rationalize the observed dataset $\mathcal{O}$, we have $x \succ y$.
- *it is robust to infer that an alternative x attracts attention* in constraint set $B \in \Omega$ at time $t \in \mathcal{T}$ if $x \in \Gamma_t(B)$ for all $((\Gamma_t)_{t \in \mathcal{T}}, \succ)$ that LA(i)-rationalize $\mathcal{O}$.
- *it is robust to infer that an alternative x does not attract attention* in constraint set $B \in \Omega$ at time $t \in \mathcal{T}$ if $x \notin \Gamma_t(B)$ for all $((\Gamma_t)_{t \in \mathcal{T}}, \succ)$ that LA(i)-rationalize $\mathcal{O}$.

This definition is an analog of the definitions of inference of preference, attention, and inattention in Masatlioglu et al. (2012). The difference is that for inference of attention and inattention in this model, we must specify the time when an alternative attracts attention or not. Moreover, note that the definition of robust inference is contingent on the type of evolution of consideration sets. For example, it may be the case that an alternative x is robustly inferred to be preferred to y under type-II evolution of attention, but not under type-I evolution.

Note from the definition above that inference of preference, attention, and inattention is useful only when the observed dataset $\mathcal{O}$ is LA-rationalizable with a particular type of evolution of attention. Thus in the following sections, we first study the characterization of choice with evolving attention. That is, we study the condition that is necessary and sufficient for the observed dataset $\mathcal{O}$ to be LA(i)-rationalizable for i $\in$ {I, II, III}. Next, we study inference of preference, attention, and inattention under the assumption that the observed dataset is LA(i)-rationalizable.

3 Type-I : $\Gamma_t(B) \supseteq \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$

In this section we closely study the model of decision making under limited attention where the consideration set evolves in the following way: $\Gamma_t(B) \supseteq \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$ for all constraint sets $B \in \Omega$ and time t. That is, the agent's attention grows over time under the restriction that she pays attention to the actual choices that she made in the past. We first give an example that illustrates LA-rationalizability with type-I evolution of attention.

Example 1. Consider the following choice function with three alternatives. Let the choices with the underline be the realized choices.

$$c_1(\{x,y,z\}) = x, \quad c_1(\{x,y\}) = x, \quad c_1(\{y,z\}) = y, \quad \underline{c_1(\{x,z\}) = z},$$
$$\underline{c_2(\{x,y,z\}) = z}, \quad c_2(\{x,y\}) = x, \quad c_2(\{y,z\}) = z, \quad c_2(\{x,z\}) = z.$$

The observed dataset is $\mathcal{O} = \{(c_1, \{x,z\}), (c_2, \{x,y,z\})\}$. Under the assumption of type-I evolution of attention, it follows that the agent's preference can be identified to be $z \succ x \succ y$. Then we infer that alternative z did not attract attention in constraint sets $\{x,y,z\}$ and $\{y,z\}$ at time period 1. Note that by the way the consideration set mappings evolve, at period 2, z must attract attention in $\{x,y,z\}$ and $\{y,z\}$. Below we give an example of the consideration set mappings that LA(I)-rationalizes the data with the preference above.

time		consideration set mapping			
		$\{x,y,z\}$	$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
time 1	Γ_1	$\{x,y\}$	$\{x\}$	$\{y\}$	$\{z\}$
time 2	Γ_2	$\{x,y,z\}$	$\{x\}$	$\{y,z\}$	$\{z\}$

3.1 Characterization

In this section, we answer the following question: what condition on the observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ is necessary and sufficient for the LA(I)-rationalizablity of $\mathcal{O}$? To do this, we first study what is implied when the observed dataset $\mathcal{O}$ is LA-rationalizable with type-I evolution of attention. Thus, assume that $\mathcal{O}$ is LA(I)-rationalizable. In particular, assume that it is rationalized by consideration set mappings and preference $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$.

Fix any period $t \in \mathcal{T}$ and constraint set $B \in \Omega$. We can robustly infer that $c_t(B)$, the choice made in B at time t, is preferred to any other alternative in the consideration set $\Gamma_t(B)$. Thus we can say that $c_t(B)$ is revealed preferred to an alternative $x \in B$ if we can say that x is in the consideration set in constraint set B at time t. To define revealed preference relations, we must study when we can robustly infer that an alternative is in the consideration set.

The first case where we can surely say that an alternative x attracts attention in constraint set B at time t comes from the evolution of the consideration set mapping; i.e. $\Gamma_t(B) \supseteq \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$. We can see that if an alternative x was actually chosen in reality in a period before t, i.e. $x = c_s(B_s)$ for some $s < t$, then it will attract attention in period $s+1$. Then by the way the agent's consideration set evolves, this alternative must attract attention in the future periods as well, including period t. Thus, we can define a revealed preference relation $\mathcal{P}_1$ as follows:

$$x\mathcal{P}_1 y \iff {}^{\exists}t \in \mathcal{T}, B \in \Omega \text{ s.th. } \begin{cases} x = c_t(B), \\ y \in B\backslash\{x\}, \text{ and} \\ y = c_s(B_s) \text{ for some } s < t. \end{cases} \tag{1}$$

Another case when we can surely say that an alternative is in the consideration comes from the definition of an attention filter. First note that under rationalizability of $\mathcal{O}$, the definition of an attention filter implies that an alternative x attracts attention in constraint set B at time t whenever $c_t(B) \neq c_t(B \backslash \{x\})$.[1] In addition to this, an alternative x attracts attention in constraint set B at time t if it attracts attention in any period before t. Thus we can define a revealed preference relation $\mathcal{P}_2$ as follows:

$$x \mathcal{P}_2 y \iff {}^{\exists} t \in \mathcal{T}, B \in \Omega \text{ s.th. } \begin{cases} x = c_t(B), \\ y \in B \backslash \{x\}, \text{ and} \\ c_s(B) \neq c_s(B \backslash \{y\}) \text{ for some } s \leqslant t. \end{cases} \tag{2}$$

Below we give an example to illustrate how these revealed preference relations are defined.

Example 2. Consider the following choice function with three alternatives. Let the choices with the underline be the realized choices.

$$c_1(\{x,y,z\}) = x, \quad c_1(\{x,y\}) = x, \quad c_1(\{y,z\}) = y, \quad \underline{c_1(\{x,z\}) = z},$$
$$\underline{c_2(\{x,y,z\}) = x}, \quad c_2(\{x,y\}) = x, \quad c_2(\{y,z\}) = z, \quad c_2(\{x,z\}) = z.$$

The observed dataset is $\mathcal{O} = \{(c_1, B_1), (c_2, B_2)\}$, where $B_1 = \{x, z\}$ and $B_2 = \{x, y, z\}$. Since $c_2(\{x,y,z\}) = x$ and $c_1(B_1) = z \in \{x,y,z\}$, we have $x \mathcal{P}_1 z$. Observing $x = c_1(\{x,y,z\}) \neq c_1(\{x,z\}) = z$, we have $x \mathcal{P}_2 y$. Note that $y = c_1(\{y,z\}) \neq c_1(\{z\}) = z$ trivially holds. Then, observing $z = c_2(\{y,z\})$ and $y = c_1(\{y,z\}) \neq c_1(\{z\}) = z$, we have $z \mathcal{P}_2 y$.

Summarizing above arguments, we need a revealed preference relation that captures both revealed preferences $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus we define a revealed preference relation $\mathcal{P}$ to be a union of revealed preference relations above, i.e. $x \mathcal{P} y$ if $x \mathcal{P}_i y$ for some $i \in \{1, 2\}$. In words, this revealed preference relation can be interpreted as follows: for some constraint set B and time t, the choice $c_t(B)$ is revealed preferred to any alternative y that we can robustly infer to attract attention in B at time t. Let us define $\bar{\mathcal{P}}$ to be the transitive closure of $\mathcal{P}$.

In fact, under LA(I)-rationalizability of $\mathcal{O}$, an alternative x is robustly inferred to be preferred to y if $x \mathcal{P} y$. If we have $x \mathcal{P} y$, then for some consideration set B and time t, we have $x = c_t(B)$ and y in the consideration set. Thus it follows by LA(I)-rationalizability that $x \succ y$ for any preference that rationalizes the data. Then, we see that the acyclicity of the revealed preference $\mathcal{P}$ is a necessary condition for rationalizability of observed dataset $\mathcal{O}$. We show

[1] If x does not attract attention in B at time t, then by definition of attention filters, it follows that the consideration sets are the same for constraint sets B and $B \backslash \{x\}$. Then we must have $c_t(B) = c_t(B \backslash \{x\})$.

below that this is in fact a sufficient condition as well.

PROPOSITION 1. *Observed dataset* $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ *is LA-rationalizable with type-I evolution of attention if and only if the revealed preference relation* $\mathcal{P}$ *is acyclic, where* $\mathcal{P}$ *is the union of the revealed preference relations* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*, defined in (1) and (2) respectively.*

Proof. Since necessity has been proved above, we prove sufficiency. Assuming the revealed preference relation $\mathcal{P}$ is acyclic, there exists an irreflexive, complete, and transitive order extension of $\mathcal{P}$, which we denote by $\succ$.[2]

Now define the consideration set mapping for $t \in \mathcal{T}$ and $B \in \Omega$ as;

$$\Gamma_t(B) = \{c_t(B)\} \cup \{y \in B : c_t(B) \succ y\}.$$

It is clear that we have $c_t(B) \in \Gamma_t(B)$ and $\Gamma_t(B) \subseteq B$. It is also clear from the construction of Γ that $c_t(B)$ is the unique preference maximizer in B; i.e. $c_t(B) \succ x$ for all $x \in \Gamma_t(B)\backslash\{c_t(B)\}$ under the preference $\succ$.

Now we show that $(\Gamma_t)_{t\in\mathcal{T}}$ follows the type-I evolution of attention. Fix any $t \in \mathcal{T}$ and any $B \in \Omega$. We separately show that $\Gamma_t(B) \supseteq \Gamma_{t-1}(B)$, and $\Gamma_t(B) \ni c_{t-1}(B_{t-1})$ whenever $c_{t-1}(B_{t-1}) \in B$. If we have $c_t(B) = c_{t-1}(B)$, we have $\Gamma_t(B) = \Gamma_{t-1}(B)$. If $c_t(B) \neq c_{t-1}(B)$, then we have $c_t(B)\mathcal{P}_2 c_{t-1}(B)$. This implies $c_t(B) \succ c_{t-1}(B)$, which in turn implies $\Gamma_t(B) \supseteq \Gamma_{t-1}(B)$. If $c_{t-1}(B_{t-1}) \in B$ and $c_t(B) \neq c_{t-1}(B_{t-1})$, we have $c_t(B)\mathcal{P}_1 y$. This implies $c_t(B) \succ y$, and $y \in \Gamma_t(B)$ follows. Summarizing, we have $\Gamma_t(B) \supseteq \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$.

Finally, we show that Γ_t is an attention filter. Take any $B \in \Omega$, and suppose $x \in B$ and $x \notin \Gamma_t(B)$. Note that if we have $c_t(B) \neq c_t(B\backslash\{x\})$, we must have $c_t(B)\mathcal{P}_2 x$, which in turn implies $c_t(B) \succ x$ and $x \in \Gamma_t(B)$, a contradiction. Thus it follows that $c_t(B) = c_t(B\backslash\{x\})$, and we have $\Gamma_t(B) = \Gamma_t(B\backslash\{x\})$. □

3.2 Inference of preference, attention, and inattention

In this section, we study what can be inferred about the preference, attention, and inattention of the agent, given the observable dataset is LA(I)-rationalizable.

First we characterize when we can robustly infer that an alternative is preferred to another. Recall that binary relation $\mathcal{P}$ is the union of revealed preference relations $\mathcal{P}_1$ and $\mathcal{P}_2$, and $\bar{\mathcal{P}}$

[2] Andrikopoulos (2009) gives a conprehensive survey on extension theorems relevant to economics.

is the transitive closure of $\mathcal{P}$. The intuition is that $x\bar{\mathcal{P}}y$ is a sufficient condition for robust inference of x preferred to y, given $\mathcal{O}$ is LA(I)-rationalizable. In fact this is true. Moreover, $x\bar{\mathcal{P}}y$ is also necessary for robust inference of preference. We formally show this below.

Proposition 2. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-I evolution of attention. Then it is robust to infer that an alternative x is preferred to y if and only if $x\bar{\mathcal{P}}y$.*

Proof. We first prove sufficiency. Let $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$ LA(I)-rationalize $\mathcal{O}$. By definition of $x\bar{\mathcal{P}}y$, we have a sequence of alternatives $(x^1, \ldots, x^m)$ such that $x = x^m\mathcal{P}\ x^{m-1}\mathcal{P}\ \cdots\ \mathcal{P}x^2\ \mathcal{P}x^1 = y$. That is, for all $k \in \{1, \ldots, m-1\}$ we have $x^{k+1}\mathcal{P}_i x^k$ for $i \in \{1, 2\}$. Then, there exist $t \in \mathcal{T}$ and $B \in \Omega$ such that $c_t(B) = x^{k+1}$ and $x^k \in \Gamma_t(B)\backslash\{x^{k+1}\}$. By LA(I)-rationalizability, we have $x^{k+1} \succ x^k$. Thus we have $x \succ y$, and we see that x is preferred to y.

Next we prove necessity by showing the contrapositive. Suppose that $x\bar{\mathcal{P}}y$ does not hold. Then there exists a preference that is an extension of $\bar{\mathcal{P}}$ and ranks y better than x. The proof of LA(I)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference. Thus it is not robust to infer that x is preferred to y. □

Now we characterize the cases where we can surely say that the agent pays attention, or does not pay attention to an alternative, provided that the observed dataset is LA(I)-rationalizable.

First we characterize when an alternative is robustly inferred not to be in a consideration set.

Proposition 3. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-I evolution of attention. Then it is robust to infer that an alternative x does not to attract attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if and only if $x\bar{\mathcal{P}}c_t(B)$.*

Proof. We first prove the if part. Let $x\bar{\mathcal{P}}c_t(B)$. Then under LA(I)-rationalizability, we have $x \succ c_t(B)$. Thus, we cannot have $x \in \Gamma_t(B)$.

Next we prove the only if part by showing the contrapositive. Assume that $x\bar{\mathcal{P}}c_t(B)$ does not hold for $x \in B\backslash\{c_t(B)\}$. Then, choose a preference $\succ$ that is an extension of $\mathcal{P}$ and ranks $c_t(B)$ above x. Then the proof of LA(I)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference and a family of consideration set mappings $(\Gamma_s)_{s\in\mathcal{T}}$ such that $x \in \Gamma_t(B)$. □

The example below illustrates how we can robustly infer the agent's preference, attention and inattention.

Example 3. Consider the following choice function with three alternatives. Let the choices with the underline be the realized choices.

$$c_1(\{x,y,z\}) = x, \quad c_1(\{x,y\}) = x, \quad c_1(\{y,z\}) = y, \quad \underline{c_1(\{x,z\}) = z},$$
$$\underline{c_2(\{x,y,z\}) = x}, \quad c_2(\{x,y\}) = x, \quad c_2(\{y,z\}) = z, \quad c_2(\{x,z\}) = z.$$

Thus the observed dataset is $\mathcal{O} = \{(c_1, B_1), (c_2, B_2)\}$, where $B_1 = \{x, z\}$ and $B_2 = \{x, y, z\}$. Recall from Example 2 that we have $x\mathcal{P}y$, $x\mathcal{P}z$, and $z\mathcal{P}y$. Thus it is robust to infer that the agent's preference is $x \succ z \succ y$. Moreover, since we have $z\mathcal{P}y = c_1(\{y,z\}) = y$, it is robust to infer that z does not attract attention in constraint set $\{y, z\}$ at time 1.

Recall that we discussed in section 3.1 two cases where an alternative surely attracts attention in some constraint set. Here we study further the inference of attention. While the necessary condition for robust inference of attention is an open problem, we give a sufficient condition, which is shown in the proposition below.

Proposition 4. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t \in \mathcal{T}}$ be LA-rationalizable with type-I evolution of attention. Then it is robust to infer that an alternative x attracts attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if there exists $B' \in \Omega$ (possibly equal to B) such that (a) and (b) holds, where;*

(a) $c_s(B') \neq c_s(B' \backslash \{x\})$ for some $s \leqslant t$ **or** *$x \in B' \cap (\cup_{s=1}^{t-1} c_s(B_s))$,*

(b) $y\bar{\mathcal{P}}c_t(B)$ for all $y \in B \backslash B'$ **and** *$z\bar{\mathcal{P}}c_t(B')$ for all $z \in B' \backslash B$.*

Proof. Let $\mathcal{O}$ be LA(I)-rationalized by $((\Gamma_t)_{t \in \mathcal{T}}, \succ)$. Fix any $B \in \Omega$, and take any $B' \in \Omega$ that satisfies (a) and (b). Recall from the discussion in section 3.1 that (a) implies that $x \in \Gamma_t(B')$. If $B' = B$ the proof is complete, since (b) is trivially satisfied in this case. Hence assume below that $B' \neq B$. Note that it follows from Proposition 3 that $y\bar{\mathcal{P}}c_t(B)$ implies $y \notin \Gamma_t(B)$. Thus, (b) implies that $y \notin \Gamma_t(B)$ for all $y \in B \backslash B'$, which in turn implies $\Gamma_t(B) = \Gamma_t(B \cap B')$. The last part follows from the attention filter property and the fact that $B \backslash (B \backslash B') = B \cap B'$. Analogously, (b) implies $\Gamma_t(B') = \Gamma_t(B \cap B')$. Thus we have $\Gamma_t(B) = \Gamma_t(B')$, and $x \in \Gamma_t(B)$ follows. □

4 Type-II : $\Gamma_t(B) = \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$

In this section we closely study the case where the consideration set mappings evolve in the following way: $\Gamma_t(B) = \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$ for all consideration sets $B \in \Omega$ and time t. That is, at each time period, the choice that the agent actually chose in the previous period is added to what she paid attention to previously. Note that this model is a special case of type-I evolution, since when the agent has type-II evolution of attention, her attention will trivially follow a type-I evolution.

Note that the dataset given in Example 1 is LA-rationalizable with type-II evolution of attention, with the same preference and consideration set mappings as those given in the example. To illustrate the difference between type-I evolution and type-II evolution of attention, we give below an example of a dataset that is LA(I)-rationalizable but not LA(II)-rationalizable.

Example 4. Consider the following choice function with three alternatives. Let the choices with the underline be the realized choices.

$$c_1(\{x,y,z\}) = y, \quad c_1(\{x,y\}) = y, \quad c_1(\{y,z\}) = y, \quad \underline{c_1(\{x,z\}) = z},$$
$$\underline{c_2(\{x,y,z\}) = z}, \quad c_2(\{x,y\}) = x, \quad c_2(\{y,z\}) = z, \quad c_2(\{x,z\}) = z.$$

The observed dataset is $\mathcal{O} = \{(c_1, \{x,z\}), (c_2, \{x,y,z\})\}$. This dataset can be LA(I)-rationalized with preference $z \succ x \succ y$, and consideration set mappings as in the table below.

However, this dataset is not LA(II)-rationalizable. To see this, focus on the constraint set $\{x,y\}$, and suppose by way of contradiction that the dataset is LA(II)-rationalized by $((G_1, G_2), \succ)$ Since $c_1(\{x,y\}) = y$, it must follow that $y \in G_1(\{x,y\})$. Similarly, since $c_2(\{x,y\}) = x$, it follows that $x \in G_2(\{x,y\})$. Under type-II evolution of attention and the fact that the actual choice $c_1(B_1) = z \notin \{x,y\}$, we must have $G_1(\{x,y\}) = G_2(\{x,y\}) = \{x,y\}$. This implies $x \succ y$ and $y \succ x$, which is a contradiction.

time		consideration set mapping			
		$\{x,y,z\}$	$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
time 1	Γ_1	$\{y\}$	$\{y\}$	$\{y\}$	$\{z\}$
time 2	Γ_2	$\{y,z\}$	$\{x,y\}$	$\{y,z\}$	$\{z\}$

4.1 Characterization

In this section we answer the following question: what condition on the observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ is necessary and sufficient for the LA(II)-rationalizablity of $\mathcal{O}$? To do this, we first study what is implied when the observed dataset $\mathcal{O}$ is LA-rationalizable with type-II evolution of attention. Thus assume below that $\mathcal{O}$ is LA(II)-rationalizable. In particular, assume that there exist consideration set mappings $(\Gamma_t)_{t\in\mathcal{T}}$ and a strict preference $\succ$ on X that satisfy the definition of LA(II)-rationalizability.

Fix any period $t \in \mathcal{T}$ and constraint set $B \in \Omega$. We can robustly infer that $c_t(B)$, the choice made in B at time t, is preferred to any other alternative in the consideration set $\Gamma_t(B)$. Thus we can say that $c_t(B)$ is revealed preferred to an alternative $x \in B$ if we can say that x is in the consideration set in constraint set B at time t. To define revealed preference relations, we must study when we can robustly infer that an alternative is in the consideration set.

The first case where we can surely say that an alternative x attracts attention in constraint set B at time t comes from the evolution of the consideration set mapping: $\Gamma_t(B) = \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\}$. We can see that if an alternative x was actually chosen in reality in a period before t, i.e. $x = c_s(B_s)$ for some $s < t$, then it will attract attention in period $s + 1$. Then by the way that the agent's consideration set evolves, this alternative must attract attention in the future periods as well, including period t. Note that this is the same case as in type-I evolution of attention, so we can use the revealed preference relation $\mathcal{P}_1$ defined in equation (1) in the previous section.

Another case when we can surely say that an alternative is in the consideration set comes from the definition of an attention filter. First note that under rationalizability of $\mathcal{O}$, the definition of an attention filter implies that an alternative x attracts attention in constraint set B at time t whenever $c_t(B) \neq c_t(B\backslash\{x\})$. In addition to this, an alternative x attracts attention in constraint set B at time t if it attracts attention in any period before t. Since this case is the same as in the type-I evolution model, we can use the revealed preference relation $\mathcal{P}_2$ defined in equation (2).

Up to this point, the revealed preference relations defined in this model of evolving attention is equivalent to the type-I evolution model. However, there is another way that we can infer that an alternative attracts attention, and thus we can define an additional revealed preference relation in this model. Taking a look at the way that the consideration set mappings evolve, there are cases where we can infer that an alternative is in the consideration set of B at time t by studying the consideration set of some future period s. For example, when we can infer that an alternative is in the consideration set of B at time $s > t$, and that alternative is not actually chosen in periods $(t, t+1, \ldots, s-1)$, then we can infer that the alternative is in the consideration set of B at time t. Formally, we define a revealed preference relation $\mathcal{P}_3$ for any distinct x and y as follows:

$$x\mathcal{P}_3 y \iff {}^\exists t \in \mathcal{T}, B \in \Omega \text{ s.th. } \begin{cases} x = c_t(B), \\ c_s(B) \neq c_s(B \backslash \{y\}) \text{ and} \\ y \notin \cup_{k=t}^{s-1} c_k(B_k) \text{ for some } s > t. \end{cases} \tag{3}$$

Summarizing above arguments, we need a revealed preference relation that captures revealed preference relations $\mathcal{P}_1$, $\mathcal{P}_2$,and $\mathcal{P}_3$. Thus we define a revealed preference relation $\mathcal{Q}$ to be a union of revealed preference relation above, i.e. $x\mathcal{Q}y$ if $x\mathcal{P}_i y$ for some $i \in \{1,2,3\}$. In words, this revealed preference relation can be interpreted as follows: for some constraint set B and time t, the choice $c_t(B)$ is revealed preferred to any alternative y that we can robustly infer to attract attention in B at time t. Let us define $\bar{\mathcal{Q}}$ to be the transitive closure of $\mathcal{Q}$.

In fact, under LA(II)-rationalizability of $\mathcal{O}$, an alternative x is robustly inferred to be preferred to y if $x\mathcal{Q}y$. If we have $x\mathcal{Q}y$, then for some constraint set B and time t, we have $x = c_t(B)$ and y in the consideration set. Thus it follows by LA(II)-rationalizability that $x \succ y$ for any preference that rationalizes the dataset. Thus $x\mathcal{Q}y$ implies $x \succ y$ for any preference. Then, we see that the acyclicity of the revealed preference $\mathcal{Q}$ is a necessary condition for LA(II)-rationalizability of observed dataset $\mathcal{O}$. We show that, in fact, the acyclicity of $\mathcal{Q}$ is sufficient for LA(II)-rationalizability as well. To show this, we use the following lemma.

Lemma 1. *Let the revealed preference relation $\mathcal{Q}$ be acyclic. Then, for any $t \in \{2, \ldots, T\}$, $c_t(B) = c_{t-1}(B_{t-1})$ whenever $c_t(B) \neq c_{t-1}(B)$.*

Proof. Let $c_t(B) \neq c_{t-1}(B)$ and assume by way of contradiction that $c_t(B) \neq c_{t-1}(B_{t-1})$. By $c_t(B) \neq c_{t-1}(B)$ we have $c_t(B)\mathcal{P}_2 c_{t-1}(B)$. Moreover, by $c_t(B) \neq c_{t-1}(B_{t-1})$ we have $c_{t-1}(B)\mathcal{P}_3 c_t(B)$. This is a contradiction of acyclicity of $\mathcal{Q}$. □

This lemma says that, fixing any constraint set $B \in \Omega$, when we observe the agent's choice changing, the agent's new choice must be the alternative that she actually chose in reality in the previous period. This is a feature that did not hold under the type-I evolution of attention. Note that the statement in this lemma is not satisfied in Example 4, where the observed dataset is not LA(II)-rationalizable.

The proposition below gives a characterization of LA-rationalizability with type-II evolution of attention.

Proposition 5. *Observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ is LA-rationalizable with type-II evolution of attention if and only if the revealed preference relation $\mathcal{Q}$ is acyclic, where $\mathcal{Q}$ is the union of the revealed preference relations $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, defined in (1), (2), and (3) respectively.*

Proof. Since necessity has been proved above, we prove sufficiency. Assuming that the revealed preference relation $\mathcal{Q}$ is acyclic, there exists an irreflexive, complete, and transitive order extension of $\mathcal{Q}$, which we denote by $\succ$.

Now define the consideration set mappings as below;

$$\Gamma_1(B) = \{c_1(B)\} \cup \{x \in B : c_1(B) \succ x\},$$
$$\Gamma_t(B) = \Gamma_{t-1}(B) \cup \{B \cap c_{t-1}(B_{t-1})\} \text{ for } t \geqslant 2.$$

It is clear by the construction that the consideration set mappings evolve as required, and $\Gamma_t(B) \subseteq B$.

We first show that $c_t(B) \in \Gamma_t(B)$ for any t and B. We show this by induction. First of all, note that we have $c_1(B) \in \Gamma_1(B)$. Then assume that $c_{t-1}(B) \in \Gamma_{t-1}(B)$ and consider time t. If we have $c_t(B) = c_{t-1}(B)$, then we have $c_t(B) \in \Gamma_t(B)$ by construction of the consideration set mappings. If $c_t(B) \neq c_{t-1}(B)$, by Lemma 1, we have $c_t(B) = c_{t-1}(B_{t-1})$, and thus $c_t(B) \in \Gamma_t(B)$.

Next we show by induction that $c_t(B)$ is the unique preference maximizer in $\Gamma_t(B)$ for any t and B, under the preference $\succ$. For time period 1, this is obvious. Assume that $c_{t-1}(B) \succ \Gamma_{t-1}(B)\backslash\{c_{t-1}(B)\}$, and consider time t. If $c_t(B) = c_{t-1}(B)$, we have $c_t(B) \succ x$ for all $x \in \Gamma_{t-1}(B)\backslash\{c_t(B)\}$. Note that for any $y \in \{B \cap c_{t-1}(B_{t-1})\}$ such that $y \neq c_t(B)$, we have $c_t(B)\mathcal{P}_1 y$, which in turn implies $c_t(B) \succ y$. Thus, when $c_t(B) = c_{t-1}(B)$, we have $c_t(B) \succ x$ for all $x \in \Gamma_t(B)\backslash\{c_t(B)\}$. Now consider the case where $c_t(B) \neq c_{t-1}(B)$. Then we have $c_t(B)\mathcal{P}_2 c_{t-1}(B)$ and $c_t(B) \succ c_{t-1}(B)$, and thus $c_t(B) \succ x$ for all $x \in \Gamma_{t-1}(B)$. On the other hand, by Lemma 1, we have $c_t(B) = c_{t-1}(B_{t-1})$. Thus we conclude that $c_t(B)$ is the unique preference maximizer in $\Gamma_t(B)$.

Finally we show that Γ_t is an attention filter for any time t. Note that Γ_1 is an attention filter. Assume by way of induction that Γ_{t-1} is an attention filter, and consider time t. Fix constraint set $B \in \Omega$ and $x \in B$ but $x \notin \Gamma_t(B)$. It follows from $x \notin \Gamma_t(B)$ that $x \notin \Gamma_{t-1}(B)$. Then since Γ_{t-1} is an attention filter, $\Gamma_{t-1}(B) = \Gamma_{t-1}(B\backslash\{x\})$. On the other hand, $x \notin \Gamma_t(B)$ also implies that $x \neq c_{t-1}(B_{t-1})$. Thus, we conclude that $\Gamma_t(B) = \Gamma_t(B\backslash\{x\})$. □

4.2 Inference of preference, attention, and inattention

In this section, we study what can be inferred about the preference, attention, and inattention of the agent, given that the observed dataset is LA(II)-rationalizable.

First we characterize when we can robustly infer that an alternative is preferred to another. Recall that the binary relation $\mathcal{Q}$ is the union of revealed preference relations $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, and $\bar{\mathcal{Q}}$ is the transitive closure of $\mathcal{Q}$. The intuition is that $x\bar{\mathcal{Q}}y$ is a sufficient condition for robust inference of x being preferred to y, given $\mathcal{O}$ is LA(II)-rationalizable. This is true. Moreover, $x\bar{\mathcal{Q}}y$ is also a necessary condition for robust inference of preference. We formally show this below.

PROPOSITION 6. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-II evolution of attention. Then it is robust to infer that an alternative x is preferred to y if and only if $x\bar{\mathcal{Q}}y$.*

Proof. We first prove sufficiency. Let $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$ LA(II)-rationalize $\mathcal{O}$. By definition of $x\bar{\mathcal{Q}}y$, we have a sequence of alternatives $(x^1, \ldots, x^m)$ such that $x = x^m\mathcal{Q}x^{m-1}\mathcal{Q}\cdots\mathcal{Q}x^2\mathcal{Q}x^1 = y$. That is, for all $k \in \{1, \ldots, m-1\}$ we have $x^{k+1}\mathcal{P}_i x^k$ for $i \in \{1, 2, 3\}$. Then, there exist $t \in \mathcal{T}$ and $B \in \Omega$ such that $c_t(B) = x^{k+1}$ and $x^k \in \Gamma_t(B)\backslash\{x^{k+1}\}$. By rationalizability, we have $x^{k+1} \succ x^k$. Thus we have $x \succ y$, and we see that x is robustly inferred to be preferred to y.

Next we prove necessity by showing the contrapositive. Suppose that $x\bar{\mathcal{Q}}y$ does not hold. Then there exists a preference that is an extension of $\bar{\mathcal{Q}}$ and ranks y better than x. The proof of LA(II)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference. Thus it is not robust to infer that x is preferred to y. □

Now we characterize the cases where we can surely say that the agent pays attention, or does not pay attention to an alternative, provided that the observed dataset is LA(II)-rationalizable.

First we study when we can robustly infer that an alternative is not in a consideration set.

PROPOSITION 7. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-II evolution of attention. Then it is robust to infer that an alternative $x \in B\backslash\{c_t(B)\}$ does not to attract attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if and only if $x\bar{\mathcal{Q}}c_1(B)$ and $x \neq c_s(B_s)$ for all $s < t$.*

Proof. We first prove sufficiency. Under LA(II)-rationalizability, $x\bar{\mathcal{Q}}c_1(B)$ implies $x \succ c_1(B)$, and thus we must have $x \notin \Gamma_1(B)$. Since $x \neq c_1(B_1)$ and $\Gamma_2(B) = \Gamma_1(B) \cup \{B \cap c_1(B_1)\}$, we have $x \notin \Gamma_2(B)$. Iterating this argument, we conclude that $x \notin \Gamma_t(B)$.

Next we prove necessity by showing the contrapositive. If we do not have $x\bar{\mathcal{Q}}c_1(B)$, then we can choose the preference $\succ$ that rationalizes the dataset to be an extension of $\mathcal{Q}$ that ranks $c_1(B)$ above x. Then the proof of LA(II)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference and a family of consideration set mappings $(\Gamma_s)_{s\in\mathcal{T}}$ such that $x \in \Gamma_1(B) \subseteq \Gamma_t(B)$.

If we have $x = c_s(B_s)$ for some $s < t$, then we have $c_t(B)\mathcal{P}_1 x$. Then we can LA(II)-rationalize $\mathcal{O}$ with consideration set mappings such that $x \in \Gamma_t(B)$, as was shown in the proof of LA(II)-rationalizability. □

Recall that we discussed in section 4.1 three cases where an alternative surely attracts attention in some constraint set. Here we study further the inference of attention. While the necessary condition for robust inference of attention is an open problem, we give a sufficient condition, which is shown in the proposition below.

PROPOSITION 8. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-II evolution of attention. Then it is robust to infer that an alternative x attracts attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if there exists $B' \in \Omega$ (possibly equal to B) such that (a) and (b) holds, where;*

(a) $c_s(B') \neq c_s(B'\backslash\{x\})$ *for some* $s \leqslant t$, **or**

$x \in \{B' \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$, **or**

$c_s(B') \neq c_s(B'\backslash\{x\})$ *for some* $s > t$ *such that* $x \notin \{B' \cap (\cup_{k=t}^{s-1} c_k(B_k))\}$,

(b) $y\bar{\mathcal{Q}}c_t(B)$ *for all* $y \in B\backslash B'$, **and**

$z\bar{\mathcal{Q}}c_t(B')$ *for all* $z \in B'\backslash B$, **and**

$y \neq c_s(B_s)$ *for all* $y \in (B\backslash B') \cup (B'\backslash B)$.

Proof. Let $\mathcal{O}$ be LA(II)-rationalized by $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$. Fix any $B \in \Omega$, and take any $B' \in \Omega$ that satisfies (a) and (b). Recall from the discussion in section 4.1 that (a) implies that $x \in \Gamma_t(B')$.

If $B' = B$ the proof is complete, since (b) is trivially satisfied in this case. Hence assume below that $B' \neq B$. Note that it follows from Proposition 7 that "$y\bar{Q}c_t(B)$" and "$y \neq c_s(B_s)$ for all $s < t$" imply $y \notin \Gamma_t(B)$. Thus, for any $y \in B \backslash B'$, "$y\bar{Q}c_t(B)$ and $y \neq c_s(B_s)$ for all $s < t$" imply $y \notin \Gamma_t(B)$ for all $y \in B \backslash B'$, which in turn implies $\Gamma_t(B) = \Gamma_t(B \cap B')$. The last part follows from the attention filter property and the fact that $B \backslash (B \backslash B') = B \cap B'$. Analogously, (b) implies $\Gamma_t(B') = \Gamma_t(B \cap B')$. Thus we have $\Gamma_t(B) = \Gamma_t(B')$, and $x \in \Gamma_t(B)$ follows. □

5 Type-III : $\Gamma_t(B) \supseteq \{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$

In this section we closely study the final case of evolution of attention, in which the consideration set mappings evolve in the following way: $\Gamma_t(B) \supseteq \{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$ for all constraint sets $B \in \Omega$ and time t. In this model, the agent's consideration sets are allowed to evolve in any way with the restriction that it contains the actual choices that she made in the past. Note that this flexibility results in the fact that we may not have $\Gamma_t(B) \supseteq \Gamma_{t-1}(B)$, which is a property that holds under type-I and type-II evolution of attention. The following example is a case where the dataset is LA-rationalizable with type-III evolution of attention, but not LA-rationalizable with neither type-I nor type-II evolution of attention.

Example 5. Consider the following choice function with three alternatives. Let the choices with the underline be the realized choices.

$$c_1(\{x,y,z\}) = x, \quad c_1(\{x,y\}) = x, \quad c_1(\{y,z\}) = y, \quad \underline{c_1(\{x,z\}) = z},$$
$$\underline{c_2(\{x,y,z\}) = x}, \quad c_2(\{x,y\}) = y, \quad c_2(\{y,z\}) = z, \quad c_2(\{x,z\}) = z.$$

The observed dataset is $\mathcal{O} = \{(c_1, B_1), (c_2, B_2)\}$, where $B_1 = \{x, z\}$ and $B_2 = \{x, y, z\}$. This dataset can be LA(III)-rationalized by preference $x \succ y \succ z$. An example of a consideration set mapping that LA(III)-rationalizes the dataset is given below.

time		consideration set mapping			
		$\{x,y,z\}$	$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
time 1	Γ_1	$\{x,y\}$	$\{x,y\}$	$\{y,z\}$	$\{z\}$
time 2	Γ_2	$\{x,y,z\}$	$\{y\}$	$\{z\}$	$\{z\}$

Note that the agent's attention does not necessarily grow: $\Gamma_1(\{x,y\})$ is not a subset of

$\Gamma_2(\{x,y\})$, and $\Gamma_1(\{y,z\})$ is not a subset of $\Gamma_2(\{y,z\})$.

In fact, the dataset above is neither LA(I)-rationalizable nor LA(II)-rationalizable. Since $x = c_1(\{x,y,z\}) \neq c_1(\{x,z\}) = z$, it follows that $x\mathcal{P}_2 y$. Furthermore, since $c_1(\{x,y\}) = x$ and $c_2(\{x,y\}) = y$, we have $y\mathcal{P}_2 x$. This results in a cycle with respect to $\mathcal{P}$ (and $\mathcal{Q}$), and this implies that the dataset is neither LA(I)-rationalizable nor LA(II)-rationalizable.

5.1 Characterization

Here we answer the following question: what condition on the observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ is necessary and sufficient for LA(III)-rationalizability of $\mathcal{O}$? To do this, we first study what is implied when the observed dataset $\mathcal{O}$ is LA-rationalizable with type-III evolution of attention. Thus assume below that there exist consideration set mappings $(\Gamma_t)_{t\in\mathcal{T}}$ and a strict preference $\succ$ on X that satisfy the definition of LA(III)-rationalizability.

Fix any period $t \in \mathcal{T}$ and constraint set $B \in \Omega$. We can robustly infer that $c_t(B)$, the choice made in B at time t, is preferred to any other alternative in the consideration set $\Gamma_t(B)$. Thus we can say that $c_t(B)$ is revealed preferred to an alternative $x \in B$ if we can infer that x is in the consideration set B at time t. To define revealed preference relations, we must study when we can robustly infer that an alternative is in the consideration set.

The first case where we can surely say that an alternative x attracts attention in constraint set B at time t comes from the evolution of the consideration set mapping: $\Gamma_t(B) \supseteq \{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$. We can see that if an alternative $x \in B$ was actually chosen in reality in a period before t, i.e. $x = c_s(B_s)$ for some $s < t$, then it will attract attention at period t in constraint set B. Note that this is the same as in the type-I evolution model, so we can use the revealed preference relation $\mathcal{P}_1$ defined in equation (1).

Another case when we can surely say that an alternative is in the consideration set comes from the definition of an attention filter. Note that under LA(III)-rationalizability of $\mathcal{O}$, the definition of an attention filter implies that an alternative x attracts attention in constraint set B at time t whenever $c_t(B) \neq c_t(B\backslash\{x\})$. Thus we can define a revealed preference relation $\mathcal{P}_4$ as follows:

$$x\mathcal{P}_4 y \iff {}^{\exists}t \in \mathcal{T}, B \in \Omega \text{ s.th. } \begin{cases} x = c_t(B), \\ y \in B\backslash\{x\}, \text{ and} \\ c_t(B) \neq c_t(B\backslash\{y\}). \end{cases} \tag{4}$$

Summarizing above arguments, we need a revealed preference relation that captures revealed preference relations $\mathcal{P}_1$ and $\mathcal{P}_4$. Thus we define a revealed preference relation $\mathcal{R}$ to be a union of the revealed preference relations above, i.e. $x\mathcal{R}y$ if $x\mathcal{P}_iy$ for some $i \in \{1, 4\}$. In words, this revealed preference relation can be interpreted as follows: for some constraint set B and some time t, the choice $c_t(B)$ is revealed preferred to any alternative y that we can robustly infer to attract attention in B at time t. Let us define $\bar{\mathcal{R}}$ to be the transitive closure of $\mathcal{R}$.

In fact, under LA(III)-rationalizability of $\mathcal{O}$, an alternative x is robustly inferred to be preferred to y if $x\mathcal{R}y$. If we have $x\mathcal{R}y$, then for some constraint set B and time t, we have $x = c_t(B)$ and y in the consideration set. Thus it follows by LA(III)-rationalizability that $x \succ y$ for any preference that rationalizes the dataset. Thus $x\mathcal{R}y$ implies $x \succ y$ for any preference. Then, we see that the acyclicity of revealed preference $\mathcal{R}$ is a necessary condition for LA(III)-rationalizability of observed dataset $\mathcal{O}$. We show that, in fact, the acyclicity of $\mathcal{R}$ is sufficient for LA(III)-rationalizability as well.

Proposition 9. *Observed dataset $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ is LA-rationalizable with type-III evolution of attention if and only if the revealed preference relation $\mathcal{R}$ is acyclic, where $\mathcal{R}$ is the union of the revealed preference relations $\mathcal{P}_1$ and $\mathcal{P}_4$, defined in (1) and (4) respectively.*

Proof. Since necessity has been proved above, we prove sufficiency. Since the revealed preference relation $\mathcal{R}$ is acyclic, there exists an irreflexive, complete, and transitive order extension of $\mathcal{R}$, which we denote by $\succ$.

Now define the consideration set mapping for $t \in \mathcal{T}$ and $B \in \Omega$ as;

$$\Gamma_t(B) = \{c_t(B)\} \cup \{y \in B : c_t(B) \succ y\}.$$

It is clear that we have $c_t(B) \in \Gamma_t(B)$ and $\Gamma_t(B) \subseteq B$. It is also clear from the construction of Γ that $c_t(B)$ is the unique preference maximizer in B; i.e. $c_t(B) \succ x$ for all $x \in \Gamma_t(B)\backslash\{c_t(B)\}$ under the preference $\succ$.

Now we show that $(\Gamma_t)_{t\in\mathcal{T}}$ follows the type-III evolution of attention. Fix any $t \in \mathcal{T}$ and any $B \in \Omega$. For any alternative $y \in \{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\}$ such that $y \neq c_t(B)$, we have $c_t(B)\mathcal{P}_1y$. This implies $c_t(B) \succ y$, and $y \in \Gamma_t(B)$ follows. Thus we have $\Gamma_t(B) \supseteq \left\{B \cap (\cup_{s=1}^{t-1} c_s(B_s))\right\}$.

Finally, we show that Γ_t is an attention filter. Take any $B \in \Omega$, and suppose $x \in B$ and $x \notin \Gamma_t(B)$. Note that if we have $c_t(B) \neq c_t(B\backslash\{x\})$, we must have $c_t(B)\mathcal{P}_4x$, which in turn implies $c_t(B) \succ x$ and $x \in \Gamma_t(B)$, a contradiction. Thus it follows that $c_t(B) = c_t(B\backslash\{x\})$, and we have $\Gamma_t(B) = \Gamma_t(B\backslash\{x\})$. □

5.2 Inference of preference, attention,and inattention

First we characterize when we can robustly infer that an alternative is preferred to another. Recall that binary relation $\mathcal{R}$ is the union of revealed preference relations $\mathcal{P}_1$ and $\mathcal{P}_4$, and $\bar{\mathcal{R}}$ is the transitive closure of $\mathcal{R}$. The intuition is that $x\bar{\mathcal{R}}y$ is a sufficient condition for robust inference of x being preferred to y, given $\mathcal{O}$ is LA(III)-rationalizable. This is true. Moreover, $x\bar{\mathcal{R}}y$ is also a necessary condition for robust inference of preference. We formally show this below.

Proposition 10. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-III evolution of attention. Then it is robust to infer that an alternative x is preferred to y if and only if $x\bar{\mathcal{R}}y$.*

Proof. We first prove sufficiency. Let $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$ rationalize $\mathcal{O}$. By definition of $x\bar{\mathcal{R}}y$, we have a sequence of alternatives $(x^1, \ldots, x^m)$ such that $x = x^m\mathcal{R}x^{m-1}\mathcal{R}\cdots\mathcal{R}x^2\mathcal{R}x^1 = y$. That is, for all $k \in \{1, \ldots, m-1\}$ we have $x^{k+1}\mathcal{P}_i x^k$ for $i \in \{1, 4\}$. Then there exist $t \in \mathcal{T}$ and $B \in \Omega$ such that $c_t(B) = x^{k+1}$ and $x^k \in \Gamma_t(B)\backslash\{x^{k+1}\}$. By LA(III)-rationalizability, we have $x^{k+1} \succ x^k$. Thus we have $x \succ y$, and we see that x is preferred to y.

Next we prove necessity by showing the contrapositive. Suppose that $x\bar{\mathcal{R}}y$ does not hold. Then there exists a preference that is an extension of $\bar{\mathcal{R}}$ and ranks y better than x. The proof of LA(III)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference. Thus it is not robust to infer that x is preferred to y. □

Now we characterize the cases where we can surely say that the agent pays attention, or does not pay attention to an alternative, provided that the observed dataset is rationalizable.

First we study when we can robustly infer that an alternative is not in a consideration set.

Proposition 11. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-III evolution of attention. Then it is robust to infer that an alternative x does not to attract attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if and only if $x\bar{\mathcal{R}}c_t(B)$.*

Proof. We first prove sufficiency. Assume that $x\bar{\mathcal{R}}c_t(B)$. Then under LA(III)-rationalizability, we have $x \succ c_t(B)$. Thus, we cannot have $x \in \Gamma_t(B)$.

Next we prove necessity by showing the contrapositive. Assume that $x\bar{\mathcal{R}}c_t(B)$ does not hold for $x \in B\backslash\{c_t(B)\}$. Then, choose a preference $\succ$ that is an extension of $\mathcal{R}$ and ranks $c_t(B)$ above x. Then the proof of LA(III)-rationalizability shows that $\mathcal{O}$ can be rationalized by such a preference and a family of consideration set mappings $(\Gamma_s)_{s\in\mathcal{T}}$ such that $x \in \Gamma_t(B)$. □

Recall that we discussed in section 5.1 two cases where an alternative surely attracts attention in some constraint set. Here we study further the inference of attention. While the necessary condition for robust inference of attention is an open problem, we give a sufficient condition, which is shown in the proposition below.

PROPOSITION 12. *Let $\mathcal{O} = \{(c_t, B_t)\}_{t\in\mathcal{T}}$ be LA-rationalizable with type-III evolution of attention. Then it is robust to infer that an alternative x attracts attention in $B \in \Omega$ at time $t \in \mathcal{T}$ if there exists $B' \in \Omega$ (possibly equal to B) such that (a) and (b) holds, where;*

(a) $c_t(B') \neq c_t(B'\backslash\{x\})$ **or** $x \in B' \cap (\cup_{s=1}^{t-1} c_s(B_s))$,

(b) $y\bar{\mathcal{R}}c_t(B)$ *for all* $y \in B\backslash B'$ **and** $z\bar{\mathcal{R}}c_t(B')$ *for all* $z \in B'\backslash B$.

Proof. Let $\mathcal{O}$ be LA(III)-rationalized by $((\Gamma_t)_{t\in\mathcal{T}}, \succ)$. Fix any $B \in \Omega$, and take any $B' \in \Omega$ that satisfies (a) and (b). Recall from the discussion in section 5.1 that (a) implies that $x \in \Gamma_t(B')$. If $B' = B$ the proof is complete, since (b) is trivially satisfied in this case. Hence assume below that $B' \neq B$. Note that it follows from Proposition 11 that $y\bar{\mathcal{R}}c_t(B)$ implies $y \notin \Gamma_t(B)$. Thus "$y\bar{\mathcal{R}}c_t(B)$ for all $y \in B\backslash B'$" implies "$y \notin \Gamma_t(B)$ for all $y \in B\backslash B'$," which in turn implies $\Gamma_t(B) = \Gamma_t(B \cap B')$. The last part follows from the attention filter property and the fact that $B\backslash(B\backslash B') = B \cap B'$. Analogously, "$z\bar{\mathcal{R}}c_t(B')$ for all $z \in B'\backslash B$" implies $\Gamma_t(B') = \Gamma_t(B \cap B')$. Thus we have $\Gamma_t(B) = \Gamma_t(B')$, and $x \in \Gamma_t(B)$ follows. □

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