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Random reduction consistency of Weber set and the core*

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Abstract

We provide an axiomatic foundation to Weber set on the set of all TU cooperative games. It can be shown that consistency conditions on probabilistic assessments of a reduction process, which we call the *random reduction consistency*, together with some other known conditions characterise Weber set. The result is beyond the mere mathematical characterisation on which any previous study on Weber set focused, and allows us to regard Weber set as a solution concept. We also provide similar characterisation for the core and compare their differences.

KEYWORDS: Weber set; The core; TU game; Consistency; Characterisation

1 Introduction

Although the core is widely accepted as one of the most prominent solution concepts in cooperative game theory, it has a drawback, that is, the core frequently becomes an empty set. On the other hand, Weber set is always non empty. Moreover, it is a superset of the core. [1] In this respect, Weber set can be a substitute of the core when it is an empty set.

Curiously, no previous study regards Weber set as a solution concept. Instead, Weber set is used as a tool to characterise a certain class of games. For example, Shapley [7] and Ichiishi [3] use Weber set to characterise a class of convex games; a game is convex if and only if

*Quite preliminary: please do not circulate without permission from the authors

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[1]Monderer, Samet and Shapley [4] shows more strengthening result; Weber set always contains the set of weighted values, which is a superset of the core.

its core and Weber set coincides. Another study on Weber set investigates its mathematical properties; Sagara [6] characterises Weber set in terms of Clarke superdifferentials.

In the present study, we provide an axiomatic characterisation of Weber set. It can be shown that consistency requirements on probabilistic assessments of a reduction process, which we call *random reduction consistency*, together with some other known conditions characterise Weber set. This result demonstrates that Weber set possesses a certain consistency requirement and allows us to regard Weber set as a solution concept.

In our characterisation, we avoid the use of non-emptieness property. Instead we require a solution to be non-empty on certain class of games, which we call zero non-positive games. In this fashion, like Hwang and Sudhölter [2], we can provide a characterisation of the core via similar random reduction consistency on entire class of games, not on balanced games. Consequently, we obtain a characterisation of the core which is comparable to above mentioned characterisation of Weber set.

The arrangement of this paper is as follows. We start basic definitions and notations in Section 1. Our main result is given in Section 2, which characterises Weber set. We also show similar characterisation for the core in Section 3.

2 Preliminary

Let $\mathcal{N}$ be a set of players. A finite subset N of $\mathcal{N}$ is called a *coalition*. A *coalition function* on N is a set function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. We call a pair (N, v) as a *game*. In this case, we call N as the *grand coalition*. The set of all games is denoted as Γ.

For a given coalition N and $T \subseteq N$, the *unanimity game of* T, denoted as (N, u_T) is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If we write Γ_N as the all games whose grand coalition is N, it is well known that $(u_T)_{T \subseteq N}$ forms a base of Γ_N.

For $x \in \mathbb{R}^N$ we write $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$. For any game $(N, v) \in \Gamma$, $\alpha > 0$, and $\beta \in \mathbb{R}^N$, we define $(N, \alpha v + \beta) \in \Gamma$ as $(\alpha v + \beta)(S) = \alpha v(S) + \beta(S)$ for each $S \subseteq N$.

A *preimputation* of a game (N, v) is $x \in \mathbb{R}^N$ which satisfies $x(N) = v(N)$. We denote the set of all preimputations of a game (N, v) as $PI(N, v)$. A *solution* is a mapping which prescribes a subset of $PI(N, v)$ to each game (N, v).

One of the two solutions of our interest is the core which is defined as follows.

Definition 2.1. The *core* $C(N, v)$ of a game (N, v) is defined as

$$C(N, v) = \{x \in PI(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}. \tag{2.2}$$

Let us fix some complete, asymmetric, transitive binary relation $\prec$ on $\mathcal{N}$. For $i, j \in \mathcal{N}$, $i \prec j$ means that i is a *predecessor* of j. With slight abuse of notation, for a coalition N, we also denote the restriction of $\prec$ on N as $\prec$. A bijective mapping $\pi : N \rightarrow N$ is called a *permutation* of N. The set of all permutations of N is denotes as $\Pi(N)$. For $\pi \in \Pi(N)$, $f(\pi)$ is the first player with respect to π, that is

$$f(\pi) = i \iff \pi(i) \prec \pi(j) \text{ for all } j \neq i. \tag{2.3}$$

For $\pi \in \Pi(N)$ and $i \in N$, we denote all predecessors of i after the permutation as S_i^π, that is

$$S_i^\pi = \{j \in N \mid \pi(j) \prec \pi(i)\}. \tag{2.4}$$

Then, the *marginal contribution* $m_i^\pi(N, v)$ of a player i is defined as

$$m_i^\pi(N, v) = v(S_i^\pi \cup \{i\}) - v(S_i^\pi). \tag{2.5}$$

We call $m^\pi(N, v) = (m_i^\pi(N, v))_{i \in N}$ as the *marginal vector*. Another solution of our interest is Weber set which is defined as follows.

Definition 2.2. *Weber set* $W(N, v)$ of a game (N, v) is defined as

$$W(N, v) = co\{m^\pi(N, v) \mid \pi \in \Pi(N)\} \tag{2.6}$$

where for a set A, coA stands for convex hull of A.

3 A characterisation of Weber set

Definition 3.1. Let $(N, v) \in \Gamma$ and $i \in N$. The *marginal game* $(N \setminus \{i\}, v^M)$ is defined as [2]

$$v^M(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(T \cup \{i\}) - v(\{i\}) & \text{if } T \subseteq N \setminus \{i\}, T \neq \emptyset. \end{cases} \tag{3.1}$$

[2]From a game $(N, v) \in \Gamma$, we can construct $|N|$ different marginal games. In this respect, we should write $v^M_{N \setminus \{i\}}$ to indicate which player is reduced. However we omit this subscript for notational convenience. Instead, we always write a game as a pair of a coalition and a coalitional function, so any confusion can be avoided.

In what follows, we will introduce two properties which are key to our characterisation of Weber set. Note that for any finite set A, we denote $\Delta(A) = \{p \in [0,1]^A \mid \sum_{i \in A} p_i = 1\}$.

Definition 3.2. A solution σ satisfies the *M-Random reduction consistency (M-RRC)* if for any $(N, v) \in \Gamma$ with $|N| \geq 2$ and $x \in \sigma(N, v)$, there exists $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N\setminus\{i\}, v^M)$ for each $i \in N$ such that

$$x_i = \sum_{j \neq i} p_j y_i^{N\setminus\{j\}} + p_i v(\{i\}), \text{ for all } i \in N. \tag{3.2}$$

The requirements of M-RRC is intended to describe the following situation. Suppose that a player $i \in N$ is decided to be reduced from a game (N, v) with having $v(\{i\})$ of payoff for a compensation of such reduction. Then players $N \setminus \{i\}$ allocate the remaining payoff $v^M(N \setminus \{i\}) = v(N) - v(\{i\})$ according to σ, so that an allocation $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^M)$ is realised. If players do not know who is reduced from the game in advance, but know each player is possibly reduced with the probability according to $p \in \Delta(N)$, then the solution for the original game (N, v) should respect the fact that each player can be reduced. Hence M-RRC requires each allocation $x \in \sigma(N, v)$ of original game can be written as the expected value of such random reduction-and-allocation process.

Note that if we restrict $p \in \Delta(N)$ in the requirement of M-RRC as $p_i = 1$ for some $i \in N$ and $p_j = 0$ for $j \neq 0$, which corresponds to the situation that player i is reduced for sure, then M-RRC coincides with the usual reduced game property.

Lemma 3.1. *Weber set W satisfies M-RRC.*

Proof. Let $(N, v) \in \Gamma$ with $|N| \geq 2$ and $x \in W(N, v)$. Since $x \in W(N, v)$, there exists $t \in \Delta(\Pi(N))$ such that

$$x = \sum_{\pi \in \Pi(N)} t_\pi m^\pi(N, v). \tag{3.3}$$

For any $i \in N$, we define p_i by

$$p_i = \sum_{\pi \in \Pi(N): f(\pi) = i} t_\pi. \tag{3.4}$$

Note that $\sum_{i=1}^n p_i = 1$, that is, $p = (p_i)_{i \in N} \in \Delta(N)$. For any $\pi \in \Pi(N)$ and $i \in N$, let us define $\pi_i \in \Pi(N \setminus \{i\})$ as

$$\pi_i(j) \prec \pi_i(k) \iff \pi(j) \prec \pi(k) \text{ for all } j, k \in N \setminus \{i\}. \tag{3.5}$$

Notice that π_i is well defined and $\Pi(N \setminus \{i\}) = \{\pi_i \mid \pi \in \Pi(N),\ f(\pi) = i\}$. We define $y^{N\setminus\{i\}}$ by

$$y^{N\setminus\{i\}} = \sum_{\pi\in\Pi(N):f(\pi)=i} \frac{t_\pi}{p_i} m^{\pi_i}(N \setminus \{i\}, v^M). \tag{3.6}$$

Since $\sum_{\pi\in\Pi(N):f(\pi)=i} \frac{t_\pi}{p_i} = 1$, $y^{N\setminus\{i\}} \in W(N \setminus \{i\}, v^M)$. Note that, for $i \in N$ and $\pi \in \Pi(N)$, if $f(\pi) \neq i$, we have $m_i^\pi(N, v) = m_i^{\pi_{f(\pi)}}(N \setminus \{f(\pi)\}, v^M)$, and if $f(\pi) = i$, we have $m_i^\pi(N, v) = v(\{i\})$. Consequently, for each $i \in N$, we have,

$$\begin{aligned}
x_i &= \sum_{\pi\in\Pi(N)} t_\pi m_i^\pi(N, v) && (3.7)\\
&= \sum_{j\neq i} \sum_{\pi\in\Pi(N):f(\pi)=j} t_\pi m_i^\pi(N, v) + \sum_{\pi\in\Pi(N):f(\pi)=i} t_\pi m_i^\pi(N, v) && (3.8)\\
&= \sum_{j\neq i} \sum_{\pi\in\Pi(N):f(\pi)=j} t_\pi m_i^{\pi_j}(N \setminus \{j\}, v^M) + \sum_{\pi\in\Pi(N):f(\pi)=i} t_\pi v(\{i\}) && (3.9)\\
&= \sum_{j\neq i} p_j \sum_{\pi\in\Pi(N):f(\pi)=j} \frac{t_\pi}{p_j} m_i^{\pi_j}(N \setminus \{j\}, v^M) + p_i v(\{i\}) && (3.10)\\
&= \sum_{j\neq i} p_j y_i^{N\setminus\{j\}} + p_i v(\{i\}), && (3.11)
\end{aligned}$$

so the claim is proven. □

The next property is the converse of M-RRC.

Definition 3.3. A solution σ satisfies the *M-Converse random reduction consistency (M-CRRC)* if the following property holds; Let $(N, v) \in \Gamma$ with $|N| \geq 2$ and $x \in PI(N, v)$. If we can find $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^M)$ for each $i \in N$ so that $x_i = \sum_{j\neq i} p_i y_i^{N\setminus\{j\}} + p_i v(\{i\}))$ for all $i \in N$, then we have $x \in \sigma(N, v)$.

Interpretation of M-CRRC is just the converse of M-RRC. It requires that an preimputation $x \in PI(N, v)$ can be constructed as the expected value of a random reduction-and-allocation process, then it should be a solution of the original game.

Lemma 3.2. *Weber set W satisfies M-CRRC.*

Proof. Let $(N, v) \in \Gamma$ with $|N| \geq 2$ and $x \in PI(N, v)$. Suppose that we have found $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^M)$ for each $i \in N$ so that $x_i = \sum_{j\neq i} p_i y_i^{N\setminus\{j\}} + p_i v(\{i\})$ for all $i \in N$. Take any player $i \in N$. Since $y^{N\setminus\{i\}} \in W(N \setminus \{i\}, v^M)$, there exists $t^i \in \Delta(\Pi(N \setminus \{i\}))$ such that

$$y^{N\setminus\{i\}} = \sum_{\pi\in\Pi(N\setminus\{i\})} t_\pi^i m^\pi(N \setminus \{i\}, v^M). \tag{3.12}$$

We define $\bar{t} \in \Delta(\Pi(N))$ as follows: for any $\pi \in \Pi(N)$ with $f(\pi) = i$, set

$$\bar{t}_\pi = p_i t^i_{\pi_i}. \tag{3.13}$$

Note that

$$\sum_{\pi \in \Pi(N)} \bar{t}_\pi = \sum_{i \in N} \sum_{\pi \in \Pi(N): f(\pi)=i} p_i t^i_\pi = \sum_{i \in N} p_i = 1. \tag{3.14}$$

In addition,

$$\begin{aligned}
\sum_{\pi \in \Pi(N)} \bar{t}_\pi m^\pi_i(N, v) &= \sum_{j \neq i} \sum_{\pi \in \Pi(N): f(\pi)=j} p_j t^j_{\pi_j} m^\pi_i(N, v) + \sum_{\pi \in \Pi(N): f(\pi)=i} p_i t^i_{\pi_i} m^\pi_i(N, v) && (3.15)\\
&= \sum_{j \neq i} p_j \sum_{\pi \in \Pi(N): f(\pi)=j} t^j_{\pi_j} m^{\pi_j}_i(N \setminus \{j\}, v^M) + p_i \sum_{\pi \in \Pi(N): f(\pi)=i} t^i_{\pi_i} v(\{i\}) && (3.16)\\
&= \sum_{j \neq i} p_j y^{N \setminus \{j\}}_i + p_i v(\{i\}) && (3.17)\\
&= x_i, && (3.18)
\end{aligned}$$

for each $i \in N$. Therefore we have that $x \in W(N, v)$. □

The following property is a slight modification of zero inessential property (ZIG) introduced in Hwang and Sudhölter [2]. We say a game $(N, v) \in \Gamma$ is *zero non-positive* if

- $v(\{i\}) = 0$ for all $i \in N$,
- $v(S) \leq 0$ for $S \subsetneq N$ with $|S| \geq 2$, and
- $v(N) = 0$.

Definition 3.4. A solution σ satisfies *Zero non-positive game property (ZNP)* if for any zero non-positive game $(N, v) \in \Gamma$, we have $\sigma(N, v) \neq \emptyset$.

Since Weber set is always nonempty, it trivially satisfies ZNP.

Note that if we restrict our attention to only two-person zero non-positive games, then ZNP coincides with ZIG. In Hwang and Sudhölter [2], a characterisation of the core by employing ZIG is provided. The advantage of characterisation using ZIG is that it allows us to drop non-emptieness property on the domain of games under the consideration. Consequently, a characterisation of the core on entire games, not just on the class of balanced games, is possible. As a result, comparison of the properties of the core and prenucleolus can be made.

Analogically, by employing ZNP, we can provide characterisation of both the core and Weber set on entire domain of games, so that comparison of the properties they possess can be accomplished.

The following property is quite common (see, for example, Peleg and Sudhölter [5]).

Definition 3.5. A solution σ satisfies *covariance (COV)* if for any $(N, v) \in \Gamma$ with $\sigma(N, v) \neq \emptyset$, $\alpha > 0$, and $\beta \in \mathbb{R}^N$, we have

$$\sigma(N, \alpha v + \beta) = \{\alpha x + \beta \mid x \in \sigma(N, v)\}. \tag{3.19}$$

Since Weber set is the convex hull of marginal vectors, which are linear, it trivially satisfies COV.

We are now in position to state our main result.

Theorem 3.1. *A solution σ satisfies ZNP, COV, M-RRP and M-CRRP if and only if $\sigma = W$.*

Proof. We already have shown the if part.

We prove only if part. Let σ be a solution which satisfies all the properties listed in the statement. Consider the game $(\{i\}, v^0) \in \Gamma$ defined as $v^0(\{i\}) = 0$. By ZNP, we have $\sigma(\{i\}, v^0) \neq \emptyset$. Since a solution prescribes a subset of Pareto optimal allocation, and only Pareto optimal allocation for $(\{i\}, v^0)$ is 0, we have that $\sigma(\{i\}, v^0) = \{0\}$. From COV, for any 1-person game, we have $\sigma(\{i\}, v) = \{v(\{i\})\} = W(N, v)$.

Assume that the result holds for $(N, v) \in \Gamma$ with $|N| \leq n$ $(n \geq 1)$. We show the result holds for $(N, v) \in \Gamma$ with $|N| = n + 1$.

We prove that $\sigma(N, v) \subseteq W(N, v)$. Let $x \in \sigma(N, v)$. From M-RRC, there exists $p \in \Delta(N)$ and $y^{N \setminus \{i\}} \in \sigma(N \setminus \{i\}, v^M)$ for each $i \in N$ such that $x_i = \sum_{j \neq i} p_j y^{N \setminus \{i\}} + p_i v(\{i\})$ for all $i \in N$. From the induction hypothesis, $y^{N \setminus \{i\}} \in W(N \setminus \{i\}, v^M)$. From M-CRRC of W, we have $x \in W(N, v)$. The proof of $\sigma(N, v) \supseteq W(N, v)$ can be obtained in a parallel manner. $\square$

4 A characterisation of the core

In this subsection, we provide a characterisation of the core by using the random reduction consistency. The following two games have been used for characterisations of the core in the literature (see Funaki and Yamato [1]).

Definition 4.1. Let $(N, v) \in \Gamma$ be a game, $i \in N$ be a player, and $x \in PI(N, v)$ be a preimputation. The *complement game* $(N \setminus \{i\}, v_x^C)$ of v is

$$v_x^C(S) = \begin{cases} v(S \cup \{i\}) - x_i & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset. \end{cases} \tag{4.1}$$

We shall omit x and write v^C when context makes clear.

Definition 4.2. Let $(N, v) \in \Gamma$ be a game, $i \in N$ be a player, and $x \in PI(N, v)$ be a preimputation. The *projection game*, denoted as $(N \setminus \{i\}, v_x^P)$, of v is

$$v_x^P(S) = \begin{cases} v(N) - x_i & \text{if } S = N \setminus \{i\} \\ v(S) & \text{if } S \subsetneq N \setminus \{i\}. \end{cases} \tag{4.2}$$

We shall omit x and write v^P when context makes clear.

We provide a variant of the random reduction consistency, which is analogous to Definition 3.2, by employing the above two reduced games.

Definition 4.3. A solution σ satisfies the *CP-Random reduction consistency (CP-RRC)* if for any $(N, v) \in \Gamma$ with $n \geq 3$, and $x \in \sigma(N, v)$, there exists $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^C) \cap \sigma(N \setminus \{i\}, v^P)$ such that $x_i = \sum_{j \neq i} p_j y_i^{N\setminus\{j\}} + p_i x_i$ for all $i \in N$.

Interpretation for CP-RRC is similar to those for M-RRC.

The following result is an immediate consequence that the core satisfies two reduced game properties, one uses the complement game and the other used the projection game (see Funaki and Yamato [1]).

Lemma 4.1. *The Core C satisfies CP-RRC.*

We also give the converse of CP-RRC, just analogous to Definition 3.3

Definition 4.4. A solution σ satisfies the *CP-Converse random reduction consistency (CP-CRRC)* if the following property holds; Let $(N, v) \in \Gamma$ with $|N| \geq 3$ and $x \in PI(N, v)$. If we can find $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^C) \cap \sigma(N \setminus \{i\}, v^P)$ for each $i \in N$ so that $x_i = \sum_{j \neq i} p_i y_i^{N\setminus\{j\}} + p_i x_i$ for all $i \in N$, then we have $x \in \sigma(N, v)$.

Note that, in Definition 4.3 and Definition 4.4, we only require the property holds for games with $|N| \geq 3$. This is because in the case of $|N| = 2$, the definition requires $x_1 = y_1^2$, which deviates from the original intension of random reduction consistency.

Lemma 4.2. *The core C satisfies CP-CRRC.*

Proof. Let $(N,v) \in \Gamma$ with $|N| \geq 3$ and $x \in PI(N,v)$. Suppose that we have found $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in C(N \setminus \{i\}, v^C) \cap C(N \setminus \{i\}, v^P)$ for each $i \in N$ so that $x_i = \sum_{j \neq i} p_i y_i^{N\setminus\{j\}} + p_i x_i$ for all $i \in N$. Our objective is to show $x \in C(N,v)$.

First, for a singleton coalition S, say $S = \{i\}$, we have

$$x_i = \sum_{j \neq i} \frac{p_j}{1-p_i} y_i^{N\setminus\{j\}} \tag{4.3}$$

$$\geq \sum_{j \neq i} \frac{p_j}{1-p_i} v^P(\{i\}) \tag{4.4}$$

$$= \sum_{j \neq i} \frac{p_j}{1-p_i} v(\{i\}) = v(\{i\}), \tag{4.5}$$

where the first equality holds by our assumption, the inequality holds because $y^{N\setminus\{j\}} \in C(N \setminus \{j\}, v^P)$. On the other hand, for a coalition of size $|S| = |N| - 1$, say $S = N \setminus \{i\}$, then we have

$$x_i = \sum_{j \neq i} \frac{p_j}{1-p_i} y_i^{N\setminus\{j\}} \tag{4.6}$$

$$\leq \sum_{j \neq i} \frac{p_j}{1-p_i} \left(v^C(N \setminus \{j\}) - v^C(N \setminus \{i,j\}) \right) \tag{4.7}$$

$$= \sum_{j \neq i} \frac{p_j}{1-p_i} (v(N) - x_j - v(N \setminus \{i\}) + x_j) \tag{4.8}$$

$$= v(N) - v(N \setminus \{i\}), \tag{4.9}$$

where the inequality holds because $y^{N\setminus\{j\}} \in C(N \setminus \{j\}, v^C)$. [3] Since $x \in PI(N,v)$ we have $v(N) = x(N)$, which, together with the above inspection, implies $v(N \setminus \{i\}) \leq x(N \setminus \{j\})$.

[3] Indeed, we have

$$y_i^{N\setminus\{j\}} = y^{N\setminus\{j\}}(N \setminus \{j\}) - y^{N\setminus\{j\}}(N \setminus \{i,j\}) \leq v^C_{N\setminus\{j\}}(N \setminus \{j\}) - v^C_{N\setminus\{j\}}(N \setminus \{i,j\}),$$

for each i.

Finally, for $S \subseteq N$ with $2 \le |S| \le |N| - 2$, we have

$$v(S) - x(S) = v(S) - \sum_{i \in S} \left(\sum_{j \neq i} p_j y_i^{N \setminus \{j\}} + p_i x_i \right) \tag{4.10}$$

$$= v(S) - \sum_{i \in S} \left(\sum_{j \in S \setminus \{i\}} p_j y_i^{N \setminus \{j\}} + \sum_{j \in N \setminus S} p_j y_i^{N \setminus \{j\}} + p_i x_i \right) \tag{4.11}$$

$$= v(S) - \sum_{j \in S} p_j \left(\sum_{i \in S \setminus \{j\}} y_i^{N \setminus \{j\}} + x_j \right) - \sum_{j \in N \setminus S} p_j \sum_{i \in S} y_i^{N \setminus \{j\}} \tag{4.12}$$

$$\le v(S) - \sum_{j \in S} p_j \left(v^C(N \setminus \{j\}) + x_j \right) - \sum_{j \in N \setminus S} p_j v^P(S) \tag{4.13}$$

$$= v(S) - \sum_{j \in S} p_j v(S) - \sum_{j \in N \setminus S} p_j v(S) = 0, \tag{4.14}$$

where the inequality holds because $y^{N \setminus \{j\}} \in C(N \setminus \{j\}, v^C) \cap C(N \setminus \{j\}, v^P)$ for each $j \in N$. Consequently, we conclude that $x \in C(N, v)$. □

The following property is standard.

Definition 4.5. A solution σ satisfies the *Individual rationality (IR)* if for any $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$, we have $x_i \ge v(\{i\})$.

Lemma 4.3. *If a solution σ satisfies ZNP and IR, then $\sigma(N, v) = \{\mathbf{0}\}$ for any zero non-positive game (N, v).*

Proof. $\sigma(N, v) \neq \emptyset$ for any zero non-positive game by ZNP. If $x \in \sigma(N, v)$, then by IR $x \ge \mathbf{0}$. Since $\sigma(N, v) \subseteq PI(N, v)$, $x(N) = 0$. Therefore $x = \mathbf{0}$. □

The following result, which is analogous to Theorem 3.1, shows that there are two major differences between properties that the core possesses and those that Weber set possesses. One is reduced game employed for RRC and CRRC. The other is the presence of IR.

Theorem 4.1. *A solution σ satisfies ZNP, IR, COV, CP-RRC, and CP-CRRC if and only if $\sigma = C$.*

Proof. We have shown that the core C satisfies CP-RRC and CP-CRRC. The core satisfies IR by its definition. It is well known that the core satisfies COV (see Peleg and Sudhölter [5]). It is evident that $\mathbf{0} \in C(N, v)$ for any zero non-positive game (N, v), so the core satisfies ZNP.

Next we show only if part of the theorem. Suppose that a solution σ satisfies all properties stated in the theorem. Notice that for a one-person game (N, v), $\sigma(N, v) = C(N, v)$ can be shown by mimicking corresponding part of the proof of Theorem 3.1. Therefore our task is to demonstrate $\sigma(N, v) = C(N, v)$ holds for more than two-person games.

We fist show $\sigma(N, v) \subseteq C(N, v)$ for all $(N, v) \in \Gamma$. $\sigma(N, v) \subseteq C(N, v)$ for two-person games by IR. [4] Assume that the result holds for (N, v), $|N| \leq n$ with $n \geq 2$, and we prove the result when $|N| = n + 1$.

Take $x \in \sigma(N, v)$. From CP-RRC, there exists $p \in \Delta(N)$ and $y^{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, v^C) \cap \sigma(N \setminus \{i\}, v^P)$ for each $i \in N$ such that $x_i = \sum_{j \neq i} p_j y^{N\setminus\{i\}} + p_i x_i$ for all $i \in N$.

By the induction hypothesis, $\sigma(N \setminus \{i\}, v^C) \subseteq C(N \setminus \{i\}, v^C)$ and $\sigma(N \setminus \{i\}, v^P) \subseteq C(N \setminus \{i\}, v^P)$ for each i, we have $y^{N\setminus\{i\}} \in C(N \setminus \{i\}, v^C) \cap C(N \setminus \{i\}, v^P)$ for each $i \in N$. Since $x \in PI(N, v)$, we can apply CP-CRRC of the Core, so we have $x \in C(N, v)$.

Next we show $C(N, v) \subseteq \sigma(N, v)$ for all $(N, v) \in \Gamma$. Take an arbitrary two-person game (N, v). Let us write $N = \{i, j\}$. Since unanimity games $(u_T)_{T \subseteq N}$ form a base of Γ_N, there exists $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ such that $v = \alpha u_N + \beta_1 u_{\{i\}} + \beta_2 u_{\{j\}}$. If we set $\beta = (\beta_1, \beta_2)$, it is evident that $v = \alpha u_N + \beta$. By COV, we have

$$C(N, v) = C(N, \alpha u_N) + \beta. \tag{4.15}$$

Since σ also satisfies COV, $C(N, v) \subseteq \sigma(N, v)$ if and only if $C(N, \alpha u_N) \subseteq \sigma(N.\alpha u_N)$. Therefore, for our purpose, it is sufficient to show that $C(N, \alpha u_N) \subseteq \sigma(N, \alpha u_N)$ for $\alpha \in \mathbb{R}$.

If $\alpha < 0$, $C(N, \alpha u_N)$ becomes an empty set, so the result holds trivially. When $\alpha = 0$, $(N, \alpha u_N)$ becomes a zero non-positive game, so $C(N, \alpha u_N) = \{\mathbf{0}\} = \sigma(N, \alpha u_N)$ by Lemma 4.3.

Assume that $\alpha > 0$. Take $x \in C(N, \alpha u_N)$. Note that $x \geq \mathbf{0}$. Let (N, v^0) be a game such that $N = \{i, j\}$, $v^0(S) = 0$ for all $S \subseteq N$. Again, by Lemma 4.3, we have $\sigma(N, v^0) = \{\mathbf{0}\}$. It follows that

$$\sigma(N, x_1 u_{\{1\}} + x_2 u_{\{2\}}) = \sigma(N, v^0) + (x_1, x_2) = \{(x_1, x_2)\} \tag{4.16}$$

from COV.

[4] Note that $\sigma(N, v) \subseteq PI(N, v)$.

Let $M = N \cup \{k\}$. Define a game (M, w) by

$$w(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } 3 \\ -x(S \cap N) & \text{if } |S| = 2. \end{cases} \tag{4.17}$$

Since (M, w) is a zero non-positive game, we have $\sigma(M, w) = \{\mathbf{0}\}$ by Lemma 4.3. Define $\bar{w} : 2^M \to \mathbb{R}$ as $\bar{w} = w + (x_i, x_j, 0)$. Then by COV, we have $\sigma(M, \bar{w}) = \{(x_i, x_j, 0)\}$. By CP-RRC, there exists $p \in \Delta(M)$ and $y \in \sigma(N, \bar{w}^P) \cap \sigma(N, \bar{w}^C)$, $y' \in \sigma(M \setminus \{j\}, \bar{w}^P) \cap \sigma(M \setminus \{j\}, \bar{w}^C)$, $y'' \in \sigma(M \setminus \{i\}, \bar{w}^P) \cap \sigma(M \setminus \{i\}, \bar{w}^C)$ such that

$$x_i = p_i x_i + p_j y'_i + p_k y_i, \tag{4.18}$$

$$x_j = p_i y''_j + p_j x_j + p_k y_j, \tag{4.19}$$

$$0 = p_i y''_k + p_j y'_k. \tag{4.20}$$

Note that $(N, \bar{w}^P) = (N, x_i u_{\{i\}} + x_j u_{\{j\}})$. By (4.16), $\sigma(N, \bar{w}^P) = \{(x_i, x_j)\}$. Therefore $y = (x_i, x_y)$. It follows that $(x_1, x_2) = y \in \sigma(N, \bar{w}^C) = \sigma(N, \alpha u_N)$.

For games with $|N| \geq 3$, the assertion can be shown similar way as we have done to show $\sigma(N, v) \subseteq C(N, v)$ with $|N| \geq 3$. $\square$

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