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Testable implications of potential games*

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Abstract

In this paper, we study exact and ordinal potential games from the viewpoint of testable implications. In general, a direct product of the feasible sets of agents is said to be a joint action set. A collective choice function defined on all possible joint action sets is Nash rationalizable, if for every joint action set, the value of it can be seen as the set of Nash equilibria. We provide two sets of necessary and sufficient conditions for a given collective choice function to be rationalized by some potential game: one requires the existence of a congruent selection from a joint choice function, while the other requires the cyclical consistency on a type of aggregate revealed preference. Our result also shows that if a joint choice function is consistent with Nash equilibrium of some potential game, then it cannot be consistent with Nash equilibrium of any non-potential game.

KEYWORDS: Revealed preference; Potential games; Testable implications; Collective choice function; Rationalizability

JEL CLASSIFICATION: C72, D71

1 Introduction

The general treatments on the revealed preference test for Nash equilibrium was first studied by Sprumont (2000), and Ray and Zhou (2001). They answered the question; what are the conditions for a given collective choice function to be consistent with the set of Nash equilibria or Subgame perfect Nash equilibria? A more recent paper by Lee (2012) posed and answered the same question for the case of zero-sum games. In this paper, as another important subclass of noncoorperative games, we aim to clarify a necessary and sufficient condition for a collective choice function to be consistent with the Nash equilibria of some potential game.

The notion of potential games is introduced by Monderer and Shapley (1996a), and it is well-known that potential games have a number of attractive properties.[1] Amongst others, in a potential game, the preference (or payoff function) of each agent can be replaced with a single real-valued function (so called a *potential function*), and the maximization of such a function leads to a subset of Nash equilibria of

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[1]There are some variations of the potential games introduced by Monderer and Shanpley (1996a). One can find them, for instance, in Voonerveld (2000), Dubey et al. (2006), and Uno (2007).

the game. Furthermore, it is known that the maximization of a potential function derives the robust equilibrium, provided that the equilibrium is unique (Ui, 2001). In addition, the existence of a potential function ensures the “Fictitious Play Property,” ensuring the convergence of one-side better learning (see Monderer and Shapley, 1996a and 1996b).

To deal with the testable implications of potential games, the properties argued above play key roles. Let us assume that a given choice function is consistent with Nash equilibrium of some potential game. Then, it must contain the set of maximizers of a potential function on every feasible set. Recalling the classical result by Richter (1966), such a selection must satisfy the “congruence axiom,” which is a kind of acyclicity of the observed choices. As long as the consistency with Nash equilibrium is assured, this is obviously a necessary condition for the existence of a potential function, while the sufficiency is not clear. We show that the existence of such a selection *is* sufficient as well. In addition, we also show another characterization for the consistency between a joint choice function and potential games, which is based on the characterization of Nash rationalizability by Galambos (2012). He showed that an extended congruence axiom on the individual revealed preferences is necessary and sufficient for Nash rationalizability of a joint choice function. We show that, in addition to Galambos' condition, a stronger acyclicity is required to find a potential game that rationalizes the observed choice function: the union of revealed preferences of agents must be acyclic.

In verifying the existence of a potential game that rationalizes a choice function, some additional results follow. First, our conditions are respectively common necessary and sufficient conditions for the consistency with two different types of potential games—*ordinal potential games* and *exact potential games.* This implies that these two-types of potential games are “observationally” equivalent, while theoretically not (the former contains the latter as a special case). In addition, it turns out that if a choice function is consistent with Nash equilibrium of some potential game, then it cannot be consistent with Nash equilibrium of any non-potential game.

The rest of this paper is arranged as follows. In section 2, we briefly introduce the notion of potential games. Subsequent to it, in section 3, we show the main result of this paper. In particular, in section 3.1, we argue the consistency between a choice function and Nash equilibrium, and, in section 3.2, the additional conditions for potential games are introduced.

2 Potential games

Let $N = \{1, ..., n\}$ be the set of agents, and X_i be a finite set of *conceivable actions* of agent i. We denote the set of *joint actions* by $X = \times_{i=1}^{n} X_i$. In addition, for the notational simplicity, we let $J := N \backslash \{i\}$ be the set of all agents other than agent i. If each agent has a complete and transitive preference $\succeq_i$ on X, or equivalently, a payoff function $u_i(x_i, x_J)$, we can consider a game $G = [N, (X_i)_{i \in N}, (u_i)_{i \in N}]$.

We say that a function $\varphi : X \to \mathbb{R}$ is an *ordinal potential* for a game G, if for all agents $i \in N$, and for all $x_J \in X_J$, $u_i(x_i, x_J) \geq u_i(y_i, x_J)$ if and only if $\varphi(x_i, x_J) \geq \varphi(y_i, x_J)$ for all $x_i, y_i \in X_i$. In addition, a function φ is said to be an *exact potential* for a game G if it satisfies, for all $i \in N$, and for all $x_J \in X_J$, $u_i(x_i, x_J) - u_i(y_i, x_J) = \varphi(x_i, x_J) - \varphi(y_i, x_J)$ for all $x_i, y_i \in X_i$.

Definition 1. A strategic form game $G = [N,\ (X_i)_{i \in N}, (u_i)_{i \in N}]$ is an *ordinal (exact) potential game* if there exists an ordinal (exact) potential for the game G.

Let us assume that a game $G = [N, (X_i)_{i \in N}, (u_i)_{i \in N}]$ admits an ordinal (exact) potential function φ. Then, it is known that the set of all pure strategy Nash equilibria coincides with that of the coordination game $G' = [N, (X_i)_{i \in N}, (\varphi, \varphi, ..., \varphi)]$, where all agents' payoff functions are replaced by the potential. In addition, the maximization of φ on X derives a subset of the set of all pure strategy Nash equilibria. We formally state this fact for future references;

Proposition 1. (Monderer and Shapley, 1996a) *Let* $NE\,[(X_i)_{i\in N},(u_i)_{i\in N}]$ *be the set of all pure strategy Nash equilibria of the game* $G=[N,(X_i)_{i\in N},(u_i)_{i\in N}]$*. If* G *admits an ordinal (exact) potential function* φ*, then it follows that* $NE\,[(X_i)_{i\in N},(u_i)_{i\in N}]=NE\,[(X_i)_{i\in N},(\varphi,\varphi,\dots,\varphi)]$*, where the latter stands for the set of all pure strategy Nash equilibria of the coordination game* $G'=[N,(X_i)_{i\in N},(\varphi,\varphi,...,\varphi)]$*. Moreover,* $\operatorname{argmax}_{x\in X}\varphi(x)\subset NE\,[(X_i)_{i\in N},(u_i)_{i\in N}]$.

In Voonerveld and Norde (1997) and Monderer and Shapley (1996a), they provided order theoretic characterizations respectively for ordinal and exact potential games as follows. A sequence of action profiles $(x^1,x^2,\dots)\subseteq X$ is said to be a *path* if for any k there exists a unique deviator, i.e. for all k, x^k and x^{k+1} differs in only one coordinate. If a path $(x^1,...,x^m)$ satisfies $x^1=x^m$, then it is referred to as a *cycle*. We say that a sequence of action profiles $(x^1,x^2,\dots)$ is an *improving path* if it is a path, and $u_{i(k)}(x^{k+1})\geq u_{i(k)}(x^k)$ for all $k\in\{1,2,\dots\}$, and there exists at least one k' such that the strict inequality holds, where $i(k)$ is the unique deviator at the k^{th} element. If an improving path is a cycle, then it is referred to as an *improving cycle*.

Proposition 2. (Voonerveld and Norde, 1997) *A game* G *is an ordinal potential game, if and only if there exists no improving cycle.*

Proposition 3. (Monderer and Shapley, 1996a) *A game* G *is an exact potential game, if and only if for every cycle* $(x^1,...,x^m)$*, it follows that* $\sum_{k=1}^m[u_{i(k)}(x^{k+1})-u_{i(k)}(x^k)]=0$.[2]

As seen from the preceding propositions, if a game admits an exact potential then it admits an ordinal potential, but not vice versa. On the other hand, in section 3.2, we will see that these two concepts are "observationally" equivalent in the sense that the "choice based" characterization of them are equivalent with each other.

3 Potential Rationalization

3.1 Choice function and Nash rationalizability

Following Sprumont (2000), we consider that at any period of 'time,' the agents have actions that are actually *feasible* because of technological, legal, or budgetary constraints, which are subsets of the *conceivable* actions X. A feasible set of agent i is denoted by $A_i\subset X_i$. Then, the class of logically possible *joint feasible sets* can be represented as

$$\mathcal{A}:=\{A=\times_{i=1}^n A_i\ :\ \emptyset\neq A_i\subset X_i\ \forall i\in N\}.$$

For any $A,A'\in\mathcal{A}$, let $A\vee A'$ be the smallest element of $\mathcal{A}$ that contains both A and A'. We say $A\in\mathcal{A}$ is a *line* if A_i is singleton for all agents but one. A choice function f is defined on $\mathcal{A}$ such that for every $A\in\mathcal{A}$, $f(A)$ denotes the observed joint choices from set A. In other words, for every $A\in\mathcal{A}$, $f(A)$ is the choice made by the agents in a particular situation where the set A_i was *feasible* for each agent i.

In Sprumont (2000), he showed a necessary and sufficient condition under which an observed choice function f can be seen as Nash equilibria in the folloing sense;

Definition 2. An observed joint choice function f is *Nash rationalizable* if there exist payoff functions u_i $(i=1,...,n)$ on X such that $f(A)$ coincides with the set of Nash equilibria of the strategic form game $G\mid_A=[N,(A_i)_{i\in N},(u_i)_{i\in N}]$ for every $A\in\mathcal{A}$.

[2]They also showed that in fact it suffices to check cycles with the length 4.

Proposition 4. (Sprumont, 2000) *An observed choice function f is Nash rationalizable if and only if it satisfies*

- Persistency under Contraction (PC)
  1. (PC 1): For every $A, A' \in \mathcal{A}$ with $A' \subset A$, $f(A) \cap A' \subset f(A')$.
  2. (PC 2): If A is a line, $A' \subset A$ and $f(A) \cap A' \neq \emptyset$ implies $f(A') \subset f(A)$.
- Persistency under Expansion (PE): For all $A, A' \in \mathcal{A}$, $f(A) \cap f(A') \subset f(A \vee A')$.

As an alternative approach to Nash rationalizability, Galambos (2012) proposed a condition in terms of revealed preference relations that are derived from a choice function. Formally, for each $i \in N$, a relation V_i on X is defined as;

$$xV_iy \iff \text{there exists } A \in \mathcal{A} \text{ such that } x, y \in A,\ x_J = y_J, \text{ and } x \in f(A).$$

In words, x is directly revealed preferred to y for some agent if x is observed in a situation where both x and y were feasible, holding other agents' actions fixed. Let $T(V_i)$ be the transitive closure of V_i, which can be interpreted as an indirect revealed preference.

Proposition 5 (Galambos, 2012). *A choice function f is Nash rationalizable, if and only if it satisfies for every $A \in \mathcal{A}$ and $x \in A$,*

- N-Congruence (NC): For every$i \in N$, and every $y_i \in A_i$, $xT(V_i)\,(y_i, x_J)$ implies $x \in f(A)$.

Condition (NC) requires that if an action profile is indirectly revealed preferred by every agent to any other feasible action, the action profile should be observed via the choice function. Although this might look just like a paraphrase of the definition of Nash equilibrium, it makes the proof for our main theorem much simpler than employing the joint of (PC) and (PE). Needless to say, (NC) is equivalent with the joint of (PC) and (PE) in our setting.[3]

3.2 Potential rationalizability

We say that a choice function f is *potential rationalizable* if there exists a potential game such that the observed choices coincide with the Nash equilibria of it. Formally;

Definition 3. A choice function f is *ordinal (exact) potential rationalizable* if there exist payoff functions u_i $(i = 1, ..., n)$ on X such that $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$ is an ordinal (exact) potential game, and $f(A)$ coincides with the set of Nash equilibria of the strategic form game $G\,|_A = [N, (A_i)_{i\in N}, (u_i)_{i\in N}]$ for every $A \in \mathcal{A}$.

Note that, although the preceding definition only requires the existence of an ordinal (exact) potential function for the game G, it is not difficult to check that if G is an ordinal (exact) potential game, then for every $A \in \mathcal{A}$, the game $G\,|_A$ is also an ordinal (exact) potential game by the same potential function with G. It is clear that potential rationalizability implies Nash rationalizability, but not vice versa (see Example A1 in Appendix). Therefore, we need additional restrictions to say that the agents are interacting in a potential game.

[3]In fact, Condition (NC) is still a necessary and sufficient condition for Nash rationalizability, even if we alleviate the finiteness of X. See Galambos (2012) for details.

First, we consider a necessary condition for the ordinal potential rationalizability by employing the properties of Nash equilibria of potential games. In particular, we use the fact that the maximizers of a potential function must be contained in the set of Nash equilibria. Suppose that a choice function f is ordinal potential rationalizable. Then, by definition, there exists an ordinal potential game G such that the set of Nash equilibria of $G\mid_A$, $NE\left[(A_i)_{i\in N},(u_i)_{i\in N}\right]$ coincides with $f(A)$ for every $A\in\mathcal{A}$. Let φ be an ordinal potential of G, and let $g(A):=\operatorname{argmax}_{x\in A}\varphi(x)$. Then, by Proposition 1, it follows that $g(A)\subset f(A)$ for every A. Since $g(A)$ is the set of maximizers of a real-valued function, it must satisfy the well-known *Congruence Axiom*, which is introduced in the seminal paper by Richter (1966). The formal definition is given as follows. Let us define the binary relation $\mathcal{R}$ on X such that

$$x\mathcal{R}y \iff \text{there exists } A\in\mathcal{A} \text{ such that } x,y\in A \text{ and } x\in g(A).$$

Then, a set function $g(\cdot)$ satisfies the congruence axiom if $\mathcal{R}$ satisfies the cyclical consistency, i.e., for every cycle $(x^1,...,x^m)$, $x^{k+1}\mathcal{R}x^k$ for all $k=1,...,m-1$ implies that $x^k\mathcal{R}x^{k+1}$ for all $k=1,...,m-1$. In addition, if A is a line, then it must hold that $g(A)=f(A)$, which follows from the fact that $\operatorname{argmax}_{x\in A}\varphi(x)=\operatorname{argmax}_{x_i\in A_i}u_i(x_i,x_J)$. Put all together, if a choice function f is ordinal potential rationalizable, it must satisfy;

- Congruent Selection Property (CSP)[4]: There exists a set function g such that
  1. $g(A)\subset f(A)$ for every $A\in\mathcal{A}$
  2. $g(A)=f(A)$, if A is a line
  3. g satisfies the congruence axiom.

Then, in turn, let us consider a necessary condition for the existence of an ordinal potential function, by using a direct revealed preference relation V_i. If a given choice function f is ordinal potential rationalizable, there exists an ordinal potential game G whose potential function is φ. By definition of an ordinal potential, it must follow that if xV_iy, then $\varphi(x)\geq\varphi(y)$, since the former reveals that $u_i(x)\geq u_i(y)$. Since this holds for every $i\in N$, by defining the binary relation V on X such that

$$xVy \iff \text{there exists } i\in N \text{ such that } xV_iy,$$

it follows that $xVy\Longrightarrow\varphi(x)\geq\varphi(y)$. It is clear that for every cycle $(x^1,...,x^m)$ with $x^{k+1}Vx^k$, it must follow that $\varphi(x^k)=\varphi(x^{k+1})$ for every $k=1,...,m-1$. Then, we can conclude that the binary relation V must be acyclic. Formally;

- Collective Acyclicity (CA): If a cycle $(x^1,...,x^m)$ satisfies $x^{k+1}Vx^k$ for every $k=1,...,m-1$, then it also holds that x^kVx^{k+1} for every $k\in\{1,...,m-1\}$.

Now, we have two necessary conditions for the potential rationalizability of a choice function. It should be noted that the conditions (CSP) and (CA) are, in general, independent (see Example A2 in Appendix). However, if a choice function f is Nash rationalizable, then (CSP) and (CA) are equivalent with each other, and they are sufficient for the existence of an ordinal potential. In addition, from the viewpoint of rationalizability, once the existence of an ordinal potential is assured, then the existence of an exact potential also follows.

THEOREM 1. *The following statements are equivalent;*

1. *A choice function f is ordinal potential rationalizable.*
2. *A choice function f is exact potential rationalizable.*

[4] By the second and third requirements, this implies (PC2), while it is independent of (PC1).

3. *A choice function f satisfies (PC), (PE), and (CSP).*
4. *A choice function f satisfies (NC), and (CA).*

PROOF: It is easy to confirm that 1.$\Longrightarrow$ 2. Indeed, if a given choice function is ordinal potential rationalizable, then f coincides with the set of all pure strategy Nash equilibria of some ordinal potential game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$ or its restriction $G\mid_A$. Let φ be an ordinal potential of G. Then, by Proposition 3, the set of Nash equilibria of $G\mid_A$ coincides with the one of the game $G'\mid_A = [N, (A_i)_{i\in N}, (\varphi \ldots \varphi)]$. That is, if G rationalizes the choice function f, then the game G' also does. The latter is obviously an exact potential game whose potential function is equal to φ itself. In addition, 2.$\Longrightarrow$ 3. has been already shown. Therefore, it suffices to show that 3.$\Longrightarrow$ 4. $\Longrightarrow$ 1.

(3.$\Longrightarrow$ 4.) In view of Propositions 4 and 5, it is obvious that f satisfies (NC). Hence, it suffices to show that f satisfies (CA). Since (CSP) holds, there exists a set function g that is congruent and coincides with f on every line. In particular, by the former, $g(A)$ is the set of maximizers of some real-valued function φ. Then, we have $xVy \Longrightarrow \varphi(x) \geq \varphi(y)$ provided that f is Nash rationalizable, which is guaranteed by (PC) and (PE). To verify this, it is sufficient to show there exists a line A with $x, y \in A$ such that $x \in f(A)$. Indeed, by the fact that g is equal to f on every line, we then have $x, y \in A$ and $x \in g(A) = \mathrm{argmax}_{z\in A}\,\varphi(z)$.

Now let us suppose xVy. Assume, by way of contradiction, there is no line A with $x, y \in A$ such that $x \in f(A)$. In particular, since $\{x, y\}$ is a line, we must have $f(\{x, y\}) = \{y\}$. By letting the unique deviator in $\{x, y\}$ as i, this implies $u_i(y) > u_i(x)$, for any utility function which can be used to rationalize f. On the other hand, since we have assumed xVy, there exists $B \in \mathcal{A}$ with $x, y \in B$ such that $x \in f(B)$. Again, by the fact that x and y are in the same line, we have $u_i(x) \geq u_i(y)$ for any utility function which can be used to rationalize f, which contradicts the previous assertion.

Thus, we have shown $xVy \Longrightarrow \varphi(x) \geq \varphi(y)$. If a cycle $(x^1, ..., x^m)$ satisfies $x^{k+1}Vx^k$ for all $k = 1, ..., m-1$, then x^kVx^{k+1} must hold for all $k = 1, ..., m-1$. Indeed, if there exists k such that x^kVx^{k+1} does not hold, we have $\{x^{k+1}\} = f(\{x,^k, x^{k+1}\}) = g(\{x,^k, x^{k+1}\}) = \mathrm{argmax}_{z\in\{x^k,x^{k+1}\}}\,\varphi(z)$ (note that $\{x^k, x^{k+1}\}$ is a line since $(x^1, ..., x^m)$ is a cycle), implying that $\varphi(x^1) \leq \varphi(x^2) \leq \cdots \leq \varphi(x^k) < \varphi(x^{k+1}) \leq \ldots \varphi(x^1)$, which is a contradiction. Hence, f must satisfy (CA).

(4.$\Longrightarrow$ 1.) By Proposition 5, if a choice function f satisfies (NC), then it is Nash rationlizable. Thus, there exists a strategic form game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$, such that $f(A) = NE\,[(A_i)_{i\in N}, (u_i)_{i\in N}]$ for every $A \in \mathcal{A}$. Thus, by Proposition 2, our claim follows, if this game G does not admit any improving cycle. Assume by way of contradiction that there exists a sequence of action profiles $(x^1, \ldots, x^m) \subset X$ which is an improving cycle such that,

- $x^1 = x^m$,
- $\forall k \in \{1, \ldots, m-1\}, \exists! i(k) \in N;\ u_{i(k)}(x^k_{i(k)}, x^{k+1}_{J(k)}) \geq u_{i(k)}(x^k_{i(k)}, x^k_{J(k)})$,
- $\exists \bar{k} \in \{1, \ldots, m-1\};\ u_{i(\bar{k})}(x^{\bar{k}+1}) > u_{i(\bar{k})}(x^{\bar{k}})$.

It follows that $x^{k+1} \in f(\{x^k_{i(k)}, x^{k+1}_{i(k)}\} \times \{x^k_{J(k)}\})$ for all k, whence $x^{k+1}Vx^k$ for all k, or $x^mT(V)x^1$. Since f is Nash rationalizable, there cannot exist $A \in \mathcal{A}$ such that $x^{\bar{k}}, x^{\bar{k}+1} \in A$ and $x^{\bar{k}} \in f(A)$. Therefore $\neg[x^{\bar{k}}Vx^{\bar{k}+1}]$ holds, which contradicts the assumption that f satisfies (CA). QED

The definition of potential rationalizability itself merely requires the existence of a rationalizing potential game. However, as seen from the proof of the preceding theorem, in particular, the proof for 4.$\Longrightarrow$ 1. implies a stronger result: if a choice function is potential rationalizable, then any non-potential game cannot rationalize it.

Corollary 1. *Suppose that a choice function f is potential rationalizable. Then, every strategic form game that rationalizes f is an ordinal potential game.*

Two remarks are in order. First, in the choice based theory, we cannot know the "true" preference of each agent even if an observed choice function can be rationalized as the set of Nash equilibria. In particular, if a given choice function is potential rationalizable, then it cannot be refuted that the preferences of all agents are the same with each other. In particular, a congruent selection g that meets the requirements in (CSP) can be seen as the set of maximizers of their common interests on every $A \in \mathcal{A}$. This means that if a choice function is potential rationalizable, then it cannot be refuted that $f(A)$ contains a Pareto-efficient action profile as Nash equilibrium of some strategic form game, although the general conditions for "Pareto rationalizability" of a choice function is still unknown (see Sprumont, 2000).

Second, at least in our framework, verifying whether a joint choice function is Nash rationalizable means that we assume the existence of a pure strategy Nash equilibrium on every feasible action set, as pointed out in Lee (2012). That is, we must *á priori* restrict our attention into a (relatively small) subclass of strategic form games that possess Nash equilibrium on every feasible set. On the other hand, when testing the rationalizability by potential games, this existence assumption is innocent, since every potential game has a pure strategy Nash equilibrium on every feasible set.

Appendix

Here, we provide two examples: the first is an example of a choice function that is Nash rationalizable, but not potential rationalizable, while the second shows that, without Nash rationalizability, the conditions (CSP) and (CA) are independent with each other.

Example A1: We give an example of a choice function that is Nash rationalizable but not potential rationalizable.

- $N = \{1, 2\}$
- $X_1 = \{U, D\}, \quad X_2 = \{L, R\}$
- $\mathcal{A} = \{A = A_1 \times \cdots \times A_n \ : \ \emptyset \neq A_i \subset X_i \ \ \forall i \in N\}$

Consider the choice function f and a selection g given as below. In the matrices below, let $\bigcirc$ denote the observed outcomes via choice function f, and $\bigotimes$ the selection g from f.

$1\backslash 2$	L	R
U		$\otimes$
D		

$1\backslash 2$	L
U	
D	$\otimes$

$1\backslash 2$	R
U	$\otimes$
D	

$1\backslash 2$	L	R
U	$\otimes$	$\otimes$

$1\backslash 2$	L	R
D		$\otimes$

$$
\begin{aligned}
& f(X) = g(X) = \{(U, R)\} && (1)\\
& f(\{(U, D)\} \times \{L\}) = g(\{(U, D)\} \times \{L\}) = \{(D, L)\} && (2)\\
& f(\{(U, D)\} \times \{R\}) = g(\{(U, D)\} \times \{R\}) = \{(U, R)\} && (3)\\
& f(\{U\} \times \{(L, R)\}) = g(\{U\} \times \{(L, R)\}) = \{(U, L), \ (U, R)\} && (4)\\
& f(\{D\} \times \{(L, R)\}) = g(\{D\} \times \{(L, R)\}) = \{(D, R)\} && (5)
\end{aligned}
$$

It can be seen that the choices above satisfy (NC), or (PC) and (PE), hence Nash rationalizability follows. We show that the choices above are not potential rationalizable.

Consider sequence of action profiles $((D,L),\ (D,R),\ (U,R),\ (U,L),\ (D,L))$. Then it follows by (2)−(5) that $(D,L)\mathcal{R}(U,L)\mathcal{R}(U,R)\mathcal{R}(D,R)\mathcal{R}(D,L)$. It can be seen that $(D,L)\mathcal{R}(D,R)$ never holds, violating the congruence axiom. Thus g does not satisfy (CSP), which implies that the choices above are not potential rationalizable.

It can be seen analogously that the choices above violate (CA) as well.

Example A2: First we show an example in which (CSP) is satisfied but (CA) is violated.

- $N = \{1,2\}$
- $X_1 = \{U,D\},\ X_2 = \{L,C,R\}$
- $\mathcal{A} = \{A = A_1 \times \cdots \times A_n\ :\ \emptyset \neq A_i \subset X_i\ \ \forall i \in N\}$

Consider the choice function f and a selection g given as below.

$1\backslash 2$	L	C	R
U	$\otimes$		
D			$\bigcirc$

$\cdots X$

$1\backslash 2$	L	C
U	$\otimes$	
D		

$1\backslash 2$	L	R
U	$\otimes$	$\bigcirc$
D		

$\cdots A^1$

$1\backslash 2$	C	R
U		
D	$\otimes$	

$1\backslash 2$	L	C	R
U	$\otimes$		

$1\backslash 2$	L	C	R
D	$\otimes$		

$\cdots A^2$

$1\backslash 2$	L	C
U	$\otimes$	

$1\backslash 2$	L	R
U	$\otimes$	

$1\backslash 2$	C	R
U	$\otimes$	

$1\backslash 2$	L	C
D	$\otimes$	

$1\backslash 2$	L	R
D	$\otimes$	

$1\backslash 2$	C	R
D	$\otimes$	

$1\backslash 2$	L
U	$\otimes$
D	

$1\backslash 2$	C
U	
D	$\otimes$

$1\backslash 2$	R
U	$\otimes$
D	$\otimes$

It can be seen that (CSP) is satisfied.

To see that (CA) is violated, consider a sequence of action profiles $((U,L),\ (U,R),\ (D,R),\ (D,L),\ (U,L))$. By observing the choices in matrices X, A^1, and A^2, it follows that;

$$(U,L)V(D,L),\ (D,L)V(D,R),\ (D,R)V(U,R),\ (U,R)V(U,L).$$

This is a cycle with respect to V. Since there is no $A \in \mathcal{A}$ such that $(U,L),(D,L) \in A$ and $(D,L) \in f(A)$, (CA) is violated.

Now we give an example in which (CA) is satisfied but (CSP) is not.

- $N = \{1,2\}$
- $X_1 = \{U,D\},\quad X_2 = \{L,R\}$
- $\mathcal{A} = \{A = A_1 \times \cdots \times A_n\ :\ \emptyset \neq A_i \subset X_i\ \ \forall i \in N\}$

Consider the choice function f and a selection g given as below.

$1\backslash 2$	L	R
U	$\otimes$	$\bigcirc$
D	$\bigcirc$	$\bigcirc$

$\cdots X$

$1\backslash 2$	L
U	
D	$\otimes$

$1\backslash 2$	R
U	$\otimes$
D	

$1\backslash 2$	L	R
U	$\otimes$	

$1\backslash 2$	L	R
D		$\otimes$

It can be seen that (CA) is satisfied, since it follows by the observed choices in matrix X, xVy and yVx holds for all $x, y \in X$ on the same line.

To see that (CSP) is violated, consider a sequence of action profiles $((U, L)$, (D, L), (D, R), (U, R), $(U, L))$. Following the observations of the matrices above, it follows that;

$$(U, L)\mathcal{R}(U, R),\ (U, R)\mathcal{R}(D, R),\ (D, R)\mathcal{R}(D, L),\ (D, L)\mathcal{R}(U, L).$$

This is a cycle with respect to $\mathcal{R}$, and since $(U, R)\mathcal{R}(U, L)$ does not hold, the congruence axiom is violated. Thus (CSP) is not satisfied.

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