Waseda Economics
Working Paper Series

Strong Addition Invariance and axiomatization of the weighted Shapley value

YOKOTE, Koji
Graduate School of Economics
Waseda University

Graduate School of Economics
Waseda University

1-6-1, Nishiwaseda, Shinjuku-ku, Tokyo

169-8050 JAPAN

Waseda Economics Working Paper Series

Strong Addition Invariance and axiomatization of the weighted Shapley value

YOKOTE, Koji
Graduate School of Economics
Waseda University

Number 13-1
May 2013

Graduate School of Economics
Waseda University

1-6-1, Nishiwaseda, Shinjuku-ku, Tokyo

169-8050 JAPAN

Strong Addition Invariance and axiomatization of the weighted Shapley value

Koji Yokote*

Abstract

This paper shows a new axiomatization of the Shapley value by using two axioms. First axiom is Dummy Player Property and second axiom is Strong Addition Invariance. Strong Addition Invariance states that the payoff vector of a game does not change even if we add some specific games to the game. By slightly changing the definition of Strong Addition Invariance, we can also axiomatize the weighted Shapley value. Moreover, based on the previous axiomatization of the Shapley value, we axiomatize the family of weighted Shapley values.

JEL classification: C71
Keywords: Cooperative games; Weighted Shapley value; Axiomatization

1 Introduction

Axiomatization of the Shapley value has been intensively discussed since Shapley (1953) first axiomatized the value. Throughout those efforts, it was shown that the Shapley value satisfies many desirable properties as a solution function. Moreover the solution concept was extended to games with coalition structures or NTU games, based on axiomatic approach.

In order to discuss axioms of the Shapley value further, we divide them into three types. First type of axioms exactly determines the payoff of some players in a single game. For instance, Null Player Property requires players who do not make any contribution to obtain 0 in a game. Another example is Equal Treatment Property, which requires symmetric players to obtain

*The author thanks Yukihiko Funaki and Yoshio Kamijo for their helpful comments and cooperation.

the same payoff in a game. Second type of axioms prescribes the relationship between games where the number of players is different. For instance, Hart and Mas-Colell (1989) characterized the Shapley value by using Consistency, which prescribes the relationship between a game and its reduced game. Myerson (1980) proved that the Shapley value is a unique efficient solution function that satisfies Balanced Contribution Property, which prescribes the relationship between games where a player 'disappeared'. By weakening the property, Kamijo and Kongo (2010) introduced Balanced Cycle Contributions Property, which dealt with the disappearance of players in a cyclic manner. Third type of axioms indicates how a solution function changes a payoff vector when the worths of a game change. One example of this type is Strong Monotonicity introduced by Young (1985). The axiom requires a solution function to increase the payoff of players whose marginal contribution increased. Another example is Fairness introduced by van den Brink (2001), under which players who are symmetric in a change are equally treated.

Regarding the second type of axioms, Kamijo and Kongo (2012) focused on the disappearance of players such that a solution function prescribes the same payoff vector to remaining players after the disappearance. In this paper, on the other hand, we consider the third type of axioms and focus on the change in the worths of coalitions such that a solution function prescribes the same payoff vector after the change.

The axioms concerning the invariance of a payoff vector have been previously investigated by Beal, Remila and Solal (2012). They proved that the Shpaley value is the unique solution function which satisfies Uniform Addition Invariance, Uniform Transfer Invariance and Dummy Player Property. While their axioms concerning invariance could axiomatize not only the Shapley value but also the Equal Division or the Equal Surplus Division, our new axiom can, by slightly changing its definition, axiomatize the weighted Shapley value as well. Moreover, based on the axiomatization by van den Brink, Funaki and Ju (2013), we axiomatize the family of weighted Shapley values. For other approaches of the axiomatization of the family, see Kalai and Samet (1987), Hart and Mas-Colell (1989) or Chun (1991).

This paper is organized as follows. Section 2 gives notations and definitions. In Section 3, we give the definition and the interpretation of our new axiom, Strong Addition Invariance, and axiomatize the Shapley value and the weighted Shapley value. In Section 4, we axiomatize the family of weighted Shapley values. Section 5 gives concluding remarks.

2 Preliminaries

For any set A, B, $A \subset B$ means that A is a proper subset of B, and $A \subseteq B$ means that $A \subset B$ or $A = B$. $|A|$ is the cardinality of the set A. Let $N \subset \mathbb{N}$ be a finite set of players, and we define $|N| = n$. A real-valued function $v : 2^N \to \mathbb{R}$ is a characteristic function which satisfies $v(\emptyset) = 0$. A pair (N, v) is called a TU cooperative game, or simply a game. For notational convenience, we write v instead of (N, v), if the set of players is clear. We call $S \subseteq N, S \neq \emptyset$ and $v(S)$ as a coalition, and the worth of a coalition, respectively. Let Γ^N denote the set of all games where the set of players is N. For any $v, w \in \Gamma^N$, we define the sum of games $v + w \in \Gamma^N$ as follows: $(v+w)(S) = v(S)+w(S)$ for all $S \subseteq N, S \neq \emptyset$. A solution function $\psi : \Gamma^N \to \mathbb{R}^n$ prescribes a n-dimensional vector $\psi(v)$ such that $\sum_{i\in N} \psi_i(v) \leq v(N)$ to each $v \in \Gamma^N$.

If $v(S) = 0$ for all $S \subseteq N$, then $v \in \Gamma^N$ is called a null game and denoted as v_0. Let $e_S \in \Gamma^N$ denote the elementary game of some coalition $S \subseteq N, S \neq \emptyset$.

$$e_S(T) = \begin{cases} 1 & \text{if } T = S, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_S \in \Gamma^N$ denote the unanimity game of some coalition $S \subseteq N, S \neq \emptyset$.

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega \in \mathbb{R}^n$ be a positive weight such that $\omega_i > 0$ for $i = 1, \dots, n$. We define the k-intersection game with positive weight ω of some coalition $S \subseteq N, S \neq \emptyset$, as follows:

$$\chi_S^{k\omega}(T) = \begin{cases} \sum_{i\in T\cap S} \omega_i & \text{if } |T \cap S| = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq k \leq |S|$. If we write χ_S^k, it is the k-intersection game with positive weight $\omega = (1, \dots, 1)$. Note that $\chi_S^{|S|} = |S|u_S$.

The Shapley value was first introduced by Shapley (1953).

$$\phi_i(N, v) = \sum_{S\subseteq N: i\in S} \frac{(n - |S|)!(|S| - 1)!}{n!}(v(S) - v(S\backslash\{i\})) \text{ for all } i \in N.$$

We can generalizes the value by dropping the symmetric property. Given a positive weight $\omega \in \mathbb{R}^n, \omega_i > 0$ for $i = 1, \dots, n$, we define $\phi^\omega(u_S), S \subseteq N, S \neq$

$\emptyset$ as follows:

$$\phi_i^\omega(u_S) = \begin{cases} \omega_i / \sum_{j \in S} \omega_j & \text{if } i \in S, \\ 0 & \text{if } i \in N \backslash S. \end{cases}$$

Since the set of unanimity games is a linear basis of Γ^N, we can express any game $v \in \Gamma^N$ uniquely by a linear combination of unanimity games. Suppose that

$$v = \sum_{S \subseteq N : S \neq \emptyset} \delta_S u_S,$$

where $(\delta_S)_{\emptyset \neq S \subseteq N}$ are coefficients in the linear combination.[1] Then, we define the weighted Shapley value with positive weight ω as follows:

$$\phi^\omega(v) = \sum_{S \subseteq N : S \neq \emptyset} \delta_S \phi^\omega(u_S).$$

A player i is called a null player in $v \in \Gamma^N$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. A player i is called a dummy player in $v \in \Gamma^N$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. For any $v \in \Gamma^N$, $S \subset N, S \neq \emptyset$ and any solution function ψ, we define the reduced game v_S^ψ on S by following Hart and Mas-Colell (1989).

$$v_S^\psi(T) = v(T \cup S^c) - \sum_{j \in S^c} \psi_j(T \cup S^c, v) \text{ for all } T \subseteq S,$$

where $S^c = N \backslash S$ and $(T \cup S^c, v)$ is a restriction of (N, v) on $(T \cup S^c, v)$.

We list the basic axioms imposed on a solution function ψ.

Efficiency $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \Gamma^N$.

Null Player Property Take any $v \in \Gamma^N$. If $i \in N$ is a null player in v, then $\psi_i(v) = 0$.

Dummy Player Property Take any $v \in \Gamma^N$. If $i \in N$ is a dummy player in v, then $\psi_i(v) = v(\{i\})$.

Linearity $\psi(\lambda v + \mu w) = \lambda \psi(v) + \mu \psi(w)$ for all $v, w \in \Gamma^N$ and $\lambda, \mu \in \mathbb{R}$.

Axiomatizations in this paper only focus on the axioms which fix the number of players in its definition. In the terminology of the division of types in Section 1, we only use first type of axioms and third type of axioms. However, we list one axiom of second type which deals with games with different

[1]The coefficients can be expressed by the dividend, which was introduced by Harsanyi (1959).

number of players, since we use the mathematical property of the axiom in the proof later.

Consistency (Hart and Mas-Colell (1989)) A solution function ψ satisfies Consistency if, for any game $v \in \Gamma^N$ with $n \geq 2$ and any coalition $S \subset N, S \neq \emptyset$, we have

$$\psi_i(S, v_S^{\psi}) = \psi_i(N, v), \text{ for all } i \in S.$$

Both ϕ and ϕ^{ω} satisfy all axioms above.

3 Strong Addition Invariance and characterization

We first revisit the axioms introduced by Beal, Remila and Solal (2012).

Uniform Addition Invariance For any $v \in \Gamma^N$ with $n \geq 2$, $1 \leq k \leq n-1$, $\lambda \in \mathbb{R}$, we have $\psi(v + \lambda\chi_N^k) = \psi(v)$.

This axiom states that if the worths of all coalitions with size k change by the same amount and the worth of the grand coalition does not change, then the payoff vector should not change.

Uniform Transfer Invariance For any $v \in \Gamma^N$ with $n \geq 3$, $2 \leq k \leq n-1$, $\lambda \in \mathbb{R}$, and any two distinct coalitions S, S' with $|S| = |S'| = k$, we have $\psi_i(v + \lambda e_S - \lambda e_{S'}) = \psi_i(v)$ for all $i \in S \cap S'$.

This axiom states that if the worths of two coalitions with the same size change by the same amount, but in the opposite direction, then the payoff of players who belong to both coalitions should not change.

Remark (Theorem 3 of Beal, Remila and Solal (2012)) *A solution function ψ satisfies Uniform Addition Invariance, Uniform Transfer Invariance and Dummy Player Property if and only if ψ is the Shapley value.*

We now introduce a new axiom which is stronger than Uniform Addition Invariance. Although the definition of Uniform Addition Invariance only allows the addition of χ_N^k, we allow the addition of χ_S^k for all $S \subseteq N, 2 \leq |S| \leq n-1$.

Strong Addition Invariance* For any $v \in \Gamma^N$ with $n \geq 2$, $S \subseteq N$ with $|S| \geq 2$, $1 \leq k \leq |S|-1$, $\lambda \in \mathbb{R}$, we have $\psi(v + \lambda\chi_S^k) = \psi(v)$.

This axiom states that if the worths of all coalitions whose intersection with S is $k, 1 \leq k \leq |S|-1$, change by the same amount, then the payoff vector should not change. We can interpret this axiom as follows. First, we can naturally expect that the payoff of players in S should change equally, since they are symmetric in the change. Similarly, the payoff of players in $N \backslash S$ should also change equally. Moreover, since $\lambda\chi_S^1(S \cup T) - \lambda\chi_S^1(T) = 0$ for all $T \subseteq N \backslash S$, marginal contribution of coalition S does not change. Then, it seems reasonable to conclude that the total payoff of players in coalition S should not change. As a result, under Efficiency, the remaining possibility is that the payoff of all players does not change.

Now, consider the following axiom which restricts k in the definition of Strong Addition Invariance* to 1.

Strong Addition Invariance For any $v \in \Gamma^N$ with $n \geq 2$, $S \subseteq N$ with $|S| \geq 2$, $\lambda \in \mathbb{R}$, we have $\psi(v + \lambda\chi_S^1) = \psi(v)$.

Then, the following proposition holds.

Proposition 1 *Strong Addition Invariance* and Strong Addition Invariance are equivalent.*

For the proof, see Lemma 7 in Appendix.[2] From this lemma, we do not need to care about which axiom to use.

Example We give an example which illustrates the contents of Strong Addition Invariance for 3-person game. The axiom states that even if we add the following games by multiplying a real number, the payoff vector does not change.

	$\chi^1_{\{1,2\}}$	$\chi^1_{\{1,3\}}$	$\chi^1_{\{2,3\}}$	χ^1_N
$\{1\}$	1	1	0	1
$\{2\}$	1	0	1	1
$\{3\}$	0	1	1	1
$\{1,2\}$	0	1	1	0
$\{1,3\}$	1	0	1	0
$\{2,3\}$	1	1	0	0
N	0	0	0	0

[2]Although the statement of Lemma 7 includes positive weight, we can get the result of this lemma by letting $\omega = (1, \cdots, 1)$.

Please note that Strong Addition Invariance states the invariance of payoff vector, and not the invariance of payoff of some specific players. Among the axioms concerning invariance, we call an axiom which prescribes the invariance of payoff vector as the axiom of global invariance. On the other hand, we call an axiom which prescribes the invariance of payoff of some specific players as the axiom of local invariance. Many previous axiomatizations of the Shapley value focused on the axiom of local invariance. For instance, in the axiomatization by Beal, Remila and Solal (2012), Uniform Transfer Invariance is the axiom of local invariance. In Young's (1985) axiomatization, Marginality played a central role, and the axiom dealt with the invariance of payoff of players whose marginal contribution does not change. Coalitional Strategic Equivalence by Chun (1989) stated the invariance of payoff of players in $N \backslash S$ after the addition of λu_S for some $S \subset N, S \neq \emptyset$ and $\lambda \in \mathbb{R}$. Looking back the history of axiomatization, the axiom of global invariance was not used, or used with the axiom of local invariance. This fact illustrates the interesting characteristic of the following theorem, which axiomatizes the Shapley value by the axiom of global invariance without using an axiom of local invariance.

Theorem 1 *A solution function ψ satisfies Dummy Player Property and Strong Addition Invariance if and only if ψ is the Shapley value.*

Instead of proving this theorem, we prove Theorem 2, since the result of Theorem 1 can be obtained by letting $\omega = (1, \cdots, 1)$.

Another characteristic of this axiomatization is that we can also axiomatize the weighted Shapley value by slightly changing the definition of Strong Addition Invariance. The weighted version is given as follows:

ω-Strong Addition Invariance* For any $v \in \Gamma^N$ with $n \geq 2$, $S \subseteq N$ with $|S| \geq 2$, $1 \leq k \leq |S| - 1$, $\lambda \in \mathbb{R}$, we have $\psi(v + \lambda \chi_S^{k\omega}) = \psi(v)$.

The restriction of k to 1 also works.

ω-Strong Addition Invariance For any $v \in \Gamma^N$ with $n \geq 2$, $S \subseteq N$ with $|S| \geq 2$, $\lambda \in \mathbb{R}$, we have $\psi(v + \lambda \chi_S^{1\omega}) = \psi(v)$.

Proposition 2 *ω-Strong Addition Invariance* and ω-Strong Addition Invariance are equivalent.*

For the proof, see Lemma 7 in Appendix.

We are ready to prove the main theorem.

Theorem 2 *ψ satisfies Dummy Player Property and ω-Strong Addition Invariance if and only if ψ is the weighted Shapley value with positive weight ω.*

Proof. **If Part:** From the assumption, $\psi = \phi^\omega$. We only need to show that ϕ^ω satisfies ω-Payoff Invariance. We first prove a lemma.

Lemma 1 *If $n \geq 2$, then $\phi^\omega(N, \chi_N^{1\omega}) = \mathbf{0}$.*

Proof. Let $N = \{i, j\}$. Then,

$$\chi_N^{1\omega} = \omega_i u_{\{i\}} + \omega_j u_{\{j\}} - (\omega_i + \omega_j) u_N.$$

From the definition of ϕ^ω,

$$\phi^\omega(N, \chi_N^{1\omega}) = (\omega_i, 0) + (0, \omega_j) - (\omega_i, \omega_j) = (0, 0),$$

and the statement holds. Suppose that the statement holds for all N' such that $2 \leq |N'| \leq m - 1$, and we show the result for $|N| = m, m \geq 3$.

Let $j \in N$ and consider the reduced game $(N \backslash \{j\}, (\chi_N^{1\omega})_{N\backslash\{j\}}^{\phi^\omega})$. Take any $T \subset N \backslash \{j\}, T \neq \emptyset$ and let us calculate $(\chi_N^{1\omega})_{N\backslash\{j\}}^{\phi^\omega}(T)$. First,

$$\chi_N^{1\omega}(T \cup \{j\}) = 0.$$

Moreover, from the induction hypothesis,

$$\phi_j^\omega(T \cup \{j\}, \chi_N^{1\omega}) = 0.^3$$

Then, we have

$$(\chi_N^{1\omega})_{N\backslash\{j\}}^{\phi^\omega}(T) = 0.$$

And from the definition,

$$(\chi_N^{1\omega})_{N\backslash\{j\}}^{\phi^\omega}(N \backslash \{j\}) = -\phi_j^\omega(N, \chi_N^{1\omega}).$$

Namely, the reduced game $(N \backslash \{j\}, (\chi_N^{1\omega})_{N\backslash\{j\}}^{\phi^\omega})$ is the game such that only the worth of grand coalition is $-\phi_j^\omega(N, \chi_N^{1\omega})$ and the worths of all other coalitions are 0.

[3]First, note that $|T \cup \{j\}| \leq m - 1$. Moreover, in the game of the restriction of $\chi_N^{1\omega}$ on $T \cup \{j\}$, only the worths of coalitions $\{l\}, l \in T \cup \{j\}$ are w_l and the worths of other coalitions are 0. It follows that the game is equal to $(T \cup \{j\}, \chi_{T\cup\{j\}}^{1\omega})$, so the induction hypothesis works.

Assume that $\phi_j^\omega(N,\chi_N^{1\omega})>0$, and we show a contradiction. In this case, we must have

$$\begin{aligned}\phi_i^\omega(N,\chi_N^{1\omega}) &= \phi_i^\omega(N\backslash\{j\},(\chi_N^{1\omega})^{\phi^\omega}_{N\backslash\{j\}})\\ &= \phi_i^\omega(N\backslash\{j\},-\phi_j^\omega(N,\chi_N^{1\omega})u_{N\backslash\{j\}})<0,\ \text{for all } i\in N\backslash\{j\}, \quad (1)\end{aligned}$$

where the first equality holds from Consistency. Take any $k\in N\backslash\{j\}$ and consider the reduced game $(N\backslash\{k\},(\chi_N^{1\omega})^{\phi^\omega}_{N\backslash\{k\}})$. By letting player k play the role of player j, we have

$$\begin{aligned}\phi_i^\omega(N,\chi_N^{1\omega}) &= \phi_i^\omega(N\backslash\{k\},(\chi_N^{1\omega})^{\phi^\omega}_{N\backslash\{k\}})\\ &= \phi_i^\omega(N\backslash\{k\},-\phi_k^\omega(N,\chi_N^{1\omega})u_{N\backslash\{k\}})>0,\ \text{for all } i\in N\backslash\{k\}. \quad (2)\end{aligned}$$

Then, $\phi_i^\omega(N,\chi_N^{1\omega})<0$ from equation (1), and $\phi_i^\omega(N,\chi_N^{1\omega})>0$ from equation (2) for all $i\in N\backslash\{j,k\}$, which is a contradiction. We can get the same contradiction when we first assume that $\phi_j^\omega(N,\chi_N^{1\omega})<0$. As a result, we must have $\phi_j^\omega(N,\chi_N^{1\omega})=0$. Since j was an arbitrary player, we have $\phi_i^\omega(N,\chi_N^{1\omega})=0$ for all $i\in N$, which completes the proof. □

We return to the proof of If Part. Since the case of $n=2$ is proved, take any player set $N, n\geq 3$, and fix. Consider the game $\chi_S^{1\omega}$, $S\subseteq N$, $|S|\geq 2$. If $S=N$, from Lemma 1, $\phi_i^\omega(N,\chi_N^{1\omega})=0$ for all $i\in N$.

Take a coalition S such that $2\leq |S|<n$. We first determine the payoff of player $j\in N\backslash S$. For any $j\in N\backslash S$, we have $|T\cap S|=|(T\cup\{j\})\cap S|$ for all $T\subseteq N\backslash\{j\}$. It follows that, from the definition of $\chi_S^{1\omega}$, player j is a null player. From Null Player Property, $\phi_j^\omega(N,\chi_S^{1\omega})=0$ for all $j\in N\backslash S$.

Next, we determine the payoff of player $i\in S$. Consider the reduced game $(S,(\chi_S^{1\omega})^{\phi^\omega}_S)$. We first calculate the worth of $T\subseteq S, T\neq\emptyset$.

Case 1 If $|T|=1$, let $T=\{k\}$, $k\in S$.

$$(\chi_S^{1\omega})^{\phi^\omega}_S(\{k\})=\chi_S^{1\omega}(\{k\}\cup S^c)-\sum_{j\in S^c}\phi_j^\omega(\{k\}\cup S^c,\chi_S^{1\omega})=\omega_k.$$

Note that $j\in S^c$ is a null player in the game $(\{k\}\cup S^c,\chi_S^{1\omega})$.

Case 2 If $|T|\geq 2$,

$$(\chi_S^{1\omega})^{\phi^\omega}_S(T)=\chi_S^{1\omega}(T\cup S^c)-\sum_{j\in S^c}\phi_j^\omega(T\cup S^c,\chi_S^{1\omega})=0.$$

Namely, $(S,(\chi_S^{1\omega})^{\phi^\omega}_S)=(S,\chi_S^{1\omega})$. Together with Lemma 1 and Consistency, we have

$$\phi_i^\omega(N,\chi_S^{1\omega})=\phi_i^\omega(S,\chi_S^{1\omega})=0,$$

for all $i \in S$. It follows that

$$\phi^{\omega}(N, \chi_S^{1\omega}) = \mathbf{0},$$

for all $S \subseteq N, |S| \geq 2$. Linearity completes the proof.

Only If Part: If $n = 1$, the proof is obvious from Dummy Player Property. Suppose $n \geq 2$ throughout the remaining part. We define $u_S^{\omega} \in \Gamma^N$ for any $S \subseteq N, S \neq \emptyset$ as follows:

$$u_S^{\omega}(T) = \begin{cases} \sum_{i \in S} \omega_i & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the set of games $(u_S^{\omega})_{\emptyset \neq S \subseteq N}$ is a linear basis of Γ^N.

Lemma 2 *For any $S \subseteq N, S \neq \emptyset$, we have*

$$\chi_S^{1\omega} = \sum_{k=1}^{|S|} (-1)^{k-1} \sum_{T \subseteq S : |T| = k} u_T^{\omega}.$$

Proof. We calculate the worth of $R \subseteq N, R \neq \emptyset$. If $r = 0$, the worth of R is 0 from the definition.[4] So, suppose that $1 \leq |R \cap S| \leq |S|$. Let $r = |R \cap S|$.

$$\begin{aligned} \sum_{k=1}^{|S|} (-1)^{k-1} \sum_{T \subseteq S : |T| = k} u_T^{\omega}(R) &= \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \sum_{i \in R \cap S} \omega_i \\ &= \sum_{i \in R \cap S} \omega_i \cdot \sum_{k=0}^{r-1} \frac{(r-1)!}{k!(r-k-1)!} (-1)^k \\ &= \sum_{i \in R \cap S} \omega_i \cdot (1-1)^{r-1} \\ &= \begin{cases} \sum_{i \in R \cap S} \omega_i & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the third equality holds from the binomial theorem. The resulting value is exactly $\chi_S^{1\omega}(R)$. □

[4] Look carefully at the right-hand side of the equation in the statement. Then, it will be clear that R gains a non-zero value only if a game u_T^{ω} such that $T \subseteq S, T \subseteq R$ is chosen in the second summation. In this case, $T \subseteq R \cap S$. So, if $|R \cap S| = 0$, R necessarily gains 0. This point of view also explains the first equality in the transformation below. For any player $i \in R \cap S$, ω_i is added when a coalition $T \subseteq R \cap S$ such that $i \in T$ is chosen. Such a coalition T is determined by choosing k, $0 \leq k \leq r-1$, players from $(R \cap S) \setminus \{i\}$.

Lemma 3 *The set of games* $\{u_{\{i\}} : i \in N\} \cup \{\chi_S^{1\omega} : S \subseteq N, |S| \geq 2\}$ *is a linear basis of* Γ^N.

Proof. Assume not. Then, there exists $(\alpha_S)_{\emptyset \neq S \subseteq N} \neq \mathbf{0}$ such that

$$\sum_{S \subseteq N : S \neq \emptyset} \alpha_S \tilde{u}_S^\omega = 0, \tag{3}$$

where $\tilde{u}_S^\omega = u_S$ if $|S| = 1$ and $\tilde{u}_S^\omega = \chi_S^{1\omega}$ if $|S| \geq 2$. Let $N(\alpha)$ denote $\{S \subseteq N : \alpha_S \neq 0\}$. If $|N(\alpha)| = 1$, then equation (3) is an obvious contradiction. So, suppose that $|N(\alpha)| \geq 2$.

Equation (3) can be rewritten as follows.

$$\sum_{S \in N(\alpha)} \alpha_S \tilde{u}_S^\omega = 0, \tag{4}$$

where we define $\alpha_\emptyset = 0$. And we also define $\tilde{u}_\emptyset^\omega = u_\emptyset = u_\emptyset^\omega = v_0$. Take a coalition $R \in N(\alpha)$ such that $|R| \geq |S|$ for all $S \in N(\alpha)$ and solve equation (4) with respect to $\tilde{u}_R^\omega$.

$$\begin{aligned}
\tilde{u}_R^\omega &= -\frac{1}{\alpha_R} \sum_{S \in N(\alpha) \setminus \{R\}} \alpha_S \tilde{u}_S^\omega \\
&= -\frac{1}{\alpha_R} \Big\{ \sum_{\{i\} \in N(\alpha) \setminus \{R\}} \alpha_{\{i\}} u_{\{i\}} + \sum_{S \in N(\alpha) \setminus \{R\} : |S| \geq 2} \alpha_S \sum_{k=1}^{|T|} (-1)^{k-1} \sum_{T \subseteq S : |T| = k} u_T^\omega \Big\},
\end{aligned}$$

where the second equality holds from Lemma 2. Suppose that $|R| = 1$. Then, from the definition, $\tilde{u}_R^\omega = u_R$ and the right-hand side contains only unanimity games with singleton coalitions except R. This contradicts the fact that unanimity games are linearly independent. So, suppose that $|R| \geq 2$.

Again, by applying Lemma 2 to $\tilde{u}_R^\omega$, we have

$$\sum_{k=1}^{|R|}(-1)^{k-1}\sum_{T\subseteq R:|T|=k}u_T^\omega$$

$$= -\frac{1}{\alpha_R}\Big\{\sum_{\{i\}\in N(\alpha)\setminus\{R\}}\alpha_{\{i\}}u_{\{i\}} + \sum_{S\in N(\alpha)\setminus\{R\}:|S|\geq 2}\alpha_S\sum_{k=1}^{|T|}(-1)^{k-1}\sum_{T\subseteq S:|T|=k}u_T^\omega\Big\},$$

$$(-1)^{|R|-1}u_R^\omega$$

$$= -\frac{1}{\alpha_R}\Big\{\sum_{\{i\}\in N(\alpha)\setminus\{R\}}\alpha_{\{i\}}u_{\{i\}} + \sum_{S\in N(\alpha)\setminus\{R\}:|S|\geq 2}\alpha_S\sum_{k=1}^{|T|}(-1)^{k-1}\sum_{T\subseteq S:|T|=k}u_T^\omega\Big\}$$

$$-\sum_{k=1}^{|R|-1}(-1)^{k-1}\sum_{T\subseteq R:|T|=k}u_T^\omega.$$

From the definition, the right-hand side never contains u_R^ω, which contradicts the fact that the set of games $\{u_{\{i\}} : i \in N\} \cup \{u_S^\omega : S \subseteq N, |S| \geq 2\}$ is a linear basis of Γ^N. □

We now return to the proof of Only If part. From Lemma 3, for any $v \in \Gamma^N$, there exists $(\beta_S)_{\emptyset\neq S\subseteq N}$ such that

$$v = \sum_{i\in N}\beta_{\{i\}}u_{\{i\}} + \sum_{S\subseteq N:|S|\geq 2}\beta_S\chi_S^{1\omega}.$$

First, from Dummy Player Property,

$$\psi\left(\textstyle\sum_{i\in N}\beta_{\{i\}}u_{\{i\}}\right) = (\beta_{\{1\}}, \ldots, \beta_{\{n\}}).$$

From ω-Strong Addition Invariance, adding $\beta_S\chi_S^{1\omega}, S \subseteq N, |S| \geq 2$ never changes the payoff vector. Hence,

$$\psi\left(\textstyle\sum_{i\in N}\beta_{\{i\}}u_{\{i\}} + \sum_{S\subseteq N:|S|\geq 2}\beta_S\chi_S^{1\omega}\right) = (\beta_{\{1\}}, \ldots, \beta_{\{n\}}).$$

It follows that the payoff vector is uniquely determined. □

We give a conclusion of this section. Since many other solution functions do not satisfy Strong Addition Invariance, we can explain the difference between the Shapley value and others from this axiom. The interesting feature of Strong Addition Invariance is: if the marginal contribution of a coalition does not change, then the total payoff of players in the coalition does not change. So, we can conclude that the Shapley value tends to regard the change in the marginal contribution of coalitions as an important criterion to judge the payoff of players. Moreover, this tendency clearly explains the difference between the Shapley value and other solution functions.

4 Axiomatization of the family of weighted Shapley values

In this section, we axiomatize the family of weighted Shapley values. Although we do not use the axioms concerning invariance, the proof of main theorem in this section highly relies on the proof in the previous section; we properly construct a linear basis, and consider the relationship between the linear basis and axioms.

Let us introduce an additional notation. For any $v \in \Gamma^N$ and $S \subseteq N$, let $\Delta_i v(S)$ denote

$$\Delta_i v(S) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S, \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S. \end{cases}$$

Let $\Delta_i v$ denote the vector whose each coordinate is $\Delta_i v(S), S \subseteq N$. For any $v, w \in \Gamma^N$, $\Delta_i v \geq \Delta_i w$ means that $\Delta_i v(S) \geq \Delta_i w(S)$ for all $S \subseteq N$.

The following axioms are used in the previous literature.

Weak Monotonicity (van den Brink et al. (2013)) Take any $v, w \in \Gamma^N$ and $i \in N$ such that $v(N) \geq w(N)$ and $\Delta_i v \geq \Delta_i w$. Then, we have $\psi_i(v) \geq \psi_i(w)$.

Covariance $\psi(w) = \alpha\psi(v) + \beta$ for all $v \in \Gamma^N, \alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$, where w is given by $w(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for all $S \subseteq N$.

Anonymity $\phi_i(v) = \phi_{\pi(\{i\})}(\pi v)$ for all $v \in \Gamma^N$ and all permutations π on N, where game $\pi v \in \Gamma^N$ is defined by $\pi v(\cup_{i \in S}\{\pi(\{i\})\}) = v(S)$ for all $S \subseteq N, S \neq \emptyset$.

The above axioms, together with Efficiency, axiomatize the Shapley value.

Remark (Corollary 4.6 of van den Brink et al. (2013)) *ψ satisfies Efficiency, Anonymity, Weak Monotonicity and Covariance if and only if ψ is the Shapley value.*

Based on this axiomatization, we introduce a new axiomatization of the family of weighted Shapley values. First, we define a variation of Weak Monotonicity.

Strict Weak Monotonicity Let $v, w \in \Gamma^N, i \in N$. If $v(N) = w(N)$ and $\Delta_i v \geq \Delta_i w$, then we have $\psi_i(v) \geq \psi_i(w)$, where $\psi_i(v) = \psi_i(w)$ if and only if $\Delta_i v = \Delta_i w$.

The next axiom, Partnership, was used by Kalai and Samet (1987), and Chun (1991) for the axiomatization of the family of weighted Shapley values. For any $v \in \Gamma^N$, we call $S \subset N, S \neq \emptyset$ as a coalition of partners in v if the following condition is satisfied.

$$v(T \cup R) = v(R) \text{ for all } T \subset S \text{ and } R \subseteq N \backslash S.$$

Partnership Let $v \in \Gamma^N$. If $S \subset N, S \neq \emptyset$ is a coalition of partners in v, then we have $\psi_i(v) = \psi_i\big(\sum_{j \in S} \phi_j(v) u_S\big)$ for all $i \in S$.

The main theorem of this section is given as follows.

Theorem 3 *ψ satisfies Efficiency, Partnership, Strict Weak Monotonicity and Covariance if and only if there exists a positive weight ω such that ψ is the weighted Shapley value with positive weight ω.*

Proof. Since If Part is obvious from previous axiomatizations, we only prove Only If part. Note first that Covariance implies $\psi(v_0) = \mathbf{0}$; assume, on the contrary, that $\psi(v_0) = a \neq \mathbf{0}$. Then, from Covariance, $\lambda a = \psi(\lambda v_0) = \psi(v_0) = a$ for any $\lambda \in \mathbb{R}$, which contradicts the definition of a solution function.

Take any player set N. If $n = 1$, then the proof is obvious from Efficiency. Suppose $n \geq 2$. For any positive weight ω and $S \subseteq N, S \neq \emptyset$, we define

$$\hat{u}_S^\omega = \sum_{i \in S} \omega_i u_{\{i\}} - \sum_{i \in S} \omega_i u_S.$$

Then, the following lemma holds:

Lemma 4 *The set of games $\{u_{\{i\}} : i \in N\} \cup \{\hat{u}_S^\omega : S \subseteq N, |S| \geq 2\}$ is a linear basis of Γ^N.*

Proof. Assume not. Then, there exists $(\alpha_T)_{\emptyset \neq T \subseteq N} \neq \mathbf{0}$ such that

$$\sum_{T \subseteq N : T \neq \emptyset} \alpha_T \tilde{u}_T^\omega = \mathbf{0},$$

where $\tilde{u}_T^\omega = u_T$ if $|T| = 1$, $\tilde{u}_T^\omega = \hat{u}_T^\omega$ if $|T| \geq 2$. Let $N(\alpha)$ denote $\{T \subseteq N : \alpha_T \neq 0, T \neq \emptyset\}$. If $|N(\alpha)| = 1$, then the equation above is an obvious contradiction. So, suppose that $|N(\alpha)| \geq 2$.

Take a coalition $R \in N(\alpha)$ such that $|R| \geq |S|$ for all $S \in N(\alpha)$, and solve the equation with respect to $\tilde{u}^{\omega}_{R}$.

$$-\tilde{u}^{\omega}_{R} = \frac{1}{\alpha_R} \sum_{T \in N(\alpha) \setminus \{R\}} \alpha_T \tilde{u}^{\omega}_{T}.$$

First, suppose that $|R| = 1$. Then, from the definition of R, the right-hand side contains only unanimity games with singleton coalitions except R. This contradicts the fact that unanimity games are linearly independent. So, suppose that $|R| \geq 2$. Then, by rearranging the equation, we have

$$\sum_{i \in R} \omega_i u_R = \sum_{i \in R} \omega_i u_{\{i\}} + \frac{1}{\alpha_R} \sum_{T \in N(\alpha) \setminus \{R\}} \alpha_T \tilde{u}^{\omega}_{T}.$$

From the definition of R and $\tilde{u}^{\omega}_{T}$, the right-hand side never contains u_R. This equation again contradicts the fact that the set of unanimity games is a linear basis of Γ^N. □

Lemma 5 *If ψ satisfies Efficiency, Covariance and Strict Weak Monotonicity, then there exist a positive weight ω such that $\psi(\hat{u}^{\omega}_{N}) = \mathbf{0}$.*

Proof. Let $A := \{a \in \mathbf{R}^n : a_i \geq 0, i = 1, \cdots, n, \sum_{i=1}^{n} a_i = 1\}$. We first show that the function $\psi(\hat{u}^{a}_{N}) : A \to \mathbf{R}^n$ is a continuous function. Consider an arbitrary sequence $\{a^k\}_{k=1}^{\infty} \subseteq A$, such that $a^k \to \bar{a}$ as $k \to \infty$. Then,

$$\begin{aligned}
\lim_{k\to\infty} \psi(\hat{u}^{a^k}_{N}) &= \lim_{k\to\infty} \psi\Big(\sum_{i\in N} a^k_i u_{\{i\}} - \sum_{i \in N} a^k_i u_N\Big) \\
&= \lim_{k\to\infty} \Big(a^k - \sum_{i\in N} a^k_i \psi(u_N)\Big) \\
&= \bar{a} - \sum_{i \in N} \bar{a}_i \psi(u_N) \\
&= \psi(\bar{u}^{\bar{a}}_{N}),
\end{aligned}$$

where the second equality holds from Covariance. We define a new function $\Psi : A \to A$ as follows:[5]

$$\Psi_i(a) = \frac{a_i + \max\{0, a_i - \psi_i(\hat{u}^{a}_{N})\}}{1 + \sum_{i=1}^{n} \max\{0, a_i - \psi_i(\hat{u}^{a}_{N})\}} \text{ for all } i \in N.$$

[5]In the construction of this function, we follow the way by Jehle and Reny (2011). See the proof of Theorem 7.2 on page 317.

Since Ψ is a continuous function and A is non-empty and compact, from Brouwer's fixed point theorem, there exists a fixed point, $\hat{a} \in A$. Let $X := \sum_{i=1}^{n} \max\{0, \hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}})\} \geq 0$. Then, we have

$$\hat{a}_i X = \max\{0, \hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}})\}. \tag{5}$$

Case 1: If $X = 0$, then $\hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}}) \leq 0$ for all $i \in N$. By summing up, together with Efficiency, we have

$$\sum_{i \in N} \hat{a}_i \leq \sum_{i \in N} \psi_i(\hat{u}_N^{\hat{a}}),$$
$$1 \leq 0,$$

which is a contradiction.

Case 2: If $\hat{a}_i = 0$ for some i, then $0 \leq \psi_i(\hat{u}_N^{\hat{a}})$. Consider the marginal contribution of player i.

$$\hat{u}_N^{\hat{a}}(S) - \hat{u}_N^{\hat{a}}(S \setminus \{i\}) = 0 \text{ for all } S \text{ such that } i \in S, S \neq N.$$
$$\hat{u}_N^{\hat{a}}(N) - \hat{u}_N^{\hat{a}}(N \setminus \{i\}) = 0 - \sum_{j \in N \setminus \{i\}} \hat{a}_j = -1.$$

Namely, $\Delta_i \hat{u}_N^{\hat{a}} \leq \Delta_i v_0$ and $\Delta_i \hat{u}_N^{\hat{a}}(S) \neq \Delta_i v_0$. Since $\hat{u}_N^{\hat{a}}(N) = v_0(N) = 0$, Strict Weak Monotonicity implies that $\psi_i(\hat{u}_N^{\hat{a}}) < \psi_i(v_0) = 0$, which is a contradiction.

Case 3: The remaining possibility is that $X > 0$ and $\hat{a}_i > 0$ for all $i \in N$. It follows that $\hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}}) > 0$ for all $i \in N$. Efficiency implies

$$X = \sum_{i \in N} \max\{0, \hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}})\} = \sum_{i \in N} \big(\hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}})\big) = 1.$$

Then, equation (5) reduces to

$$\hat{a}_i = \hat{a}_i - \psi_i(\hat{u}_N^{\hat{a}}),$$
$$\psi_i(\hat{u}_N^{\hat{a}}) = 0,$$

for all $i \in N$. By letting $\omega = \hat{a}$, the proof is completed. □

Lemma 6 *If ψ satisfies the four axioms, then there exists a positive weight ω such that $\psi(u_S) = \phi^{\omega}(u_S)$ for all $S \subseteq N, S \neq \emptyset$.*

Proof. From Lemma 5, there exist a positive weight ω such that $\sum_{i=1}^{n} \omega_i = 1$ and $\psi(\hat{u}_N^{\omega}) = \mathbf{0}$. From Covariance,

$$0 = \psi_i(\hat{u}_N^{\omega}) = \psi_i\Big(\sum_{k \in N} \omega_k u_{\{k\}} - \sum_{k \in N} \omega_k u_N\Big)$$
$$= \omega_i - \psi_i(u_N),$$
$$\psi_i(u_N) = \omega_i \text{ for all } i \in N.$$

Take any $S \subset N, S \neq \emptyset$ and consider the game $\hat{u}^\omega_S$. First, we determine $\psi_j(u_S)$, $j \in N \backslash S$. Consider the marginal contribution of j to coalition $T \subseteq N$.

Case 1 If $|T \cap S| = 0$, $\hat{u}^\omega_S(T \cup \{j\}) - \hat{u}^\omega_S(T) = 0 - 0 = 0$.

Case 2 If $1 \leq |T \cap S| < |S|$,

$$\hat{u}^\omega_S(T \cup \{j\}) - \hat{u}^\omega_S(T) = \sum_{k \in T \cap S} \omega_k - \sum_{k \in T \cap S} \omega_k = 0.$$

Case 3 If $|T \cap S| = |S|$, $\hat{u}^\omega_S(T \cup \{j\}) - \hat{u}^\omega_S(T) = 0 - 0 = 0$.

Namely, $\Delta_j \hat{u}^\omega_S = \Delta_j v_0$ and $\hat{u}^\omega_S(N) = v_0(N) = 0$. Strict Weak Monotonicity implies $\psi_j(\hat{u}^\omega_S) = 0$ for all $j \in N \backslash S$. Moreover, from Covariance,

$$\begin{aligned} 0 = \psi_j(\hat{u}^\omega_S) &= \psi_j\Big(\sum_{k \in S} \omega_k u_{\{k\}} - \sum_{k \in S} \omega_k u_S\Big) \\ &= 0 - \sum_{k \in S} \omega_k \psi_j(u_S) \text{ for all } j \in N \backslash S. \end{aligned}$$

From $\sum_{k \in S} \omega_k > 0$, we have $\psi_j(u_S) = 0$ for all $j \in N \backslash S$.

Next, we determine $\psi_i(u_S), i \in S$. Since S is a coalition of partners in u_N, Partnership implies

$$\psi_i(u_N) = \psi_i\Big(\sum_{k \in S} \psi_k(u_N) u_S\Big).$$

Covariance implies

$$\begin{aligned} \omega_i = \psi_i(u_N) &= \sum_{k \in S} \omega_k \psi_i(u_S), \\ \psi_i(u_S) &= \omega_i \Big/ \sum_{k \in S} \omega_k \text{ for all } i \in S. \end{aligned}$$

□

We return to the proof of the theorem. In the remaining part, we follow the proof by Young (1985) and Chun (1991). Take a positive weight ω which satisfies the statement of Lemma 6. For any $v \in \Gamma^N$, from Lemma 4, there exists $(\gamma_S)_{\emptyset \neq S \subseteq N}$ such that

$$v = \sum_{S \subseteq N : S \neq \emptyset} \gamma_S \tilde{u}^\omega_S,$$

where $\tilde{u}^\omega_S = u_S$ if $|S| = 1$ and $\tilde{u}^\omega_S = \hat{u}^\omega_S$ if $|S| \geq 2$. Let $N(\gamma)$ denote $\{T \subseteq N : T \neq \emptyset, \gamma_T \neq 0\}$. If $|N(\gamma)| = 0$, $v = v_0$. From Covariance, we have

$$\psi(v) = \phi^\omega(v) = \mathbf{0}.$$

If $|N(\gamma) = 1|$, we can write $v = \gamma_S \tilde{u}^\omega_S$ for some $S \subseteq N, S \neq \emptyset$. Since we already showed that $\psi = \phi^\omega$ in unanimity games, together with Covariance, we have

$$\psi(v) = \psi(\gamma_S \tilde{u}^\omega_S) = \gamma_S \psi(\tilde{u}^\omega_S) = \gamma_S \phi^\omega(\tilde{u}^\omega_S) = \phi^\omega(v).$$

We explain why the third equality holds. First, if $|S| = 1$, then $\tilde{u}^\omega_S$ is a unanimity game, and the equality holds. Next, if $|S| \geq 2$, then $\tilde{u}^\omega_S = \hat{u}^\omega_S = \sum_{i \in S} \omega_i u_{\{i\}} - \sum_{i \in S} \omega_i u_S$. So, Covariance implies the equality.

Take any $t \geq 2$. Suppose that $\psi(v) = \phi^\omega(v)$ for $0 \leq |N(\gamma)| \leq t - 1$. We show that $\psi(v) = \phi^\omega(v)$ for $|N(\gamma)| = t$. We first rewrite the game as follows:

$$v = \sum_{k=1}^{t} \gamma_{R_k} \tilde{u}^\omega_{R_k},$$

where $R_k \in N(\gamma)$ for $k = 1, 2, \ldots, t$. Let $\bar{R} := \cap_{k=1}^t R_k$ and suppose that $j \notin \bar{R}$. Then, there exists at least one coalition R_k such that $j \notin R_k$. Without loss of generality, assume that $R_k, k = 1, \ldots, s$ contain j, and $R_l, l = s + 1, \ldots, t$ do not. We can express v as follows.

$$v = \sum_{k=1}^{s} \gamma_{R_k} \tilde{u}^\omega_{R_k} + \sum_{l=s+1}^{t} \gamma_{R_l} \tilde{u}^\omega_{R_l}.$$

From the induction hypothesis, $\psi(\sum_{k=1}^s \gamma_{R_k} \tilde{u}^\omega_{R_k})$ is uniquely determined. Moreover, consider the case of adding $\gamma_{R_l} \tilde{u}^\omega_{R_l}, l = s+1, \ldots, t$. If $|R_l| = 1$, then from Covariance, j's payoff does not change. If $|R_l| \geq 2$, then $j's$ marginal contribution and the worth of the grand coalition do not change. Strict Weak Monotonicity implies that j's payoff again does not change. Namely, in both cases, we have

$$\psi_j(v) = \phi^\omega_j(v) \text{ for all } j \notin \bar{R}.$$

Consider a player $i \in \bar{R}$. If $|\bar{R}| = 1$, Efficiency completes the proof. Suppose that $|\bar{R}| \geq 2$. Then, v can be expressed by not using $u_{\{i\}}, i = 1, \ldots, n$. We can rewrite v as follows.

$$
\begin{aligned}
v &= \sum_{k=1}^{t} \gamma_{R_k} \hat{u}^{\omega}_{R_k} \\
&= \sum_{k=1}^{t} \gamma_{R_k} \Big(\sum_{m \in R_k} \omega_m u_{\{m\}} - \sum_{m \in R_k} \omega_m u_{R_k} \Big) \\
&= \sum_{k=1}^{t} \sum_{m \in R_k} \gamma_{R_k} \omega_m u_{\{m\}} - \sum_{k=1}^{t} \sum_{m \in R_k} \gamma_{R_k} \omega_m u_{R_k}.
\end{aligned}
$$

We define

$$
v' = -\sum_{k=1}^{t} \sum_{m \in R_k} \gamma_{R_k} \omega_m u_{R_k}.
$$

Since both ψ and ϕ^{ω} satisfy Covariance, the proof will be completed if we show that $\psi(v') = \phi^{\omega}(v')$. In the remaining part of the proof, we follow the proof by Chun (1991). For any $C \subset \bar{R}$ and $S \subseteq N \backslash \bar{R}$, we have

$$
v'(C \cup S) = v'(S) = 0,
$$

since a coalition gets a nonzero worth in v' only if the coalition includes $\bar{R}$. Namely, $\bar{R}$ is a coalition of partners in v'. By Partnership and Covariance, we have

$$
\psi_i(v') = \psi_i\Big(\sum_{m \in \bar{R}} \psi_m(v') u_{\bar{R}} \Big) = \sum_{m \in \bar{R}} \psi_m(v') \psi_i(u_{\bar{R}}) \text{ for all } i \in \bar{R}.
$$

Since ϕ^{ω} also satisfies Partnership and Covariance, we have

$$
\phi^{\omega}_i(v') = \sum_{m \in \bar{R}} \phi^{\omega}_m(v') \phi^{\omega}_i(u_{\bar{R}}) \text{ for all } i \in \bar{R}.
$$

From Efficiency of both solution functions and the fact that $\psi_j(v') = \phi^{\omega}_j(v')$ for all $j \in N \backslash \bar{R}$, we have $\sum_{m \in \bar{R}} \phi^{\omega}_m(v') = \sum_{m \in \bar{R}} \psi_m(v')$. As we saw, both solution functions coincide in unanimity games, which completes the proof.

□

5 Concluding remarks

We point out the remarkable property of our new linear basis. When we express a game by a linear combination of the linear basis in Lemma 3, the coefficient of $u_{\{i\}}$ is exactly the weighted Shapley value of player i, as we saw in the proof of Theorem 1. Together with Lemma 4, we give a theorem which summarizes the discussion above.

Theorem 4 *Let ω be any positive weight. Then, the set of games $\{u_{\{i\}} : i \in N\} \cup \{\chi_S^{1\omega} : |S| \geq 2\}$ is a linear basis of Γ^N, where $n \geq 2$. Moreover, when we express a game by a linear combination of the linear basis, the coefficient of $u_{\{i\}}$ is equal to the weighted Shapley value with positive weight ω of $i \in N$.*

In Yokote, Funaki and Kamijo (2013a), we investigate the linear basis in the case of $\omega = (1, \cdots, 1)$ and discuss the null space and the inverse problem of the Shapley value. In Yokote, Funaki and Kamijo (2013b), we apply this linear basis to the analyze of the relationship between the Shapley value and other solution concepts.

Appendix

We prove Proposition 1 and 2. The equivalence will be proved from the following lemma.

Lemma 7 *If ψ satisfies ω-Strong Addition Invariance, ψ also satisfies the following property: For any $v \in \Gamma^N$ with $n \geq 2$, $S \subseteq N$ with $|S| \geq 2$, $1 \leq k \leq |S| - 1$, $\lambda \in \mathbb{R}$, we have*

$$\psi(v) = \psi(v + \lambda\chi_S^{k\omega}).$$

Proof. Suppose that $n = 2$. Then the statement holds from ω-Strong Addition Invariance.

Suppose that $n \geq 3$. If $|S| = 2$, the statement holds from ω-Strong Addition Invariance. Assume that the statement holds for $|S| = s - 1$, and we prove that it also holds for $|S| = s$, $s \geq 3$. From ω-Strong Addition Invariance, if $k = 1$, the equation holds. Assume that the statement holds for $k = k' - 1$, and we prove that it also holds for $k = k', 2 \leq k' \leq s - 1$. From the induction hypothesis, the proof will be completed if we prove that the following equality holds.

$$\lambda\chi_S^{k'\omega} = \frac{\lambda}{k' - 1}\left(\sum_{i\in S}\chi_{S\setminus\{i\}}^{(k'-1)\omega} - (s - k' + 1)\chi_S^{(k'-1)\omega}\right).$$

Take any coalition $T \subseteq N, T \neq \emptyset$ and let us calculate the worth of T of both sides.

Case1 $0 \le |T \cap S| \le k' - 2$.

It is obvious that $\chi_S^{k'\omega}(T) = \chi_S^{(k'-1)\omega}(T) = 0$. Consider the game $\chi_{S\backslash\{i\}}^{(k'-1)\omega}$, and $i \in S$.

If $i \in T$, $|(S\backslash\{i\}) \cap T| \le k' - 3$,
If $i \notin T$, $|(S\backslash\{i\}) \cap T| \le k' - 2$.

It follows that $\chi_{S\backslash\{i\}}^{(k'-1)\omega}(T) = 0$ for all $i \in S$.

Case 2 $k' + 1 \le |T \cap S| \le s$.

It is obvious that $\chi_S^{k'\omega}(T) = w_S^{(k'-1)\omega}(T) = 0$. Consider the game $\chi_{S\backslash\{i\}}^{(k'-1)\omega}$, and $i \in S$.

If $i \in T$, $|(S\backslash\{i\}) \cap T| \ge k'$,
If $i \notin T$, $|(S\backslash\{i\}) \cap T| \ge k' + 1$.

It follows that $\chi_{S\backslash\{i\}}^{(k'-1)\omega}(T) = 0$ for all $i \in S$.

Case 3 $|T \cap S| = k' - 1$.

Take any $i \in S$. It is obvious that $\chi_S^{k'\omega}(T) = 0$.

If $i \in T$, $|(S\backslash\{i\}) \cap T| = k' - 2$,
If $i \notin T$, $|(S\backslash\{i\}) \cap T| = k' - 1$.

Namely, $\sum_{i\in S} \chi_{S\backslash\{i\}}^{(k'-1)\omega}(T) = \big(s - (k'-1)\big) \sum_{j\in T\cap S} \omega_j$. Together with $-(s-k'+1)\chi_S^{(k'-1)\omega}(T) = -(s-k'+1)\sum_{j\in T\cap S} \omega_j$, the right-hand side is equal to 0, which is equal to the left-hand side.

Case 4 $|T \cap S| = k'$.

It is obvious that $\chi_S^{(k'-1)\omega}(T) = 0$. Take any $i \in S$.

If $i \in T$, $|(S\backslash\{i\}) \cap T| = k' - 1$,
If $i \notin T$, $|(S\backslash\{i\}) \cap T| = k'$.

Namely, $\sum_{i\in S} \chi_{S\backslash\{i\}}^{(k'-1)\omega}(T) = (k'-1)\sum_{j\in T\cap S} \omega_j$.[6] By multiplying $\frac{\lambda}{k'-1}$, the right-hand side is equal to $\lambda \sum_{j\in T\cap S} \omega_j$, which is equal to the left-hand side.

□

[6] We explain why this equation holds. Take any $j \in T \cap S$ and fix. In the summation of the left-hand side, if $k \in (T \cap S)\backslash\{j\}$ is chosen, then w_j is added. Namely, each w_j is added by $|T \cap S| - 1 = k' - 1$ times.

References

[1] Beal, S., E. Remila and P. Solal (2012) "Axioms of invariance for TU-games," Working Paper.

[2] Chun, Y. (1989): "A new axiomatization of the Shapley value," Games and Economic Behavior, 1, 119-130.

[3] Chun, Y. (1991): "On the symmetric and weighted Shapley values," International Journal of Game Theory, 20, 183-190.

[4] Harsanyi, J.C. (1959): "A bargaining model for cooperative n-person games," In: Tucker AW, Luce RD (ed) Contributions to the theory of games IV. Princeton UP, Princeton, pp 325-355.

[5] Hart, S., and A. Mas-Colell (1989): "Potential, value and consistency," Econometrica, 57, 589-614.

[6] Jehle, G. A., and P. J. Reny (2011): Advanced microeconomic theory, 3rd ed. Prentice Hall.

[7] Kamijo, Y., and T. Kongo (2010): "Axiomatization of the Shapley value using the balanced cycle contributions property," International Journal of Game Theory, 39, 563-571.

[8] Kamijo, Y., and T. Kongo (2012): "Whose deletion does not affect your payoff? The difference between the Shapley value, the egalitarian value, the solidarity value and the Banzhaf value," European Journal of Operations Research," 216, 638-646.

[9] Kalai, E., and D. Samet (1987): "On weighted Shapley values," International Journal of Game Theory, 16, 205-222.

[10] Myerson R.B. (1980): "Conference structures and fair allocation rules," International Journal of Game Theory, 9, 169-182.

[11] Peleg, B., and P. Sudhölter (2007): Introduction to the theory of cooperative games, 2nd ed. Springer-Verlag.

[12] Schmeidler, D. (1969): "The nucleolus of a characteristic function game," SIAM Journal of Applied Mathematics, 17, 1163-1170.

[13] Shapley, L.S. (1953): "A value for n-person games," In: Roth AE (ed) The Shapley value. Cambridge University Press, Cambridge, pp 41-48.

[14] van den Brink, R. (2001): "An axiomatization of the Shapley value using a fairness property," International Journal of Game Theory, 30, 309-319.

[15] van den Brink, R., Y. Funaki and Y. Ju (2013): "Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values," Social Choice and Welfare, 40, 693-714.

[16] Yokote, K., Y. Funaki and Y. Kamijo (2013a): "Linear basis approach to the Shapley value," Working Paper.

[17] Yokote, K., Y. Funaki and Y. Kamijo (2013b): "Relationship between the Shapley value and other solution concepts," Working Paper.

[18] Young, H.P. (1985): "Monotonic solutions of cooperative games," International Journal of Game Theory, 14, 65-72.