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Testable implications of the core in market game with transferable utility

Yasushi Agatsuma*†

Abstract

This paper clarifies testable implications of the core on allocation data in cooperative market games with transferable utility (TU market games). By employing the revealed preference approach, we provide a finite system of linear inequalities whose solvability is equivalent to an allocation data set to be consistent with the core of some TU market game. It turns out that the core rationalizability is equivalent to the Pareto rationalizability, that is, the core and Pareto optimality are observationally equivalent in TU market games. In addition we investigate the relationship between the testability of the core and that of the competitive equilibrium.

Keywords: Testable implications; Core rationalization; Pareto rationalization; Market games; Revealed preference
JEL Classification: C71, D51

1 Introduction

This paper clarifies testable implications of the core on allocation data in cooperative market games with transferable utility (TU market games). Specifically, we answer the following question; when can we justify observed allocations as a result of players choosing the core allocations in some TU market game?

To this end, we follow the revealed preference approach pioneered by Afriat [1]. According to Afriat [1], the utility maximizing hypothesis can be tested from given collection of price-consumption observations by checking the solvability of a particular system of linear inequalities, now known as the Afriat inequalities. We will show that our testable implications of the core has similar structure; observed allocations can be rationalized as the core allocations if and only if a particular system of linear inequalities, which is similar to the Afriat inequalities, has a solution. Moreover, it will be demonstrated that the solvability of the inequality system is also equivalent to rationalizability by Pareto optimality. Consequently, we will see that the core allocations and Pareto optimal allocations are indistinguishable from observation, that is, they are observationally equivalent.

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By employing the market game model, we can investigate not only the testable implications of the core itself, but also its relationship between the testable implications of the competitive equilibrium. Since any competitive equilibrium allocation belongs to the core of a market game, it is natural to expect that the testable implications of the two concepts are closely related. For the comparison, we take a result given in Brown and Matzkin [3], which studies the testable implications of the competitive equilibrium in pure exchange economies on the equilibrium manifold. We will see that relationship between these testable implications reminiscent duality. On the one hand, our testable implications of the core states that, from allocation observations, we need to find supergradients of utility functions which meet above mentioned inequality conditions. On the other hand, the testable implications of the competitive equilibrium states that, from market price observations, we need to find allocations which meet the same inequality conditions. Moreover, the supergradients and the observed market price play exactly the same role in the inequality conditions.

1.1 Related literature

The core is one of the most prominent solution concepts in cooperative game theory, and it is also widely accepted as an important concept in economic analysis. From this perspective, it is natural to be curious about what testable implications, if any, the core has. Recent studies given by Echenique [7], Echenique et al. [8], and Chambers and Echenique [4] show the testable implications of the stability on the set of matchings in the two-sided matching model with/without transfers.[1]

The transferable utility hypothesis is a popular assumption in both cooperative game theory and economic theory, although it is considered as a strong assumption. Literature investigating the testable implications of transferable utility is quite limited. Some examples are; Brown and Calsamiglia [2] and Sákovics [9], investigating the testable implications of a consumer having quasi-linear utility, which implies transferable utility, Chiappori [6], deriving a necessary and sufficient condition on the aggregate demand function which is compatible with Pareto optimality and transferable utility within the group, and Cherchye et al. [5], which provides a revealed preference test on observational data for household's consumption behaviour to be consistent with Pareto optimality and transferable utility.

1.2 The arrangement of this paper

The rest of this paper is arranged as follows. We begin with a brief introduction of the model, market games, in section 2. Section 3 contains the main result of this paper; we show a necessary and sufficient condition for observed allocations to be seen as the core allocations. In section 4, we investigate relationship between testable implications of the core and the competitive equilibrium.

[1]Since the stable matching of two-sided matching model is the core with admissible coalition size is restricted to two, these studies can be categorised as investigation of the testable implications of the core.

2 Market games

Let $N = \{1, \ldots, n\}$ be a finite set of players. A *game* is a set function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Each subset A of N is called a *coalition.* $v(A)$ represents the *worth* of the coalition A. A *preimputation* is $U \in \mathbb{R}^N$ which satisfies $\sum_{i \in N} U_i = v(N)$. The *core* of a game v, denoted $c(v)$, is defined as

$$c(v) = \left\{ U \in \mathbb{R}^N \mid \sum_{i \in A} U_i \geq v(A) \text{ for all } A \subseteq N, \sum_{i \in N} U_i = v(N) \right\}. \tag{2.1}$$

A market $M = (N, \mathbb{R}^L_+, (\omega, \xi), (u_i)_{i \in N})$ consists of four elements. Each of the components is interpreted as follows:

- $\mathbb{R}^L_+$ is the commodity space,
- $(\omega, \xi) = (\omega_i, \xi_i)_{i \in N}$ where $(\omega_i, \xi_i) \in \mathbb{R}^L_+ \times \mathbb{R}$ is understood as a pair of the initial endowment of commodities and numeraire,
- $u_i : \mathbb{R}^L_+ \to \mathbb{R}_+$ is a continuous, concave, and increasing subutility function for i.

Utility for player i who possesses $x_i \in \mathbb{R}^L_+$ of commodities and $y_i \in \mathbb{R}$ of numeraire is given by $U_i(x_i, y_i) = u_i(x_i) + y_i$.

For a coalition $A \subseteq N$, $(x, y) = (x_i, y_i)_{i \in A} \in (\mathbb{R}^L_+ \times \mathbb{R})^A$ is a *feasible allocation for* A if $\sum_{i \in A}(x_i, y_i) \leq \sum_{i \in A}(\omega_i, \xi_i)$ holds. We denote X_A as the set of feasible commodity allocations for A.

Definition 2.1. A TU game (N, v_M) is called the *market game generated by* $M = (N, \mathbb{R}^L_+, (\omega, \xi), (u_i)_{i \in N})$ if

$$v_M(A) = \max_{x \in X_A} \sum_{i \in A} u_i(x_i) + \sum_{i \in A} \xi_i. \tag{2.2}$$

Definition 2.2. A feasible allocation $(x_i, y_i)_{i \in N}$ is a *core allocation* if

$$(U_i(x_i, y_i))_{i \in N} \in c(v_M). \tag{2.3}$$

In words, a TU game is the market game if the worth of each coalition is given by the maximal total utility that the coalition can achieve. We say a feasible allocation $(x_i, y_i)_{i \in N}$ is a core allocation of the market M if the preimputation derived from utility values from the allocation is in the core of the market game generated by M.

3 The core rationalization

Suppose that we have observed a finite number of allocations $\{(x^t, y^t)\}_{t=1}^T$, where $(x^t, y^t) = (x_i^t, y_i^t)_{i \in N} \in (\mathbb{R}^L_+ \times \mathbb{R})^N$ for all t. We refer to $\{(x^t, y^t)\}_{i=1}^T$ as *finite allocation data.*[2]

[2] Here we assume that commodity allocations as well as numeraire allocations can be observed. This assumption opposes the most of the literature investigating testable implications of transferable utility. For example, Cherchye et al. [5] provides both cases when transfers are observable and are not, noting that the unobservable case is more realistic when it comes to empirical applications. Their argument seems reasonable since in their setting, a household participates in the market and utility transfers are made among household's members, indicating the transfers occur outside the market. On the other hand, in the present setting, market participants are individuals and utility transfers are also made among these individuals. Therefore we can interpret that utility transfers, which is equivalent to numeraire transfers in our setting, occur inside the market. However, the observability assumption on numeraire allocation does not alter our result, which will be seen in Theorem 3.1 and the argument following it.

Our purpose in this section is to investigate a condition under which given finite allocation data $\{(x^t, y^t)\}_{i=1}^T$, each of the observed allocations (x^t, y^t) can be seen as a core allocation of some market. In what follows, we will provide two tests of hypothesis, one it the core rationalization and the other is the Pareto rationalization. We state the formal definitions of rationalizablity as follow.

Definition 3.1. An allocation data $\{(x^t, y^t)\}_{t=1}^T$ is *core rationalizable* if there exist concave, continuous, and increasing subutility functions $(u_i(\cdot))_{i\in N}$, and initial endowments $(\omega^t, \xi^t) = (\omega_i^t, \xi_i^t)_{i\in N} \in (\mathbb{R}_+^L \times \mathbb{R})^N$ for each t such that

- $\sum_{i\in N}(\omega_i^t, \xi_i^t) = \sum_{i\in N}(x_i^t, y_i^t)$ for all $t = 1, \ldots, T$
- $(U(x^t, y^t))_{i\in N} \in c(v_{M^t})$ for all $t = 1, \ldots, T$, where $M^t = (N, \mathbb{R}_+^L, (\omega^t, \xi^t), (u_i)_{i\in N})$.

Definition 3.2. An allocation data $\{(x^t, y^t)\}_{t=1}^T$ is *Pareto rationalizable* if there exist concave, continuous, and increasing subutility functions $(u_i(\cdot))_{i\in N}$ and initial endowments $(\omega^t, \xi^t) = (\omega_i^t, \xi_i)_{i\in N} \in (\mathbb{R}_+^L \times \mathbb{R})^N$ such that

- $\sum_{i\in N}(\omega_i^t, \xi_i^t) = \sum_{i\in N}(x_i^t, y_i^t)$ for all $t = 1, \ldots, T$
- $\sum_{i\in N} u_i(x_i^t) = v_{M^t}(N)$ for all $t = 1, \ldots, T$, where $M^t = (N, \mathbb{R}_+^L, (\omega^t, \xi^t), (u_i)_{i\in N})$.

Note that for Pareto rationalizability, only the observed commodity allocations matter; observing numeraire and specifying particular initial endowments are in fact redundant. To see this, suppose that $\{(x^t, y^t)\}_{t=1}^T$ is Pareto rationalizable with initial endowments $\{(\omega^t, \xi^t)\}_{t=1}^T$. Then for each t, $\sum_{i\in N} U_i(x_i^t, y_i^t) = v_{M^t}(N)$ holds, which is equivalent with (x^t, y^t) being a solution of

$$\begin{aligned} \max \sum_{i\in N} U_i(x_i, y_i) \\ \text{s.t.} \sum_{i\in N}(x_i, y_i) = \sum_{i\in N}(\omega_i^t, \xi_i^t). \end{aligned} \tag{3.1}$$

Since $\sum_{i\in N}(\omega_i^t, \xi_i^t) = \sum_{i\in N}(x_i^t, y_i^t)$ and $U_i(x_i, y_i) = u_i(x_i) + y_i$, the above optimization problem is equivalent with x^t being a solution of

$$\begin{aligned} \max \sum_{i\in N} u_i(x_i) \\ \text{s.t.} \sum_{i\in N} x_i = \sum_{i\in N} x_i^t. \end{aligned} \tag{3.2}$$

It is clear that, when the observed data is core rationalizable, it is also Pareto rationalizable. The following result states that the converse is also true; the core rationalizability and the Pareto rationalizability are observationally equivalent. Theorem 3.1 also says that the core rationalizability can be tested via verifying the existence of a solution for a particular system of linear inequalities. In order to give a formal statement, we need some additional concepts.

For a given commodity allocation data $\{x^t\}_{t=1}^T$ and a player $i \in N$, let $X_i = co\{x_i^t\}_{t=1}^T$ be the convex hull of observed allocations for the player. We say $u_i(\cdot)$ is *mixture linear on* X_i if for any $x, x' \in X_i$ and for all $\alpha \in [0, 1]$ it holds that $u_i(\alpha x + (1-\alpha)x') = \alpha u_i(x) + (1-\alpha)u_i(x')$. It is evident that u_i is mixture linear if and only if for any k, $x_1, \dots, x_k \in X_i$ and $\alpha_1, \dots, \alpha_k \in [0,1]$ with $\sum_j \alpha_j = 1$, we have $u_i(\sum_j \alpha_j x_j) = \sum_j \alpha_j u_i(x_j)$. We say u_i is *mixture non-linear on* X_i if it is not mixture linear on X_i.

The reason why we impose mixture non-linearity of $u_i(\cdot)$ on X_i is to guarantee the testing power of core rationalizability. In other words, without mixture non-linearity, any allocation data can be rationalized as the core allocations.[3] This is analogous to utility functions are assumed to be locally non-satiated in the literature of revealed preference test for utility maximization.

Theorem 3.1. *The following statements are equivalent.*

1. *An allocation data* $\{(x^t, y^t)\}_{t=1}^T$ *is core rationalizable by a set of continuous, concave, increasing, and mixture non-linear subutility functions* $\{u_i(\cdot)\}_{i\in N}$.

2. *An allocation data* $\{(x^t, y^t)\}_{t=1}^T$ *is Pareto rationalizable by a set of continuous, concave, increasing, and mixture non-linear subutility functions* $\{u_i(\cdot)\}_{i\in N}$.

3. *There exist numbers* $u_i^t \in \mathbb{R}_+$, $\lambda^t \in \mathbb{R}^L_{++}$ $(i \in N; t = 1, \dots, T)$ *such that,*

$$u_i^s - u_i^t \leq \langle \lambda^t, x_i^s - x_i^t \rangle, \ \forall i \in N, \ \forall s, t = 1, \dots, T, \tag{3.3}$$

where for each i *and* t *there exists an index* s *such that the above inequality is satisfied strictly.*

Proof. It is evident that 1 implies 2. Hence we only need to show that 2 implies 3, and 3 implies 1.

[2 $\Rightarrow$ 3] Suppose that an allocation data set $\{(x^t, y^t)\}_{t=1}^T$ is Pareto rationalizable. Then there exist concave, continuous, mixture non-linear and increasing subutility functions $u_i(\cdot)$, $(i \in N)$ such that each x^t is a Pareto optimal allocation. Then $x^t = (x_i^t)_{i\in N}$ is a solution of the problem

$$\max \sum_{i\in N} u_i(x_i) \tag{3.4}$$

$$\text{s.t.} \sum_{i\in N} x_i \leq \sum_{i\in N} x_i^t. \tag{3.5}$$

It can be shown that there exists $\lambda^t \in \mathbb{R}^L_{++}$ such that $\lambda^t \in \bigcap_{i\in N} \partial u_i(x_i^t)$, where $\partial u_i(x_i^t)$ is the set of supergradients of u_i at x_i^t (see Lemma A.2 in Appendix). Set numbers $u_i^t = u_i(x_i^t)$ $(t = 1, \dots, T, \ i \in N)$. Then it is easy to verify these numbers satisfy inequalities (3.3).

It remains to show that for each i and t, (3.3) holds strictly for some s. Take i and t arbitrarily. Suppose, on the contrary, that for any s we have $u_i^s - u_i^t = \langle \lambda^t, x_i^s - x_i^t \rangle$. This implies, by the definition of u_i^t, that $u_i(x_i^s) =$

[3] For example, the subutility function

$$u_i(x_i) = \begin{cases} \langle \mathbf{1}, x_i \rangle & \text{if } \sum_\ell x_{i\ell} \leq \sum_\ell \bar{x}_\ell, \\ \langle \theta, x_i - \bar{x} \rangle + \langle \mathbf{1}, \bar{x} \rangle & \text{otherwise.} \end{cases}$$

for all i, where $\theta = (\theta_\ell)_{\ell \in L}$ with $\theta_\ell \leq 1$ for all ℓ, and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^L$, rationalizes any allocation data. This indicates that imposing merely non-linearity is insufficient for core rationalization to possess testing power.

$u_i(x_i^t) + \langle \lambda^t, x_i^s - x_i^t \rangle$ for any observed allocation x_i^s for i. Take $\alpha_s \in [0,1]$ for $s = 1, \ldots, T$ with $\sum_s \alpha_s = 1$ as coefficients of the convex combination, and set $x_i^\alpha = \sum_s \alpha_s x_i^s$. Since $\lambda^t \in \partial u_i(x_i^t)$, we have

$$u_i(x_i^\alpha) - u_i(x_i^t) \leq \langle \lambda^t, x_i^\alpha - x_i^t \rangle = \sum_s \alpha_s \langle \lambda^t, x_i^s - x_i^t \rangle = \sum_s \alpha_s (u_i(x_i^s) - u_i(x_i^t)), \tag{3.6}$$

which implies

$$u_i(x_i^\alpha) - \sum_s \alpha_s u_i(x_i^s) \leq 0. \tag{3.7}$$

By the concavity of u_i, we obtain,

$$u_i(x_i^\alpha) = \sum_s \alpha_s u_i(x_i^s). \tag{3.8}$$

Now let $x, x' \in X_i$ be arbitrary. Since x and x' are in the convex hull generated by the observed allocations for i, we can find $\alpha_s, \alpha'_s \in [0,1]$ for $s = 1, \ldots, T$ with $\sum_s \alpha_s = \sum_s \alpha'_s = 1$ such that $x = \sum_s \alpha_s x_i^s$ and $x' = \sum_s \alpha'_s x_i^s$. Let $\beta \in [0,1]$ be arbitrary. Then,

$$\begin{aligned}
u_i(\beta x + (1-\beta)x') &= u_i\left(\beta \sum_s \alpha_s x_i^s + (1-\beta) \sum_s \alpha'_s x_i^s\right) && (3.9)\\
&= u_i\left(\sum_s (\alpha_s \beta + \alpha'_s (1-\beta)) x_i^s\right) && (3.10)\\
&= \sum_s (\alpha_s \beta + \alpha'_s (1-\beta)) u_i(x_i^s) && (3.11)\\
&= \beta \sum_s \alpha_s u_i(x_i^s) + (1-\beta) \sum_s \alpha'_s u_i(x_i^s) && (3.12)\\
&= \beta u_i(x) + (1-\beta) u_i(x'), && (3.13)
\end{aligned}$$

where the third and the last equality are followed by (3.6). However, this means that u_i is mixture linear on X_i, which contradicts the assumption.

[3 ⇒ 1] Now suppose that we have found a set of numbers satisfying (3.3). For each i, define $u_i : \mathbb{R}_+^L \to \mathbb{R}$ as

$$u_i(x_i) = \min_t \left\{ u_i^t + \langle \lambda^t, x_i - x_i^t \rangle \right\}.$$

Since $u_i(\cdot)$ is a minimum of affine functions, it is concave and continuous. By construction, it is clear that $u_i(\cdot)$ is increasing and $u_i(x_i^t) = u_i^t$. Take $(\omega^t, \xi^t) \in \mathbb{R}_+^L \times \mathbb{R}$ such that $\sum_i (\omega_i^t, \xi_i^t) = \sum_i (x_i^t, y_i^t)$ and $\langle \lambda^t, \omega_i^t - x_i^t \rangle + \xi_i^t \leq y_i^t$ for each $i \in N$ and each t (for example, $(\omega^t, \xi^t) = (x^t, y^t)$ meets the requirement). We will show that (x^t, y^t) is a core allocation in $M^t = (N, \mathbb{R}_+^L, (\omega^t, \xi^t), (u_i)_{i \in N})$.

Take $x = (x_i)_{i \in N}$ arbitrarily which satisfies $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i^t$. Then we have

$$\begin{aligned}
\sum_{i \in N} u_i(x_i) &\leq \sum_{i \in N} (u_i^t + \langle \lambda^t, x_i - x_i^t \rangle) && (3.14)\\
&\leq \sum_{i \in N} (u_i^t + \langle \lambda^t, \omega_i^t - x_i^t \rangle) = \sum_{i \in N} u_i^t = \sum_{i \in N} u_i(x_i^t), && (3.15)
\end{aligned}$$

where the first inequality holds by the construction of u_i and the second holds because $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i^t$, and the third equality holds since $\sum_{i \in N} \omega_i^t = \sum_{i \in N} x_i^t$. Therefore, it has shown that x^t is a Pareto optimal allocation.

Let $A \subsetneq N$ and take $(x,y) = (x_i, y_i)_{i\in A}$ arbitrarily which satisfies $\sum_{i\in A}(x_i, y_i) \leq \sum_{i\in A}(\omega_i^t, \xi_i^t)$. For such A and (x,y), we have

$$\sum_{i\in A}(u_i(x_i) + y_i) \leq \sum_{i\in A}(u_i^t + \langle \lambda^t, x_i - x_i^t\rangle + y_i) \tag{3.16}$$

$$\leq \sum_{i\in A}(u_i^t + \langle \lambda^t, \omega_i^t - x_i^t\rangle + \xi_i^t) \leq \sum_{i\in A}(u_i^t + y_i^t) = \sum_{i\in A} U_i(x_i^t, y_i^t), \tag{3.17}$$

where the first inequality holds by the construction of u_i, the second holds because $\sum_{i\in A}(x_i, y_i) \leq \sum_{i\in A}(\omega_i^t, \xi_i^t)$, and the third follows from $\langle \lambda^t, \omega_i^t - x_i^t\rangle + \xi_i^t \leq y_i^t$ for all i. This shows that (x^t, y^t) is a core allocation.

Now what remains to show is the mixture non-linearity of $u_i(\cdot)$ for every $i \in N$. Let $\alpha_s \in (0,1)$ for $s = 1, \ldots, T$ such that $\sum_s \alpha_s = 1$. Note that by assumption, there is no index t such that $u_i^s - u_i^t = \langle \lambda^t, x_i^s - x_i^t\rangle$ for all s. Define $x^\alpha = \sum_s \alpha_s x_i^s$. Let $t \in \{1, \ldots, T\}$ be such that $u_i(x^\alpha) = u_i^t + \langle \lambda^t, x^\alpha - x_i^t\rangle$. Then

$$u_i(x^\alpha) - \left(\sum_s \alpha_s u_i(x_i^s)\right) = u_i^t + \sum_s \alpha_s \langle \lambda^t, x_i^s - x_i^t\rangle - \sum_s \alpha_s u_i^s \tag{3.18}$$

$$= \sum_s \alpha_s((u_i^t - u_i^s) + \langle \lambda^t, x_i^s - x_i^t\rangle) \tag{3.19}$$

$$> \sum_s \alpha_s(\langle \lambda^t, x_i^t - x_i^s\rangle + \langle \lambda^t, x_i^s - x_i^t\rangle) = 0, \tag{3.20}$$

where the inequality holds because $u_i^s - u_i^t \leq \langle \lambda^t, x_i^s - x_i^t\rangle$ for all s, and the strict inequality holds at least for one s. This shows the mixture non-linearity of u_i.

□

As noted before, for Pareto rationalization, we only need to observe commodity allocations $\{x^t\}_{t=1}^T$. The equivalence between Pareto rationalization and core rationalization provided in the preceding result indicates that the same is in fact true for core rationalization. Precisely, it is also true that only observed commodity allocations $\{x^t\}_{t=1}^T$ matter for core rationalization. This can be seen in the equivalent inequality condition; no condition on $\{y^t\}_{t=1}^T$ is involved. Therefore, even if we cannot observe numeraire allocations $\{y^t\}_{t=1}^T$, testable implications of the core does not alter, and exactly the same result as Theorem 3.1 holds.[4]

The following example shows that there exists an allocation data which is not core rationalizable, indicating that the core rationalizability is testable.

Example. Suppose $N = \{1,2\}$, $L = \{1,2\}$ and $T = 2$. Consider the following allocation data:

$$x_1^1 = (1,2),\ x_1^2 = (4,6),\ x_2^1 = (9,5),\ x_2^2 = (6,1). \tag{3.21}$$

This data is not core rationalizable. Suppose, on the contrary, that the data is core rationalizable. Then by Theorem 3.1, there exists $\lambda^t \in \mathbb{R}^2_{++}$ and $u_i^t \in \mathbb{R}$ for $t = 1,2$ and $i = 1,2$ such that,

$$u_i^1 - u_i^2 < \langle \lambda^2, x_i^2 - x_i^1\rangle, \text{ and } u_i^2 - u_i^1 < \langle \lambda^1, x_i^1 - x_i^2\rangle \tag{3.22}$$

[4] A similar redundancy result of numeraire observation can be inferred by comparing Brown and Calsamiglia [2] and Sákovics [9].

for $i = 1, 2$, which implies

$$\langle \lambda^1, x_i^2 - x_i^1 \rangle < \langle \lambda^2, x_i^2 - x_i^1 \rangle \tag{3.23}$$

for $i = 1, 2$. However, this is impossible since it requires that

$$3\lambda_1^1 + 4\lambda_2^1 < 3\lambda_1^2 + 4\lambda_2^2, \text{ and } -3\lambda_1^1 - 4\lambda_2^1 < -3\lambda_1^2 - 4\lambda_2^2. \tag{3.24}$$

4 Relationship between testable implications of the core and the competitive equilibrium

In this section, we investigate the relationship between testable implications of the core rationalizability and the rationalizabitlity of the competitive equilibrium, in the sense of Brown and Matzkin [3]. We will see a kind of dual relationship between testable implications of the core rationalizability and the rationalizabitlity of the competitive equilibrium.

Brown and Matzkin [3] clarifies the testable implications of the competitive equilibrium in pure exchange economies on the equilibrium manifold. In other words, it shows a condition under which observed market prices can be seen as competitive equilibrium prices. Hence our observations change from allocations to market prices. We first recall setting given in Brown and Matzkin [3].[5]

Suppose that we have observed $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$, where $p^t \in \mathbb{R}_{++}^L$ is the market price of commodities and $(\omega^t, \xi^t) = (\omega_i^t, \xi_i^t)_{i \in N}$ with $(\omega_i^t, \xi_i^t) \in \mathbb{R}_+^L \times \mathbb{R}$ is the initial endowment of commodities and numeraire. We refer $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ as a *market situation*. We say a market situation $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is *rationalizable as competitive equilibrium* if there exist allocations $(\bar{x}^t, \bar{y}^t) = (\bar{x}_i^t, \bar{y}_i^t)_{i \in N}$ for each t such that $(p^t, 1, \bar{x}^t, \bar{y}^t)$ constitutes a competitive equilibrium. The formal definition is as follows.

Definition 4.1. A market situation $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is *rationalizable as competitive equilibrium* if there exist concave, continuous, increasing and mixture non-linear subutility function $u_i(\cdot)$ for each $i \in N$ and allocation $(\bar{x}^t, \bar{y}^t) = (\bar{x}_i^t, \bar{y}_i^t)_{i \in N} \in (\mathbb{R}_+^L \times \mathbb{R})^N$ for each $t = 1, \ldots, T$ such that the following are satisfied;

- Utility maximization: $\langle p^t, x_i \rangle + y_i \leq \langle p^t, \omega_i^t \rangle + \xi_i^t$ implies $U_i(x_i, y_i) \leq U_i(\bar{x}_i^t, \bar{y}_i^t)$,
- Feasibility: $\sum_{i \in N} (\bar{x}_i^t, \bar{y}_i^t) = \sum_{i \in N} (\omega_i^t, \xi_i^t)$.[6]

Theorem 4.1 (Brown and Matzkin [3], Brown and Calsamiglia [2]). *The following statements are equivalent.*

1. *A market situation $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is rationalizable as competitive equilibrium of some TU pure exchange economy.*

[5] A slight modification is made in order to incorporate their argument in our TU setting.

[6] As before $U_i(x_i, y_i) = u_i(x_i) + y_i$.

2. *There exist numbers $u_i^t \in \mathbb{R}_+$ and $(\bar{x}_i^t, \bar{y}_i^t) \in \mathbb{R}_+^L \times \mathbb{R}$ $(t = 1, \ldots, T; i \in N)$ such that*

$$u_i^t - u_i^s \leq \langle p^s, \bar{x}_i^t - \bar{x}_i^s \rangle \qquad \forall i \in N,\ \forall s, t = 1, \ldots, T, \tag{4.1}$$

$$\langle p^t, \bar{x}^t \rangle + \bar{y}_i^t = \langle p^t, \omega_i^t \rangle + \xi_i^t \qquad \forall i \in N,\ \forall t = 1, \ldots, T, \tag{4.2}$$

$$\sum_{i \in N} (\bar{x}_i^t, \bar{y}_i^t) = \sum_{i \in N} (\omega_i^t, \xi_i^t) \qquad \forall t = 1, \ldots, T, \tag{4.3}$$

where for each i and t, there exists an index s such that (4.1) holds strictly.

Proof. It is evident that the feasibility and (4.3) is equivalent. Brown and Calsamiglia [2] shows that (4.1) and (4.2) are equivalent to the existence of $u_i(\cdot)$ which is concave, continuous, and increasing such that $(\bar{x}_i^t, \bar{y}_i^t)$ maximizes $U_i(x_i, y_i)$ under the budget constraint $\langle p^t, x_i \rangle + y_i \leq \langle p^t, \omega_i^t \rangle + \xi_i^t$. We have shown, in Theorem 3.1, that mixture non-linearity is equivalent to the condition that, for each i and t, there exists an index s such that (4.1) holds strictly. □

When $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is rationalized as competitive equilibrium, we can find $(\bar{x}^t, \bar{y}^t)$ for each t so that $(p^t, 1, \bar{x}^t, \bar{y}^t)$ constitutes a competitive equilibrium. Since equilibrium allocations are core allocations, it must be the case that $\{(\bar{x}^t, \bar{y}^t)\}_{t=1}^T$ is core rationalizable. Our inequality conditions correspond this fact; letting $\lambda^t = p^t$ for all t in inequalities (4.1) leads to the inequalities in Theorem 3.1.

Remarkably, we can say the converse in the following sense: When $\{(x^t, y^t)\}_{t=1}^T$ is core rationalizable, we can find (ω^t, ξ^t) and p^t for each t such that $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is rationalized as competitive equilibrium. Moreover, in order to satisfy the inequality condition in Theorem 4.1, we can take $(\bar{x}^t, \bar{y}^t)$ to be the same as the originally observed core allocation, i.e., $(\bar{x}^t, \bar{y}^t) = (x^t, y^t)$. It turns out, therefore, that a core rationalizable allocation (x^t, y^t) can be rationalized as the competitive equilibrium allocation as well. We formally state this result as follows.

Theorem 4.2. 1. *If a market situation $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ is rationalized as competitive equilibrium, then we can find (x^t, y^t) such that $\{(x^t, y^t)\}_{t=1}^T$ is core rationalizable. In particular, as λ^t in inequality (3.3), we can take the observed price, i.e., $\lambda^t = p^t$.*

2. *If a set of allocations $\{(x^t, y^t)\}_{t=1}^T$ is core rationalizable, then we can find a market situation $\{(p^t, \omega^t, \xi^t)\}_{t=1}^T$ which is rationalized as competitive equilibrium. In particular, as $(\bar{x}^t, \bar{y}^t)$ in inequality (4.1), we can take the observed allocations, i.e., $(\bar{x}^t, \bar{y}^t) = (x^t, y^t)$.*

Proof. 1. Follows from the inspection given preceding to the statement of this result.

2. The strategy of this proof is essentially the same as we show [3 ⇒ 1] in Theorem 3.1, but for the sake of completeness, we repeat it here explicitly. Since $\{(x^t, y^t)\}_{t=1}^T$ is core rationalizable, we can find $\lambda^t \in \mathbb{R}_{++}^L$, $u_i^t \in \mathbb{R}_+$ for each t and i, such that inequality (3.3) holds. Let (ω^t, ξ^t) be such that $\sum_i (\omega_i^t, \xi_i^t) = \sum_i (x_i^t, y_i^t)$ and $\langle \lambda^t, \omega_i^t - x_i^t \rangle + \xi_i^t = y_i^t$ for each i. By defining $p^t = \lambda^t$ and $(\bar{x}^t, \bar{y}^t) = (x^t, y^t)$, it is easily verified that $(p^t, \omega^t, \xi^t, \bar{x}^t, \bar{y}^t, u_i^t)$ meets the inequality conditions (4.1)-(4.3). Therefore by Theorem 4.1, $\{(p^t, \omega^t)\}_{t=1}^T$ is rationalized as competitive equilibrium. □

Note that, since the Pareto rationalizability and the core rationalizability are equivalent, as we have shown in Theorem 3.1, the statement remains true even if we replace the term "core rationalizability" with "Pareto rationalizability" in the statement 2 of Theorem 4.2.

The relationship between rationalizability by the core and the competitive equilibrium reminiscent duality. For the core rationalization, observations are allocations and we need to find suitable supergradients of utility functions. On the other hand, for the rationalization by competitive equilibrium, observations are market prices and we need to find suitable allocations. These two rationalizations are connected by the inequality condition (3.3) (or equivalently (4.1)); the supergradients and the observed market price play exactly the same role in the inequality conditions.

A Appendix

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. The set

$$\partial f(x) = \{\lambda \in \mathbb{R}^n \mid f(y) \leq f(x) + \langle \lambda, y - x \rangle \ \forall y \in \mathbb{R}^n\} \tag{A.1}$$

is called the *superdifferential* of f at x. An element $\lambda \in \partial f(x)$ is called a *supergradient* of f at x. If f is concave, then $\partial f(x)$ is non-empty (see Schirotzek [10]).

For $x, \xi \in \mathbb{R}^n$, we call

$$f'(x;\xi) = \lim_{\theta \downarrow 0} \frac{f(x + \theta\xi) - f(x)}{\theta} \tag{A.2}$$

as the *directional Gâteau derivative* of f at x in the direction ξ. If f is convex, $f'(x;\xi)$ always exists and

$$\partial f(x) = \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \xi \rangle \geq f'(x;\xi) \ \forall \xi \in \mathbb{R}^n\} \tag{A.3}$$

holds (see Shirotzek [10]).

Lemma A.1. *Let $f : (\mathbb{R}^L_+)^N \to \mathbb{R}$ be defined as $f(x) = \sum_{i \in N} u_i(x_i)$ for $x = (x_i)_{i \in N}$. Then we have,*

$$\partial f(x) = \times_{i \in N} \partial u_i(x_i). \tag{A.4}$$

Proof. Let $\lambda \in \times_{i \in N} \partial u_i(x_i)$, or in other words, $\lambda = (\lambda_i)_{i \in N}$, $\lambda_i \in \partial u_i(x_i)$ for all $i \in N$. We have $u(y_i) \leq u_i(x_i) + \langle \lambda_i, y_i - x_i \rangle$ for all $y_i \in \mathbb{R}^L$ for all i. Sum both sides of the inequality over $i \in N$, we obtain

$$f(y) = \sum_{i \in N} u_i(y_i) \leq \sum_{i \in N} u_i(x_i) + \sum_{i \in N} \langle \lambda_i, y_i - x_i \rangle \tag{A.5}$$

$$= f(x) + \langle \lambda, y - x \rangle, \tag{A.6}$$

for all $y = (y_i)_{i \in N}$. Therefore $\lambda \in \partial f(x)$.

Let $\lambda = (\lambda_i)_{i \in N} \in \partial f(x)$. Take $i \in N$ and $\xi_i \in \mathbb{R}^L$ arbitrarily. Let $\zeta = (\zeta_j)_{j \in N} \in (\mathbb{R}^L)^N$ be $\zeta_j = 0$ for $j \neq i$ and $\zeta_i = \xi_i$. Then we have

$$f'(x;\zeta) = \lim_{\theta \downarrow 0} \frac{1}{\theta}(f(x + \theta\zeta) - f(x)) \tag{A.7}$$

$$= \lim_{\theta \downarrow 0} \frac{1}{\theta}(u_i(x_i + \theta\xi_i) - u_i(x_i)) \tag{A.8}$$

$$= u_i'(x_i;\xi_i) \leq \langle \lambda, \zeta \rangle = \langle \lambda, \xi_i \rangle \tag{A.9}$$

where the inequality holds because $\lambda \in \partial f(x)$. Since x_i and i was arbitrary, we have $\lambda_i \in \partial u_i(x_i)$ for all $i \in N$. □

Lemma A.2. *If $x^* = (x_i^*)_{i \in N}$ is a solution of the following problem:*

$$\max \sum_{i \in N} u_i(x_i)$$
$$s.t. \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i^t, \tag{P}$$

then there exists $\lambda \in \mathbb{R}^L_{++}$ such that $\lambda \in \bigcap_{i \in N} \partial u_i(x_i^*)$.

Proof. Let $f(x) = \sum_{i \in N} u_i(x_i)$ for $x = (x_i)_{i \in N}$. For $l \in L$, define $a_l \in (\mathbb{R}^L)^N$ by $a_{l,i} = \chi_{\{l\}} \in \mathbb{R}^L$ where $\chi_{\{l\}}(k) = 1$ if $k = l$ and $\chi_{\{l\}}(k) = 0$ if $k \neq l$. Define also $\beta_l = \sum_{i \in N} \omega_{i,l}$ for each $l \in L$. Let $C_l = \{x \in (\mathbb{R}^L)^N \mid g_l \geq 0\}$ where $g_l(x) = \beta_l - \langle a_l, x \rangle$, and $C = \bigcap_{l \in L} C_l$. Then maximization problem (P) is equivalently written as

$$\max f(x)$$
$$\text{s.t.} x \in C. \tag{P'}$$

Then, by the Kuhn-Tucker theorem, $x^* = (x_i^*)_{i \in N}$ is a solution of (P') if and only if there exists $b \in \partial f(x^*)$ and $\lambda = (\lambda_l)_{l \in L} \in \mathbb{R}^L_{++}$ such that

$$b = \sum_{l \in L} \lambda_l a_l. \tag{A.10}$$

By definition of a_l, we have $b = (\lambda, \dots, \lambda)$. Now by Lemma A.1, $b \in \times_{i \in N} \partial u_i(x_i^*)$. Therefore we have $\lambda \in \bigcap_{i \in N} u_i(x_i^*)$. □

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