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Abstract

The literature on endogenous growth cycles with R&D predicts counter-cyclical allocation of resources to R&D. However, this prediction is not supported by empirical studies. This paper considers the R&D-based growth model with endogenous fluctuations introducing population growth and negative externality that works in R&D. We show that this simple modification makes R&D investment pro-cyclical along sustained business cycles using both an overlapping generations framework and an infinitely-lived agents framework.
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1 Introduction

This paper examines the how R&D activity varies over the business cycle using the framework of the R&D-based growth models. Studies of the interaction between R&D and endogenous fluctuations were pioneered by Judd (1985) and Deneckere and Judd (1992). They found fluctuating equilibrium paths in the variety-expansion model without capital accumulation by applying a bifurcation theorem. Matsuyama (1999, 2001) has modified the model in Deneckere and Judd (1992) by introducing capital accumulation and intertemporal utility maximization, and investigated endogenous

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growth with fluctuations.[1] One of the main predictions of these theoretical models was counter-cyclical behavior of R&D over the sustained growth cycles, that is, with low growth, resource allocation to R&D is high. In addition, this counter-cyclicality of R&D made productivity improvements counter-cyclical. Francois and Lloyd-Ellis (2003) have studied the endogenous growth model with the endogenous fluctuations, which was based on the quality-ladder framework of Grossman and Helpman (1991) and the theory of implementation cycle of Shleifer (1986). In their model, productivity improvements were pro-cyclical; however R&D expenditure was still counter-cyclical. Other several literature, such as Wälde (2002), Bental and Peled (1996), and Francois and Lloyd-Ellis (2008), predicted the counter-cyclical behavior of R&D.[2]

However, the prediction that R&D expenditure is counter-cyclical is difficult to justify from empirical studies. Wälde and Woitek (2004) have studied cyclical properties of R&D in G7 countries using annual data for 1973 to 2000. They found that aggregate R&D expenditure tended to be pro-cyclical, and argued that the prediction of Matsuyama (1999, 2001) was counter-factual. Fatás (2000) and Comin and Gertler (2006) also have found a highly pro-cyclical tendency of R&D expenditure using U.S. data. In particular, Comin and Gertler (2006) focused on longer term oscillations than conventional business cycles. They called such oscillations as the " medium-term cycle", which was defined as including frequencies between a half year and 50 years. In this respect, there is close relationship between their empirical study and our theoretical analysis. Further, Geroski and Walters (1995) argued that productivity improvement was also pro-cyclical from U.K. data. Barlevy (2007), using data from both the NSF and Standard & Poor's Compustat database of publicly traded companies, found a

[1] More precise studies of the dynamics of Matsuyama's model were presented by Mitra (2001), Mukherji (2005), and Gardini et al. (2008). Gardini et al. (2008) showed that no stable cycle can exist except for period-2 cycles.

[2] Francois and Lloyd-Ellis (2008) did not interpret the activity that was a source of productivity improvements as R&D, but as "entrepreneurial search". However, its process was formally identical to the R&D process in the earlier model such as Grossman and Helpman (1991) and Francois and Lloyd-Ellis (2003).

positive correlation between the growth rate of R&D at the industry level as well as the aggregate level.

The main purpose of this paper is to provide the endogenous fluctuations model with pro-cyclical behavior of R&D. We modify the variety-expansion model in Matsuyama (1999, 2001), introducing population growth and a negative externality that affects the productivity of R&D. In the model developed here, we assume that finding new knowledge becomes more difficult as economies become technologically more advanced, as in the semi-endogenous growth model in Jones (1995) and Segerstrom (1998).[3] This assumption has been first proposed to eliminate the scale effect, which is serious counter-factual prediction in the first-generation R&D-based endogenous growth models such that the economy with a large population grows faster.[4]

The literature that has a common purpose with this paper includes Wälde (2005), Francois and Lloyd-Ellis (2009), Comin and Gertler (2006), and Barlevy (2007). Francois and Lloyd-Ellis (2009) have studied the endogenous business cycle model based on Francois and Lloyd-Ellis (2003), and decomposed the innovation process into three distinct stages: R&D, commercialization, and innovation, to modify their previous result. Their model provided the pro-cyclical movement of R&D, and counter-cyclical movement of commercialization played central role for this new result. Further, they showed that the total expenditure for innovation, which was defined by the sum of expenditure for R&D and commercialization, moved pro-cyclically in their numerical example. Wälde (2005) also showed pro-cyclical R&D behavior, by using a quality-ladder framework with capital accumulation. Above two models had a similar property to our model in that they assumed the negative externality of knowledge accumulation and derive non-scale growth with endogenous fluctuations. On the other hand,

[3] Jones (1995) calls such externality the fishing out effect.

[4] Jones (1995) was also the study based on the variety-expansion model in Romer (1990), however, its balanced growth path has a saddle property, and no endogenous fluctuation occurs as proved by Arnold (2006). Note that in order to examine the dynamics analytically, Arnold (2006) assumed constant returns to labor in R&D, which was not assumed in Jones' original model.

Comin and Gertler (2006) and Barlevy (2007) have discussed the cyclicality of R&D over the business cycles that were caused by exogenous shocks. The former was based on a variety-expansion framework, and uses similar approach to Francois and Lloyd-Ellis (2009), i.e., decomposing the innovation process.[5] The later, using a quality-ladder framework, showed that equilibrium R&D was pro-cyclical in a decentralized market, however, optimal R&D was counter-cyclical by a central planner's problem.

As can be seen from these studies, theoretical explanation of pro-cyclicality of R&D is one of the most controversial topics in the studies of R&D and business cycles. This paper achieves the pro-cyclical R&D behavior under more simple assumption than Francois and Lloyd-Ellis (2009) and Wälde (2005), and does not requires the exogenous shocks unlike in the Comin and Gertler (2006) and Barlevy (2007).

The rest of this paper is organized as follows. The next section sets up the model used in our theoretical investigation, and derives the law of motion which characterizes the equilibrium path of the economy. Section 3 examines the dynamical property of the model, and shows that the equilibrium path fluctuates endogenously. Section 4 focuses on period-2 cycles, and studies the cyclicality of R&D investment. Section 5 studies the model with infinitely lived agents to show robustness of our results. Section 6 provides conclusions.

2 The Model

We consider a dynamic model based on Matsuyama (1999). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. There is a single final good taken as a numeraire, which is produced using intermediate goods and labor. It can be consumed or invested. A new variety of intermediate goods is invented by allocating capital for R&D activities, and inventors enjoy a one-period monopoly by patent protection. The available intermediate goods are produced by multiple intermediate firms using capital. Finally, we assume two-period-lived overlapping generations households, who supply labor

[5] They introduced the stage of "adoption" instead of commercialization.

inelastically when young.

Final goods. We assume that perfect competition prevails in the final goods market. The production function is given by

$$Y_t = AL_t^{1-\alpha} \int_0^{N_t} x_t(z)^\alpha dz, \quad 0 < \alpha < 1, \quad A > 0. \tag{2.1}$$

where Y_t is the amount of final output, L_t is labor which supplied inelastically, $x_t(z)$ is the amount of the intermediate good indexed by z, and $1/1-\alpha$ denotes the elasticity of substitution between every pair of intermediate goods. N_t is the number of available intermediate goods in period t, which represents a technology level of the economy.

Profit maximization yields $w_t = (1-\alpha)Y_t/L_t$ and the inverse demand function for each intermediate good z as

$$p_t(z) = \alpha AL_t^{1-\alpha} x_t(z)^{-(1-\alpha)}, \quad \text{for } z \in (0, N_t], \tag{2.2}$$

where w_t is the real wage rate and $p_t(z)$ is the price of the intermediate good z.

Intermediate goods. Each intermediate good is produced by using one unit of capital. Because of limited patent protection, the "old" intermediate goods, $[0, N_{t-1})$, are supplied competitively, and hence the price is equal to marginal cost, $p_t(z) = r_t$, for $z \in (0, N_{t-1}]$. However, the "new" intermediate goods invented in period $t-1$, $(N_{t-1}, N_t]$, are supplied monopolistically, and sold at monopoly price, $p_t(z) = r_t/\alpha$, for $z \in (N_{t-1}, N_t]$. All intermediate goods enter symmetrically into the production of the final good, i.e. $x_t(z) = x_{ct}$, for $z \in [0, N_{t-1}]$, and $x_t(z) = x_{mt}$, for $z \in (N_{t-1}, N_t]$. From (2.2), we can easily show that $x_{mt} = \alpha^{\frac{1}{1-\alpha}} x_{ct}$ holds, and the maximized monopoly profit is

$$\Pi_t(z) = \Pi_t \equiv \frac{1-\alpha}{\alpha} x_{mt} r_t, \quad \text{for } z \in (N_{t-1}, N_t]. \tag{2.3}$$

Considering these results of profit maximization of the intermediate goods firms, we can rewrite the production function (2.1) as

$$Y_t = AL_t^{1-\alpha} (\alpha^{\frac{1}{1-\alpha}} x_{ct})^\alpha N_{t-1} \left[\frac{N_t}{N_{t-1}} - 1 + \alpha^{-\frac{\alpha}{1-\alpha}} \right]. \tag{2.4}$$

R&D. The number of intermediate goods, N, expands according to the following equation: [6]

$$N_t - N_{t-1} = \eta \frac{R_t}{N_{t-1}^{\phi}}, \quad N_0 > 0, \quad \phi > 0, \quad \eta > 0,$$

where R_t is the amount of the capital allocated to R&D. Following the formation adopted in Jones (1995), we assume that the past discoveries make more difficulty to invent the new machine. This effect is captured by $\phi > 0$.

Each inventor enjoys one-period monopoly and earns the profit Π_t. Therefore, in equilibrium, the following free-entry condition must be hold:

$$\Pi_t^m \leq \eta^{-1} N_{t-1}^{\phi} r_t, \quad \text{with an equality whenever } N_t > N_{t-1}. \tag{2.5}$$

A break-even point of x_{mt} is given by $\bar{x}_{mt} \equiv \frac{\alpha}{1-\alpha}\eta^{-1}N_{t-1}^{\phi}$. It becomes larger for large value of ϕ since R&D becomes costlier for given N_{t-1}.

Finally, the capital market clearing requires

$$K_{t-1} = R_t + (N_t - N_{t-1})x_{mt} + N_{t-1}x_{ct}, \tag{2.6}$$

where K_{t-1} is the amount of capital which is accumulated in period $t-1$ and available in period t. Available capital is used by (1)R&D, (2) producing monopolistic intermediate goods, and (3)producing competitive intermediate goods, as shown on the right-hand side of (2.6).

Consumers. Each consumer lives for two period. When young, she supplies one unit of labor and earns the wage w_t, which is divided into saving and consumption. When old, she only consumes using her saving. Let c_{1t} and c_{2t+1} denote the consumptions in periods t and $t+1$ of the consumers born in period t, respectively. Each consumer chooses c_{1t} and c_{2t+1} that maximizes the utility $U_t = (1-s)\log c_{1t} + s \log c_{2t+1}$, subject to the budget constraint, $c_{2t+1} = (w_t - c_{1t})r_{t+1}$.

[6] This specification is based on Rivera-Batiz and Romer (1991) "the lab equipment model".

The solution to this simple maximization problem is characterized by the following linear saving function.

$$K_t = sw_tL_t = s(1-\alpha)Y_t, \tag{2.7}$$

where L_t represents the number of consumers born in period t, which grows at the exogenous rate n, i.e., $L_t = (1+n)L_{t-1}$.

2.1 Equilibrium

Substituting (2.3), (2.5), and $x_{ct}/x_{mt} = \alpha^{-\frac{1}{1-\alpha}}$ into (2.6) yields

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \mu(k_{t-1}) \equiv \max\{0, \alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)\}, \tag{2.8}$$

$$x_{ct} = \max\left\{\frac{K_t}{N_{t-1}}, \alpha^{-\frac{1}{1-\alpha}}\bar{x}_{mt}\right\}, \tag{2.9}$$

where we define $k_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta K_t/N_{t-1}^{1+\phi}$. If $k_{t-1} > 1$ holds, i.e., the economy has the sufficient stock of capital relative to the technological level, the positive amount of capital is allocated R&D and $N_t > N_{t-1}$ holds. On the other hand, if $k_{t-1} \leq 1$, no R&D occurs and no technological progress arises.

Substituting (2.8) and (2.9) into (2.4) shows that the total output is equal to

$$Y_t = \frac{AL_t^{1-\alpha}N_{t-1}^{1+\alpha\phi}}{[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta]^\alpha}\psi(k_{t-1}), \quad \psi(k_{t-1}) \equiv \begin{cases} k_{t-1}^\alpha, & \text{if } k_{t-1} \leq 1, \\ k_{t-1}, & \text{if } k_{t-1} > 1. \end{cases} \tag{2.10}$$

To describe the equilibrium path of this economy, we define the following a new variable :

$$\ell_{t-1} \equiv [s(1-\alpha)A]^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\eta\frac{L_t}{N_{t-1}^\phi}.$$

Summarizing (2.7), (2.8) and (2.10) yields the following two-dimensional dynamical system:

$$\ell_t = f^\ell(k_{t-1}, \ell_{t-1}) = \begin{cases} (1+n)\ell_{t-1}, & \text{if } k_{t-1} \leq 1 \\ \dfrac{(1+n)\ell_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^\phi}, & \text{if } k_{t-1} > 1 \end{cases}$$
$$k_t = f^k(k_{t-1}, \ell_{t-1}) = \begin{cases} \ell_{t-1}^{1-\alpha} k_{t-1}^{\alpha}, & \text{if } k_{t-1} \leq 1 \\ \dfrac{\ell_{t-1}^{1-\alpha} k_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{1+\phi}}, & \text{if } k_{t-1} > 1 \end{cases} \tag{2.11}$$

If the initial values of k_0 and ℓ_0 are given, the law of motion (2.11) characterizes the equilibrium path, $\{k_t, \ell_t\}_0^\infty$, whose properties are dependent on parameter values, α, n, and ϕ.

3 Dynamics

The law of motion (2.11) has a unique positive fixed point, (k^*, ℓ^*), where

$$k^* = 1 + \alpha^{\frac{\alpha}{1-\alpha}}[(1+n)^{\frac{1}{\phi}} - 1] > 1, \quad \ell^* = (1+n)^{\frac{1+\phi}{\phi(1-\alpha)}}. \tag{3.1}$$

In the long-run, $k \leq 1$ is not sustainable by an exogenous population growth, therefore, (k^*, ℓ^*) is a unique non-trivial fixed point of the dynamical system (2.11).[7] At this fixed point, K, Y (or per capita output $y \equiv Y_t/(L_{t-1} + L_t)$), and N grow at constant rates, i.e., the economy achieves the balanced growth. The balanced growth rates of per capita output is derived as $g_y^* = (1+n)^{\frac{1}{\phi}} - 1$, which is independent of population, L.

Stability. The two-dimensional system (2.11) has two predetermined variable, k and ℓ. If the fixed point is a sink, it is locally stable.

Proposition 3.1 There is a unique bifurcation pointof ϕ, ϕ_b, which satisfies $B(\phi_b) - \Lambda(\phi_b) = 0$. The fixed point (k^*, ℓ^*) is a sink for $\phi < \phi_b$, whereas is a saddle point for $\phi > \phi_b$, where $B(\alpha, \phi)$ and $\Lambda(\phi)$ are defined as follows:

$$B(\phi) \equiv \frac{2 - \phi(1+\alpha)}{2 + \phi(1+\alpha)}, \quad \Lambda(\phi) \equiv \frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}.$$

[7] Substituting $\ell_t = \ell_{t-1} > 0$ into $\ell_t = f^\ell(k_{t-1}, \ell_{t-1})$, and solving for k_{t-1}, we obtain k^* uniquely. There uniquely exists ℓ^* corresponding to k^*.

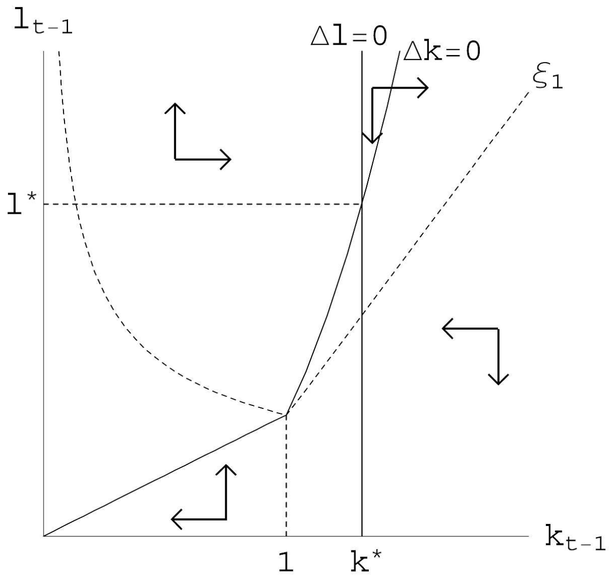

Figure 1: The Phase Diagram.

Further, in the sufficiently small neighborhood of (k^*, ℓ^*), the system (2.11) has a periodic orbit of period-2 on one side of the bifurcation point ϕ_b.

proof. See Appendix A. □

Proposition 3.1 argues that the unique fixed point loses its stability for sufficiently large ϕ. Moreover, a flip bifurcation (period-doubling bifurcation) occurs by slightly changing a bifurcation parameter ϕ. If this bifurcation is supercritical, there is stable period-2 cycles for $\phi > \phi_b$ in the neighborhood of ϕ_b. On the other hand, if the bifurcation is subcritical, period-2 cycles with a saddle property for $\phi < \phi_b$.

Phase diagram. Figure 1 shows the phase diagram of the dynamical system (2.11). $\Delta\ell = 0$ curve is vertical line at $k_{t-1} = k^*$. $\Delta k = 0$ curve, which is upward sloping with a kink at $k_{t-1} = 1$, is given by

$$\ell_{t-1} = \xi_0(k_{t-1}) \equiv \begin{cases} k_{t-1}, & \text{for } k_{t-1} \leq 1, \\ [1 + \alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1} - 1)]^{\frac{1+\phi}{1-\alpha}}, & \text{for } k_{t-1} > 1. \end{cases}$$

The dotted curve is given by

$$\ell_{t-1} = \xi_1(k_{t-1}) = \begin{cases} \xi_{1,k\leq 1}(k_{t-1}) \equiv k_{t-1}^{-\frac{\alpha}{1-\alpha}}, & \text{for } k_{t-1} \leq 1, \\ \xi_{1,k>1}(k_{t-1}) \equiv \left\{ \frac{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{1+\phi}}{k_{t-1}} \right\}^{\frac{1}{1-\alpha}}, & \text{for } k_{t-1} > 1, \end{cases}$$

and if and only if (k_{t-1}, ℓ_{t-1}) is below this curve, $k_t < 1$ holds.

3.1 Cyclical properties of R&D

In Proposition 3.1, we have shown the existence of fluctuating equilibrium paths. For the remainder of this section, we will study how R&D investment and a growth rate move along such fluctuating equilibrium path.

Let $g_{yt} \equiv (y_t - y_{t-1})/y_{t-1}$ denote the growth rate of per capita output in period t. The level of R&D investment in period t is given by $R_t \equiv \max\{0, N_{t-1}^{1+\phi}\alpha^{-\frac{\alpha}{1-\alpha}}\eta(k_{t-1} - 1)\}$. Comparing periods t and $t+1$, when $(g_{yt} - g_{yt-1})(R_t - R_{t-1}) > 0$ holds, then R&D investment is pro-cyclical , that is, capital allocation to R&D is high with a high growth.

Whether R&D investment in periods t and $t+1$ is pro-cyclical or counter-cyclical depends on a pair of k_{t-1} and ℓ_{t-1}. We define S as follows:

$$S = S_1 \cup S_2 \cup S_3 \cup S_4.$$

If (k_{t-1}, ℓ_{t-1}) belongs to S, R&D investment is pro-cyclical for periods t and $t + 1$. S_1 – S_4 are defined as

$$\begin{aligned}
S_1 &= \left\{(k_{t-1}, \ell_{t-1}) \mid \xi_{1,k\geq 1}^{-1}(\ell_{t-1}) \leq k_{t-1} < \xi_2^{-1}(\ell_{t-1})\right\}, \\
S_2 &= \left\{(k_{t-1}, \ell_{t-1}) \mid \max\{k^*, \xi_3^{-1}(\ell_{t-1})\} < k_{t-1} < \xi_{1,k>1}^{-1}(\ell_{t-1})\right\}, \\
S_3 &= \left\{(k_{t-1}, \ell_{t-1}) \mid 1 < k_{t-1} < \min\{\xi_3^{-1}(\ell_{t-1}), k^*\}\right\}, \\
S_4 &= \left\{(k_{t-1}, \ell_{t-1}) \mid \max\{(1+n)^{-1}, \xi_{1,k\leq 1}^{-1}(\ell_{t-1})\} < k_{t-1} < 1\right\},
\end{aligned}$$

	S_1	S_2	S_3	S_4
k_{t-1}	$k_{t-1} > 1$			$k_{t-1} \leq 1$
k_t	$k_t \leq 1$	$k_t > 1$		
R &D	$R_t > R_{t+1} = 0$	$R_t > R_{t+1} > 0$	$R_{t+1} > R_t > 0$	$R_{t+1} > R_t = 0$
growth	$g_{yt} > g_{yt+1}$		$g_{yt+1} > g_{yt}$	

Table 1: Behaviors of R&D investment and growth rate for $(k_{t-1}, \ell_{t-1}) \in S$.

where

$$\xi_2(k) \equiv \left[\frac{(1+n)[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k-1)]}{k} \right]^{\frac{1}{1-\alpha}},$$

$$\xi_3(k) \equiv \left[\frac{k-1+[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k-1)]^{1+\phi}}{k} \right]^{\frac{1}{1-\alpha}},$$

and these domains are $k > 1$. ξ_i^{-1} denotes the inverse function of ξ_i. Since $\xi_3(k)$, $\xi_2(k)$, $\xi_{1,k<1}(k)$, and $\xi_{1,k\geq 1}(k)$ are strictly monotonic functions, these inverse functions exist.

Figure 2 shows S on the (k_{t-1}, ℓ_{t-1}) plane. Note that $\xi_3(k_{t-1})$ satisfies $\xi_0(k_{t-1}) > \xi_3(k_{t-1}) > \xi_1(k_{t-1})$ for $\forall k_{t-1} > 1$, and $\xi_2(k_{t-1})$ intersects $\xi_1(k_{t-1})$ at $k_{t-1} = k^*$. S_1, S_2, and S_3 are located in the region with $k_{t-1} > 1$, whereas S_4 is located in the region with $k_{t-1} \leq 1$.

For $(k_{t-1}, \ell_{t-1}) \in S_1$, $\ell_{t-1} < \xi_1(k_{t-1})$, then $k_t < 1$. Therefore, no R&D occurs in period $t+1$, that is, $R_t > R_{t+1} = 0$ holds. In this case, the growth rates of two periods are derived as $y_{t+1} = (1+n)^{-1}\ell_{t-1}^{1-\alpha}$ and $y_t = (1+n)^{-1}\ell_t^{1-\alpha}k_t^{\alpha-1} = (1+n)^{-1}[f^\ell(k_{t-1}, \ell_{t-1})]^{1-\alpha}[f^k(k_{t-1}, \ell_{t-1})]^{\alpha-1}$, and $g_{yt} > g_{yt+1}$ holds when $\ell_{t-1} > \xi_2(k_{t-1})$. As for S_2, S_3, and S_4, we summarize in Table 1. If the equilibrium paths continue to fluctuate among these regions, R&D moves pro-cyclically in the long-run. In the next section, we will investigate this issue focusing on the period-2 cycles.

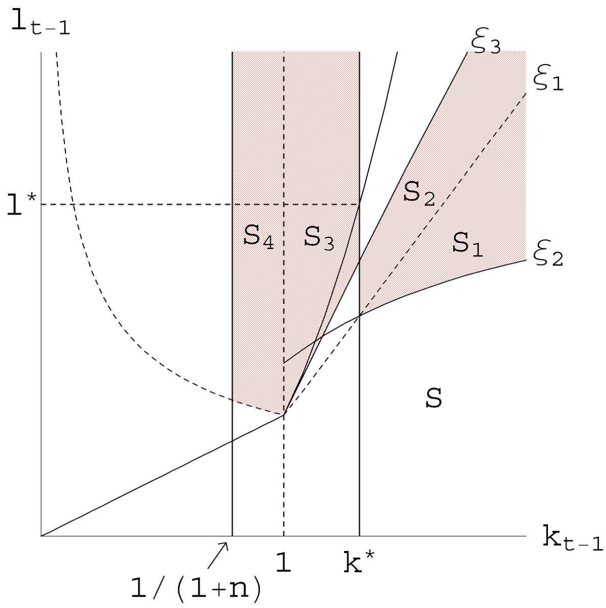

Figure 2: The set of (k_{t-1}, ℓ_{t-1}) that R&D moves pro-cyclically.

4 Period-2 Cycles

The existence of the period-2 cycles is verified in the following way.

Proposition 4.1 If and only if $\phi \geq \bar{\phi}_1$, the system (2.11) has a pair of the fixed point of period-2, (k^H, ℓ^H) and (k^L, ℓ^L), such that $k^L \leq 1 < k^H$, where $\bar{\phi}_1$ is defined as follows:

$$\bar{\phi}_1 \equiv \frac{\log(1+n)}{\log\left[(1+n)^{\frac{1+\alpha}{2}} + \sqrt{(1+n)^{1+\alpha} - 4\alpha^{\frac{\alpha}{1-\alpha}}\left(1-\alpha^{\frac{\alpha}{1-\alpha}}\right)}\right] - \log\left(2\alpha^{\frac{\alpha}{1-\alpha}}\right)}.$$

proof. Solving $\ell^H = f^L(f^K(k^H, \ell^H), f^L(k^H, \ell^H))$ for $k^s H$, we obtain

$$k^H \equiv 1 + \alpha^{\frac{\alpha}{1-\alpha}}[(1+n)^{\frac{2}{\phi}} - 1] > 1.$$

Further, k^L, ℓ^L, ℓ^H are

$$k^L \equiv [k^H(1+n)^{\frac{1-\alpha}{2}-\frac{1+\phi}{\phi}}]^{\frac{2}{1+\alpha}},$$
$$\ell^L \equiv (k^H)^{\frac{1}{1+\alpha}}(1+n)^{\frac{2\alpha}{1-\alpha^2}\left(\frac{1+\phi}{\phi}-\frac{1-\alpha}{2}\right)},$$

and $\ell^H = (1+n)\ell^L$. Some algebra shows that $k^L \leq 1$ is satisfied if and only if $\phi \geq \bar{\phi}_1$. $\square$

When the parameters satisfy the conditions of Proposition 4.1, the system (2.11) has the period-2 cycles moving back and forth between two phases, as shown in Deneckere and Judd (1992) or Matsuyama (1999). In one phase, capital is allocated to R&D, and new intermediate goods are invented. In the other phase, all capital is allocated to the intermediate goods sector, and no invention occurs. We shall refer to each phase as the *R&D phase* and the *no R&D phase*, respectively. The average growth rates of per capita output over the cycles are given by $g_y^{cycle} = (1+n)^{\frac{1}{\phi}} - 1$, which is equal to the growth rate along BGP, g^*.

Our main purpose is to make clear whether R&D investment is pro-cyclical or counter-cyclical over business cycles. Let g_X^1 denote the growth rate of the variable X in the R&D phase that is characterized in Proposition 4.1. As similar, g_X^0 denotes the growth rate of X in the no-R&D phase.

Proposition 4.2 (a) If $\alpha > 1/2$, there exists the threshold, $\bar{\phi}_2$, such that $g_y^1 > g_y^0$ holds for $\phi \in [\bar{\phi}_1, \bar{\phi}_2)$, while $g_y^1 < g_y^0$ holds for $\phi > \bar{\phi}_2$, where

$$\bar{\phi}_2 \equiv \frac{\log(1+n)}{\log(1-\alpha^{\frac{\alpha}{1-\alpha}}) - \log \alpha^{\frac{\alpha}{1-\alpha}}} > \bar{\phi}_1.$$

(b) If $\alpha \leq 1/2$, $g_y^1 > g_y^0$ holds for any $\phi \geq \bar{\phi}_1$.

$g_y^1 > g_y^0$ means that the R&D phase achieves a faster growth than the no-R&D phase, that is, R&D investment is pro-cyclical. In contrast, when $g_y^1 < g_y^0$, R&D investment is high with a low growth. According to the results showed in the previous section, considering parameter conditions for (k^H, ℓ^H) belongs to S_1 (or, equivalently, (k^L, ℓ^L) belongs to S_4), we can prove the above proposition.

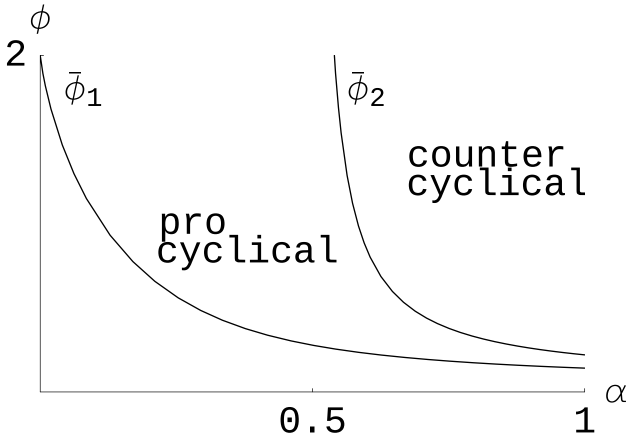

Figure 3: the cyclicality of R&D.

The result of Propositions 4.1 and 4.2 appears in Figure 3 with α on the horizontal axis, ϕ on the vertical axis, and two downward curves. The region above the graph of $\bar{\phi}_1$ corresponds to the set of α and ϕ for which period-2 cycles that are described in Proposition 4.1 exists. Further, that region is separated into two regions by the graph of $\bar{\phi}_2$, and the lower region corresponds to the set of parameters associated with pro-cyclical R&D investment, while the upper region corresponds to the set of parameters with counter-cyclical R&D investment. We can see that when $\alpha \leq 1/2$, in which case the elasticity of substitution between each intermediate good is low, or the mark-up of monopoly price is high, there does not exist the value of ϕ that cause counter-cyclical R&D investment. Whereas, when $\alpha > 1/2$, the sign of the inequality between g_y^1 and g_y^0 may change depending the values of ϕ.

Examples. We assume that the parameter values are $\alpha = \{1/3, 0.9\}$, $n = (1.012)^{10} - 1 \simeq 0.1267$. The rate of population growth chosen means that the population growth rate of 1.2%/*year* and the patent length of 10 years.

Example 1 If $\alpha = 0.9$, $n = 0.1267$, and $\phi = 0.27$, the fixed point (k^*, ℓ^*) is a saddle point, and there exist the fixed points of period-2. The growth rates of each phase are $(g_y^1, g_y^0) = (0.5552, 0.5558)$. Therefore, R&D investment is counter-cyclical.

Example 2 If $\alpha = 0.9,\ n = 0.1267,$ and $\phi = 0.25$, the fixed point (k^*, ℓ^*) is a saddle point, and there exist the fixed points of period-2. The growth rates of each phase are $(g_y^1, g_y^0) = (0.6118, 0.6111)$. Therefore, R&D investment is pro-cyclical.

The value of parameter, $\alpha = 0.9$, corresponds to the mark-up of monopoly price, $1/\alpha \simeq 1.1111$. For example, Rotemberg and Woodford (1995) estimated as 1.115. $\alpha = 0.9$ is consistent with their estimation. The threshold values of ϕ are $\phi_b = 0.1572$, $\bar{\phi}_1 = 0.1007$, and $\bar{\phi}_2 = 0.2604$.

Example 3 If $\alpha = 1/3,\ n = 0.1267$, and $\phi = 0.61$, the fixed point (k^*, ℓ^*) is a saddle point, and there exist the fixed points of period-2. The growth rates of each phase are $(g_y^1, g_y^0) = (0.2458, 0.1869)$. Therefore, R&D investment is pro-cyclical.

In this case, $\phi_b = 0.4298$ and $\bar{\phi}_1 = 0.4280$, and R&D moves pro-cyclically regardless of the value of ϕ.

4.1 Decomposition of growth rates.

When R&D occurs in period t, R&D affects the amount of output, Y_t, through the three channels: introducing the new goods, monopoly distortion, and decreasing resources allocated to the manufacturing sector. The effects of monopoly distortion can be decomposed into two effects. First, the direct effect: the patented goods are supplied lower than the competitive level by the monopolistic competition. Second, the indirect effect: the resource constraint is relaxed by decreasing the supply of the patented goods by the direct effect. Then, available capital per variety increases.

As similar, the effects by introducing new varieties can be decomposed into the direct effect and the indirect effect. The manufacturing sector is more productive by the direct effect. Whereas, available capital per variety decreases by the indirect effect. These effects work in opposite directions, however, the direct effect always dominates. Further, the direct effect of introducing new goods is weakened by the direct effect of monopoly distortion. The similar relation applies to the indirect effects.

The production function (2.1) can be rewritten as

$$\begin{aligned} Y_t =& A L_t^{1-\alpha} \left(\frac{K_{t-1}}{N_{t-1}} \right)^{\alpha} \\ & \times \underbrace{[1+\alpha^{\frac{\alpha}{1-\alpha}}\mu_t]}_{\text{the direct effects}} \times \underbrace{\frac{1}{(1+\alpha^{\frac{1}{1-\alpha}}\mu_t)^{\alpha}}}_{\text{the indirect effects}} \times \underbrace{\frac{K_{t-1}-R_t}{K_{t-1}}}_{\substack{\text{the loss by allocating} \\ \text{capital to R\&D}}} \end{aligned}$$

where $\mu_t \equiv \mu(k_{t-1}) = (N_t - N_{t-1})/N_{t-1}$, which captures the effect of introducing new goods. The effects of monopoly distortion, which are captured by $\alpha^{\frac{\alpha}{1-\alpha}} < 1$ and $\alpha^{\frac{1}{1-\alpha}} < 1$, are amplified by a larger value of μ_t. If $\mu_t = R_t = 0$, the second line of the above equation equals to 1.

Then, the gross growth rate of output in period t, $1 + g_{Yt} = Y_t/Y_{t-1}$, is written as

$$\begin{aligned} \frac{Y_t}{Y_{t-1}} =& \underbrace{[1+\mu_t\alpha^{\frac{\alpha}{1-\alpha}}]\left[\frac{1}{1+\mu_t\alpha^{\frac{1}{1-\alpha}}}\right]^{\alpha}}_{\substack{\text{the effect of} \\ \text{introducing new goods}}} \times \underbrace{\frac{1+\mu_{t-1}}{1+\mu_{t-1}\alpha^{\frac{\alpha}{1-\alpha}}}\left[\frac{1+\alpha^{\frac{1}{1-\alpha}}\mu_{t-1}}{1+\mu_{t-1}}\right]^{\alpha}}_{\substack{\text{dissolution of} \\ \text{monopoly distortion}}} \\ & \times \underbrace{\left[\frac{(K_{t-1}-R_t)/K_{t-1}}{(K_{t-2}-R_{t-1})/K_{t-2}}\right]^{\alpha}}_{\substack{\text{variation of the rate of capital} \\ \text{using in manufacturing sector}}} \times \underbrace{\left(\frac{K_{t-1}}{K_{t-2}}\right)^{\alpha}\left(\frac{L_t}{L_{t-1}}\right)^{1-\alpha}}_{\substack{\text{accumulation of} \\ \text{production factor}}}. \end{aligned} \tag{4.1}$$

In the R&D-based growth model with temporally patent protection, the offspring of R&D spreads for two stages. The first stage is introducing new goods, which corresponds to the first part of right-hand side of (4.1). In this stage, however, the supply of new goods are lower than the competitive level. The second stage is dissolution of such monopoly distortion. This corresponds to the second part of (4.1). Therefore, the growth rate in period t is affected by R&D in periods t and $t-1$.

The first part is constructed by the direct effect and indirect effect as shown above, and it is larger than 1 when $\mu_t > 0$, whereas equals to 1 when $\mu_t = 0$. As similar, the second part is larger than 1 as long as $\mu_{t-1} > 0$, whereas equals to 1 for $\mu_{t-1} = 0$. The third part captures the variation of the rate of capital allocated to the intermediate goods sector between periods t and $t-1$. The fourth part captures capital accumulation and growth of labor supply.

We will study the case of the period-2 cycles characterized in Proposition 3.1. In the R&D phase, which assumed in period T, no R&D occurs in the previous period, i.e., $\mu_{T-1} = R_{T-1} = 0$. Then, (4.1) is rewritten for $t = T$ as

$$\frac{Y_T}{Y_{T-1}} = [1 + \mu_T \alpha^{\frac{\alpha}{1-\alpha}}] \left[\frac{1}{1 + \mu_T \alpha^{\frac{1}{1-\alpha}}}\right]^\alpha \left[\frac{K_{T-1} - R_T}{K_{T-1}}\right]^\alpha \left(\frac{K_{T-1}}{K_{T-2}}\right)^\alpha \left(\frac{L_T}{L_{T-1}}\right)^{1-\alpha}. \tag{4.2}$$

On the other hand, in the no R&D phase, period $T+1$, the gross growth rate is

$$\frac{Y_{T+1}}{Y_T} = \frac{1 + \mu_T}{1 + \mu_T \alpha^{\frac{\alpha}{1-\alpha}}} \left[\frac{1 + \alpha^{\frac{1}{1-\alpha}} \mu_T}{1 + \mu_T}\right]^\alpha \left[\frac{K_{T-1}}{K_{T-1} - R_T}\right]^\alpha \left(\frac{K_T}{K_{T-1}}\right)^\alpha \left(\frac{L_{T+1}}{L_T}\right)^{1-\alpha}. \tag{4.3}$$

Larger k_{T-1} means that larger μ_T. Therefore, in (4.2), the effect of introducing new goods works strongly, and the rate of capital allocated to the intermediate goods sector falls. Moreover, the effect of capital accumulation $(K_{T-1}/K_{T-2})^\alpha$ weakens because the production function has diminishing returns for capital in period $T-1$.[8] These effects cancel each other out, and (4.2) is independent from k_{T-1} after all. Whereas, k_{T-1} has a positive influence on (4.3) through the release from monopoly distortion and the

[8]We should recall that $K_{T-1}/K_{T-2} = sY_{T-1}/K_{T-2}$ and k_{T-2} has a positive relation to k_{T-1}.

loss of allocating capital to R&D.[9] Therefore, R&D tend to move counter-cyclically for large k_{T-1}.

Both (4.2) and (4.3) depend on ℓ_{t-1} through the effect of capital accumulation. However, ℓ_{t-1} has a larger influence on (4.2) than (4.3). Therefore, R&D tend to move pro-cyclically for large ℓ_{T-1}.

Summarizing these analyses, we can explain the reason why $\xi_2(k)$ is upward sloping in Figure 2. Taking ϕ that (k^H, ℓ^H) belongs to a northwest region of $\xi_2(k)$, we obtain the period-2 cycle with pro-cyclical R&D. In addition, if $\phi = 0$ and $n = 0$, $\xi_1(k) = \xi_2(k)$ holds for $\forall k > 1$. Therefore, $S_1 = \emptyset$, i.e., the pair of k_{t-1} and ℓ_{t-1} that achieves pro-cyclical R&D does not exist. This case corresponds to Matsuyama's original case.

5 Model with Infinitely lived Agents

In this section, we consider an infinitely lived agents economy instead of an OLG framework, and show robustness of our results. Using an OLG framework is not desirable in the temporally patent model as in Matsuyama (1999) or Aloi and Lasselle (2007), because the one period of the discrete time has two distinct interpretations: patent length and a half of lifetime.[10] There is no reason that they are identical.

5.1 The model

We suppose the same structure as in section 2 regarding the final goods sector, the intermediate sector, and R&D. Therefore, (2.1) – (2.6) and (2.8) – (2.10) hold, where L_t is the number of infinitely lived households, who supply one unit of labor inelastically, and grows at n. With other parameters and variables, we define in the same way we did in sections 2.

As for consumers or households, assuming infinitely lived agents makes

[9] K_T/K_{T-1} is independent from k_{T-1}, since the production function has constant returns for capital in the R&D regime.

[10] Matsuyama (1999) did not present utility maximization explicitly, however, his saving function can be derived from the conventional OLG assumptions. Matsuyama (2001) have studied an infinitely lived agents economy.

that the optimal consumption path is characterized by an Euler equation instead of the saving function (2.7). Each household chooses a consumption path that maximizes the discounted utility, $\sum_{t=0}^{\infty} \beta^t \log \hat{c}_t$, subject to the budget constraint, $\hat{k}_{t+1} = w_t + r_t \hat{k}_t - \hat{c}_t - n\hat{k}_{t+1}$, where $\beta \in (0,1)$ is the discount factor. $\hat{c}_t = C_t/L_t$ and $\hat{k}_t = K_{t-1}/L_t$ are per capita consumption and stock of capital, and the final goods market clears when

$$Y_t = K_t + C_t. \tag{5.1}$$

The solution to this simple maximization problem is characterized by an Euler equation and a transversality condition as follows:

$$\frac{\hat{c}_t}{\hat{c}_{t-1}} = \frac{\beta r_t}{1+n}, \tag{5.2}$$

$$\lim_{T\to\infty} \beta^T \frac{\hat{k}_{T+1}}{\hat{c}_T} = 0. \tag{5.3}$$

Equilibrium. We define the following new variables:

$$\hat{\ell}_{t-1} \equiv A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)\eta \frac{L_t}{N_{t-1}^{\phi}}, \quad c_{t-1} \equiv \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)\eta \frac{C_{t-1}}{N_{t-1}^{1+\phi}}.$$

In equilibrium, $Y_t = r_t K_t + w_t L_t$ holds, therefore the rate of return on capital is

$$r_t = \alpha Y_t / K_t = \alpha \hat{\ell}_{t-1}^{1-\alpha} \psi(k_{t-1}) k_{t-1}^{-1}. \tag{5.4}$$

Summarizing (2.8), (2.10), (5.1), (5.2), and (5.4) gives the three-dimensional dynamical system as follows:

$$\begin{aligned}
k_t &= \hat{f}^k(k_{t-1}, \hat{\ell}_{t-1}, c_{t-1}) \equiv \begin{cases} \hat{\ell}_{t-1}^{1-\alpha} k_{t-1}^{\alpha} \left[1 - \alpha\beta \frac{c_{t-1}}{k_{t-1}}\right], & \text{for } k_{t-1} \le 1, \\ \frac{\hat{\ell}_{t-1}^{1-\alpha}[k_{t-1} - \alpha\beta c_{t-1}]}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{1+\phi}}, & \text{for } k_{t-1} > 1, \end{cases} \\
\hat{\ell}_t &= f^{\ell}(k_{t-1}, \hat{\ell}_{t-1}), \\
c_t &= \hat{f}^c(k_{t-1}, \hat{\ell}_{t-1}, c_{t-1}) \equiv \begin{cases} \alpha\beta \hat{\ell}_{t-1}^{1-\alpha} k_{t-1}^{-(1-\alpha)} c_{t-1}, & \text{for } k_{t-1} \le 1, \\ \frac{\alpha\beta \hat{\ell}_{t-1}^{1-\alpha} c_{t-1}}{[1+\alpha^{-\frac{\alpha}{1-\alpha}}(k_{t-1}-1)]^{1+\phi}}, & \text{for } k_{t-1} > 1, \end{cases}
\end{aligned} \tag{5.5}$$

where $f^{\ell}(\cdot,\cdot)$ was defined in (2.11). If the initial value of $(k_0, \hat{\ell}_0)$ is given, the equilibrium path, $\{k_t, \hat{\ell}_t, c_t\}_0^{\infty}$, is characterized by the law of motion (5.5) and the transversality condition (5.3).

5.2 Dynamics

The law of motion (5.5) has a unique positive fixed point, $(k^*, \hat{\ell}^*, c^*)$, where

$$\hat{\ell}^* = \left[\frac{(1+n)^{\frac{1+\phi}{\phi}}}{\alpha\beta} \right]^{\frac{1}{1-\alpha}}, \quad c^* = k^* \left(\frac{1-\alpha\beta}{\alpha\beta} \right).$$

At this fixed point, the economy achieves the balanced growth. Moreover, since $\hat{k}$ and $\hat{c}$ grow at the same rate along BGP, the transversality condition (5.3) is satisfied as long as $\beta < 1$.

The three-dimensional system (5.5) has two predetermined variable, k and $\hat{\ell}$, and one non-predetermined variable, c. The local saddle path stability requires a two-dimensional locally stable manifold. We can verify the following proposition through a local stability analysis.

Proposition 5.1 There is a unique threshold of ϕ, $\hat{\phi}_b$, which satisfies $\hat{B}(\hat{\phi}_b) - \Lambda(\hat{\phi}_b) = 0$. If $\phi > \hat{\phi}_b$, the fixed point $(k^*, \hat{\ell}^*, c^*)$ is locally unstable, where $\hat{B}(\phi)$ is defined as follows:

$$\hat{B}(\phi) \equiv \frac{2(1+\alpha\beta) - \phi[\alpha^2\beta + \alpha(1+\beta) + 1]}{2(1+\alpha\beta) + \phi[\alpha^2\beta + \alpha(1+\beta) + 1]}.$$

proof. See Appendix B. □

According to the proposition, for the sufficiently large value of ϕ, there exists only the one-dimensional locally stable manifold. Therefore, the economy that starts close to the fixed point will move away from it. Since a trajectory cannot approach the unique fixed point asymptotically, the equilibrium dynamics of the economy exhibits endogenous fluctuation for almost all initial condition.

Period-2 Cycles. With respect to the existence of the period-2 cycles, we can show a similar result with Proposition 4.1, i.e., the three-dimensional dynamical system (5.5) has the period-2 cycles fluctuating between R&D regime and no-R&D regime exist for $\phi > \bar{\phi}_1$. Such periodic orbits satisfy the transversality condition. As for cyclical properties of R&D, it is possible to show that the identical result shown in Proposition 4.2 and Figure 3. The discount factor β does not affect these results.

6 Conclusion

This paper has examined the cyclicality of the R&D investment over the business cycles by using the variety expansion model with limited patent protection. We have shown that the unique fixed point loses its stability and there exists period-2 cycles moving back and forth between the R&D phase and the no-R&D phase. Moreover, we have examined the possibility and conditions that R&D investment is pro-cyclical over the period-2 cycles, and have proved the existence of the parameters set that achieves pro-cyclical R&D, which is shown in many empirical studies. In our model, counter-cyclical R&D requires the large capital share and the sufficiently strong external effect. In other cases, R&D investment is pro-cyclical.

We assume exogenous population growth and the negative externality of the stock of knowledge that works in R&D, following the formation of the semi-endogenous growth model. [11] In our model, the parameter of this externality plays the central role in decision of the cyclicality of R&D.

A Proof of Proposition 3.1

In order to examine the local stability, we linearize the system (2.11) around the fixed point (k^*, ℓ^*).

$$\begin{bmatrix} k_t - k_t^* \\ \ell_t - \ell_t^* \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} k_{t-1} - k^* \\ \ell_{t-1} - \ell^* \end{bmatrix}, \quad \text{where} \quad \boldsymbol{J} \equiv \begin{bmatrix} f_1^{k*} & f_2^{k*} \\ f_1^{\ell*} & f_2^{\ell*} \end{bmatrix}.$$

It is well known that a stability type of the fixed point depends on the trace $(\operatorname{tr}\boldsymbol{J})$ and the determinant $(\det \boldsymbol{J})$ of the Jacobian matrix. We define seven region separated by three lines, $\det \boldsymbol{J} = \operatorname{tr}\boldsymbol{J} - 1$, $\det = -\operatorname{tr}\boldsymbol{J} - 1$, and

[11] As a result, even if the economy grows along the fluctuating equilibrium path, the long-run growth is not endogenous and requires positive population growth, as shown by literature using the similar assumption such as Jones (1995) and Segerstrom (1998). Some literature such as Young (1998), Peretto (1998), and Howitt (1999), point out this problem, and propose models that have non-scale endogenous growth. A survey of this issue is presented by Jones (1999, 2005). Li (2000, 2002) argue that the predictions of these models depend on the knife-edge assumption and that the semi-endogenous growth prediction is more general.

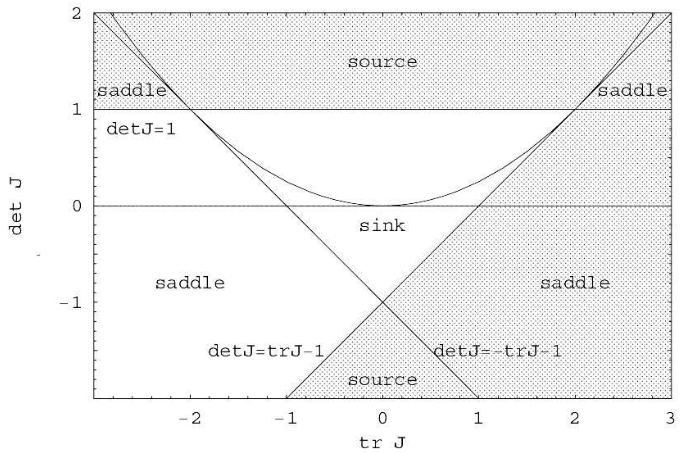

Figure 4: Local stability on the plane

$\det \boldsymbol{J} = 1$, as shown in Figure 4.[12] We also known that, if the Jacobian were somehow to move from inside the triangle with sink stability to outside, a bifurcation would occur.

$\det \boldsymbol{J}$ and $\operatorname{tr} \boldsymbol{J}$ are derived as

$$\begin{aligned} \det \boldsymbol{J} &= -\alpha\phi - (1+\alpha\phi)\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}, \\ \operatorname{tr} \boldsymbol{J} &= -\phi + 1 - (1+\phi)\frac{\alpha^{-\frac{\alpha}{1-\alpha}} - 1}{(1+n)^{\frac{1}{\phi}}}. \end{aligned} \tag{A.1}$$

It is clear that $\det \boldsymbol{J} < 1$ and $\det \boldsymbol{J} > \operatorname{tr} \boldsymbol{J} - 1$, that is, the pair of $\det \boldsymbol{J}$ and $\operatorname{tr} \boldsymbol{J}$ does not belong to the shaded region in Figure 4.[13] In addition, from (A.1), we obtain the following relation:

$$\Lambda(\phi) \lesseqqgtr B(\phi) \quad \Leftrightarrow \quad \det \boldsymbol{J} \gtreqqless -\operatorname{tr} \boldsymbol{J} - 1.$$

$\square$

[12]See Azariadis (1993, Ch.6) for further details.

[13]Therefore, the possibilities of a saddle-node bifurcation and a Hopf bifurcation can be ruled out.

B Proof of Proposition 5.1

We linearize the system (5.5) around the fixed point $(k^*, \hat{\ell}^*, c^*)$.

$$\begin{bmatrix} k_t - k^* \\ \hat{\ell}_t - \hat{\ell}^* \\ c_t - c^* \end{bmatrix} = \hat{\boldsymbol{J}} \begin{bmatrix} k_{t-1} - k^* \\ \hat{\ell}_{t-1} - \hat{\ell}^* \\ c_{t-1} - c^* \end{bmatrix}, \quad \text{where} \quad \hat{\boldsymbol{J}} = \begin{bmatrix} \hat{f}_1^{k*} & \hat{f}_2^{k*} & \hat{f}_3^{k*} \\ f_1^{\ell*} & f_2^{\ell*} & 0 \\ \hat{f}_1^{c*} & \hat{f}_2^{c*} & \hat{f}_3^{c*} \end{bmatrix}.$$

It is easily shown that $f_2^{\ell*} = \hat{f}_3^{c*} = 1$ and $\hat{f}_3^{k*} = -1$, therefore the eigenvalues of the Jacobian matrix, $\hat{\boldsymbol{J}}$, which are denoted as λ, are obtained by solving the following characteristic equation:

$$\begin{aligned} P(\lambda) \equiv |\hat{\boldsymbol{J}} - \lambda \boldsymbol{I}| = & -\lambda^3 + (\hat{f}_1^{k*} + 2)\lambda^2 + (-\hat{f}_1^{c*} + f_1^{\ell*}\hat{f}_2^{k*} - 2\hat{f}_1^{k*} - 1)\lambda \\ & + (\hat{f}_1^{k*} - f_1^{\ell*}\hat{f}_2^{c*} + \hat{f}_1^{c*} - f_1^{\ell*}\hat{f}_2^{k*}) = 0. \end{aligned} \tag{B.1}$$

Here, $\hat{f}_1^{k*}$, $\hat{f}_3^{k*}\hat{f}_1^{c*}$, $f_1^{\ell*}\hat{f}_2^{k*}$, and $f_1^{\ell*}\hat{f}_2^{c*}\hat{f}_3^{k*}$ are

$$\begin{aligned} \hat{f}_1^{k*} &= \frac{1}{\alpha\beta} - (1+\phi)(\Lambda(\phi)+1), \quad \hat{f}_1^{c*} = -(1+\phi)\left(\frac{1}{\alpha\beta} - 1\right)(\Lambda(\phi)+1), \\ f_1^{\ell*}\hat{f}_2^{k*} &= -\phi(1-\alpha)(\Lambda(\phi)+1), \quad f_1^{\ell*}\hat{f}_2^{c*} = -\phi(1-\alpha)\left(\frac{1}{\alpha\beta} - 1\right)(\Lambda(\phi)+1). \end{aligned}$$

From $\lim_{\lambda\to\infty} P(\lambda) = -\infty$ and $P(1) = -f_1^{\ell*}\hat{f}_2^{c*} > 0$, there is at least one real root that is larger than 1. On the other hand, $P(-1)$ is given by

$$P(-1) = \frac{4(\alpha\beta+1)}{\alpha\beta} - \frac{2(\alpha\beta+1) + [\alpha^2\beta + \alpha(1+\beta) + 1]\phi}{\alpha\beta}(\Lambda(\phi)+1),$$

then, $P(-1) = 0$ requires that parameters satisfy

$$\hat{B}(\phi) - \Lambda(\phi) = 0. \tag{B.2}$$

$P(-1)$ is monotonically decreasing in ϕ, and $\lim_{\phi\to 0} P(-1) = \frac{2(1+\ \alpha\beta)}{\alpha\beta} > 0$, $\lim_{\phi\to\infty} P(-1) = -\infty$. Therefore, there exists a unique value of ϕ, $\hat{\phi}_b$, that satisfies (B.2). When $\phi >$
$hatphi_b$, $P(-1) < 0$ and $\lim_{\lambda\to-\infty} P(\lambda) = \infty$ hold, therefore, (B.1) has at least one root belongs to $(-1, -\infty)$ as shown in Figure 5. As similar, from $P(1) > 1$ and $P(-1) < 0$, (B.1) has a root in $(1, -1)$.

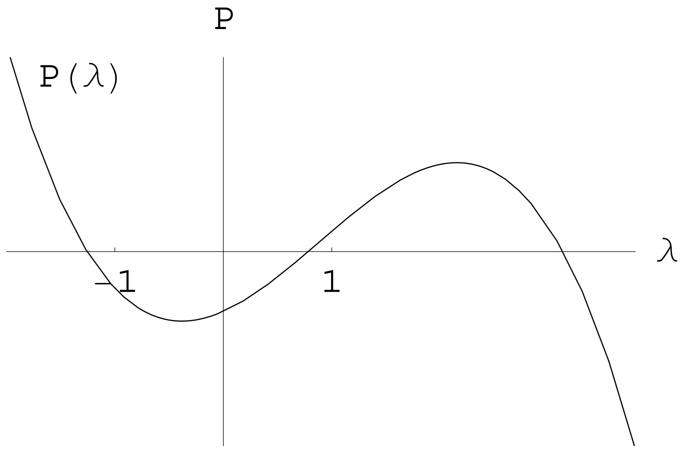

Figure 5: Characteristic equation for $\phi > \hat{\phi}_b$

Summarizing these result, we show that the Jacobian matrix, $\hat{\boldsymbol{J}}$, has the three real eigenvalues, $\lambda_0 > 1$, $\lambda_1 \in (1, -1)$, and $\lambda_2 < -1$ for $\phi > \hat{\phi}_b$. There is only one eigenvalue in an unit circle, therefore, the fixed point $(k^*, \hat{\ell}^*, c^*)$ is unstable.

$\square$

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