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Abstract

The scale effect prediction of a first-generation R&D based endogenous growth model is empirically inconsistent. This problematic prediction is due to linearity between increase in knowledge and stock of knowledge. This paper considers the variety expansion model without capital accumulation, and shows that the dynamical system has a fluctuating equilibrium path when the linearity of knowledge spillover is dropped.

JEL classification: E32, O11, O41

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1 Introduction

It is well known that the first-generation R&D-based endogenous growth models, e.g., Romer (1990), Grossman and Helpman (1991b), and Aghion and Howitt (1992), involve a serious counter-factual prediction such that the economy with a large population grows faster. These models assume that increase in knowledge is linearly proportional to stock of knowledge, and that problematic prediction called the *scale effect* is due to such a knife-edge assumption.[1]

On the other hand, Jones (1995) and Segerstrom (1998) construct R&D-based growth models in which the long-run growth rate is independent of population. They show that relaxing the knife-edge assumption eliminates

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[1] This assumption is introduced in the variety-expansion model(see Romer (1990)).

the scale effect. To achieve this purpose, they introduce the idea that finding new knowledge becomes more difficult as economies become technologically more advanced.

However, such modifications make long-run growth *semi-endogenous*. This basically means that technological change is endogenous in the sense that it requires real resources, whereas long-run growth is not endogenous. Public policies and consumers' preferences do not affect the long-run growth rate. Further, in their models, long-run positive growth requires positive population growth. Some literature, e.g., Young (1998), Peretto (1998), Howitt (1999), point out this problem, and propose models that have non-scale endogenous growth.[2] This semi-endogenizing of long-run growth is the first "side effect" obtained by eliminating scale effects.

The main purpose of this paper is to verify the existence of another side effect caused by the absence of linearity between increase in knowledge and stock of knowledge. We consider the variety-expansion model without capital accumulation, and show that the dynamic system has fluctuating equilibrium paths when the knife-edge assumption of knowledge spillover is dropped.[3] Jones (1995) also uses the framework of the variety-expansion model. However, the balanced growth path of Jones' original model has a saddle property, and no endogenous fluctuation occurs as proved by Arnold (2006).[4]

[2] A survey of this issue is presented by Jones (1999, 2005). These models assume that R&D can affect productivity through two channels: variety expansion and quality improvement. However, Li (2000, 2002) argue that the predictions of these two-R&D sector models also depends on the knife-edge assumption about knowledge spillover, and that the semi-endogenous growth prediction is more general than originally thought. On the other hand, there is another approach to scale and growth regarding fertility or human capital accumulation, as in Jones (2003) and Dalgaard and Kreiner (2001).

[3] Li (2003) also investigates endogenous fluctuations with non-scale growth in a variety expansion model. However, Li's model requires a very low rate of intertemporal substitution for fluctuations to arise. Therefore, any bifurcation cannot occur under the logarithmic utility function. In this paper, assuming a temporally patent protection like Deneckere and Judd (1992), we can verify that fluctuations occur under the very simple utility function, (2.1) below, and show the clearer analytical result.

[4] Strictly, Arnold (2006) assumes constant returns to labor in R&D, which is not assumed in Jones' original model. However, this modification does not have an influence on

Studies of the interactions between R&D and endogenous fluctuations were pioneered by Judd (1985) and Deneckere and Judd (1992).[5] They find fluctuating equilibrium paths in the variety-expansion model without capital accumulation by applying a bifurcation theorem. However, in the above mentioned model, consumers do not have the means for intertemporal resource allocation, and a sustained R&D effort does not contribute to long-run growth because it is balanced by the depreciation of knowledge.

Matsuyama (1999, 2001) modify the model in Deneckere and Judd (1992) by introducing capital accumulation and intertemporal utility maximization, and investigate endogenous growth with fluctuations. However, the prediction of these models inherits the scale effect on long-run growth rate from the first generation R&D-based endogenous growth models. Francois and Lloyd-Ellis (2003, 2008, 2009) and Wälde (2005) also investigate endogenous growth with fluctuation, by using the framework of the quality-ladder model. In particular, Wälde's model has a similar property to my model in that he derives non-scale growth with fluctuations. These lines of research on R&D and fluctuations examine Kondratiev waves.[6]

The rest of this paper is organized as follows. The next section sets up the basic model used in our theoretical investigation, and derives the solutions to the maximization problems of firms and consumers. Sections 3 and 4 provide the law of motion of the economy and characterize the dynamic equilibrium in the knife-edge case (section 3) and others (section4). Section 5 studies the properties of the fluctuating equilibrium path characterized in section 4. Section 6 provides a conclusion.

the scale effect.

[5]Shleifer (1986) also proposes the theoretical model with technical progress and endogenous fluctuation, but he does not explicitly consider the R&D effort.

[6]For other works, see Benhabib et al. (1994), Bental and Peled (1996), Gale (1996), and Boldrin and Levine (2002). On the other hand, Comin and Gertler (2006) and Barlevy (2007) investigate the fluctuating R&D caused by exogenous productivity shocks.

2 Basic Model

We consider a dynamic model based on Grossman and Helpman (1991a, ch.3). Time is discrete and indexed by $t = 0, 1, 2, \ldots$. There is a single final good taken as a numeraire, which is produced using intermediate goods. A new variety of intermediate goods is invented by allocating labor for R&D activities, and inventors enjoy a one-period monopoly by temporary patent protection. The available intermediate goods are produced by multiple intermediate firms using labor. The economy is populated by infinitely lived agents, and therefore the optimal consumption path is characterized by an Euler equation and a transversality condition.

Households There are L_t households, who supply one unit of labor inelastically in period t. Each household chooses a consumption path that maximizes the following discounted utility,

$$\sum_{t=0}^{\infty} \beta^t \log c_t, \tag{2.1}$$

subject to the following budget constraint, $a_{t+1} = w_t + (1+r_t)a_t - c_t - na_{t+1}$, where c_t is per capita consumption, a_t is per capita stock of asset, r_t is the rate of return on asset, w_t is the wage rate, and $\beta \in (0, 1)$ is the discount factor. Households own assets as shares of intermediate firms that produce patented goods. The rate of population growth is given as an exogenous parameter n, the rate at which the number of households grows, i.e., $L_t = (1 + n)L_{t-1}$.

The solution to this simple maximization problem is characterized by an Euler equation and a transversality condition, as follows:

$$\frac{c_{t+1}}{c_t} = \frac{\beta(1 + r_{t+1})}{1 + n}, \tag{2.2}$$

$$\lim_{T\to\infty} \beta^T u'(c_T)a_{T+1} = 0. \tag{2.3}$$

Final goods We assume that perfect competition prevails in the final goods market. The production function is given by

$$C_t = \left[\int_0^{N_t} x_t(z)^\alpha dz\right]^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1, \tag{2.4}$$

where $x_t(z)$ is the amount of the intermediate good indexed by z, and $1/1-\alpha$ denotes the elasticity of substitution between every pair of intermediate goods. C_t is the amount of final output, and the final goods market is clear when $C_t = c_t L_t$. N_t is the number of intermediate goods available at period t, and represents the technology level of there economy.

Profit maximization yields the following demand functions for intermediate goods $z \in [0, N_t]$:

$$x_t(z) = C_t \frac{p_t(z)^{-\frac{1}{1-\alpha}}}{\int_0^{N_t} p_t(j)^{-\frac{\alpha}{1-\alpha}} dj}, \tag{2.5}$$

where $p_t(z)$ is the price of the intermediate good z.

Intermediate goods Each intermediate good is produced using one unit of labor. Because of temporary patent protection, the "old" intermediate goods, $[0, N_{t-1})$, are supplied competitively, and hence price is equal to marginal cost, $p_t(z) = w_t$, for $z \in (0, N_{t-1}]$. However, the "new" intermediate goods, invented at period $t-1$, $(N_{t-1}, N_t]$, are supplied monopolistically, and sold at monopoly price, $p_t(z) = w_t/\alpha_t$, for $z \in (N_{t-1}, N_t]$.

Taking into account that all intermediate goods enter symmetrically into the production of the final good, i.e., $x_t(z) = x_{ct}$, for $z \in [0, N_{t-1}]$, and $x_t(z) = x_{mt}$, for $z \in (N_{t-1}, N_t]$. From (2.5), we get

$$x_{mt} = \alpha^{\frac{1}{1-\alpha}} x_{ct}, \tag{2.6}$$

and the maximized monopoly profit is given by

$$\Pi_t(z) = \Pi_t = \frac{1-\alpha}{\alpha} w_t x_{mt}, \quad \text{for } z \in (N_{t-1}, N_t]. \tag{2.7}$$

Substituting (2.6), we can rewrite the demand function (2.5) as

$$x_t(z) = \frac{p_t(z)^{-\frac{1}{1-\alpha}} C_t}{(1 + \alpha^{\frac{\alpha}{1-\alpha}} g_{t-1}) w_t^{-\frac{\alpha}{1-\alpha}} N_{t-1}},$$

where $g_{t-1} \equiv (N_t - N_{t-1})/N_{t-1}$.[7] If the patented intermediate goods exist,

$$x_{mt} = \frac{\alpha^{\frac{1}{1-\alpha}} C_t}{w_t N_{t-1}(1 + \alpha^{\frac{\alpha}{1-\alpha}} g_{t-1})}. \tag{2.8}$$

R&D The number of intermediate goods expands according to the following equation:

$$N_{t+1} - N_t = \eta^{-1} R_t N_t^{\phi}, \quad N_0 > 0, \quad \phi \leq 1, \tag{2.9}$$

where R_t denotes the amount of labor allocated to R&D, and $\eta > 0$ is the parameter that reflects the productivity of R&D. Following the formalism adopted in much of the literature, we assume that the stock of existing knowledge has an effect on the productivity of present R&D, and that it is captured by N_t^{ϕ}.

Potential inventors compare the present value of the monopoly profit with the unit cost of R&D to decide whether to enter R&D or not. In the equilibrium, the following free-entry condition must be satisfied:

$$\frac{\Pi_{t+1}}{1 + r_{t+1}} \leq \frac{\eta w_t}{N_t^{\phi}} \quad \text{with equality whenever } g_t > 0. \tag{2.10}$$

Finally, labor market clearing requires

$$L_t = N_{t-1} x_{ct} + (N_t - N_{t-1}) x_{mt} + R_t. \tag{2.11}$$

3 Growth with Scale Effect

In this section, we consider the model with the following knife-edge assumption:[8]

Assumption 1 $\phi = 1$ and $n = 0$.

Under Assumption 1, (2.9) is rewritten as

$$N_{t+1} - N_t = \eta^{-1} R_t N_t. \tag{3.1}$$

[7]Note that g_{t-1} denotes the growth rate of N at period t.

[8]This special case is analyzed by Furukawa (2007b).

That is, the new machines created are linear in the stock of knowledge. (3.1) typifies the R&D equation in the first-generation R&D-based endogenous growth model.

Substituting (2.8) and (3.1) into (2.11), we obtain the following equation:

$$\frac{C_t}{w_t(L-\eta g_t)} = \xi(g_{t-1}), \quad \text{where} \quad \xi(g_{t-1}) \equiv \frac{1+\alpha^{\frac{\alpha}{1-\alpha}} g_{t-1}}{1+\alpha^{\frac{1}{1-\alpha}} g_{t-1}}, \tag{3.2}$$

where L (without a subscript) denotes the population, which is constant under Assumption 1. It is easily shown that $\xi'(g) > 0,\ \xi''(g) < 0,\ \xi(0) = 1$.

Substituting (2.7) and (2.8) into (2.10), and using the Euler equation (2.2) to eliminate $1 + r_{t+1}$, we get the new free-entry condition as follows:

$$\frac{C_t}{w_t} \leq \eta\Lambda(1+\alpha^{\frac{\alpha}{1-\alpha}} g_t) \quad \text{with equality whenever } g_t > 0, \tag{3.3}$$

where $\Lambda \equiv [\beta(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}]^{-1}$. Eliminating C_t/w_t from (3.2) and (3.3), we obtain

$$\frac{\Lambda(1+\alpha^{\frac{\alpha}{1-\alpha}} g_t)}{(L/\eta) - g_t} \geq \xi(g_{t-1}) \quad \text{with equality whenever } g_t > 0. \tag{3.4}$$

The minimum value of the left-hand side of (3.4) is $\eta\Lambda/L$ for $g_t = 0$, whereas the range of $\xi(g_{t-1})$, on the right-hand side, is $[1, 1/\alpha)$. Therefore, there exist the following three cases:

- If $\eta\Lambda > L/\alpha$, $g_t = 0$ holds in the equilibrium, regardless of the value of g_{t-1},
- If $\eta\Lambda < L$, g_t is positive in the equilibrium, regardless of the value of g_{t-1},

and

- if $\eta\Lambda \in (L, L/\alpha)$, there is the following threshold of g_{t-1}:

  $$\bar{g} = (\eta\Lambda - L)/[\alpha^{\frac{\alpha}{1-\alpha}}(L - \alpha\eta\Lambda)] > 0,$$

  such that $g_t = 0$ holds for $g_{t-1} < \bar{g}$, and g_t is positive for $g_{t-1} > \bar{g}$.

3.1 Dynamics

Solving for g_t, we can rewrite (3.4) as

$$g_t = \max\{0, \hat{\gamma}(g_{t-1})\}, \quad \hat{\gamma}(g_{t-1}) \equiv \frac{\xi(g_{t-1})(L/\eta) - \Lambda}{\alpha^{\frac{\alpha}{1-\alpha}}\Lambda + \xi(g_{t-1})}. \tag{3.5}$$

If the initial value g_0 is given, the equilibrium path, $\{N_t, g_t\}_0^\infty$, is characterized by the law of motion (3.5) and the transversality condition (2.3). Considering that $\xi'(g_{t-1}) > 0$ and $\xi''(g_{t-1}) < 0$, we get $\hat{\gamma}'(g_{t-1}) > 0$ and $\hat{\gamma}''(g_{t-1}) < 0$.[9]

Balanced growth path If $\eta\Lambda < L$, the one-dimensional system (3.5) has a positive fixed point, $g^* \equiv [(L/\eta) - \Lambda]/(\alpha^{\frac{1}{1-\alpha}}\Lambda + 1)$ (See Figure 1) . Along this balanced growth path, N grows at the constant rate, g^*, and the growth rate of output is $g_c^* = (c_{t+1} - c_t)/c_{t-1} = (1 + g^*)^{\frac{1-\alpha}{\alpha}} - 1$ from (2.4). Figure 1 shows monotonic convergence to the fixed point g^* for any initial conditions $g_0 \geq 0$. Such equilibrium paths satisfy the transversality condition.

Growth trap If $\eta\Lambda \geq L$, the system (3.5) does not have a fixed point in the positive orthant[10] (See Figure 2). The trajectories for any initial conditions $g_0 \geq 0$ converge to the origin $g = 0$. The economy is trapped in the steady state without long-run growth.

If $\eta\Lambda \in [L, L/\alpha)$, there is a threshold, $\bar{g} \geq 0$, such that $\hat{\gamma}(g_{t-1}) \gtreqless 0 \Leftrightarrow g_{t-1} \gtreqless \bar{g}$. Thus, the equilibrium path starting from $g_0 > \bar{g}$ achieves a positive growth rate for some periods. However, if $\eta\Lambda > L/\alpha$, $\hat{\gamma}(g) \leq 0$ holds for any $g \geq 0$. Therefore, the economy cannot achieve a positive growth rate, except in the initial period, i.e., $g_t = 0, \forall t \geq 1$. Both of these paths satisfy the transversality condition (2.3).

The above results are summarized as follows:

Proposition 3.1 Let Assumption 1 hold,

[9] $\hat{\gamma}'(g_{t-1}) = [(\alpha^{\frac{\alpha}{1-\alpha}}(L/\eta) + 1)\Lambda\xi'(g_{t-1})]/[\alpha^{\frac{\alpha}{1-\alpha}}\Lambda + \xi(g_{t-1})]^2 > 0$

[10] If $\eta\Lambda \geq L$, $\hat{\gamma}(0) = [(L/\eta) - \Lambda]/(\alpha^{\frac{\alpha}{1-\alpha}}\Lambda + 1) \leq 0$ and $\hat{\gamma}'(g) \leq \hat{\gamma}'(0) = [\alpha^{\frac{\alpha}{1-\alpha}}(1 - \alpha)\Lambda]/(\alpha^{\frac{\alpha}{1-\alpha}}\Lambda + 1) < 1$ hold for any $g \geq 0$.

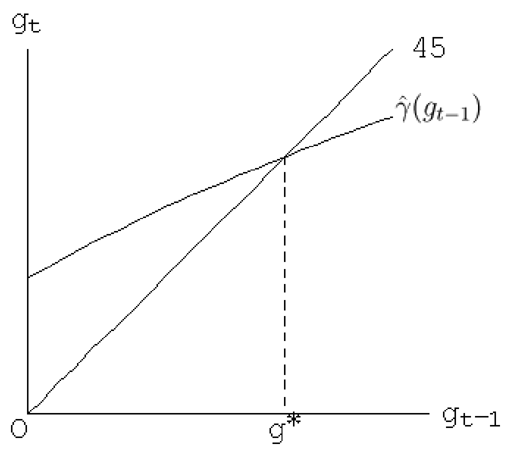

Figure 1: Balanced growth path

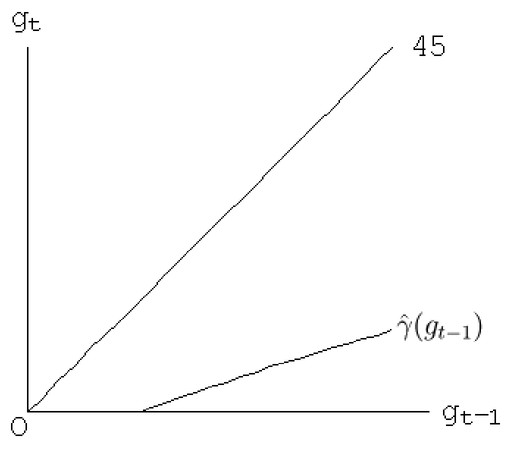

Figure 2: Growth trap

(a) if $\eta\Lambda < L$, $\lim_{T\to\infty} g_T = g^* > 0$ holds for any initial condition, $g_0 \geq 0$. The economy converges monotonically to the balanced growth path.

(b) if $\eta\Lambda \geq L$, , $\lim_{T\to\infty} g_T = 0$ holds for any initial condition, $g_0 \geq 0$.

The equilibrium path starting from g_0 converges to the origin or g^*, and no fluctuations can be observed in either cases. However, we should note that g_0 is not given and only N_0 is given. There is no endogenous mechanism to determine uniquely the initial condition g_0 and the equilibrium path that satisfies (3.4) and the transversality condition, i.e., the indeterminacy arises.[11]

4 Growth without Scale Effect

In this section, we analyze the dynamics of the model with the following assumption:

Assumption 2 $\phi < 1$ and $n > 0$.

Under Assumption 2, the value of ϕ is decided by two externalities in the R&D process, the *standing on shoulder effect* and the *fishing out effect*. If the former dominates, we get $\phi > 0$ which corresponds to the case in which previous discoveries raise the productivity of the current R&D. In contrast, if the latter effect is dominant, ϕ is negative, and past discoveries make it more difficult to invent a new machine. We should note that the knife-edge assumption, $\phi = 1$ (Assumption 1), is the special case of $\phi > 0$ with measure zero, and is not considered in this section.

To analyze the economy with population growth, we define a new variable, $\ell_t \equiv L_t/(\eta N_t^{1-\phi})$. From $L_t = (1+n)L_{t-1}$, the following relation holds between ℓ_t and ℓ_{t-1}:

$$\ell_t = \lambda(g_{t-1}, \ell_{t-1}) \equiv \frac{1+n}{(1+g_{t-1})^{1-\phi}}\ell_{t-1}. \tag{4.1}$$

[11] See Furukawa (2007b, Proposition 1). Benhabib et al. (1994), Evans et al. (1998), and Furukawa (2007a) also analyze the indeterminacy using the framework of the variety expansion model.

Dividing both sides of (2.11) by $\eta N_t^{1-\phi}$, and substituting (2.6) and (2.9), we obtain

$$\ell_t = \frac{N_{t-1}}{\eta N_t^{1-\phi}} x_{mt} \alpha^{-\frac{1}{1-\alpha}} (1 + \alpha^{\frac{1}{1-\alpha}} g_{t-1}) + g_t. \tag{4.2}$$

Substituting (2.8) in the above equation yields

$$\frac{C_t}{w_t(\ell_t - g_t)} = \xi(g_{t-1}) \eta N_t^{1-\phi}, \tag{4.3}$$

where $\xi(g_{t-1})$ were defined in (3.2).

As in the previous section, substituting (2.7) and (2.8) into (2.10) and using (2.2) to eliminate $1 + r_{t+1}$, we obtain the following new free-entry condition:

$$\frac{C_t}{w_t} \leq \Lambda(1 + \alpha^{\frac{\alpha}{1-\alpha}} g_t) \eta N_t^{1-\phi} \quad \text{with equality whenever } g_t > 0, \tag{4.4}$$

where $\Lambda \equiv [\beta(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}]^{-1}$ as defined in the previous section. Eliminating C_t/w_t from (4.3) and (4.4), we can derive the following relation among g_{t-1}, ℓ_t, and g_t:

$$\frac{\Lambda(1 + \alpha^{\frac{\alpha}{1-\alpha}} g_t)}{\ell_t - g_t} \geq \xi(g_{t-1}) \quad \text{with equality whenever } g_t > 0. \tag{4.5}$$

Note that if

$$\Lambda \geq \xi(g_{t-1}) \lambda(g_{t-1}, \ell_{t-1}) \tag{4.6}$$

is satisfied, there is no positive g_t such that (4.5) holds with equality, and $g_t = 0$ holds in the equilibrium.[12]

4.1 Dynamics

Summarizing (4.1) and (4.5) yields the following two-dimensional system:

$$\begin{aligned} g_t &= \max\{0, \gamma(g_{t-1}, \ell_{t-1})\}, \\ \ell_t &= \lambda(g_{t-1}, \ell_{t-1}), \end{aligned} \tag{4.7}$$

[12] We can calculate,

$$\frac{\partial}{\partial g_{t-1}}[\xi(g_{t-1})\lambda(g_{t-1}, \ell_{t-1})] = \xi(g_{t-1})\lambda(g_{t-1}, \ell_{t-1}) \left[\frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{(1+\alpha^{\frac{1}{1-\alpha}} g_{t-1})(1+\alpha^{\frac{\alpha}{1-\alpha}} g_{t-1})} - \frac{1-\phi}{1+g_{t-1}} \right],$$

and this sign is indeterminate for general parameters. Assuming $\phi < \bar{\phi}$ (see Proposition 4.1 below), we can verify that this sign is negative.

where

$$\gamma(g_{t-1}, \ell_{t-1}) \equiv \frac{\xi(g_{t-1})\lambda(g_{t-1}, \ell_{t-1}) - \Lambda}{\alpha^{\frac{\alpha}{1-\alpha}}\Lambda + \xi(g_{t-1})}.$$

The equilibrium path, $\{g_t, \ell_t\}_0^\infty$, is characterized by the law of motion (4.7) and the transversality condition (2.3).

Balanced growth path The system (4.7) has a unique positive fixed point (g^*, ℓ^*), which is derived as follows:[13]

$$\begin{aligned} g^* &= (1+n)^{\frac{1}{1-\phi}} - 1, \\ \ell^* &= \Lambda(1 + \alpha^{\frac{1}{1-\alpha}} g^*) + g^*. \end{aligned} \tag{4.8}$$

The long-run equilibrium with $g = 0$, as in the case of the growth trap in the previous section, is ruled out.[14] Therefore, this balanced growth path, (g^*, ℓ^*), is the unique steady state for any parameters. We also see that the condition to maintain the positive growth rate, $\xi(g^*)\lambda(g^*, \ell^*) = (1 + \alpha^{\frac{\alpha}{1-\alpha}} g^*)\Lambda + \xi(g^*)g^* > \Lambda$, is satisfied in (g^*, ℓ^*).

From (2.4), we get the following equation:

$$c_t L_t = N_{t-1}^{\frac{1}{\alpha}-\phi} N_{t-1}^{\phi} x_{mt} \alpha^{-\frac{1}{1-\alpha}} (1 + \alpha^{\frac{\alpha}{1-\alpha}} g_{t-1})^{\frac{1}{\alpha}}. \tag{4.9}$$

From (4.2), if ℓ and g are constant, $N_{t-1}^{\phi} x_{mt}$ is also constant, therefore, $c_t / N_{t-1}^{\frac{1-\alpha}{\alpha}}$ is constant. Thus, the growth rate of per capita output along the balanced growth path is derived as $g_c^* = (1+g^*)^{\frac{1-\alpha}{\alpha}} - 1 = (1+n)^{\frac{1-\alpha}{\alpha(1-\phi)}} - 1$.

Stability We define a threshold of ϕ as $\bar{\phi} \equiv 1 - 2\alpha^{\frac{\alpha}{1-\alpha}}[1 + (1-\alpha)(1+\beta)]$. $\bar{\phi}$ positively depends on α, and its value can range from $-3 - 2\beta$ to $1 - 2/e$. Therefore, there uniquely exists $\alpha_c \in (0, 1)$ defined by $\alpha_c^{\frac{\alpha_c}{1-\alpha_c}}[1 + (1-\alpha_c)(1+\beta)] = 1$, such that $\bar{\phi}$ is larger than -1 for $\alpha > \alpha_c$. We can verify the following proposition through a local stability analysis.

[13] Substituting $\ell_t = \ell_{t-1} > 0$ into $\ell_t = \lambda(g_{t-1}, \ell_{t-1})$, and solving for g_{t-1}, g^* is uniquely derived. $\gamma(g_{t-1}, \ell_{t-1})$ is monotonically increasing in ℓ_{t-1}; therefore, there uniquely exists ℓ^* corresponding to g^*.

[14] If the economy continued to stay in equilibrium with $g = 0$, the right-hand side of (4.6) would grow at the population growth rate, n. Hence, (4.6) will be violated.

Proposition 4.1 Let Assumption 2 holds.

(a) When $\alpha > \alpha_c$ and $\phi \in (-1, \bar{\phi})$, there is a unique bifurcation point of n, $n_1 > 0$, such that (g^*, ℓ^*) is a sink for $n > n_1$, and is a saddle point for $n < n_1$.

(b) When $\alpha < \alpha_c$ and $\phi \in (\bar{\phi}, -1)$, there is a unique bifurcation point of n, $n_2 > 0$, such that (g^*, ℓ^*) is a sink for $n < n_2$, and is a saddle point for $n > n_2$.

Further, if the bifurcation point n_1 or n_2 exists, in the sufficiently small neighborhood of (g^*, ℓ^*), the system (4.7) has a periodic orbit of period-2 on one side of the bifurcation point.[15]

Proof. See Appendix.

Proposition 4.1 argues that a flip bifurcation (period-doubling bifurcation) occurs by slightly changing a bifurcation parameter n. If this bifurcation is supercritical, there is a stable period-2 cycle for $n < n_1$ or $n > n_2$ in the neighborhood of the bifurcation point. On the other hand, if the bifurcation is subcritical, there is a period-2 cycle with a saddle property for $n > n_1$ or $n < n_2$. The above proposition does not tell us which cases occur. It is possible to show that these periodic orbits satisfy the transversality condition.

The two-dimensional system (4.7) has one predetermined variable, ℓ, and one non-predetermined variable, g. Therefore, when the fixed point is a sink, local indeterminacy again arises. In addition, even if the fixed point is a saddle point, the possibilities of the cyclical equilibrium make the equilibrium path indeterminate. If the bifurcation is supercritical, the trajectories toward the cyclical equilibrium coexist with the saddle path which converges to the fixed point, and the behavior of the economy is decided depending on what the agents expect.

For the remainder of this paper, we focus on the case of $\alpha > \alpha_c$. Note that $\lim_{\beta \to 1} \alpha_c = 1/2$ and $\alpha_c < 1/2$ holds for $\beta < 1$. In other words, if

[15] From (4.8), it is clear that there exists a one-to-one correspondence between the fixed point (g^*, ℓ^*) and n.

$\alpha > 1/2$, $\alpha > \alpha_c$ holds regardless of the value of β. There is considerable validity in this parameter restriction. For example, in this model, α is equal to a inverse of the mark-up of monopoly price, which is obviously larger than 1/2.[16]

5 Period-2 Cycles

In this section, we investigate the period-2 cycles. First, the existence of these cycles is justified in the following way.

Proposition 5.1 Let Assumption 2 and $\phi < \bar{\phi}$. There is a pair of the fixed points of period-2, (g^1, ℓ^1) and (g^2, ℓ^2), such that $g^1 = \gamma(g^2, \ell^2)$, $g^2 = 0$, $\ell^1 = \lambda(g^2, \ell^2)$, and $\ell^2 = \lambda(g^1, \ell^1)$ are satisfied, for sufficiently small n.

Proof. See Appendix.

If the conditions of Proposition 5.1 are satisfied, the system (4.7) has the period-2 cycles moving back and forth between two phases. In one phase, labor is allocated to R&D, and new intermediate goods are invented. In the other phase, all labor is allocated to the intermediate goods sector, and no invention occurs. We shall refer to each phase as the *R&D phase* and the *no R&D phase*, respectively. Note that there is a one-period lag between inventing goods and introducing them into the manufacturing sector. Therefore, the economy enjoys a higher productivity growth rate, g^1 , in the no-R&D phase as a result of the previous R&D effort. On the other hand, when the economy is in the R&D phase, productivity does not grow, i.e., $g^2 = 0$.

The fixed points of period-2 are calculated as follows:

$$\begin{aligned} g^1 &= (1+n)^{\frac{2}{1-\phi}} - 1, \quad \ell^1 = (1 + \alpha^{\frac{\alpha}{1-\alpha}}\Lambda)g^1 + \Lambda, \\ g^2 &= 0, \quad \ell^2 = \frac{(1 + \alpha^{\frac{\alpha}{1-\alpha}}\Lambda)g^1 + \Lambda}{1+n}. \end{aligned} \tag{5.1}$$

[16]See also footnote 20.

From (5.1), it becomes clear that the average growth rate of N over these cycles is equal to g^*.[17]

We can also derive the growth rate of per capita output, c, in each phase.[18]

$$g_c^1 = \frac{(1+\alpha^{\frac{\alpha}{1-\alpha}} g^1)^{\frac{1-\alpha}{\alpha}} (1+\alpha^{\frac{\alpha}{1-\alpha}} g^1 + g^1/\Lambda)}{1+\alpha^{\frac{1}{1-\alpha}} g^1} - 1,$$
$$g_c^2 = (1+g^1)^{\frac{1-\alpha}{\alpha}} \frac{1+\alpha^{\frac{1}{1-\alpha}} g^1}{(1+\alpha^{\frac{\alpha}{1-\alpha}} g^1)^{\frac{1-\alpha}{\alpha}} (1+\alpha^{\frac{\alpha}{1-\alpha}} g^1 + g^1/\Lambda)} - 1, \tag{5.2}$$

where g_c^1 (g_c^2) is the growth rate of c in the no-R&D phase (the R&D phase). Similar to the above, the average growth rate over the cycles is equal to g_c^*.

Moreover, we can show that $g_c^1 > g_c^2$ holds, i.e., that the no-R&D phase grows faster than the R&D phase. In other words, R&D investments are counter-cyclical. With low growth, resource allocation to R&D is high. However, productivity improvements are pro-cyclical as shown above.[19]

Further, g_c^2 may be negative if α and β satisfy the following condition:

$$\frac{\alpha}{(1-\alpha) - \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha^2)} > \Lambda \quad \Leftrightarrow \quad \alpha^{-\frac{1}{1-\alpha}}[1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)] < \beta.$$

Note that if $\alpha < 1/2$, $\alpha^{-\frac{1}{1-\alpha}}[1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)]$ is larger than 1, therefore, the above condition never be satisfied. During the R&D phase, the labor allocated to the intermediate sector is smaller than in the previous period, because the R&D sector also requires labor. When the above condition holds true, this effect is very large, and therefore, per capita outputs decrease. The economy falls into negative growth during recession.

[17]The average growth rate of N is given by $\sqrt{(1+g^2)(1+g^1)} - 1 = (1+n)^{\frac{1}{1-\phi}} - 1$.

[18]The appendix provides a detailed derivation.

[19]These results are similar to the prediction of Francois and Lloyd-Ellis (2003). Empirically, pro-cyclical productivity improvements are comparatively easy to justify, whereas the prediction that R&D expenditures are counter-cyclical is not easy to defend. See for example Comin and Gertler (2006), Barlevy (2007), Wälde and Woitek (2004), and Fatás (2000). Francois and Lloyd-Ellis (2003) justifies this prediction from firms' "labor hoarding" during recession.

5.1 Examples

Example 5.1 If $\alpha = 0.9$, $\beta = 1/(1.05)^{20}$, and $\phi = 0$, a flip bifurcation arises at $n_b \simeq 0.2636$.[20] Letting $n = 0.2636 - 0.001$, the fixed point is a saddle point, and the system (4.7) has a period-2 cycle with $g^1 \simeq 0.5942$, $g^2 = 0$, $\ell^1 \simeq 84.8462$, and $\ell^2 \simeq 67.1988$. The growth rate is $g_c^1 \simeq 0.0305$, $g_c^2 \simeq 0.0029$.

Example 5.2 If $\alpha = 0.9$, $\beta = 1/(1.03)^{10}$, and $\phi = 0$, a flip bifurcation arises at $n_b \simeq 0.1922$. Letting $n = 0.1922 - 0.001$, the fixed point is a saddle point, and the system (4.7) has a period-2 cycle with $g^1 \simeq 0.4215$, $g^2 = 0$, $\ell^1 \simeq 40.7743$, and $\ell^2 \simeq 34.1995$. The growth rate is $g_c^1 \simeq 0.0276$, $g_c^2 \simeq -0.0023$.

6 Conclusion

In this paper, we have studied knowledge spillover, fluctuations, and the scale effect of long-run growth, using the variety expansion model without capital accumulation and the infinitely lived patent. When the knife-edge condition of knowledge spillover, based on the first-generation R&D-based endogenous growth models, is assumed, every equilibrium path converges to the unique steady state, and no bifurcation occurs. In contrast, after relaxing the knife-edge condition, and following the formalism proposed in the semi-endogenous growth models, e.g., Jones (1995), the economy has a fluctuating equilibrium path by a bifurcation theorem. In other words, we have shown that the loss of the linearity of knowledge spillover is the important factor in causing fluctuations.

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[20] The value of the parameter, $\alpha = 0.9$, corresponds to the mark-up of monopoly price, $1/\alpha \simeq 1.1111$. For example, Matsuyama (1999) estimates the mark-up at $1.05 - 1.25$, and Rotemberg and Woodford (1995) estimates at 1.115. $\alpha = 0.9$ is consistent with these findings.

Appendix

A Proof of Proposition 4.1

To analyze the stability property, we linearize the system (4.7) around the fixed point (g^*, ℓ^*).

$$\begin{bmatrix} g_t - g^* \\ \ell_t - \ell^* \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} g_{t-1} - g^* \\ \ell_{t-1} - \ell^* \end{bmatrix}, \quad \text{where } \boldsymbol{J} \equiv \begin{bmatrix} \gamma_1^* & \gamma_2^* \\ \lambda_1^* & \lambda_2^* \end{bmatrix}.$$

The elements of the Jacobian matrix $\boldsymbol{J}$ are given by:

$$\begin{aligned} \gamma_1^* &= \frac{\psi_0 - \xi(g^*)\psi_1}{\psi_2}, & \gamma_2^* &= \frac{\xi(g^*)}{\psi_2}, \\ \lambda_1^* &= -\psi_1, & \lambda_2^* &= 1, \end{aligned} \tag{A.1}$$

where ψ_0, ψ_1, and ψ_2 are defined as follows:

$$\begin{aligned} \psi_0 &\equiv \xi'(g^*)\Lambda(1 + \alpha^{\frac{1}{1-\alpha}} g^*) = \frac{\Lambda \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{1 + \alpha^{\frac{1}{1-\alpha}} g^*} > 0, \\ \psi_1 &\equiv \frac{1-\phi}{1+g^*}\ell^* > 0, \quad \psi_2 \equiv \alpha^{\frac{\alpha}{1-\alpha}}\Lambda + \xi(g^*) > 0. \end{aligned}$$

It is well known that a stability type of the fixed point depends on the trace ($\operatorname{tr} \boldsymbol{J}$) and the determinant ($\det \boldsymbol{J}$) of the Jacobian matrix. We define seven region, separated by three lines, $\det \boldsymbol{J} = \operatorname{tr} \boldsymbol{J} - 1$, $\det \boldsymbol{J} = -\operatorname{tr} \boldsymbol{J} - 1$, and $\det \boldsymbol{J} = 1$, as shown in Figure 3.[21] We also know that, if the Jacobian were somehow to move from inside the triangle with sink stability to outside, a bifurcation would occur.

First, we will verify that $\det \boldsymbol{J} > \operatorname{tr} \boldsymbol{J} - 1$ and $\det \boldsymbol{J} < 1$ hold true for any parameters, that is, the pair of $\det \boldsymbol{J}$ and $\operatorname{tr} \boldsymbol{J}$ does not belong to the dotted region in Figure 3.[22] Since $\lambda_2^* = 1$, $\xi(g^*) > 0$, and $\psi_1 > 0$, the following relation holds between $\det \boldsymbol{J}$ and $\operatorname{tr} \boldsymbol{J}$:

$$\det \boldsymbol{J} = \frac{\psi_0}{\psi_2} > \frac{\psi_0 - \xi(g^*)\psi_1}{\psi_2} = \operatorname{tr} \boldsymbol{J} - 1.$$

[21] See Azariadis (1993, Ch.6) for further detail.

[22] Thus, the possibilities of a saddle-node bifurcation and a Hopf bifurcation can be ruled out.

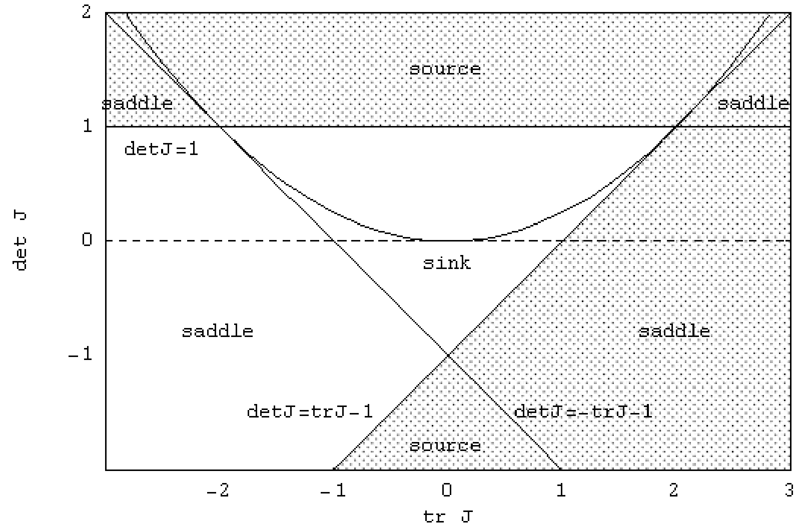

Figure 3: Local stability on the plane

We also show that $\det \boldsymbol{J} < 1$ as follows:

$$\det \boldsymbol{J} = \frac{\psi_0}{\psi_2} = \frac{1-\alpha}{1+\alpha^{\frac{1}{1-\alpha}} g^*} \times \frac{\alpha^{\frac{\alpha}{1-\alpha}} \Lambda}{\alpha^{\frac{\alpha}{1-\alpha}} \Lambda + \xi(g^*)} < 1.$$

Second, we will show that there exists a bifurcation point of n. From (A.1), we obtain

$$\det \boldsymbol{J} - (-\operatorname{tr} \boldsymbol{J} - 1) = \frac{2\psi_0}{\psi_2} - \frac{\xi(g^*)\psi_1}{\psi_2} + 2 = \frac{q(g^*)}{\psi_2(1+\alpha^{\frac{1}{1-\alpha}} g^*)(1+g^*)}.$$

We define $q(g^*) \equiv \mu_1 (g^*)^2 + \mu_2 g^* + \mu_3$, where

$$\begin{aligned}
\mu_1 \equiv& (1+\phi)\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha^{\frac{1}{1-\alpha}}\Lambda), \\
\mu_2 \equiv& 2[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\Lambda + 1 + \alpha^{\frac{\alpha}{1-\alpha}}\Lambda + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha^{\frac{1}{1-\alpha}}\Lambda)] \\
&- (1-\phi)[1+\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)\Lambda], \\
\mu_3 \equiv& 2[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\Lambda + 1 + \alpha^{\frac{\alpha}{1-\alpha}}\Lambda] - (1-\phi)\Lambda.
\end{aligned}$$

Therefore, $\det \boldsymbol{J} - (-\operatorname{tr} \boldsymbol{J} - 1)$ has the identical sign with $q(g^*)$.[23]

[23]Since μ_1, μ_2, μ_3, and g^* positively depend on ϕ, $q(g^*)$ is monotonically increasing in

(a) When $\alpha > \alpha_c$, $\bar{\phi} > -1$ holds. From $\phi > -1$, μ_1 is positive, then, $q(g^*) > 0$ for large enough g^*. On the other hand, as long as $\phi < \bar{\phi}$ is satisfied, $\lim_{g^* \to 0} q(g^*) = \mu_3 < 0$. Because $q(g^*)$ is continuous, there uniquely exists $\bar{g}_1^* \in (0, \infty)$ such that satisfies $q(\bar{g}_1^*) = 0$, at where the sign of $q(g^*)$ changes, i.e., $q(g) > 0$ for $g > \bar{g}_1^*$ and $q(g) < 0$ for $g < \bar{g}_1^*$. Since there exists the one-to-one correspondence between the fixed point (g^*, ℓ^*) and n by (4.8), there exists $n_1 > 0$ such that $(1 + n_1)^{\frac{1}{1-\phi}} - 1 = \bar{g}_1^*$ holds. This is the bifurcation point of a flip bifurcation, and the system has periodic orbits in its neighborhood.

(b) When $\alpha < \alpha_c$, $\bar{\phi} < -1$ holds. For $\phi \in (\bar{\phi}, -1)$, $\mu_1 < 0$ and $\mu_3 > 0$. As similar above, we can show that the quadratic equation, $q(g^*) = 0$, has a unique positive root, $\bar{g}_2^*$, and a unique bifurcation value, $n_2 = (1 + \bar{g}_2^*)^{1-\phi} - 1$ exists. [24] [25]

□

B Proof of Proposition 5.1

Solving $\ell^1 = \lambda(0, \lambda(g^1, \ell^1))$, $g^1 = \gamma(0, \lambda(g^1, \ell^1))$ and $\ell^2 = \lambda(g^1, \ell^1)$ yields the fixed points of period-2, (g^1, ℓ_1), and (g^2, ℓ^2), as (5.1). Here, we adopt the notations, $g^1(n)$, $\ell^1(n)$, and $\ell^2(n)$, to emphasize that they are functions of n.

If these points are the equilibrium path of the system (4.7), following conditions must be satisfied:

$$\xi(g^2(n))\lambda(g^2(n), \ell^2(n)) > \Lambda \quad \text{and} \quad \xi(g^1(n))\lambda(g^1(n), \ell^1(n)) \leq \Lambda. \qquad \text{(B.1)}$$

ϕ. Further, $\lim_{\phi \to 1} q(g^*) > 0$ and $\lim_{\phi \to -\infty} q(g^*) < 0$. Therefore, if we choose ϕ as a bifurcation parameter, it can be shown that there is a unique bifurcation value for a flip bifurcation.

[24] We can derive $\bar{g}_1^* = \frac{-\mu_2 + \sqrt{(\mu_2)^2 - 4\mu_1\mu_3}}{2\mu_1}$ and $\bar{g}_2^* = \frac{-\mu_2 - \sqrt{(\mu_2)^2 - 4\mu_1\mu_3}}{2\mu_1}$.

[25] In this proof, the possibilities of bifurcations are not ruled out for $\phi > \max\{-1, \bar{\phi}\}$ or $\phi < \min\{-1, \bar{\phi}\}$. However, even if bifurcation points exist for such ϕ, they are not unique. Moreover, we can show that no bifurcation occurs for sufficiently large ϕ.

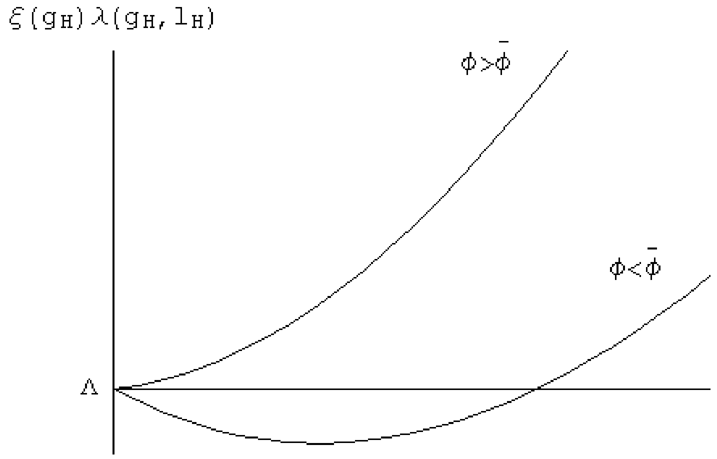

Figure 4: Graph of $\xi(g^1(n))\lambda(g^1(n),\ell^1(n))$

Since $\xi(g^2(n))\lambda(g^2(n),\ell^2(n)) = \ell^1(n)$, the former relation holds true for any $n > 0$.

For $\phi < \bar{\phi}$, the following equation holds:

$$\begin{aligned}&\frac{d}{dn}\xi(g^1(0))\lambda(g^1(0),\ell^1(0))\\&= (g^1)'(n)[\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\Lambda - \frac{1-\phi}{2}\Lambda + 1 + \alpha^{\frac{\alpha}{1-\alpha}}\Lambda] < 0,\end{aligned}$$

i.e., $\xi(g^1(n))\lambda(g^1(n),\ell^1(n))$ has negative slope at $n = 0$.
Since $\xi(g^1(0))\lambda(g^1(0),\ell^1(0)) = \Lambda$, there exists small enough $n > 0$ which satisfies the latter condition of (B.1) (See Figure 4).

C Derivation of (5.2)

Let $g_{T-1} = g_{T+1} = 0$, $g_T = g_{T-2} = g^1$, $\ell_{T-1} = \ell_{T+1} = \ell^2$, and $\ell_T = \ell^1$. From (4.2), we obtain

$$N_t^{\phi} x_{ct} = N_t^{\phi}\alpha^{-\frac{1}{1-\alpha}} x_{mt} = \frac{\eta(\ell_t - g_t)(1 + g_{t-1})}{1 + \alpha^{\frac{1}{1-\alpha}} g_{t-1}}.$$

Therefore, for periods T and $T+1$,

$$\begin{aligned} N_T^{\phi} x_{cT} &= \eta(\ell^1 - g^1), \\ N_{T+1}^{\phi} x_{cT+1} = N_{T-1}^{\phi} x_{cT-1} &= \frac{\eta \ell^2 (1+g^1)}{1+\alpha^{\frac{1}{1-\alpha}} g^1}. \end{aligned} \tag{C.1}$$

From (4.9), we obtain

$$\begin{aligned} c_T L_T &= N_{T-1}^{\frac{1}{\alpha}} x_{cT}, \\ c_{T+1} L_{T+1} &= N_T^{\frac{1}{\alpha}} x_{cT+1} (1+\alpha^{\frac{\alpha}{1-\alpha}} g^1)^{\frac{1}{\alpha}}, \end{aligned} \tag{C.2}$$

and $c_{T-1}L_{T-1}$ is similar to period $T+1$. Noting that $N_{T-1}/N_{T-2} = N_{T+1}/N_T = (1+g^1)$ and $N_T/N_{T-1} = 1$ holds, and substituting (C.1), (C.2), and (5.1) to $g_c^H \equiv (c_{T+1} - c_T)/c_T$ and $g_c^L \equiv (c_T - c_{T-1})/c_{T-1}$, we obtain (5.2).

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