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A Lattice Theoretical Existence Theorem for Cournot-Walras Equilibria*

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We establish an existence result of Cournot-Walras equilibria in the same framework with the counter example of Roberts and Sonnenschein. Our result is based on the lattice theoretical game theory and do not depend on the convex-valuedness of best responses of imperfect competitors. All our assumptions are on the fundamental datum of economies. *JEL Classification Numbers*: C72, D43, D51.

Keywords: Cournot-Walras equilibrium, Lattice, Supermodular.

1 Introduction

In this paper, we consider the Cournot-Walras equilibrium, which is one of the most standard general equilibrium concepts when taking account of imperfectly competitive markets. It is, however, well known through the counter example presented by Roberts and Sonnenschein(1977) that even in a simple model we might have a possibility of non-existence of such an equilibrium. The difficulty can be attributed to the fact that the convex-valuedness of best responses of firms cannnot be derived from the assumptions on the fundamental datum of economies. We should note here that the concern about the convex-valuedness is caused by the intention of applying the Kakutani's fixed point theorem.

Thus, we abandon the application of the Kakutani's fixed point theorem. Instead, we use the lattice theoretical approach initiated by Milgrom

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and Shannon(1994) to obtain a desirable consequence. Specifically, through that specific approach, we circumvent the convex-valuedness to show an existence theorem for the equilibrium in the same model as Roberts and Sonnenschein's, only assuming some mild conditions on fundamental datum. We present an example in the text that the convex-valuedness of best responses of firms is dispensable to gain the equilibrium.

2 Results

Consider the economies consisting of 1-consumer, 2-monopolistic firms and 3-goods. The consumer is assumed to be a price-taker and each monopolistic firm faces to the inverse demand curve and maximizes its profit by quantity setting. Among three goods, good $j(=1,2)$ is produced by firm j respectively. And good 3 plays a role of a numéraire. We exclude the trades between firms hence the market demand for each good is coincident with the consumer's demand.

In the following, let $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ denote the consumption vector and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3_{++}$ the initial endownment of the consumer. And $y_j \geq 0$ designates the production level of firm j. The price vector is denoted by $p = (p_1, p_2, 1) \in \mathbb{R}^2_+ \times \{1\}$. We impose the following assumptions.

Assumption 1; The consumer's preference is represented by a CES-utility function

$$u(x) = [\alpha_1 x_1^{-\rho} + \alpha_2 x_2^{-\rho} + \alpha_3 x_3^{-\rho}]^{1/\rho}.$$

Note that $\alpha_i > 0$ $(i = 1,2,3)$ and $-1 < \rho < \infty$. When $\rho = 0$, we can consider $u(\cdot)$ is a Cobb-Douglas function.

Assumption 2; The technology of each firm j can be represented by a cost function $C^j : \mathbb{R} \to \mathbb{R}$, which is lower semi-continuous, non-decreasing, $C^j(y_j) = 0 \iff y_j = 0$.

Since the consumer is a price-taker, he\she decides the consumption level in a way that satisfies $MRS_{j3} = p_j$ $(j = 1,2)$. By our hypotheses, the inverse demand curve to which each firm j faces can be written as

$$p_j(y_1, y_2) = MRS_{j3}(y_1, y_2) = \frac{\alpha_j}{\alpha_3}\left(\frac{\omega_3 - C^1(y_1) - C^2(y_2)}{\omega_j + y_j}\right)^{1+\rho}.$$

Then, the profit of firm j is

$$\pi_j(y_1, y_2) = p_j(y_1, y_2) y_j - C^j(y_j).$$

We can provide the definition of the equilibrium as follows.

Definition 1: A profile (y_1^*, y_2^*) is Cournot-Walras equilibrium[1] if

$$\pi_1(y_1^*, y_2^*) \geq \pi_1(y_1, y_2^*) \quad \forall y_1$$

$$\pi_2(y_1^*, y_2^*) \geq \pi_2(y_1^*, y_2) \quad \forall y_2.$$

Remark 1: In the cases with $\rho = \infty$, i.e., the Leontief utility function, or $\rho = -1$, i.e., the linear utility function, we cannot define the inverse demand curve for each good as a function of both firms' production profiles. That's why we assume $-1 < \rho < \infty$.

Now we proceed to proving the existence of the equilibrium defined in Definition 1. See Appendix for the lattice theoretical backgrounds.

Lemma 1: *The inverse demand curve for each produced good satifies log decreasing-differences (Definition A in Appendix).*

Proof.) To simplify the notation, we consider $j = 1$ and assume $\frac{\alpha_1}{\alpha_3} = 1$. Then,

$$p_1(y_1, y_2) = (\frac{\omega_3 - C^1(y_1) - C^2(y_2)}{\omega_1 + y_1})^{1+\rho}.$$

Taking the logarithm of the both sides,

$$\log p_1(y_1, y_2) = (1 + \rho)[\log(\omega_3 - C^1(y_1) - C^2(y_2)) - \log(\omega_1 + y_1)].$$

Since the function $\log(\cdot)$ is monotone increasing and concave, and $(\omega_3 - C^1(y_1) - C^2(y_2))$ is monotone and has decreasing-differences, we can apply Theorem A in Appendix to obtain the result. []

Lemma 2: *The profit of each firm j satisfies strictly dual single-crossing property (Definition B in Appendix.) in $(y_j; y_k)$.*

Proof.) With the results of the previous lemma, this can be shown in the same way as Amir(1996) [Theorem 2.1, pp143.] []

Then we can prove the existence result.

[1] See Gabszewicz and Vial(1972), for the definition of the equilibria in more general models

Proposition 1: *Under Assumption 1 and 2, Cournot-Walras equilibrium exists.*

Proof.) Since each firm j can produce at most $C^{j^{-1}}(\omega_3)$, taking a sufficiently large number $K > 0$, we may assume $y_j \in [0, K]$ $(j = 1, 2)$. By Lemma 2 and Therem B in Appendix, every selection from the best response of each firm is nonincreasing. Let $b^j(y_k)$ denote a selection. Then, the composite mapping

$$B : [0, K] \to [0, K]$$
$$B(y_1) = (b^1 \circ b^2)(y_1)$$

is monotone nondecreasing and we can apply the Tarski's fixed point theorem (Theorem C in Appendix.) to B. Let y_1^* be a fixed point of B and $y_2^* = b^2(y_1^*)$. We have

$$\pi_1(y_1^*, y_2^*) \geq \pi_1(y_1^*, y_2^*) \ \forall y_1 \in [0, K]$$
$$\pi_2(y_1^*, y_2^*) \geq \pi_2(y_1^*, y_2^*) \ \forall y_2 \in [0, K].$$

To prove that (y_1^*, y_2^*) is Cournot-Walras equilibrium, it is sufficient to show that $C^1(y_1^*) + C^2(y_2^*) \leq \omega_3$, which is straightforward. []

Note that our assumptions are on preference or technologies, which are the fundamental datum of economies. The following example implies that our existence result do not rely on the convex-valuedness of best responses.

Example 1; Consider the consumer in the economy has utility function $u(x) = x_1 x_2 x_3$. And let the cost function of the firm 1 be

$$\begin{aligned} C^1(y_1) &= y_1^2 & 0 \leq y_1 \leq 2 \\ &= y_1^{1/2} + (4 - \sqrt{2}) & 2 < y_1 \leq 4 \\ &= y_1 + (2 - \sqrt{2}) & 4 < y_1 \end{aligned}$$

and that of the firm 2 be $C^2(y_2) = y_2^2$. Set the initial endowment $\omega = (1, 1, 50)$. In this economy, the best response of the firm 1 is not continuous, whence not convex-valued. Indeed, at the point $y_2 \approx 5.109$, $b^1(y_2) \approx \{4.000\} \cup \{1.317\}$. However, according to Proposition 1, Cournot-Walras equilibrium exists. Actually, it is $(y_1^*, y_2^*) \approx (4.000, 2.297)$.

3 Conclusion

In this paper, we are based on the lattice programming as a methodology. Generally, by using this mathematics one can prove the existence of pure

Nash equilibrium of nomal form games without the convex-valuedness of best responses, which give us a prospect of getting over the difficulty presented in the counter example by Roberts and Sonnenschein. Indeed, we have succeed in showing the existence of the Cournot-Walras equilibrium only with the assumptions on the fundamental datum of economies. Our model is intrinsically the same as the counter example, whence our result may lead to the construction of the general theory of imperfect competition in the general equilibrium model without suffering from the non-existence of the equilibria.

Finally, we should refer to two works related to ours; Amir(1996) and Bloch and Ferrer(1998). In the former, the existence of Cournot equilibrium is shown by using lattice theory. Our result can be seen as an extension of his claim from the Cournot equilibrium to the Cournot-Walras equilibriu. In the latter, the existence of the interior Nash equilibrium is proven in a general equilibrium model with a bilateral oligopoly, where a CES utility function plays a special role as in our model.

Appendix

Here, we state the lattice theoretical results applied in the text. In the following let X be a compact interval and $T \subset \mathbb{R}$. And assume $x', x'' \in X$ and $x' > x''$, similarly, $t' > t''$.

Definition A.: A function $f : X \times T \to \mathbb{R}$ has decreasing-differences in $(x; t)$ if

$$f(x', t') - f(x'', t') \leq f(x', t'') - f(x'', t'').$$

When the inequality holds strictly, we say that f has strictly decreasing-differences. And if $\log f$ has decreasing-differences, f is said to be having log decreasing-differences.

Definition B.: A function $f : X \times T \to \mathbb{R}$ has dual single-crossing property in $(x; t)$ if

$$f(x', t'') \leq f(x'', t'') \Rightarrow f(x', t') \leq f(x'', t').$$

When the inequality in the right hand side holds strictly, we say f has strictly dual single-crossing property.

Then, we show three theorems which applied in Section 2.

Theorem A; *Let $g : \mathbb{R} \to \mathbb{R}$ be nondecreasing and concave and $f : X \times T \to \mathbb{R}$ be monotone and have decreasing-differences in $(x; t)$. Then, $g \circ f$ has* *decreasing-differences in $X \times T$.*

Proof.) Assume f is monotone nonincreasing. Define the function h : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$h(a,k) = g(a+k) - g(a).$$

Since g is nondecreasing and concave $h(\cdot,\cdot)$ is nonincreasing in a and nondecreasing in k. Let $a' = f(x'', t'')$, $a'' = f(x', t')$, $k' = f(x'', t') - f(x', t')$ and $k'' = f(x'', t'') - f(x', t'')$. By hypothesis, $a' \geq a''$ and $k' \geq k''$. Then we have

$$h(a', k'') = g(f(x', t'')) - g(f(x'', t'')) \leq g(f(x'', t')) - g(f(x', t')) = h(a'', k').$$

This is nothing but our statement. []

Theorem B.: *Assume $f : X \times T \to \mathbb{R}$ has strictly dual single-crossing property in $(x;t)$. Denote $M(t)$ by the solusion set of the maximization problem*

$$\max_{x \in X} f(x,t).$$

Then, the greatest and the least selection from $M(t)$ is nonincreasing function of t. If f has strictly dual single-crossing property, the every selection of $M(t)$ is nonincreasing fuction of t. Moreover, the converse is also true.

Proof.) (See Milgrom and Shannon(1994)[2].)

Theorem C.; *Let $F : X \to X$ be monotone nondecreasing. Then F has the greatest and the least fixed point.*

Proof.) (See Tarski(1955).)

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