Generalized time-invariant overtaking*

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Abstract

We present a new version of the overtaking criterion, which we call *generalized time-invariant overtaking*. The generalized time-invariant overtaking criterion (on the space of infinite utility streams) is defined by extending proliferating sequences of complete and transitive binary relations defined on finite dimensional spaces. The paper presents a general approach that can be specialized to at least two, extensively researched examples, the utilitarian and the leximin orderings on a finite dimensional Euclidean space.

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1 Introduction

Recent contributions have suggested new social welfare relations for the purpose of evaluating infinite utility streams. In particular, Basu and Mitra (2007a) extend the utilitarian ordering on a finite dimensional Euclidian space to the infinite dimensional case, while Bossert, Sprumont and Suzumura (2007) do likewise for the leximin ordering. Both these social welfare relations are incomplete, but may still be effective in the sense of selecting a small set of optimal or maximal elements for a given class of feasible infinite utility streams.

However, it is easy to construct pairs of infinite utility streams where it is clear that the one infinite stream is socially preferred to the other both from an utilitarian and egalitarian point of view, but where the streams are incomparable according to the criteria of Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007). To illustrate, consider the following two streams:

$$\begin{array}{ccccccccccc} \mathbf{u} & : & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \cdots \\ \mathbf{v} & : & 0 & \frac{3}{4} & \frac{5}{8} & \frac{9}{16} & \frac{17}{32} & \frac{33}{64} & \cdots & \frac{2^{n-1}+1}{2^n} & \cdots \end{array}$$

It is intuitively clear that $\mathbf{u}$ is socially preferred to $\mathbf{v}$ from an utilitarian perspective since the sum of utility differences between $\mathbf{u}$ and $\mathbf{v}$ is convergent and converges to $\frac{1}{2}$. Likewise, it is intuitively clear that $\mathbf{u}$ is socially preferred to $\mathbf{v}$ from an egalitarian perspective since minimal utility exists for both streams and the minimal utility of $\mathbf{u}$ ($= \frac{1}{2}$) is greater than the minimal utility of $\mathbf{v}$ ($= 0$). Still, according to the criteria of Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007) these streams are incomparable since there is no cofinite set on which $\mathbf{u}$ equals or Pareto-dominates $\mathbf{v}$. *This motivates an investigation of social welfare relations for the evaluation of infinite utility streams which are more complete* than those proposed by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007), *without compromising desirable properties.*

Extensions of utilitarian and leximin orderings to the infinite horizon are normally required to satisfy the axioms of Finite Anonymity (ensuring equal treatment of generations) and Strong Pareto (ensuring sensitivity for the interests for each generation). Recent work by Lauwers (2007) and Zame (2007) confirms the following conjecture, suggested by Fleurbaey and Michel (2003): it is not possible to construct and describe a complete and transitive binary relation on the set of infinite utility streams which satisfies the axioms of Finite Anonymity and Strong Pareto.[1] We will here be concerned with constructible social welfare relations satisfying Finite Anonymity and Strong Pareto, and hence completeness is an unreachable goal.

However, there might be reasons—other than such non-constructibility—why one should refrain from seeking excessive comparability. To make this argument, consider the following two infinite utility streams:

$$\begin{array}{cccccccccccc} \mathbf{x} & : & 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & \dots \\ \mathbf{y} & : & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & \dots \end{array}$$

When traditional overtaking (Atsumi, 1965; von Weizsäcker, 1965) is applied to the utilitarian or leximin ordering (in the sense of catching up in finite time, see Asheim and Tungodden, 2004), then $\mathbf{x}$ is socially preferred to $\mathbf{y}$, since the finite head of $\mathbf{x}$ is preferred to the finite head of $\mathbf{y}$ at all odd times, while they are indifferent at even times. When extended Fixed-step Anonymity (Lauwers, 1997; Mitra and Basu, 2007) is added to the criterion of Basu and Mitra (2007a) (as done by Banerjee, 2006) and to the the criterion of Bossert, Sprumont and Suzumura (2007) (as done by Kamaga and Kojima, 2008) then $\mathbf{x}$ is socially indifferent to $\mathbf{y}$. This is demonstrated by the fact that choosing a fixed-step of 2 and permuting odd and even times for $\mathbf{x}$ makes $\mathbf{x}$ identical to $\mathbf{y}$.

We argue that either conclusion is problematic. By invoking Fixed-step Anonymity, leading to social indifference between $\mathbf{x}$ and $\mathbf{y}$, Strong Pareto forces us to

[1] Existence of such a complete and transitive binary relation follows (in an non-constructive way) from Szpilrajn's (1930) Lemma; see Svensson (1980).

conclude that the former of the following two streams is preferred to the latter:

$$\begin{array}{lcccccccccccc} (0,\mathbf{x}) & : & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 & \ldots \\ (0,\mathbf{y}) & : & 0 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 & \ldots \end{array}$$

This contradicts Koopmans's (1960) Stationarity axiom (in the sense that preference over future utilities should be independent of present utility). Hence, if one considers Stationarity to be compelling, it comes at a cost to impose Fixed-step Anonymity.

A problem with the strict ranking between $\mathbf{x}$ and $\mathbf{y}$ induced by traditional overtaking (in the sense of catching up in finite time) is that it is not invariant to the sequencing of time periods. In particular, permuting odd and even times for both $\mathbf{x}$ and $\mathbf{y}$, makes $\mathbf{x}$ equal to $\mathbf{y}$ and $\mathbf{y}$ equal to $\mathbf{x}$, thereby inverting the strict ranking. Even worse, by allowing for permutations that are not of the fixed-step kind, there exists an infinite permutation matrix P such that

$$\begin{array}{lcccccccccccc} P\mathbf{x} & : & 0 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 & \ldots \\ P\mathbf{y} & : & 1 & 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 & \ldots \end{array}$$

implying that Strong Pareto implies that $P\mathbf{y}$ is socially preferred to $P\mathbf{x}$ when combined with *either* (i) Fixed-step Anonymity *or* (ii) Finite Anonymity and traditional overtaking (in the sense of catching up in finite time).[2]

It should be noted that Relative Anonymity (in the sense that social preference is not influenced by the sequencing of time periods) is weaker than ordinary Anonymity. To illustrate: the incomplete social welfare relation generated by Strong Pareto alone satisfies Strong Relative Anonymity (in the sense that social preference is invariant to any permutation of time periods), but fails to satisfy even the weakest form of Anonymity, Finite Anonymity, because Pareto-dominance can vanish when two elements of the one stream (only) are permuted.

[2]The concept of a permutation matrix is introduced in Section 2.2. The matrix P moves time 2 to time 1, all other even times two periods forwards, and all odd times two periods backwards.

It is easy to demonstrate that the utilitarian and leximin social welfare relations proposed by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007) respectively satisfy both Stationarity and Strong Relative Anonymity. It is the purpose of the present paper to extend the asymmetric parts of these binary relations without compromising Stationarity and Strong Relative Anonymity. In particular, we will present utilitarian and leximin social welfare relations that rank $\mathbf{u}$ strictly above $\mathbf{v}$, while considering $\mathbf{x}$ and $\mathbf{y}$ (and $(0, \mathbf{x})$ and $(0, \mathbf{y})$, and $P\mathbf{x}$ and $P\mathbf{y}$) to be incomparable. When evaluating the merit of this exercise one should keep in mind that it is the extension of the asymmetric part that matters if one seeks to reduce the set of maximal elements for a given class of feasible infinite utility streams.

A simple but important fact is that, for comparing infinite utility streams, all welfare criteria, whether the utilitarian criterion of Basu and Mitra (2007b), the leximin criterion of Bossert, Sprumont and Suzumura (2007), as well as other utilitarian criteria such as Catching Up and Overtaking introduced by von Weizsäcker (1965) and Atsumi (1965), and the leximin criteria defined in Asheim and Tungodden (2004), all use an infinite sequence of the standard finite version of either the utilitarian or the leximin social welfare ordering. Using this fact, and a known property of these respective sequences, namely that of being "proliferating" (to impose the criterion for any finite number of persons, it is sufficient to impose it in situations where only two persons are involved), all these criteria can be given a "generalized" formulation. This generalized formulation is meaningful for any given proliferating sequence of social welfare relations defined on finite utility streams (and usually assumed to satisfy some Anonymity and Pareto conditions). The notion of a proliferating sequence was introduced for the analysis of generalized versions of infinite-dimensional SWRs by d'Aspremont (2007).

Here we suggest a new version of the overtaking criterion within this general approach to the evaluation of infinite utility streams. We call this *generalized time-invariant overtaking*. The generalized time-invariant overtaking criterion (on the space of infinite utility streams) is defined by extending proliferating sequences of complete and transitive binary relations defined on finite dimensional spaces. Our general analysis specializes in a straightforward manner to the utilitarian and leximin cases. We establish as a general result (stated in Theorem 1) that generalized time-invariant overtaking satisfies Stationarity and Strong Relative Anonymity. Moreover, we provide methods for determining the asymmetric and symmetric parts in the special cases of the utilitarian and leximin time-invariant overtaking criteria; these methods show that both criteria rank $\mathbf{u}$ strictly above $\mathbf{v}$.

The paper is organized as follows: Section 2 contains preliminaries, Section 3 presents the concept of proliferating sequences, and Section 4 reviews different kinds of "generalized criteria". Section 5 defines and investigates the properties of generalized time-invariant overtaking, while Section 6 specializes this concept to the utilitarian and leximin cases. Section 7 offers concluding remarks.

2 Preliminaries

2.1 Notation and Definitions

Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, ...\}$ and $\mathbb{R}$ the set of real numbers. Let $\mathbf{X}$ denote the set $Y^{\mathbb{N}}$, where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0, 1] \subseteq Y$. We let $\mathbf{X}$ be the domain of utility sequences (also referred to as "utility streams" or "utility profiles"). Thus, we write $\mathbf{x} \equiv (x_1, x_2, \ldots) \in \mathbf{X}$ iff $x_n \in Y$ for all $n \in \mathbb{N}$. For $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ we will write $\mathbf{x} \geq \mathbf{y}$ iff $x_i \geq y_i$ for all $i \in \mathbb{N}$ and $\mathbf{x} > \mathbf{y}$ iff $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Whenever we write about subsets M, N of $\mathbb{N}$, we will be dealing with subsets of finite cardinality, entailing that $\mathbb{N}\backslash M$, $\mathbb{N}\backslash N$ are cofinite sets (i.e., subsets of $\mathbb{N}$ which complements are finite). For all $\mathbf{x} \in \mathbf{X}$ and any $N \subset \mathbb{N}$, we will write $\mathbf{x}$ as $(\mathbf{x}_N, \mathbf{x}_{\mathbb{N}\backslash N})$. We will denote vectors (finite as well as infinite dimensional) by bold letters; example are **x, y**, etc. The components of a vector will be denoted by normal font. Negation of a statement is indicated by the logical quantifier $\neg$.

A *social welfare relation* (SWR) is a reflexive and transitive binary relation defined on $\mathbf{X}$ (and denoted $\succsim$) or $Y^{|M|}$ for some $M \subset \mathbb{N}$ (and denoted $\succsim_M$). A *social welfare order* (SWO) is a complete SWR.

A SWR $\succsim'$ is a *subrelation* of SWR $\succsim''$ if (a) for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, ($\mathbf{x} \sim' \mathbf{y} \Rightarrow \mathbf{x} \sim'' \mathbf{y}$), and (b) for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, ($\mathbf{x} \succ' \mathbf{y} \Rightarrow \mathbf{x} \succ'' \mathbf{y}$).

2.2 Permutations

A *permutation* π is a one-to-one map from $\mathbb{N}$ onto $\mathbb{N}$. For any $\mathbf{x} \in \mathbf{X}$ and a permutation π, we write $\mathbf{x} \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) \in \mathbf{X}$. Permutations can be *represented* by a permutation matrix. A permutation matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$ is an infinite matrix satisfying the following properties:

(1) For each $i \in \mathbb{N}$, $p_{ij(i)} = 1$ for some $j(i) \in \mathbb{N}$ and $p_{ij} = 0$ for all $j \neq j(i)$.

(2) For each $j \in \mathbb{N}$, $p_{i(j)j} = 1$ for $i(j) \in \mathbb{N}$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Writing permutations in terms of mappings or matrices, unsurprisingly, turns out to be equivalent. Given any permutation π, there is a permutation matrix P such that for $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots)$ can also be written as $P\mathbf{x}$ in the usual matrix multiplication. Conversely, given any permutation matrix P, there is a permutation π defined by $\pi = P\mathbf{a}$, where $\mathbf{a} = (1, 2, 3, \dots)$. The identity matrix I is an infinite permutation matrix such that $p_{ii} = 1$ for all $i \in \mathbb{N}$. Given any infinite permutation matrix P, we denote by P' its unique inverse which satisfies $PP' = P'P = I$. We denote the set of all permutations (permutation matrices) by $\mathcal{P}$.

A finite permutation π is a permutation such that there is some $N \subset \mathbb{N}$ with $\pi(i) = i$ for all $i \notin N$. Thus, a finite permutation matrix has $p_{ii} = 1$ for all $i \notin N$ for some $N \subset \mathbb{N}$. The set of all finite permutations is denoted by $\mathcal{F}$.

Given a permutation matrix $P \in \mathcal{P}$ and $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j\in\{1,\ldots,n\}}$ by $P(n)$. Let

$$\mathcal{S} = \{P \in \mathcal{P} \mid \text{there is some } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \; P(nk) \text{ is a finite dimensional permutation matrix}\}.$$

This class of permutations was introduced in Lauwers (1997). It is easily checked that this is a group (with respect to matrix multiplication) of cyclic permutations.[3]

2.3 Axioms of Anonymity and Pareto

In this subsection we introduce the basic axioms that are repeatedly used in the rest of the paper. The first set of axioms pertains to SWRs defined on pairs from a finite-dimensional space, whereas the later set is on the space of infinite utility streams.

Let $\succsim_M$ be a SWR defined on $Y^{|M|}$. Throughout we will as assume that $\succsim_M$ satisfies the following condition as a minimal requirement. It is an anonymity condition where the same permutation applied to the two utility vectors. Hence we call it "relative anonymity". In the present intergenerational context it can be interpreted as a time invariance property.

Axiom m-**I** (m-Relative Anonymity) For all $\mathbf{x}_M$, $\mathbf{y}_M$ $\mathbf{u}_N$, $\mathbf{v}_N \in Y^m$ with $M = \{i_1, i_2, ..., i_m\} \subset \mathbb{N}$ and $N = \{j_1, j_2, ..., j_m\} \subset \mathbb{N}$ satisfying $|M| = |N| = m \geq 2$, if there exists a finite permutation $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $x_{i_{\pi(k)}} = u_{j_k}$ and $y_{i_{\pi(k)}} = v_{j_k}$ for all $k \in \{1, \ldots, m\}$, then $\mathbf{x}_M \succsim_M \mathbf{y}_M$ if and only if $\mathbf{u}_N \succsim_N \mathbf{v}_N$.

By satisfying m-**I**, $\succsim_M$ depends only on the dimension $|M|$. We will henceforth write $\succsim_m$ for an SWR on Y^m, thereby signifying that the SWR satisfies m-**I**.

[3] A permutation is *cyclic* if for each $\mathbf{e}^i = (0, \ldots, 0, 1, 0 \ldots)$ (with 1 at the i^{th} place) there exists a $k \in \mathbb{N}$ such that $\pi^k(\mathbf{e}^i) = \mathbf{e}^i$. The class of cyclic permutations is not necessarily a group, while $\mathcal{P}$ is a group which does not contain only cyclic permutations.

It is useful to compare m-**I** to the usual anonymity condition where a permutation is applied to the one utility stream only.

Axiom m-**A** (m-Anonymity) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a}$ is a permutation of $\mathbf{b}$, then $\mathbf{a} \sim_m \mathbf{b}$.

Clearly, since $\succsim_m$ is transitive, m-**A** is equivalent to having $\mathbf{a} \sim_m \mathbf{b}$ whenever there exists i, $j \in \{1, \ldots, m\}$ such that $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$.

The m-Pareto Principle ($\mathbf{a} \succsim_m^P \mathbf{b}$ if and only if $a \geq b$) illustrates that m-**I** does not imply m-**A**. However, as originally shown by d'Aspremont and Gevers (1977, Lemma 4), the former axiom implies the latter if $\succsim_m$ is complete.

Lemma 1 *If $\succsim_m$ with $m \geq 2$ is complete, then $\succsim_m$ satisfies m-**A**.*

Proof. Assume that $\succsim_m$ is complete (where the notation entails that the SWR satisfies m-**I**). Suppose by way of contradiction that there exists $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$ such that $\neg(\mathbf{a} \sim_m \mathbf{b})$. Since $\succsim_m$ is complete, we may w.l.o.g. assume that $\mathbf{a} \succ_m \mathbf{b}$. However, by permuting the ith and jth element of both $\mathbf{a}$ and $\mathbf{b}$ and invoking m-**I**, we obtain $\mathbf{b} \succ_m \mathbf{a}$, which contradicts $\mathbf{a} \succ_m \mathbf{b}$. Hence, $\mathbf{a} \sim_m \mathbf{b}$ whenever there exists i, $j \in \{1, \ldots, m\}$ such that $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$. ■

The other kind of basic axiom is the Pareto condition.

Axiom m-**P** (m-Pareto) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a} > \mathbf{b}$, then $\mathbf{a} \succ_m \mathbf{b}$.

Clearly, since $\succsim_m$ is transitive, m-**P** is equivalent to having $\mathbf{a} \succ_m \mathbf{b}$ whenever there exists $i \in \{1, \ldots, m\}$ such that $a_i > b_i$ and $a_k = b_k$ for all $k \neq i$. As a matter of notation, if it is clear from the context that an axiom on finite dimension is invoked, then we will drop the letter m from its abbreviation.

Let $\succsim$ be a SWR defined on $\mathbf{X}$. Consider the following versions of the anonymity and Pareto axioms on $\succsim$. Let $\mathcal{Q}$ be some fixed group of permutations equaling $\mathcal{F}$, $\mathcal{S}$

or $\mathcal{P}$, corresponding to the terms "Finite", "Fixed-step" and "Strong" respectively in the names of the axioms below.

Axiom $\mathcal{Q}$I (Finite/Fixed-step/Strong Relative Anonymity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and all $P \in \mathcal{Q}$, $\mathbf{x} \succsim \mathbf{y}$ iff $P\mathbf{x} \succsim P\mathbf{y}$.

Axiom $\mathcal{Q}$A (Finite/Fixed-step/Strong Anonymity) For all $\mathbf{x} \in \mathbf{X}$ and all $P \in \mathcal{Q}$, $\mathbf{x} \sim P\mathbf{x}$.

Axiom FP (Finite Pareto) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N} \backslash N$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom SP (Strong Pareto) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Clearly, since $\succsim$ is transitive, $\mathcal{F}$**A** is equivalent to having $\mathbf{x} \sim \mathbf{y}$ whenever there exist i, $j \in \mathbb{N}$ such that $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \neq i, j$. Likewise, **FP** is equivalent to having $\mathbf{x} \succ \mathbf{y}$ whenever there exists $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. This is what Basu and Mitra (2007b) refer to as Weak Dominance; hence, **FP** coincides with Weak Dominance. Note that for $\mathcal{Q} = \mathcal{F}$, $\mathcal{S}$ or $\mathcal{P}$, $\mathcal{Q}$**A** implies $\mathcal{Q}$**I**, while the converse is not true for incomplete infinite-dimensional SWRs. It is also well-known that $\mathcal{P}$**A** cannot be combined with **SP**, while $\mathcal{S}$**A** can (since it is a group of cyclic permuations, cf. Mitra and Basu, 2007).

3 Proliferating sequences

Many well-known finite-dimensional SWRs form proliferating sequences, a concept introduced to the analysis of infinite-dimensional SWRs by d'Aspremont (2007). The structure imposed by this concept on a sequence of finite-dimensional SWR enables the extension to an infinite-dimensional SWR to be analyzed at a generalized level, without considering the specific nature of the finite-dimensional counterpart. Furthermore, it allows infinite-dimensional SWRs to be defined solely on the basis of the 2-dimensional version of the underlying finite-dimensional SWR.

A SWR $\succsim$ *extends* the SWR $\succsim_m$ if, for all $M \subset \mathbb{N}$ with $|M| = m$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i = y_i$ for every $i \in \mathbb{N}\backslash M$, $\mathbf{x}_M \succ_m \mathbf{y}_M$ implies $\mathbf{x} \succ \mathbf{y}$, and $\mathbf{x}_M \sim_m \mathbf{y}_M$ implies $\mathbf{x} \sim \mathbf{y}$.

Definition 1 A sequence of SWRs, $\{\succsim_m^*\}_{m=2}^{\infty}$, is *proliferating* if any SWR $\succsim$ that extends $\succsim_2^*$ also extends $\succsim_m^*$ for every $m \geq 2$.

The following result implies that the m-Grading Principle ($\mathbf{a} \succsim_m^S \mathbf{b}$ if and only if there exists a permutation $\mathbf{c}$ of $\mathbf{b}$ such that $\mathbf{a} \geq \mathbf{c}$) is proliferating.[4]

Lemma 2 *(i) If $\succsim_2$ is an SWR on Y^2 that satisfies* ***A****, and $\succsim$ is a SWR on $\mathbf{X}$ that extends $\succsim_2$, then $\succsim$ satisfies $\mathcal{F}$**A**.*
(ii) If $\succsim_2$ is an SWR on Y^2 that satisfies ***P****, and $\succsim$ is a SWR on $\mathbf{X}$ that extends $\succsim_2$, then $\succsim$ satisfies* ***FP****.*

Proof. (i) Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and for some i, $j \in \mathbb{N}$ $(i \neq j)$, $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \neq i$, j. Set $M = \{i, j\}$. Since $\succsim_2$ satisfies **A**, $\mathbf{x}_M \sim_2 \mathbf{y}_M$. By the fact that $x_k = y_k$ for all $k \in \mathbb{N}\backslash M$ and $\succsim$ extends $\succsim_2$, $\mathbf{x} \sim \mathbf{y}$.

(ii) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for some $i \in \mathbb{N}$, $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. Set $M = \{i, k\}$ for some $k \neq i$. Since $\succsim_2$ satisfies **P**, $\mathbf{x}_M \succ_2 \mathbf{y}_M$. By the fact that $x_j = y_j$ for all $j \in \mathbb{N}\backslash M$ and $\succsim$ extends $\succsim_2$, $\mathbf{x} \succ \mathbf{y}$. ■

The utilitarian and leximin SWOs, which will be defined and analyzed in Section 6, are other important examples of proliferating sequences. In the case of such complete SWRs, the notion of proliferation yields added structure.[5]

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[4] The Grading Principle was introduced by Suppes (1966) and further analyzed by Sen (1970), Kolm (1972) and Hammond (1976, 1979). Its proliferating property is mentioned by Sen (1976, fn 26) as suggested by Hammond as a step to derive the same property for Leximin. For a proof, see Hammond (1979). The proof of d'Aspremont (1985, Lemma 3.1.1) can be immediately transposed to Y^m (in place of $\mathbb{R}^m$).

[5] Sakai (2008) refers to property (i) of the following lemma as "correlation".

Lemma 3 *A proliferating sequence* $\{\succsim_m^*\}_{m=2}^{\infty}$ *of SWOs satisfies:*

(i) Assume $x_i = y_i$ *for some* $i \in \mathbb{N}\backslash M$. *Then* $\mathbf{x}_M \succsim_{|M|}^* \mathbf{y}_M$ *iff* $\mathbf{x}_{M\cup\{i\}} \succsim_{|M|+1}^* \mathbf{y}_{M\cup\{i\}}$.

(ii) Assume that $\succsim_m^*$ *satisfies* $\boldsymbol{P}$ *for each* $m \geq 2$. *If there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \sim_{|N|}^* \mathbf{y}_N$ *for all* $N \supseteq M$, *then* $x_i = y_i$ *for all* $i \in \mathbb{N}\backslash M$.

Proof. *(i)* Let $\{\succsim_m^*\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs, and let $\succsim$ extend $\succsim_2^*$, implying that $\succsim$ extends $\succsim_m^*$ for all $m \geq 2$. Assume that $\mathbf{x}_M \succsim_{|M|}^* \mathbf{y}_M$ and $x_i = y_i$ for some $i \in \mathbb{N}\backslash M$. Let $\mathbf{z} \in \mathbf{X}$ be an arbitrarily chosen utility stream. Since $\succsim$ extends $\succsim_{|M|}^*$, this implies $(\mathbf{x}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})}) \succsim (\mathbf{y}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})})$. Suppose $\mathbf{x}_{M\cup\{i\}} \prec_{|M|+1}^* \mathbf{y}_{M\cup\{i\}}$. Since $\succsim$ extends $\succsim_{|M|+1}^*$, this implies $(\mathbf{x}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})}) \prec (\mathbf{y}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})})$, leading to a contradiction. Hence, $\neg(\mathbf{x}_{M\cup\{i\}} \prec_{|M|+1}^* \mathbf{y}_{M\cup\{i\}})$, implying since the SWO $\succsim_{|M|+1}^*$ is complete that $\mathbf{x}_{M\cup\{i\}} \succsim_{|M|+1}^* \mathbf{y}_{M\cup\{i\}}$. Likewise, $\mathbf{x}_M \succ_{|M|}^* \mathbf{y}_M$ and $x_i = y_i$ for some $i \in \mathbb{N}\backslash M$ implies that $\mathbf{x}_{M\cup\{i\}} \succ_{|M|+1}^* \mathbf{y}_{M\cup\{i\}}$, thereby establishing the converse statement.

(ii) Let $\{\succsim_m^*\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim_m^*$ satisfying $\mathbf{P}$. Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$. Suppose that $x_i \neq y_i$ for some $i \in \mathbb{N}\backslash M$; w.l.o.g. we can set $x_i > y_i$. Since $\succsim_{|M|+1}^*$ satisfies $\mathbf{P}$, it follows from part (i) that

$$\mathbf{x}_{M\cup\{i\}} \sim_{|M|+1}^* (\mathbf{y}_M, x_i) \succ_{|M|+1}^* \mathbf{y}_{M\cup\{i\}} ,$$

contradicting that $\mathbf{x}_{M\cup\{i\}} \sim_{|M|+1}^* \mathbf{y}_{M\cup\{i\}}$. Hence, $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$. ■

4 Generalized criteria

In this section we review "generalized criteria", namely infinite-dimensional SWRs that extend finite-dimensional SWRs that are both complete and proliferating. We first introduce two additional axioms on the space of infinite utility streams that will be used to differentiate these generalized criteria and in the rest of the paper.

Axiom ST (Stationarity) For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$ with $x_1 = y_1$ and, for all $i \in \mathbb{N}$, $u_i = x_{i+1}$ and $v_i = y_{i+1}$, $\mathbf{x} \succsim \mathbf{y}$ iff $\mathbf{u} \succsim \mathbf{v}$.

Axiom IPC (Time-Invariant Preference Continuity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exists $M \subset \mathbb{N}$ such that, for all $N \supseteq M$, $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\setminus N}) \succ \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom **IPC** will turn out to be sufficient to ensure strict preference between $\mathbf{u}$ and $\mathbf{v}$ of the introduction, in the utilitarian and leximin cases.

Let $\{\succsim^*_m\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim^*_m$ satisfying axiom **P** (while, by Lemma 1, axiom **A** follows from the assumption that axiom **I** is satisfied). To illustrate the axioms and the trade-offs between them, consider the following generalized criteria. The possibility results are available on request from the authors, while the impossibility results follow from the examples of Section 1 in the context of the utilitarian and leximin proliferating sequences.

- **Equality on a cofinite set.** $\succsim^*$ is the SWR defined by

  $$\mathbf{x} \succsim^* \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\setminus N} = \mathbf{y}_{\mathbb{N}\setminus N}\,.$$

  The SWR $\succsim^*$ satisfies $\mathcal{P}$**I**, $\mathcal{F}$**A**, **FP** and **ST**, but not $\mathcal{S}$**A**, **SP** and **IPC**.

- **Equality or Pareto-dominance on a cofinite set** (Basu and Mitra, 2007a; Bossert, Sprumont and Suzumura, 2007). $\succsim^*_{\mathcal{F}}$ is the SWR defined by

  $$\mathbf{x} \succsim^*_{\mathcal{F}} \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\setminus N} \geq \mathbf{y}_{\mathbb{N}\setminus N}\,.$$

  The SWR $\succsim^*_{\mathcal{F}}$ satisfies $\mathcal{P}$**I**, $\mathcal{F}$**A**, **SP**, **ST**, but not $\mathcal{S}$**A** and **IPC**.

- **Extended Anonymity** (Banerjee, 2006; Kamaga and Kojima, 2008). $\succsim^*_{\mathcal{S}}$ is the SWR defined by

  $$\mathbf{x} \succsim^*_{\mathcal{S}} \mathbf{y} \text{ iff there exists } P \in \mathcal{S} \text{ such that } \mathbf{x} \succsim^*_{\mathcal{F}} P\mathbf{y}\,.$$

  The SWR $\succsim^*_{\mathcal{S}}$ satisfies $\mathcal{S}$**I**, $\mathcal{S}$**A** and **SP**, but not $\mathcal{P}$**I**, $\mathcal{P}$**A**, **ST** and **IPC**.

- **Catching up (in finite time)** (Atsumi, 1965; von Weizsäcker, 1965) $\succsim^*_{\mathcal{C}}$ is the SWR defined by

  $$\mathbf{x} \succsim^*_{\mathcal{C}} \mathbf{y} \text{ iff there exists } m \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,n\}} \succsim^*_n \mathbf{y}_{\{1,\dots,n\}} \text{ for all } n \geq m\,.$$

  The SWR $\succsim^*_{\mathcal{C}}$ satisfies $\mathcal{F}$**I**, $\mathcal{F}$**A**, **SP**, **ST** and **IPC**, but not $\mathcal{S}$**I** and $\mathcal{S}$**A**.

- **Fixed-step catching up** (Fleurbaey and Michel, 2003). $\succsim^*_{\mathcal{SC}}$ is the SWR defined by

  $$\mathbf{x} \succsim^*_{\mathcal{SC}} \mathbf{y} \text{ iff there exists } k \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,nk\}} \succsim^*_{nk} \mathbf{y}_{\{1,\dots,nk\}} \text{ for all } n \in \mathbb{N}.$$

  The SWR $\succsim^*_{\mathcal{SC}}$ satisfies $\mathcal{S}$**I**, $\mathcal{S}$**A**, **SP** and **IPC**, but not $\mathcal{P}$**I**, $\mathcal{P}$**A** and **ST**.

We have that, for a fixed proliferating sequence of SWOs, $\{\succsim^*_m\}_{m=2}^{\infty}$, $\succsim^*$ is a subrelation of $\succsim^*_{\mathcal{F}}$, $\succsim^*_{\mathcal{F}}$ is a subrelation of each of $\succsim^*_{\mathcal{S}}$ and $\succsim^*_{\mathcal{C}}$, and $\succsim^*_{\mathcal{S}}$ is a subrelation of $\succsim^*_{\mathcal{SC}}$. Going from $\succsim^*_{\mathcal{F}}$ to $\succsim^*_{\mathcal{C}}$ we pick up **IPC**, but must weaken $\mathcal{P}$**I** all the way to $\mathcal{F}$**I**. Going from $\succsim^*_{\mathcal{F}}$ to $\succsim^*_{\mathcal{SC}}$ we strengthen $\mathcal{F}$**A** to $\mathcal{S}$**A** and pick up **IPC**, but must weaken $\mathcal{P}$**I** to $\mathcal{S}$**I** and drop **ST**. This leads to the question: Is it possible to pick up **IPC** without weakening $\mathcal{P}$**I** and dropping **ST**?[6] We show that this is indeed possible by means of generalized time-invariant overtaking.

5 A new criterion for infinite utility streams

We are now ready to state the definition of the generalized time-invariant overtaking criterion. Let $\{\succsim^*_m\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with $\succsim^*_m$ satisfying axiom **P** (while axiom **A** is implied by axiom **I**) for each $m \geq 2$.[7]

[6]The $(\mathbf{x}, \mathbf{y})$ example of Section 1 illustrates the problems of strengthening $\mathcal{F}$**A** to $\mathcal{S}$**A** while retaining **ST**. Mitra (2007) discusses the problem of combining **ST** with any kind of extended anonymity. The emphasis of the present paper is to show how the asymmetric part of $\succsim^*_{\mathcal{F}}$ can be extended, while retaining **ST**.

[7]Definition 2 is formulated as a "catching up" criterion. However, Lemmas 3(ii) and 4, showing that a formulation in terms of an "overtaking" criterion is equivalent, justify our terminology.

Definition 2 (Generalized time-invariant overtaking) The *generalized time-invariant overtaking criterion* $\succsim_{\mathcal{I}}^*$ generated by $\{\succsim_m^*\}_{m=2}^{\infty}$ satisfies, for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim_{\mathcal{I}}^* \mathbf{y} \text{ iff there exists } M \subset \mathbb{N} \text{ with } |M| \geq 2 \text{ such that } \mathbf{x}_N \succsim_{|N|}^* \mathbf{y}_N \text{ for all } N \supseteq M.$$

We can now state our main result.

Theorem 1 *Let $\{\succsim_m^*\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim_m^*$ satisfying axiom* ***P***. *Then:*

*(i) $\succsim_{\mathcal{I}}^*$ is a SWR that satisfies $\mathcal{P}$**I**, $\mathcal{F}$**A**, **SP** and **ST**.*

*(ii) A SWR $\succsim$ extends $\succsim_2^*$ and satisfies **IPC** iff $\succsim_{\mathcal{I}}^*$ is a subrelation of $\succsim$.*

In the proof of Theorem 1, we make use of the following lemmas.

Lemma 4 *The SWR $\succsim_{\mathcal{I}}^*$ satisfies:*

(i) $\mathbf{x} \succ_{\mathcal{I}}^* \mathbf{y}$ iff there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$.

(ii) $\mathbf{x} \sim_{\mathcal{I}}^* \mathbf{y}$ iff there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$.

Proof. (Only-if part of (i): $\mathbf{x} \succ_{\mathcal{I}}^* \mathbf{y}$ only if there exist $M \subset N$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$.) Assume $\mathbf{x} \succ_{\mathcal{I}}^* \mathbf{y}$ that is, (a) $\mathbf{x} \succsim_{\mathcal{I}}^* \mathbf{y}$ and (b) $\neg(\mathbf{y} \succsim_{\mathcal{I}}^* \mathbf{x})$. By (a), there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succsim_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$. Note that $\neg(\mathbf{y} \succsim_{\mathcal{I}}^* \mathbf{x})$ implies that for any $M \subset \mathbb{N}$ there is some $M' \supset M$ such that $\mathbf{x}_{M'} \succ_{|M'|}^* \mathbf{y}_{M'}$. By way of contradiction, suppose that there does not exist $M'' \subset \mathbb{N}$ such $\mathbf{x}_N \succ_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M''$. In particular, since then $\mathbf{x}_N \succ_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$ does not hold, it follows from (a) that there exists $A \supseteq M$ such that $\mathbf{x}_A \sim_{|A|}^* \mathbf{y}_A$. We claim that there exists $B \subset \mathbb{N}$ with $A \cap B = \emptyset$ such that $\mathbf{x}_{A\cup B} \succ_{|A|+|B|}^* \mathbf{y}_{A\cup B}$. That is, the statement: for all $B \subset \mathbb{N}$ with $A \cap B = \emptyset$

we must have $\mathbf{y}_{A\cup B} \succsim^*_{|A|+|B|} \mathbf{x}_{A\cup B}$ is false. This possibility is ruled out since if it were correct, we would obtain $\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x}$, which is contradicted by (b).

Since we suppose that there does not exist $M'' \subset \mathbb{N}$ such $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M''$, it does not hold that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq A \cup B$. Hence, by (a) there exists $C \subseteq \mathbb{N}$ with $(A \cup B) \cap C = \emptyset$ such that $\mathbf{x}_{A\cup B\cup C} \sim^*_{|A|+|B|+|C|} \mathbf{y}_{A\cup B\cup C}$. This leads to the first indifference in (1), while the second strict preference in (1) follows from Lemma 3(i):

$$\mathbf{y}_{A\cup B\cup C} \sim^*_{|A|+|B|+|C|} \mathbf{x}_{A\cup B\cup C} \succ^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{x}_C)\,. \tag{1}$$

By transitivity we get $(\mathbf{y}_{A\cup B}, \mathbf{y}_C) \succ^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{x}_C)$. So, $\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C$. [If $\neg(\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C)$, then $\mathbf{x}_C \succsim^*_{|C|} \mathbf{y}_C$. By Lemma 3(i), we obtain $(\mathbf{y}_{A\cup B}, \mathbf{x}_C) \succsim^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{y}_C)$.] We now get:

$$\mathbf{y}_{A\cup C} \succ^*_{|A|+|C|} (\mathbf{y}_A, \mathbf{x}_C) \sim^*_{|A|+|C|} \mathbf{x}_{A\cup C} \succsim^*_{|A|+|C|} \mathbf{y}_{A\cup C}\,, \tag{2}$$

The first strict preference in (2) is a consequence of Lemma 3(i) and $\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C$. The second indifference in (2) is a consequence of Lemma 3(i) and $\mathbf{x}_A \sim^*_{|A|} \mathbf{y}_A$. The last weak preference in (2) follows from (a) and the fact that $A \cup C \supset M$. So (2) leads us to a contradiction.This completes the proof of the only-if part of (i).

(If part of (i): $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ if there exists $M \subset N$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Then $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. By way of contradiction, suppose $\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x}$. Then there exists $M' \subset \mathbb{N}$ with $|M'| \geq 2$ such that $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$ for all $N \supseteq M'$. For $N \supseteq M' \cup M$ we must have $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ and $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$. This leads to a contradiction. Hence, $\neg(\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x})$ and, consequently, $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$.

(Only-if part of (ii): $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$ only if there exist $M \subset N$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.) Let $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$. Then there exists sets $M', M'' \subset \mathbb{N}$ such that $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M'$ and $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$ for all $N \supseteq M''$. Then for all $N \supseteq M' \cup M''$ we must have $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$, as was required.

The if part of (ii) follows directly from the definition and we omit the details. ■

Lemma 5 *The SWR $\succsim^*_{\mathcal{I}}$ satisfies $\mathcal{P}$**I**, **SP** and **ST**.*

Proof. ($\succsim^*_{\mathcal{I}}$ satisfies $\mathcal{P}$**I**.) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $P \in \mathcal{P}$. Assume $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be the equivalent representation of the infinite permutation matrix P. Clearly π is a one-to-one and onto function. Since $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ there exists $M \subset N$ with $|M| \geq 2$ such that $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Let the image of M under the function π be denoted by $\pi(M)$, that is $\pi(M) = \{i \in \mathbb{N} \mid \text{there exists } j \in M \text{ such that } \pi(j) = i\}$. Now for $N \supseteq \pi(M)$, we must have $\pi^{-1}(N) \supseteq M$, where $\pi^{-1} : \mathbb{N} \to \mathbb{N}$ is the inverse of π. Since $\succsim^*_m$ satisfies m-**I** for all $m \geq 2$, we must have for all $N \supseteq \pi(M)$, $(P\mathbf{x})_N \succsim^*_{\mathcal{I}} (P\mathbf{y})_N$. Hence, $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ implies $P\mathbf{x} \succsim^*_{\mathcal{I}} P\mathbf{y}$ for any $P \in \mathcal{P}$. The converse is established in a similar manner.

($\succsim^*_{\mathcal{I}}$ satisfies **SP**.) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ satisfy $\mathbf{x} > \mathbf{y}$. Pick $M \subset \mathbb{N}$ such that $\mathbf{x}_M \neq \mathbf{y}_M$. Since $\succsim^*_m$ satisfies **P** for all $m \geq 2$, we must have $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. By Lemma 4 (i) we can conclude $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$.

($\succsim^*_{\mathcal{I}}$ satisfies **ST**.) Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$ satisfy $x_1 = y_1$, and for all $i \in \mathbb{N}$, $u_i = x_{i+1}$ and $v_i = y_{i+1}$. Assume $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. Hence, there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Construct M' as follows: $M' = \{i \in \mathbb{N} \mid i+1 \in M\}$, with an arbitrary element added in if the number of elements in M' would otherwise be 1. Consider any $N' \subseteq M'$, and construct N as follows: $N = \{i \in \mathbb{N} \mid i - 1 \in N'\} \cup \{1\}$. Since, by construction, $N \supseteq M$, $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$. By Lemma 3(i), $\mathbf{x}_{N\setminus\{1\}} \succ^*_{|N|-1} \mathbf{y}_{N\setminus\{1\}}$ since $x_1 = y_1$. Thus, $\mathbf{u}_{N'} \succsim^*_{|N|-1} \mathbf{v}_{N'}$ since $\succsim^*_m$ satisfies m-**I** for all m. Hence, $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ implies $\mathbf{u} \succsim^*_{\mathcal{I}} \mathbf{v}$. The converse is establish in a similar manner. ■

Proof of Theorem 1. (i) It can be easily checked that $\succsim^*_{\mathcal{I}}$ is reflexive and transitive provided that $\succsim^*_m$ is reflexive and transitive for each m; hence, $\succsim^*_{\mathcal{I}}$ is a SWR on $\mathbf{X}$. The rest of part (i) follows directly from Lemma 2(i) and Lemma 5.

(Only-if part of (ii): A SWR $\succsim$ extends $\succsim^*_2$ and satisfies **IPC** only if $\succsim^*_{\mathcal{I}}$ is a subrelation of $\succsim$.) For $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, let $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$. Then using Lemma 4 (i) we must have that there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. For all $N \supseteq M$, since $\succsim$ extends $\succsim_2^*$ and $\{\succsim_m^*\}_{m=2}^\infty$ is a proliferating sequence we obtain $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\setminus N}) \succ \mathbf{y}$. Now by **IPC** we have $\mathbf{x} \succ \mathbf{y}$.

Now let $\mathbf{x} \sim_{\mathcal{I}}^* \mathbf{y}$. By Lemma 4 (ii) we must have that there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$. By Lemma 3 (ii), we have $x_i = y_i$ for all $i \in \mathbb{N}\setminus M$. Since $\succsim$ extends $\succsim_2^*$ and $\{\succsim_m^*\}_{m=2}^\infty$ is a proliferating sequence we get $\mathbf{x} \sim \mathbf{y}$.

(If part of (ii): A SWR $\succsim$ extends $\succsim_2^*$ and satisfies **IPC** if $\succsim_{\mathcal{I}}^*$ is a subrelation of $\succsim$.) We omit the straightforward proof of the result that $\succsim$ extends $\succsim_2^*$.

To show that $\succsim$ satisfies **IPC**, assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that, for all $N \supseteq M$, $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\setminus N}) \succ \mathbf{y}$. Since $\succsim$ extends $\succsim_2^*$ and $\{\succsim_m^*\}_{m=2}^\infty$ is proliferating, it follows from the completeness of the SWO $\succsim_m^*$ for every m that $\mathbf{x}_N \succ_{|N|}^* \mathbf{y}_N$ for all $N \supseteq M$. Hence, $\mathbf{x} \succ_{\mathcal{I}}^* \mathbf{y}$ by Lemma 4(i), and $\mathbf{x} \succ \mathbf{y}$ since $\succsim_{\mathcal{I}}^*$ is a subrelation of $\succsim$. This shows that $\succsim$ satisfies condition **IPC**. ■

6 Applications

In this section we study specific criteria based on particular proliferating sequences. In particular, as the utilitarian SWO and the leximin SWO defined for pairs on any subset of the m-dimensional Euclidean space define two proliferating sequences, they lay the foundation for two specializations of the generalized time-invariant overtaking criterion: utilitarian and leximin time-invariant overtaking. Furthermore, we propose methods for determining the asymmetric and symmetric parts of the utilitarian and leximin time-invariant overtaking criteria.

6.1 The Utilitarian Case

To state the definition of the utilitarian SWO defined on Y^m we first introduce some additional notation. For each $N \subset \mathbb{N}$, the partial sum $\sum_{i \in N} x_i$ is written as $\sigma(\mathbf{x}_N)$. Let $\{\succsim_m^U\}_{m=2}^\infty$ denote the sequence of utilitarian SWOs, with each $\succsim_m^U$ defined on

Y^m. Formally, for $\mathbf{a}$, $\mathbf{b} \in Y^m$,

$$\mathbf{a} \succsim_m^U \mathbf{b} \text{ iff } \sigma(\mathbf{a}) \geq \sigma(\mathbf{b}) .$$

In order to rely on a standard characterization of utilitarianism, we first state the Translation Scale Invariance axiom for finite populations social choice theory.

Axiom m-TSI (m-Translation Scale Invariance) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a} \succsim_m \mathbf{b}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$ satisfies $\mathbf{a} + \boldsymbol{\alpha} \in Y^m$ and $\mathbf{b} + \boldsymbol{\alpha} \in Y^m$, then $\mathbf{a} + \boldsymbol{\alpha} \succsim_m \mathbf{b} + \boldsymbol{\alpha}$.

This axiom says that utility differences can be compared interpersonally. A comprehensive treatment of the literature on social choice with interpersonal utility comparisons can be found in Bossert and Weymark (2004). The following characterization of finite-dimensional utilitarianism is well-known.[8]

Lemma 6 *For all $m \in \mathbb{N}$, the utilitarian SWO $\succsim_m^U$ is equal to $\succsim_m$ iff $\succsim_m$ satisfies **A**, **P** and **TSI**.*

Let $\succsim$ be a SWR defined on $\mathbf{X}$. Consider the following axiom on $\succsim$.

Axiom FTSI (Finite Translation Scale Invariance) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N}\backslash N$, if $\mathbf{x} \succsim \mathbf{y}$ and $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ satisfies that $\mathbf{x} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\mathbf{y} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\alpha_i = 0$ for all $i \in \mathbb{N}\backslash N$, then $\mathbf{x} + \boldsymbol{\alpha} \succsim \mathbf{y} + \boldsymbol{\alpha}$.

By means of this axiom we can characterize the class of SWRs extending $\succsim_2^U$:

Proposition 1 *$\succsim$ is a SWR on X that extends $\succsim_2^U$ iff $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**.*

Proof of Proposition 1. (Only-if part: $\succsim$ is a SWR on X that extends $\succsim_2^U$ only if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**.) Assume $\succsim$ is a SWR on X that extends

[8] The argument is due to Milnor (1954) in the context of individual decision under risk. For a proof in the social choice context, see d'Aspremont and Gevers (2002).

$\succsim_2^U$. It follows from Lemma 2 that $\succsim$ satisfies $\mathcal{F}\mathbf{A}$ and **FP**. To show that $\succsim$ satisfies **FTSI**, consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ for which there exists some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N}\backslash N$, and $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ which satisfies $\mathbf{x} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\mathbf{y} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\alpha_k = 0$ for all $i \in \mathbb{N}\backslash N$. Since $\succsim$ extends $\succsim_2^U$ and satisfies **FP**, it follows from Lemma 8 of the appendix that $\mathbf{x} \succsim \mathbf{y}$ iff $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ and $\mathbf{x} + \boldsymbol{\alpha} \succsim \mathbf{y} + \boldsymbol{\alpha}$ iff $\sigma(\mathbf{x}_N + \boldsymbol{\alpha}_N) \geq \sigma(\mathbf{y}_N + \boldsymbol{\alpha}_N)$. Clearly, $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ implies $\sigma(\mathbf{x}_N + \boldsymbol{\alpha}_N) \geq \sigma(\mathbf{y}_N + \boldsymbol{\alpha}_N)$, thereby establishing that $\succsim$ satisfies **FTSI**.

(If part: $\succsim$ is a SWR on X that extends $\succsim_2^U$ if $\succsim$ satisfies $\mathcal{F}\mathbf{A}$, **FP** and **FTSI**.) Assume that $\succsim$ satisfies $\mathcal{F}\mathbf{A}$, **FP** and **FTSI**. Fix $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = 2$. Construct $\succsim_2^{\mathbf{z}}$ as follows: $\mathbf{x}_M \succsim_2^{\mathbf{z}} \mathbf{y}_M$ iff $(\mathbf{x}_M, \mathbf{z}_{\mathbb{N}\backslash M}) \succsim (\mathbf{y}_M, \mathbf{z}_{\mathbb{N}\backslash M})$. Since $\succsim$ satisfies $\mathcal{F}\mathbf{A}$, **FP** and **FTSI**, it follows that $\succsim_2^{\mathbf{z}}$ satisfies **A**, **P** and **TSI**. Thus, by Lemma 6, $\succsim_2^U$ is equal to $\succsim_2^{\mathbf{z}}$. Since $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = 2$ are arbitrarily chosen, it follows that $\succsim$ extends $\succsim_2^U$. ■

Combined with Lemma 6, Proposition 1 implies the following result, which makes Theorem 1 applicable in the utilitarian case.

Proposition 2 *The sequence of utilitarian SWOs, $\{\succsim_m^U\}_{m=2}^\infty$, is proliferating.*

Proposition 2 is established by d'Aspremont (2007, Lemma 4) in the case where $Y = \mathbb{R}$. In the appendix we provide a direct proof of Proposition 2 in the present case where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0, 1] \subseteq Y$.

Since, by Proposition 2, $\{\succsim_m^U\}_{m=2}^\infty$ is proliferating, we can now state the following specialization of generalized time-invariant overtaking.

Definition 3 (Utilitarian time-invariant overtaking) The *utilitarian time-invariant overtaking criterion* $\succsim_{\mathcal{I}}^U$ satisfies, for $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$\mathbf{x} \succsim_{\mathcal{I}}^U \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ for all $N \supseteq M$.

By Propositions 1 and 2, the following characterization of utilitarian time-invariant overtaking is a direct consequence of Theorem 1 and Lemma 4:

Corollary 1 *(i)* $\succsim^U_{\mathcal{I}}$ *is a SWR that satisfies* $\mathcal{P}$***I****,* ***SP*** *and* ***ST****.*

(ii) A SWR $\succsim$ *satisfies* $\mathcal{F}$***A****,* ***FP****,* ***FTSI*** *and* ***IPC*** *iff* $\succsim^U_{\mathcal{I}}$ *is a subrelation of* $\succsim$*.*

(iii) $\mathbf{x} \succ^U_{\mathcal{I}} \mathbf{y}$ *iff there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\sigma(\mathbf{x}_N) > \sigma(\mathbf{y}_N)$ *for all* $N \supseteq M$*.*

(iv) $\mathbf{x} \sim^U_{\mathcal{I}} \mathbf{y}$ *iff there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\sigma(\mathbf{x}_N) = \sigma(\mathbf{y}_N)$ *for all* $N \supseteq M$*.*

To facilitate its use, we provide a characterization of the asymmetric and symmetric parts of the utilitarian generalized overtaking criterion.

Proposition 3 *Utilitarian time-invariant overtaking satisfies:*

(i) $\mathbf{x} \succ^U_{\mathcal{I}} \mathbf{y}$ *iff there exists* $M^+ \subseteq \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ *such that* $\sigma(\mathbf{x}_{M^+ \cup M^-}) > \sigma(\mathbf{y}_{M^+ \cup M^-})$ *for all* $M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$*.*

(ii) $\mathbf{x} \sim^U_{\mathcal{I}} \mathbf{y}$ *if and only* $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ *and* $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ *are finite sets satisfying* $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$*.*

Proof. (If part of (i).) Assume that there exists $M^+ \subseteq \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ such that $\sigma(\mathbf{x}_{M^+ \cup M^-}) > \sigma(\mathbf{y}_{M^+ \cup M^-})$ for all $M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$. Let $M = M^+$ and choose $N \supseteq M$. We can partition N into $A := \{i \in N \mid x_i - y_i \geq 0\}$ and $M^- := \{i \in N \mid x_i - y_i < 0\}$, implying that $x_i - y_i \geq 0$ for all $A \backslash M^+$. Hence,

$$\sigma(\mathbf{x}_N) - \sigma(\mathbf{y}_N) = \sigma(\mathbf{x}_{A \cup M^-}) - \sigma(\mathbf{y}_{A \cup M^-}) \geq \sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) > 0\,,$$

where the partitioning of N into A and M^- implies the first equality, $x_i - y_i \geq 0$ for all $A \backslash M^+$ implies the second weak inequality, and the premise implies the third strong inequality.

(Only-if part of (i).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) > \sigma(\mathbf{y}_N)$ for all $N \supseteq M$. Let $M^+ := M \cap \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and choose

$M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$. Note that $x_i \leq y_i$ for all $i \in M \backslash (M^+ \cap M^-)$. Hence,

$$\sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) \geq \sigma(\mathbf{x}_{M \cup M^-}) - \sigma(\mathbf{y}_{M \cup M^-}) > 0$$

by the premise since $M \cup M^- \supseteq M$.

(If part of Part (ii).) Assume that $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are finite sets satisfying $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$. Let $M = M^+ \cup M^-$ and choose $N \supseteq M$. Since $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$, it follows that

$$\sigma(\mathbf{x}_N) - \sigma(\mathbf{y}_N) = \sigma(\mathbf{x}_M) - \sigma(\mathbf{y}_M) = \sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) = 0$$

by the premise.

(Only-if part of (ii).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) = \sigma(\mathbf{y}_N)$ for all $N \supseteq M$. By Lemma 3(ii) and the fact that $\{\succsim_m^U\}_{t=2}^{\infty}$ is proliferating, it follows that $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$. Hence, $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are finite sets satisfying $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$.

The if parts can easily be amended to ensure that $|M| \geq 2$. ■

This characterization can be illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1. In this example $\{i \in \mathbb{N} \mid u_i - v_i > 0\} = \{1\}$ and $\{i \in \mathbb{N} \mid u_i - v_i < 0\} = \mathbb{N} \backslash \{1\}$. By choosing $M^+ = \{1\}$ so that $\sigma(\mathbf{u}_{M^+}) - \sigma(\mathbf{v}_{M^+}) = 1$, and noting $\sigma(\mathbf{u}_{M^-}) - \sigma(\mathbf{v}_{M^-}) < \frac{1}{2}$ for all $M^- \subset \mathbb{N} \backslash \{1\}$, it follows from Proposition 3(i) that $\mathbf{u} \succ_{\mathcal{I}}^U \mathbf{v}$.

The utilitarian criterion proposed by Basu and Mitra (2007a), which we discussed in Section 1 and denoted $\succsim_{\mathcal{F}}^U$, yields comparability only if there is equality or Pareto-dominance on a cofinite set:

$$\mathbf{x} \succsim_{\mathcal{F}}^U \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N) \text{ and } \mathbf{x}_{\mathbb{N} \backslash N} \geq \mathbf{y}_{\mathbb{N} \backslash N} .$$

It follows from Proposition 3 that $\succsim_{\mathcal{F}}^U$ is a subrelation of $\succsim_{\mathcal{I}}^U$, since the symmetric parts, $\sim_{\mathcal{I}}^U$ and $\sim_{\mathcal{F}}^U$, coincide, while $\succ_{\mathcal{I}}^U$ strictly extends $\succsim_{\mathcal{F}}^U$, as illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1.

6.2 The Leximin Case

To state a precise definition of the leximin order we introduce additional notation. For any $\mathbf{x}_M$, $(x_{(1)}, \dots, x_{(|M|)})$ denotes the rank-ordered permutation of $\mathbf{x}_M$ such that $x_{(1)} \leq \dots \leq x_{(|M|)}$, ties being broken arbitrarily. For any $\mathbf{x}_M$ and $\mathbf{y}_M$, $\mathbf{x}_M \succ^L_{|M|} \mathbf{y}_M$ iff there exists $m \in \{1, \dots, |M|\}$ such that $x_{(k)} = y_{(k)}$ for all $k \in \{1, \dots, m-1\}$ and $x_{(m)} > y_{(m)}$ and $\mathbf{x}_M \sim^L_{|M|} \mathbf{y}_M$ iff $x_{(k)} = y_{(k)}$ for all $k \in \{1, \dots, |M|\}$.

We first recall through Lemma 7 below a standard characterization of finite-dimensional leximin using the Hammond Equity axiom. This axiom states, in our intergenerational context, that if there is a conflict between two generations, with every other generation being as well off in the compared profiles, then society should weakly prefer the profile where the least favored generation is better off.

Axiom m-HE (m-Hammond Equity) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if there exist i, $j \in \{1, \dots, m\}$ such that $b_i > a_i > a_j > b_j$ and $a_k = b_k$ for all $k \neq i$, j, then $\mathbf{a} \succsim_m \mathbf{b}$.

Lemma 7 *For all $m \in \mathbb{N}$, the leximin SWO $\succsim^L_m$ is equal to $\succsim_m$ iff $\succsim_m$ satisfies **A**, **P** and **HE**.*

Let $\succsim$ be a SWR defined on $\mathbf{X}$. Consider also the **HE** axiom on $\succsim$.

Axiom HE (Hammond Equity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exist i, $j \in \mathbb{N}$ such that $y_i > x_i > x_j > y_j$ and $x_k = y_k$ for all $k \neq i$, j, then $\mathbf{x} \succsim \mathbf{y}$.

By means of this axiom we can characterize the class of SWRs extending $\succsim^L_2$:

Proposition 4 *$\succsim$ is a SWR on X that extends $\succsim^L_2$ iff $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**.*

Proof. (Only-if part: $\succsim$ is a SWR on X that extends $\succsim^L_2$ only if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**.) Assume $\succsim$ is a SWR on X that extends $\succsim^L_2$. It follows from Lemma 2 that $\succsim$ satisfies $\mathcal{F}$**A** and **FP**. To show that $\succsim$ satisfies **HE**, let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ satisfy that there exist $i, j \in \mathbb{N}$ such that $y_i > x_i > x_j > y_j$ and $x_k = y_k$ for all $k \neq i, j$. Then $\mathbf{x}_{\{i,j\}} \succsim_2^L \mathbf{y}_{\{i,j\}}$ (since $\succsim_2^L$ satisfies 2-**HE**) and $\mathbf{x} \succsim \mathbf{y}$ (since $\succsim$ extends $\succsim_2^L$). This establishes that $\succsim$ satisfies **HE**.

(If part: $\succsim$ is a SWR on X that extends $\succsim_2^L$ if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**.) Assume that $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**. Fix $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = 2$. Construct $\succsim_2^{\mathbf{z}}$ as follows: $\mathbf{x}_M \succsim_2^{\mathbf{z}} \mathbf{y}_M$ iff $(\mathbf{x}_M, \mathbf{z}_{\mathbb{N}\setminus M}) \succsim (\mathbf{y}_M, \mathbf{z}_{\mathbb{N}\setminus M})$. Since $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**, it follows that $\succsim_2^{\mathbf{z}}$ satisfies **A**, **P** and 2-**HE**. Thus, by Lemma 7, $\succsim_2^L$ is equal to $\succsim_2^{\mathbf{z}}$. Since $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = 2$ are arbitrarily chosen, it follows that $\succsim$ extends $\succsim_2^L$. ■

Combined with Lemma 7, Proposition 4 implies the following result, which makes Theorem 1 applicable in the utilitarian case.

Proposition 5 *The sequence of leximin SWOs, $\{\succsim_m^L\}_{m=2}^{\infty}$, is proliferating.*

d'Aspremont (2007, Lemma 5) proves Proposition 5 through a direct argument which is applicable also to the present case where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0, 1] \subseteq Y$.

Since, by Proposition 5, $\{\succsim_m^L\}_{m=2}^{\infty}$ is proliferating, we can now state the following specialization of generalized time-invariant overtaking.

Definition 4 (Leximin time-invariant overtaking) The *leximin time-invariant overtaking criterion* $\succsim_{\mathcal{I}}^L$ satisfies, for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim_{\mathcal{I}}^L \mathbf{y} \text{ iff there exists } M \subset \mathbb{N} \text{ with } |M| \geq 2 \text{ such that } \mathbf{x}_N \succsim_{|N|}^L \mathbf{y}_N \text{ for all } N \supseteq M.$$

By Propositions 4 and 5, the following characterization of leximin time-invariant overtaking is a direct consequence of Theorem 1 and Lemma 4:

Corollary 2 *(i) $\succsim_{\mathcal{I}}^L$ is a SWR that satisfies $\mathcal{P}$**I**, **SP** and **ST**.*

*(ii) A SWR $\succsim$ satisfies $\mathcal{F}$**A**, **FP**, **HE** and **IPC** iff $\succsim_{\mathcal{I}}^L$ is a subrelation of $\succsim$.*

(iii) $\mathbf{x} \succ_{\mathcal{I}}^{L} \mathbf{y}$ *iff there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \succ_{|N|}^{L} \mathbf{y}_N$ *for all* $N \supseteq M$.

(iv) $\mathbf{x} \sim_{\mathcal{I}}^{L} \mathbf{y}$ *iff there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \sim_{|N|}^{L} \mathbf{y}_N$ *for all* $N \supseteq M$.

We provide a characterization of the asymmetric and symmetric parts of the leximin generalized overtaking criterion. For this purpose, write $\mathbf{X}^c := \{\mathbf{x} : \mathbb{N}^{\mathbf{x}} \to Y \,|\, \mathbb{N}^{\mathbf{x}}$ is a cofinite subset of $\mathbb{N}\}$. For any $\mathbf{x} \in \mathbf{X}^c$, write $\mathbb{N}^{\mathbf{x}}_{\min} := \{i \in \mathbb{N}^{\mathbf{x}} \mid x_i = \inf_{j \in \mathbb{N}^{\mathbf{x}}} x_j\}$. Say that $\mathbf{x} \in \mathbf{X}^c$ and $\mathbf{y} \in \mathbf{X}^c$ have *the same minimum and the same number of minimal elements* if $\inf_{j \in \mathbb{N}^{\mathbf{x}}} x_j = \inf_{j \in \mathbb{N}^{\mathbf{y}}} y_j$ and $0 < |\mathbb{N}^{\mathbf{x}}_{\min}| = |\mathbb{N}^{\mathbf{y}}_{\min}| < \infty$.

Define the operator $R : (\mathbf{X}^c)^2 \to (\mathbf{X}^c)^2$ as follows, where $\mathbf{x}'$ denotes the restriction of $\mathbf{x}$ to $\mathbb{N}^{\mathbf{x}} \backslash \mathbb{N}^{\mathbf{x}}_{\min}$ and $\mathbf{y}'$ is restriction of $\mathbf{y}$ to $\mathbb{N}^{\mathbf{y}} \backslash \mathbb{N}^{\mathbf{y}}_{\min}$ if $\mathbf{x} \in \mathbf{X}^c$ and $\mathbf{y} \in \mathbf{X}^c$ satisfy that $|\mathbb{N}^{\mathbf{x}}_{\min}|$ and $|\mathbb{N}^{\mathbf{y}}_{\min}|$ are positive and finite:

$$R(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}', \mathbf{y}') & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ have the same minimum and} \\ & \text{the same number of minimal elements,} \\ (\mathbf{x}, \mathbf{y}) & \text{otherwise.} \end{cases}$$

Write $R^0(\mathbf{x}, \mathbf{y}) := (\mathbf{x}, \mathbf{y})$ and, for $n \in \mathbb{N}$, $R^n(\mathbf{x}, \mathbf{y}) := R(R^{n-1}(\mathbf{x}, \mathbf{y}))$.

Proposition 6 *Leximin time-invariant overtaking satisfies:*

(i) $\mathbf{x} \succ_{\mathcal{I}}^{L} \mathbf{y}$ *iff*

(a) there is $P \in \mathcal{F}$ *such that* $P\mathbf{x} > \mathbf{y}$, *or*

(b) there exists m *such that* $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ *for all* $n \geq m$ *and one of following is true:*

$$\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j > \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$$

$$\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j \;\; \text{and} \;\; 0 \leq |\mathbb{N}^{\mathbf{x}'}_{\min}| < |\mathbb{N}^{\mathbf{y}'}_{\min}| \leq \infty.$$

(ii) $\mathbf{x} \sim_{\mathcal{I}}^{L} \mathbf{y}$ *iff there is* $P \in \mathcal{F}$ *such that* $P\mathbf{x} = \mathbf{y}$.

Proof. Write $(\mathbf{x}^n, \mathbf{y}^n) = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq 0$.

(If part of (i).) First assume that there is $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$. By the definition of $\succsim^L_{|M|}$, there exists $M \subset \mathbb{N}$ such that $\mathbf{x}_M \succ^L_{|M|} \mathbf{y}_M$ and $x_i \geq y_i$ for all $i \in \mathbb{N}\backslash M$. Hence, $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

Then assume that there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$. Let m be the smallest such integer. Then, for all $k \in \{0, \ldots, m-1\}$, $\mathbf{x}^k$ and $\mathbf{y}^k$ have the same minimum and the same number of minimal elements. Write

$$M^{\mathbf{y}} := \bigcup_{k \in \{0,\ldots,m-1\}} \mathbb{N}^{\mathbf{y}^k}_{\min}.$$

If $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j > \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, choose $i' \in \mathbb{N}^{\mathbf{y}'}$ so that $y'_{i'} < \inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j$. Let $M = M^{\mathbf{y}} \cup \{i'\}$. Then $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. If $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ and $0 \leq |\mathbb{N}^{\mathbf{x}'}_{\min}| < |\mathbb{N}^{\mathbf{y}'}_{\min}| \leq \infty$, let $N^{\mathbf{y}'}$ be a subset of $\mathbb{N}^{\mathbf{y}'}_{\min}$ with a larger number of elements than $\mathbb{N}^{\mathbf{x}'}_{\min}$. Let $M = M^{\mathbf{y}} \cup N^{\mathbf{y}'}$. Then $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

(Only-if part of (i).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Suppose that (a) and (b) are not true. We must show that, for all $M \subset \mathbb{N}$ with $|M| \geq 2$, there exists $N \supseteq M$ such that $\mathbf{x}_N \precsim^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$, and there exists no m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$. Then, for all $n \geq 0$, $\mathbf{x}^n$ and $\mathbf{y}^n$ have the same minimum and the same number of minimal elements, and $\bigcup_{n \geq 0} \mathbb{N}^{\mathbf{y}^n}_{\min}$ is an infinite set. For any $M \subset \mathbb{N}$, one can choose $N \supseteq M$ such that N contains at least as many $\mathbb{N}^{\mathbf{x}^n}_{\min}$ elements as $\mathbb{N}^{\mathbf{y}^n}_{\min}$ elements for any $n \geq 0$, and more for some n'. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$ and that, even though there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$ and $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, we have that $|\mathbb{N}^{\mathbf{x}'}_{\min}| = |\mathbb{N}^{\mathbf{y}'}_{\min}| = \infty$. Let m be the smallest such integer. Independently of how $M^{\mathbf{y}}$ is complemented to form $M \subset \mathbb{N}$, one can always choose $N \supseteq M$ such that N in addition to including $\bigcup_{k \in \{0,\ldots,m-1\}} \mathbb{N}^{\mathbf{x}^k}_{\min}$ contains more $\mathbb{N}^{\mathbf{x}'}_{\min}$ elements than $\mathbb{N}^{\mathbf{y}'}_{\min}$ elements. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$ and that, even though there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$ and $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, we have that $|\mathbb{N}^{\mathbf{x}'}_{\min}| = |\mathbb{N}^{\mathbf{y}'}_{\min}| = 0$. Let m be the smallest such integer. Independently of how $M^{\mathbf{y}}$ is complemented to form $M \subset \mathbb{N}$, one can always choose $N \supseteq M$ such that N in addition to including $\bigcup_{k \in \{0,\dots,m-1\}} \mathbb{N}^{\mathbf{x}^k}_{\min}$ contains $i' \in \mathbb{N}^{\mathbf{x}'}$ so that $x'_{i'} < \min_{j \in N \cap \mathbb{N}^{\mathbf{y}'}} y'_j$. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose that, even though there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$, we have that (1) $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j < \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ or (2) $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ and $\infty \geq |\mathbb{N}^{\mathbf{x}'}_{\min}| > |\mathbb{N}^{\mathbf{y}'}_{\min}| \geq 0$. Then there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$, and it follows from the if-part above that $\mathbf{x} \prec^L_{\mathcal{I}} \mathbf{y}$.

(If part of (ii).) Assume that there is $P \in \mathcal{F}$ such that $P\mathbf{x} = \mathbf{y}$. By the definition of $\succsim^L_{|M|}$, there exists $M \subset \mathbb{N}$ such that $\mathbf{x}_M \sim^L_{|M|} \mathbf{y}_M$ and $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$. Hence, $\mathbf{x}_N \sim^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

(Only-if part (ii).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. By Lemma 3(ii) and the fact that $\{\succsim^L_m\}^{\infty}_{t=2}$ is proliferating, it follows that $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$. It now follows from the definition of $\succsim^L_{|M|}$ that there is $P \in \mathcal{F}$ such that $P\mathbf{x} = \mathbf{y}$.

The if parts can easily be amended to ensure that $|M| \geq 2$. ∎

This characterization can be illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1. In this example $\mathbb{N}^{\mathbf{u}} = \mathbb{N}^{\mathbf{v}} = \mathbb{N}$ and $\inf_{j \in \mathbb{N}} u_j > \inf_{j \in \mathbb{N}} v_j$ so that $\mathbf{u}$ and $\mathbf{v}$ do not have the same minimum, implying that $(\mathbf{u}, \mathbf{v}) = R^n(\mathbf{u}, \mathbf{v})$ for all $n \geq 1$. By Proposition 6(i)(b) it follows that $\mathbf{u} \succ^L_{\mathcal{I}} \mathbf{v}$.

To illustrate part (i) of Proposition 6 further, we also consider the comparison of $\mathbf{v}$ of Section 1 to

$$\mathbf{w} \quad : \quad 0 \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \dots \quad \tfrac{1}{2} \quad \dots$$

Then $\mathbf{v}$ and $\mathbf{w}$ have the same minimum and the same number of minimal element, implying that $(\mathbf{v}', \mathbf{w}') = R(\mathbf{v}, \mathbf{w})$ with $\mathbf{v}'$ and $\mathbf{w}'$ being the restrictions of $\mathbf{v}$ and $\mathbf{w}$ to

$\mathbb{N}\backslash\{1\}$. Furthermore, $\inf_{j\in\mathbb{N}\backslash\{1\}} v'_j = \inf_{j\in\mathbb{N}\backslash\{1\}} w'_j = \frac{1}{2}$ and $0 = |\mathbb{N}^{\mathbf{v}'}_{\min}| < |\mathbb{N}^{\mathbf{w}'}_{\min}| = \infty$. This entails that $(\mathbf{v}', \mathbf{w}') = R^n(\mathbf{v}, \mathbf{w})$ for all $n \geq 1$. By Proposition 6(i)(b) it follows that $\mathbf{v} \succ^L_{\mathcal{I}} \mathbf{w}$.

The leximin criterion proposed by Bossert, Sprumont and Suzumura (2007), which we discussed in Section 1 and denoted $\succsim^L_{\mathcal{F}}$, yields comparability only if there is equality or Pareto-dominance on a cofinite set:

$$\mathbf{x} \succsim^L_{\mathcal{F}} \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^L_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\backslash N} \geq \mathbf{y}_{\mathbb{N}\backslash N} .$$

It follows from Proposition 6 that $\succsim^L_{\mathcal{F}}$ is a subrelation of $\succsim^L_{\mathcal{I}}$, since the symmetric parts, $\sim^L_{\mathcal{I}}$ and $\sim^L_{\mathcal{F}}$, coincide, while $\succ^L_{\mathcal{I}}$ strictly extends $\succsim^L_{\mathcal{F}}$, as illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1.

7 Concluding remarks

We have defined the generalized time-invariant overtaking criterion $\succsim^*_{\mathcal{I}}$ and specialized this criterion to the utilitarian and leximin cases, leading to $\succsim^U_{\mathcal{I}}$ and $\succsim^L_{\mathcal{I}}$. We have shown that through $\succsim^U_{\mathcal{I}}$ and $\succsim^L_{\mathcal{I}}$ we can extend the asymmetric parts of the utilitarian and leximin criteria suggested by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007), $\succsim^U_{\mathcal{F}}$ and $\succsim^L_{\mathcal{F}}$ respectively, without compromising their desirable properties.

It is feasible to go further as indicated at the end of Section 4: $\succsim^*_{\mathcal{I}}$ is subrelation *both* of the traditional overtaking criterion (in the sense of catching up in finite time), which we denote $\succsim^*_{\mathcal{C}}$, *and* of fixed-step overtaking, which was suggested in its utilitarian version by Fleurbaey and Michel (2003) and which we denote $\succsim^*_{\mathcal{SC}}$.

Going from $\succsim^*_{\mathcal{I}}$ to $\succsim^*_{\mathcal{C}}$ entails that Strong Time Invariance must be weakened all the way to Finite Time Invariance, leading to the strict (and perhaps uncompelling) ranking of $\mathbf{x}$ above $\mathbf{y}$ in the $(\mathbf{x}, \mathbf{y})$ example of Section 1.

Going from $\succsim^*_{\mathcal{I}}$ to $\succsim^*_{\mathcal{SC}}$ entails not only that Strong Time Invariance must be weakened to Fixed-step Time Invariance, but also that Koopmans's (1960) axiom of

Stationarity must be dropped. On the other hand, Finite Anonymity is strengthened to Fixed-step Anonymity, which implies that both the symmetric and asymmetric parts of $\succsim_{\mathcal{I}}^*$ are extended. These positive properties makes it worthwhile to investigate $\succsim_{\mathcal{SC}}^*$ further; in particular, to characterize its implications for social preference in the utilitarian and leximin cases. We expect to return to this in future work.

Appendix

Lemma 8 *If the SWR $\succsim$ extends $\succsim_2^U$, then $\mathbf{x} \sim \mathbf{u}$ whenever $\mathbf{x}$, $\mathbf{u} \in \mathbf{X}$ satisfy that there exists $N \subset \mathbb{N}$ such that $u_i = \sigma(\mathbf{x}_N)/|N|$ for $i \in N$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash N$.*

Proof. The result is shown by induction. Consider the statement that $\mathbf{x} \sim \mathbf{u}$ whenever $\mathbf{x}$, $\mathbf{u} \in \mathbf{X}$ satisfy that there exists $N \subset \mathbb{N}$ such that $u_i = \sigma(\mathbf{x}_N)/|N|$ for $i \in N$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash N$.

This statement is true for all $N \subset \mathbb{N}$ with $|N| = 1$ by the reflexivity of $\succsim$.

Assume that the statement is true for all $M \subset \mathbb{N}$ with $|M| \le m$. It remains to be shown that then the statement is true for all $N \subset \mathbb{N}$ with $|N| = m + 1$, provided that $\succsim$ extends $\succsim_2^U$. This is shown in the remainder of the proof.

Suppose $\mathbf{u} \in \mathbf{X}$ satisfy that there exists $N \subset \mathbb{N}$ such that $u_i = \sigma(\mathbf{x}_N)/|N|$ for $i \in N$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash N$, where $|N| = m + 1$. W.l.o.g., $N = \{1, \ldots, m + 1\}$. Consider any $M \subset \mathbb{N}$ such that $M \subset \mathbb{N}$ and $|M| = m$. W.l.o.g., $M = \{1, \ldots, m\}$. Construct $\mathbf{v} \in \mathbf{X}$ by $v_i = \sigma(\mathbf{x}_M)/|M|$ for $i \in M$ and $v_i = x_i$ for $i \in \mathbb{N}\backslash M$.

Let the sequence $\{\mathbf{y}^k\}_{k=0}^m$, where $\mathbf{y}^k \in \mathbf{X}$ for each k, be constructed as follows:

$$\mathbf{y}_M^k = \begin{cases} \mathbf{v}_M & \text{for } k = 0 \\ (\mathbf{u}_{\{1,\ldots,k\}}, \mathbf{v}_{\{k+1,\ldots,m\}}) & \text{for } k = 1, \ldots, m-1 \\ \mathbf{u}_M & \text{for } k = m\,, \end{cases}$$

while for all k, $y_{m+1}^k = x_{m+1}^k + k(v_1 - u_1)$, and $y_i^k = u_i$ for $i \in \mathbb{N}\backslash N$. Then $\mathbf{y}^{k-1} \sim \mathbf{y}^k$ for $k \in \{1, \ldots, m\}$ by the property that $\succsim$ extends $\succsim_2^U$, since $y_k^{k-1} + y_{m+1}^{k-1} = y_k^k + y_{m+1}^k$

and $y_i^{k-1} = y_i^k$ for $i \in \mathbb{N}\backslash\{k, m+1\}$. By transitivity, $\mathbf{v} = \mathbf{y}^0 \sim \mathbf{y}^m = \mathbf{u}$. By assumption, $\mathbf{x} \sim \mathbf{v}$, leading by transitivity to the conclusion that $\mathbf{x} \sim \mathbf{u}$. ■

Direct proof of Proposition 2. Consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ for which there exists some subset $M \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$.

If $\mathbf{x}_M \sim^U_{|M|} \mathbf{y}_M$, then $\sigma(\mathbf{x}_M) = \sigma(\mathbf{y}_M)$ and, by Lemma 8, $\mathbf{x} \sim \mathbf{u} \sim \mathbf{y}$, where $u_i = \sigma(\mathbf{x}_M)/|M|$ for $i \in M$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash M$. By transitivity, $\mathbf{x} \sim \mathbf{y}$.

If $\mathbf{x}_M \succ^U_{|M|} \mathbf{y}_M$, then $\sigma(\mathbf{x}_M) > \sigma(\mathbf{y}_M)$ and, by Lemma 8 and **FP**, $\mathbf{x} \sim \mathbf{u} \succ \mathbf{v} \sim \mathbf{y}$, where $u_i = \sigma(\mathbf{x}_M)/|M|$ and $v_i = \sigma(\mathbf{y}_M)/|M|$ for $i \in M$ and $u_i = v_i = x_i = y_i$ for $i \in \mathbb{N}\backslash M$. By transitivity, $\mathbf{x} \succ \mathbf{y}$. ■

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