# A Coalition Formation Value for Games in Partition Function Form

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## 0. Preliminaries

$N = \{1, 2, 3\}$

$v(\{1, 2, 3\}) = 90$

$v(\{1, 2\}) = 80, v(\{1, 3\}) = 60, v(\{2, 3\}) = 40$

$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

Equal division:$(30, 30, 30)$

Marginal contributions:

$M_1 = 90 - 40 = 50,\ M_2 = 90 - 60 = 30,$

$M_3 = 90 - 80 = 10 \qquad (50, 30, 10)$

$N = \{1, 2, ..., n\}$: the player set

$S \subseteq N$: coalition

$v(S)$: worth of coaltion $S$

$(x_1, x_2, ..., x_n)$: payoff allocation

Efficiency: $\sum_{k \in N} x_k = v(N)$.

How to allocate the worth $v(N)$ among the players?

Equal division:

$$f_i(N, v) = \frac{v(N)}{n}$$

Equal surplus:

$$f_i(N, v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{k \in N} v(k)\right)$$

Marginal contribution of $i$: $v(N) - v(N \setminus \{i\})$

Equal surplus based on marginal contributions:

$$f_i(N, v) = (v(N) - v(N \setminus \{i\})) + \frac{1}{n} \left( v(N) - \sum_{k \in N} (v(N) - v(N \setminus \{k\})) \right)$$

Order of coalitions (contribution when a player enters):

$$\emptyset \to \{1\} \quad \to \quad \{12\} \to \{123\},$$

$$\emptyset \to \{1\} \quad \to \quad \{13\} \to \{123\},$$

$$\emptyset \to \{2\} \quad \to \quad \{12\} \to \{123\},$$

$$\emptyset \to \{2\} \quad \to \quad \{23\} \to \{123\},$$

$$\emptyset \to \{3\} \quad \to \quad \{13\} \to \{123\},$$

$$\emptyset \to \{3\} \quad \to \quad \{23\} \to \{123\},$$

| order | 1's contri. | 2's contri. | 3's contri. |
|---|---|---|---|
| $123$ | $v(1)$ | $v(12) - v(1)$ | $v(123) - v(12)$ |
| $132$ | $v(1)$ | $v(123) - v(13)$ | $v(13) - v(1)$ |
| $213$ | $v(12) - v(2)$ | $v(2)$ | $v(123) - v(12)$ |
| $231$ | $v(123) - v(23)$ | $v(2)$ | $v(23) - v(2)$ |
| $312$ | $v(13) - v(3)$ | $v(123) - v(13)$ | $v(3)$ |
| $321$ | $v(123) - v(23)$ | $v(23) - v(3)$ | $v(3)$ |
| Shapley | average | average | average |

$$v(\{1,2,3\}) = 90$$
$$v(\{1,2\}) = 80, v(\{1,3\}) = 60, v(\{2,3\}) = 40$$

| order | 1's contri. | 2's contri. | 3's contri. |
|---|---|---|---|
| 123 | 0 | 80 | 10 |
| 132 | 0 | 30 | 60 |
| 213 | 80 | 0 | 10 |
| 231 | 50 | 0 | 40 |
| 312 | 60 | 30 | 0 |
| 321 | 50 | 40 | 0 |
| Shapley | 40 | 30 | 20 |

*The Shapley value*

$$Sh_i(N,v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \setminus \{i\}))$$

where $n = |N|, s = |S|$.

## 1. Introduction

Coalition formation problem with Externalities:
Ray and Vohra (1997), Diamantoudi and Xue (2007), Funaki and Yamato (1999,2008)

Partition function form (PFF) games:
Thrall and Lucas (1963), Funaki and Yamato (1999)

Value for PFF games:
Myerson (1977), Bolger (1989), Do and Norde (2007), Clippel and Serrano (2008), Albizuri et al. (2005), Macho-Stadler et al. (2007).

Order of coalitions (contribution when a player enters):

$$\emptyset \to \{1\} \quad \to \quad \{12\} \to \{123\},$$

$$\emptyset \to \{1\} \quad \to \quad \{13\} \to \{123\},$$

$$\emptyset \to \{2\} \quad \to \quad \{12\} \to \{123\},$$

$$\emptyset \to \{2\} \quad \to \quad \{23\} \to \{123\},$$

$$\emptyset \to \{3\} \quad \to \quad \{13\} \to \{123\},$$

$$\emptyset \to \{3\} \quad \to \quad \{23\} \to \{123\},$$

Order of coalition structures (coalition formation):

$$\{1,2,3\} \quad \rightarrow \quad \{12,3\} \rightarrow \{123\},$$

$$\{1,2,3\} \quad \rightarrow \quad \{13,2\} \rightarrow \{123\},$$

$$\{1,2,3\} \quad \rightarrow \quad \{23,1\} \rightarrow \{123\},$$

Externality: $v(S,\pi)$

Order of embedded coalitions (coalition formation, contribution)

$$1\{1,2,3\} \;\rightarrow\; 12\{12,3\} \rightarrow 123\{123\},$$

$$1\{1,2,3\} \;\rightarrow\; 13\{13,2\} \rightarrow 123\{123\},$$

$$1\{1,2,3\} \;\rightarrow\; 1\{1,23\} \rightarrow 123\{123\},$$

$$2\{1,2,3\} \;\rightarrow\; 12\{12,3\} \rightarrow 123\{123\},$$

$$2\{1,2,3\} \;\rightarrow\; 23\{23,1\} \rightarrow 123\{123\},$$

$$2\{1,2,3\} \;\rightarrow\; 2\{2,13\} \rightarrow 123\{123\},$$

$$3\{1,2,3\} \;\rightarrow\; 13\{13,2\} \rightarrow 123\{123\},$$

$$3\{1,2,3\} \;\rightarrow\; 23\{23,1\} \rightarrow 123\{123\},$$

$$3\{1,2,3\} \;\rightarrow\; 3\{12,3\} \rightarrow 123\{123\},$$

We give:

(1) the Shapley senario-value

(2) the Shapley process-value

(3) the Shapley coalition formation value

We apply (2) to Cournot Oligopoly.

We give Aximatizations of (1).

## 2. Partitions and Embedded Coalitions

$N := \{1, 2, \ldots, n\}$ : set of players

$\pi := \{S_1, \ldots, S_k\}$ : partition, coalition structure

$\Pi(N)$: the set of all possible partitions of $N$

$\pi \leq \pi'$: $\pi$ is a *refinement* of $\pi'$

$(\Pi(N), \leq)$: *partition lattice*

$\pi^{\perp} := \{\{1\}, \ldots, \{n\}\}$: *the singletons coalition structure*
(the bottom element of the lattice)

$\pi^{\top} := \{N\}$: *the grand coalition structure*
(The coarsest partition)

$S\pi := (S, \pi)$: *embedded coalition* when $S \in \pi$

$\mathcal{C}(N)$: the set of embedded coalition on $N$

Notation: $12\{12, 3\}$ instead of $(\{1, 2\}, \{\{1, 2\}, \{3\}\}))$

the following order relation on embedded coalition,

$S\pi \sqsubseteq S'\pi' \Leftrightarrow S \subseteq S'$ and $\pi \leq \pi'$.

The top element is $(N, \pi^\top)$

All elements of the form $(\{i\}, \pi^\perp)$ are minimal.

For convenience, we introduce an artificial bottom element $\perp$ to $\mathcal{C}(N)$ $(:= \emptyset, \pi^\perp))$, and denote $\mathcal{C}(N)_\perp := \mathcal{C}(N) \cup \{\perp\}$.

**Definition** PFF-game

A *game in partition function form* (PFF-game) on $N$ is a mapping $v : \mathcal{C}(N)_\perp \to \mathcal{R}$, s.t. $v(\perp) = 0$. The set of all PFF-games on $N$ is denoted by $\mathcal{PG}(N)$.

We assume $v(N\{N\}) \geq \sum_{S\in\pi} v(S,\pi)$, for all $\pi \in \Pi(N)$.

Main Question: How to allocate the total worth $v(N\{N\})$ among the players.

A *chain from* $\perp$ *to* $\top$: totally ordered sequence of elements of $\mathcal{C}(N)_\perp$ including $\perp, \top$.

The chain is *maximal* if no other chain can contain it.

The set of maximal chains in $\mathcal{C}(N)$ is denoted by $\mathcal{C}(\mathcal{C}(N))$.

- The total number of elements is given below:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{C}(N)_\perp\rvert$ | 2 | 4 | 11 | 38 | 152 | 675 | 3264 | 17008 |

- The number of maximal chains from $\perp$ to $(N, \{N\})$ is $\lvert\mathcal{C}(\mathcal{C}(N)_\perp)\rvert = \frac{(n!)^2}{2^{n-1}}$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{C}(\mathcal{C}(N)_\perp)\rvert$ | 1 | 2 | 9 | 72 | 900 | 16 200 | 396 900 | 12 700 800 |

For simplicity, we put $c := \lvert\mathcal{C}(\mathcal{C}(n)_\perp)\rvert$ and it is shown in Grabisch(2008) that $\lvert\mathcal{C}(\Pi(N))\rvert = \frac{c}{n}$.

## 3. Processes and Scenarios

**Definition** coalition formation process, scenario

1. A *coalition formation process* $\mathcal{P}$: maximal chain in $\Pi(N)$. The set of all processes is $\mathcal{C}(\Pi(N))$

2. A *scenario* $\mathcal{S}$ *in a process* $\mathcal{P}$ is any maximal chain in $\mathcal{C}(N)_{\perp}$ so that the sequence of partitions corresponds to $\mathcal{P}$ (notation: $\mathcal{S} \leftarrow \mathcal{P}$). The set of all scenarios is $\mathcal{C}(\mathcal{C}(N)_{\perp})$.

For a given process $\mathcal{P}$, there are $n$ scenarios $\mathcal{S}_i$, $i \in N$, scenario $\mathcal{S}_i$ tracking the history of player $i$ in the coalition formation process.

**Example** Three processes in 3 persons game with $N = \{1, 2, 3\}$:

$$\mathcal{P}_1 : \{1, 2, 3\} \to \{12, 3\} \to \{123\}$$

$$\mathcal{P}_2 : \{1, 2, 3\} \to \{13, 2\} \to \{123\}$$

$$\mathcal{P}_3 : \{1, 2, 3\} \to \{1, 23\} \to \{123\}$$

A process describes how exactly the coalition structure evolves step by step.

Each coalition process has three scenarios. $\mathcal{P}_2$ has the following three scenarios:

$$\mathcal{S}_1 \; : \; 1\{1,2,3\} \to 13\{13,2\} \to 123\{123\}$$
$$\mathcal{S}_2 \; : \; 2\{1,2,3\} \to 2\{13,2\} \to 123\{123\}$$
$$\mathcal{S}_3 \; : \; 3\{1,2,3\} \to 13\{13,2\} \to 123\{123\}$$

Scenario $\mathcal{S}_i$ for player $i$ describes what happens to player $i$ in the process $\mathcal{P}_2$ of a coalition formation in view of player $i$ or in position of player $i$.

In a scenario $\mathcal{S}$, some elements play a special role. We consider those elements $S\pi$ such that in the sequence of elements of $\mathcal{S}$ from bottom to top, $S\pi$ is the last element with base coalition $S$. They are called *terminal elements*. For $\mathcal{S}$:

$$\mathcal{S} = \{\bot, S_1\pi_{1,1}, \ldots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \ldots, S_2\pi_{2,m_2}, \ldots, \\ S_k\pi_{k,1}, \ldots, S_k\pi_{k,m_k}, N\{N\}\},$$

with $S_1 \neq \cdots \neq S_k \neq N$, the terminal elements are $S_i\pi_{i,m_i}$, $i = 1, \ldots, k$. We denote by $\mathcal{F}(\mathcal{S})$ this family of elements.

**Example.** We consider 4 players and the following process $\mathcal{P}$:

$$\{1,2,3,4\} \to \{13,2,4\} \to \{13,24\} \to \{1234\}.$$

and the four different scenarios in $\mathcal{P}$ where terminal elements are red:

$$\mathcal{S}_1 : \mathbf{1\{1,2,3,4\}} \to 13\{13,2,4\} \to \mathbf{13\{13,24\}} \to \mathbf{N\{N\}}$$

$$\mathcal{S}_2 : 2\{1,2,3,4\} \to \mathbf{2\{13,2,4\}} \to \mathbf{24\{13,24\}} \to \mathbf{N\{N\}}$$

$$\mathcal{S}_3 : \mathbf{3\{1,2,3,4\}} \to 13\{13,2,4\} \to \mathbf{13\{13,24\}} \to \mathbf{N\{N\}}$$

$$\mathcal{S}_4 : 4\{1,2,3,4\} \to \mathbf{4\{13,2,4\}} \to \mathbf{24\{13,24\}} \to \mathbf{N\{N\}}$$

## 4. Definition of Values

In a given scenario $\mathcal{S}_i$ belonging to process $\mathcal{P}$, the contribution of player $j$, $j \in N$, is the marginal worth of $j$ in the scenario. It is somehow player $j$'s contribution in process $\mathcal{P}$ in $i$'s viewpoint.

Consider scenario $\mathcal{S}_1$ for player 1:

$$\perp \to \mathbf{1\{1,2,3,4\}} \to 13\{13,2,4\} \to \mathbf{13\{13,24\}} \to \mathbf{N\{N\}}.$$

1. Player 1 enters the game and becomes a singleton embedded coalition. Then the marginal worth goes to player 1.

2. Player 3 merges to form base coalition 13. The marginal worth goes to player 3 with coalition 1, but not in this step.

3. No player merges with the base coalition. Since the resulting embedded coalition is terminal, the marginal worth is given.

4. Two players merge together with the base coalition. The marginal worth is equally divided.

Applying this method to the above scenario $\mathcal{S}_1$,

$$\perp \to \mathbf{1\{1,2,3,4\}} \to 13\{13,2,4\} \to \mathbf{13\{13,24\}} \to \mathbf{N\{N\}}.$$

we obtain:

1. contribution of player 1: $v(1\{1,2,3,4\}) - 0$
2. contribution of player 2: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$
3. contribution of player 3: $v(13\{13,24\}) - v(1\{1,2,3,4\})$
4. contribution of player 4: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$

Applying to scenario $\mathcal{S}_2$,

$$\perp \to 2\{1,2,3,4\} \to \mathbf{2\{13,2,4\}} \to \mathbf{24\{13,24\}} \to \mathbf{N\{N\}}.$$

by the same considerations, we obtain:

1. contribution of player 1: $\frac{1}{2}(v(1234\{1234\}) - v(24\{13,24\}))$
2. contribution of player 2: $v(2\{13,2,4\}) - 0$
3. contribution of player 3: $\frac{1}{2}(v(1234\{1234\}) - v(24\{13,24\}))$
4. contribution of player 4: $v(24\{13,24\}) - v(2\{13,2,4\}))$.

Applying to scenario $\mathcal{S}_3$,

$$\perp \rightarrow \mathbf{3\{1,2,3,4\}} \rightarrow 13\{13,2,4\} \rightarrow \mathbf{13\{13,24\}} \rightarrow \mathbf{N\{N\}}.$$

by the same considerations, we obtain:

1. contribution of player 1: $v(13\{13,24\}) - v(3\{1,2,3,4\})$
2. contribution of player 2: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$
3. contribution of player 3: $v(3\{1,2,3,4\}) - 0$
4. contribution of player 4: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$

Applying to scenario $\mathcal{S}_4$,

$$\perp \to 4\{1,2,3,4\} \to \mathbf{4\{13,2,4\}} \to \mathbf{24\{13,24\}} \to \mathbf{N\{N\}}.$$

by the same considerations, we obtain:

1. contribution of player 1: $\frac{1}{2}(v(1234\{1234\}) - v(24\{13,24\}))$
2. contribution of player 2: $v(24\{13,24\}) - v(4\{13,2,4\}))$
3. contribution of player 3: $\frac{1}{2}(v(1234\{1234\}) - v(24\{13,24\}))$
4. contribution of player 4: $v(4\{13,2,4\})) - 0.$

**Definition** contribution in a scenario

The *contribution of player* $i$ *in a given scenario* $\mathcal{S}$ is given by

$$\Delta_i^{\mathcal{S}}(v) := \frac{1}{|S' \setminus S|}(v(S'\pi') - v(S\pi)),$$

where $S\pi$ is the last terminal element where $i$ is not present, and $S'\pi'$ is the next terminal element.

**Definition** scenario-value

A *scenario-value* is a mapping $\psi : \mathcal{PG} \to \mathcal{R}^{n\times|\mathcal{C}(\mathcal{C}(N)_\perp)|}$. Components of $\psi(v)$ are denoted by $\psi_i^{\mathcal{S}}(v)$ for scenario $\mathcal{S}$ and player $i$. The *Shapley scenario-value* $\phi(v) = (\phi^{\mathcal{S}}(v))_{\mathcal{S}\in\mathcal{C}(\mathcal{C}(N)_\perp)}$ is defined by

$$\phi_i^{\mathcal{S}}(v) := \Delta_i^{\mathcal{S}}(v), \quad i \in N, \mathcal{S} \in \mathcal{C}(\mathcal{C}(N)_\perp).$$

**Definition** process-value

*process-value* is a mapping $\psi : \mathcal{PG} \rightarrow \mathcal{R}^{n\times|\mathcal{C}(\Pi(N))|}$. Components of $\psi(v)$ are denoted by $\psi_i^{\mathcal{P}}(v)$ for process $\mathcal{P}$ and player $i$. Any scenario-value $\psi$ induces a process-value by:

$$\psi_i^{\mathcal{P}}(v) := \frac{1}{n}\sum_{\mathcal{S}\leftarrow\mathcal{P}} \psi_i^{\mathcal{S}}(v), \quad i \in N, \mathcal{P} \in \mathcal{C}(\Pi(N)).$$

**Definition** coalition formation value

A *value* is a mapping $\psi : \mathcal{PG} \rightarrow \mathcal{R}^n$. Components of $\psi(v)$ are denoted by $\psi_i(v)$ for player $i$. Any scenario-value or process-value $\psi$ induces a value by:

$$\psi_i(v) := \frac{n}{c}\sum_{\mathcal{P}\in\mathcal{C}(\Pi(N))} \psi_i^{\mathcal{P}}(v) = \frac{1}{c}\sum_{\mathcal{S}\in\mathcal{C}(\mathcal{C}(N)_\perp)} \psi_i^{\mathcal{S}}(v).$$

**Example.**

The formula of the Shapley coalition formation value for three persons games:

$$\begin{aligned}\phi_i(v) &= \frac{1}{3}v(N\{N\}) + \frac{1}{9}v(ij\{ij,k\}) + \frac{1}{9}v(ik\{ik,j\}) - \frac{2}{9}v(jk\{i,jk\}) \\ &\quad + \frac{1}{9}v(i\{i,jk\}) - \frac{1}{18}v(j\{j,ik\}) - \frac{1}{18}v(k\{k,ij\}) \\ &\quad + \frac{2}{9}v(i\{i,j,k\}) - \frac{1}{9}v(j\{i,j,k\}) - \frac{1}{9}v(k\{i,j,k\}).\end{aligned}$$

This is different from other values of PFF games.

If $v(S\pi) = v(S\pi')$ for any $\pi$ and $\pi'$ with $\pi \ni S, \pi' \ni S$, we have the TU game $\hat{v}$ from the PFF game $v$ by: $\hat{v}(S) := v(S\pi)$ for any $S \in 2^N$, any $\pi \ni S$. Then

$$\phi_i(\hat{v}) = \frac{1}{3}\hat{v}(N) + \frac{1}{9}\hat{v}(ij) + \frac{1}{9}\hat{v}(ik) - \frac{2}{9}\hat{v}(jk) + \frac{1}{3}\hat{v}(i) - \frac{1}{6}\hat{v}(j) - \frac{1}{6}\hat{v}(k).$$

## 5. Coalition Formation Process in Cournot Oligopoly

Consider a symmetric case: For $\pi = \{S, S_2, \ldots, S_k\}$, $v(S\pi) = v_k$. The Shapley coalition formation value equals to $\frac{v(N\{N\})}{n}$.

(1) Consider a 4-person game and two processes $\mathcal{P}$ and $\mathcal{P}'$:

$$\mathcal{P} \;:\; \{1,2,3,4\} \to \{12,3,4\} \to \{123,4\} \to \{1234\}$$

$$\mathcal{P}' \;:\; \{1,2,3,4\} \to \{12,3,4\} \to \{12,34\} \to \{1234\}$$

The Shapley process-values of $\mathcal{P}$ and $\mathcal{P}'$ are given by

$$\phi^{\mathcal{P}} = \left(\frac{2v_1 + v_2 + 3v_3}{24}, \frac{2v_1 + v_2 + 3v_3}{24}, \frac{v_1 + 5v_2 - 3v_3}{12}, \frac{3v_1 - 2v_2}{4}\right)$$

$$\phi^{\mathcal{P}'} = \left(\frac{v_1}{4}, \frac{v_1}{4}, \frac{v_1}{4}, \frac{v_1}{4}\right)$$

(2) Consider coalition formation process $\mathcal{P}$ in an $n$-person game.

$$\mathcal{P}: \{1, 2, 3, \ldots, n\} \to \{12, 3, \ldots, n\} \to \{123, \ldots, n\} \to \cdots \to \{N\}$$

The Shapley process-value becomes

$$\phi_k^{\mathcal{P}} = \frac{1}{n}\left[v_{n-k+2} + (k-1)(v_{n-k+1} - v_{n-k+2}) + \sum_{j=k}^{n-1} \frac{1}{j}(v_{n-j} - v_{n-j+1})\right]$$

for $k = 2, \ldots, n-1$, and

$$\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}}, \quad \phi_n^{\mathcal{P}} = \frac{1}{n}\left[v_2 + (n-1)(v_1 - v_2)\right].$$

(3) Consider a case of $n = 2^m$ and a process $\mathcal{P}'$:

$$
\begin{aligned}
\mathcal{P}' \quad : \quad & \{1, 2, 3, 4, \dots, n\} \to \{12, 3, 4, \dots, n\} \to \{12, 34, \dots, n\} \\
& \to \cdots \to \{12, 34, \dots, (n-1)n\} \to \{1234, \dots, (n-1)n\} \\
& \to \cdots \to \{1234, \dots, (n-3)(n-2)(n-1)n\} \to \cdots \\
& \to \{12345678, \dots, (n-3)(n-2)(n-1)n\} \\
& \to \cdots \to \{123 \cdots (2^{m-1}), (2^{m-1}+1) \cdots n\} \to \{N\}
\end{aligned}
$$

The Shapley process-value for process $\mathcal{P}'$ is given by $\phi_k^{\mathcal{P}'} = \frac{v_1}{n}$ for $k = 1, \dots, n$.

(4) Cournot oligopoly with linear demand.

Consider Cournot model with $n$ identical firms. We assume $\{N\}$ is formed.

$x_i$: firm $i$'s output $(i = 1, \dots, n)$.

$p = a - \sum_{i=1}^{n} x_i$: linear inverse demand function

$cx_i$: the total cost function of firm $i$ where $a > c > 0$.

Given $\pi = \{S_1, S_2, \dots, S_k\}$, each coalition $S_j$ chooses the total output level. Then the total profit of each coalition $S_j$ at unique Nash equilibrium is given by $\frac{(a-c)^2}{(k+1)^2}$.

We assume that $a - c = 1$. Then $v_k = \frac{1}{(k+1)^2}$ and it implies

$$\begin{aligned}
\phi_k^{\mathcal{P}} &= \frac{1}{n(n-k+3)^2} + \frac{(k-1)(2n-2k+5)}{n(n-k+2)^2(n-k+3)^2} \\
&+ \sum_{j=k}^{n-1} \frac{2n-2j+3}{nj(n-j+1)^2(n-j+2)^2} \qquad \text{for } k = 2, \ldots, n-1, \\
\phi_1^{\mathcal{P}} &= \phi_2^{\mathcal{P}}, \ \ \phi_n^{\mathcal{P}} = \frac{5n-1}{36n}, \quad \phi_k^{\mathcal{P}'} = \frac{1}{4n} \quad \text{for } k = 1, 2, \ldots, n,
\end{aligned}$$

Moreover it holds that for $k = 2, \ldots, n-1$,

$$\begin{aligned}
\phi_{k+1}^{\mathcal{P}} - \phi_k^{\mathcal{P}} &= \frac{1}{n}[v_{n-k+1} - v_{n-k+2} + (k - \frac{1}{k})(v_{n-k} - v_{n-k+1}) \\
&\quad - (k-1)(v_{n-k+1} - v_{n-k+2})] > 0.
\end{aligned}$$

Since $v_k = \frac{1}{(k+1)^2}$ is concave, the Shapley process-value $\phi_k^{\mathcal{P}}$ satisfies $\phi_1^{\mathcal{P}} = \phi_2^{\mathcal{P}} < \phi_3^{\mathcal{P}} ... < \phi_n^{\mathcal{P}}$, because $v_{n-k+1} - v_{n-k+2} > 0, k - \frac{1}{k} > k - 1$ and $v_{n-k} - v_{n-k+1} > v_{n-k+1} - v_{n-k+2} > 0$.

We give numerical results for the 4-person case.

$$\phi^{\mathcal{P}} \approx \frac{1}{4}(.138, .138, .206, .528), \;\; \phi^{\mathcal{P}'} = \frac{1}{4}(.25, .25, .25, .25).$$

## 6. Axiomatizations

### 2.1. Original Axiomatization

**Axiom (efficiency)**: $\sum_{i \in N} \phi_i(v) = v(N)$

**Axiom (null player)**: For a null player $i$, $\phi_i(v) = 0$.
Here $i$ is null $\iff$ $v(S \cup \{i\}) = v(S) \;\; \forall S \subseteq N \setminus \{i\}$

**Axiom (symmetry)**: If $i$ and $j$ are symmetric players, then $\phi_i(v) = \phi_j(v)$. Here $i$ and $j$ are symmetric $\iff$ $v(S \cup \{i\}) = v(S \cup \{j\}) \;\; \forall S \subseteq N \setminus \{i, j\}$

**Axiom (additivity)**: For any games $(N, v)$, $(N, w)$,
$\phi_i(v + w) = \phi_i(v) + \phi_i(w) \;\; \forall i \in N$.

Here $(v + w)(S) = v(S) + w(S) \;\; \forall S \subseteq N$.

**Theorem (Shapley)** $\phi$ satisfies efficiency, additivity, symmetry and null player $\iff \phi = Sh$.

**Theorem 1.**

The Shapley scenario-value is the unique scenario-value satisfying (Linear), (Senario Null), (Symmetry), (Egalitarian) and (Senario Efficent).

**Theorem 2.**

The Shapley scenario-value is the unique scenario-value satisfying (Linear), (Senario Null), (Symmetry), (Efficiency) and (Markovian and Ergodic).

We provide an explicit expression of the Shapley coalition formation value. For any game $v$, we obtain:

$$\phi_i(v) =$$

$$\frac{1}{n}v(N\{N\}) + \sum_{T\sigma\in\mathcal{C}(N) T\not\ni i, T_2\supset\{i\}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!(t_2-1)!\cdots t_k!$$

$$\Big[\frac{t+1}{t}v(T\cup i\sigma_{T\cup i}) - v(T\sigma)\Big]$$

$$- \sum_{T\sigma\in\mathcal{C}(N) T\not\ni i, T_2=\{i\}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!t!t_3!\cdots t_k!v(T\sigma)$$

where $\sigma_{T\cup i}$ is the partition obtained from $\sigma$ by moving $i \in T_2$ to $T$.