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MICROECONOMIC FOUNDATION OF LENDER OF LAST RESORT FROM THE VIEWPOINT OF PAYMENTS*

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We construct a model to clarify the mechanism by which the lender of last resort (LLR) can prevent bank runs. In our model, a bank has both the function of facilitating payments in which inside money is settled using outside money and the function of financial intermediation using a deposit contract. The deposit contract might lead to a bank run, and might even contribute to an efficient allocation. Therefore, to consider the liquidity supply by the LLR, we introduce the deposit contract as a factor of instability in the banking model. We show that the LLR can assist in the recovery of both the efficiency and stability of the financial system.
JEL Classification Numbers: E52, E58.

1. Introduction

In this paper, we present a model showing that the lender of last resort (LLR) can prevent the outbreak of bank runs. On the one hand, the Federal Reserve Bank (FRB) was criticized for triggering the Great Depression of the 1930s by not adopting a policy of liquidity supply in response to Black Thursday; on the other hand, the FRB was lauded for adopting a policy of huge liquidity supply to avoid a credit crunch in the case of Black Monday.[1] Although we have observed many cases in which a huge liquidity supply has been used in response to bank runs or financial crises, to the best of our knowledge, there has been no microeconomic model describing the use of the LLR to avoid a market breakdown after a financial crisis.

We aim to clarify the mechanism by which the LLR can prevent bank runs. For this purpose, we need a model in which outside money has value and a bank plays the role of facilitating payments. To create a situation resulting in a bank run, we adopt a model where a bank offers a deposit contract to consumers. In other words, the bank should have the function of making payments as well as that of financial intermediation. In this setup, the deposit contract might lead to a bank run, even though it may contribute to an efficient allocation. Therefore, the payments function also runs the risk of a bank run.

We construct a model consisting of two elements: the bank run model of Diamond and Dybvig (1983) and the payments model of Freeman (1996a,b). Diamond and Dybvig succeed in constructing a model to explain a bank run and examine the use of deposit

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[1] For example, Bernanke (1990) considers the latter case and argues the role of the central bank in establishing a stable system of payments.

insurance to prevent such an occurrence. In their model, the behaviour of a bank is mainly described by the deposit contract. Two types of Nash equilibria, the efficient equilibrium and the bank run, emerge as a result of the properties of this deposit contract. The study of deposit insurance has steadily progressed since the breakthrough of the model proposed by Diamond and Dybvig. Further more, although it is true that deposit insurance is one of the twin cornerstones of prudential policy, the other cornerstone, the LLR, has not been theoretically explained.

We adopt Freeman's model to explain the payments function.[2] Freeman (1996b) presents a model in which inside money is settled by outside money. Freeman (1996a) clarifies the role of the central bank as a liquidity supplier in the case where a shortage of liquidity is caused by a lag in the arrival of debtors.[3] Freeman's model creates an environment in which outside money is incorporated in the payments function, using a combination of the frictions of time, space and meeting patterns. That outside money's value is a crucial factor given the condition that the agents actually meet each other in the physical environment. Hicks (1935) has referred to this sort of restriction as "friction". The friction of time is a two-period overlapping generations model, and the friction of space is a location model in which the trading place is changed at each period. The meeting pattern implies a trading opportunity in which a double coincidence of wants cannot hold because of the defects of preferences and initial endowments (i.e. each agent has different preferences and endowments).[4] These frictions can assign value to both outside money and inside money with the latter being settled by the former.

Freeman's papers have mainly focused on the efficiency of monetary transactions through the payments system. However, we will analyse the instability in the payments system caused by financial intermediation. In our model, the bank run and banking panic stem from the risk of financial intermediation. Therefore, we embed the deposit contract as a factor of instability in the payments model to analyse the liquidity supply by the LLR.[5] We show that the LLR can assist in regaining both the efficiency and stability of the financial system.

2. Model

2.1 Agents

2.1.1 Debtor and creditor

In each period, a continuum [0, 1] of debtors and creditors is born, and each will live for two periods. There are two types of consumption goods in the economy: *c*-goods and *d*-goods.

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[2] The model is introduced by Champ and Freeman (2001). For a survey of the theoretical models of the payment system, see Zhou (2000).

[3] In addition, Freeman (1999) deals with an aggregate risk, although it is not directly related to our argument.

[4] Kiyotaki and Wright (1989) elegantly construct a model in which outside money is endogenously produced.

[5] Chakravorti (2000) studies the systemic risk without considering the payments function of outside money.

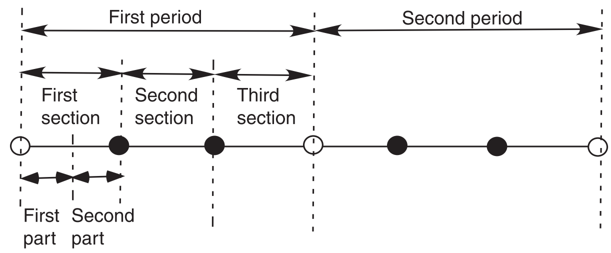

FIGURE 1. Time line

Each period consists of three sections, and each section consists of two parts. In the first part of each section, goods are traded and agents can consume them. Financial transactions such as payments occur in the second part of these sections. This time line is shown in Figure 1.

Each debtor initially holds w^d units of d-goods, and each creditor initially holds W^c units of c-goods. Goods cannot be stored over periods, and each agent cannot carry the goods that are not initially endowed over sections. This implies that agents consume goods that are obtained within the section when trade occurs.

2.1.2 Place of living and movement

The agents belonging to the same generation are born at and live in the same place; therefore, it is always possible for them to trade with each other. However, to trade with agents belonging to different generations, they must move to places inhabited by these other agents. We assume that this movement is possible only in the second section, with the young generation visiting the old generation. When agents visit other places, those belonging to the same generation cannot trade with each other (i.e. trade among the same generation can occur only in the first and third sections).

2.1.3 Pattern of consumption

The debtors in generation t derive utility from the consumption of c-goods and d-goods in the first section of period t, c_t and d_t, respectively. Let the utility function be $u(c_t) + v(d_t)$.

The creditors in generation t are of the following types: impatient type (type 1) and patient type (type 2). Agents of both types derive utility from the consumption of c-goods in the first section of period t, C_t, during which time the type of each agent is uncertain. The type becomes known to each agent after he or she consumes these goods. It is assumed that the information pertaining to the type of the agents is private. Impatient creditors derive utility from the consumption of d-goods in the second or third section of period t, D_t^1, and patient creditors derive utility from the consumption of d-goods in the second or third section of period $t + 1$, D_{t+1}^2. Let the utility function of impatient creditors be $U(C_t) + V^1(D_t^1)$ and that of patient creditors be $U(C_t) + V^2(D_{t+1}^2)$. The probability of being impatient is denoted as $\alpha \in (0, 1)$. Then, the expected utility that creditors in generation t maximize becomes $U(C_t) + \alpha V^1(D_t^1) + (1 - \alpha)V^2(D_{t+1}^2)$.[6]

[6] It is shown later that an impatient creditor consumes d-goods in the third section in period t and that a patient creditor consumes d-goods in the second section in period $t + 1$.

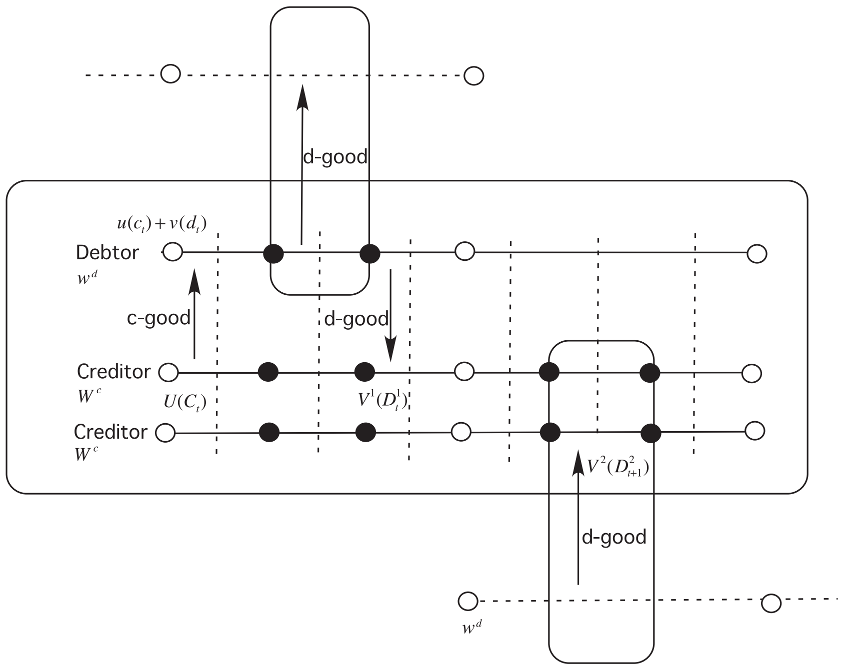

FIGURE 2. Flow of goods

In the initial period of the model (i.e. period 1), the old patient creditors of generation 0 exist in the continuum $[0, 1 - \alpha]$. These creditors derive utility from the consumption of d-goods in the second section, D_1^2. Let the utility function of such an agent be $V^2(D_1^2)$. This agent initially holds $\bar{M}$ units of money, implying that there are $(1 - \alpha)\bar{M}$ units of money in the economy.

The initial endowment and consumption patterns of these agents are summarized in Figure 2.[7] This figure also shows the flow of goods in cases where each agent consumes what he wants.

2.2 Timing of financial transaction

2.2.1 First period of life

First part of the first section. When the debtors of generation t consume c-goods in period t, they need to obtain these goods from the creditors belonging to the same generation. However, these creditors do not wish to consume d-goods at this time. Hence, the debtors issue IOUs and exchange them for the c-goods held by the creditors.[8]

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[7] In this figure, agents living in the same place are indicated by a rounded square.

[8] We assume that the creditor can redeem this IOU in the first or second period of life.

First part of the second section.

1 At the beginning of the second section, before each creditor is aware of his type, creditors establish a bank by laying out debtors' IOUs to pool risks regarding debtors' type and to manage and collect securities.[9] The bank is assumed to be an institution that can only make financial transactions.
2 In the second part of the second section, debtors redeem part of the IOUs in the form of money. To procure money for this redemption, they visit the old patient creditors and exchange *d*-goods and money. (Because this is a transaction between agents of different generations, money should be used.)

Second part of the second section. The bank receives money by collecting IOUs from debtors of the same generation. The creditors who become impatient withdraw money from the bank (to buy *d*-goods in the first part of the third section).

First part of the third section. In this part, debtors and creditors of the same generation meet to trade *d*-goods. These goods move from debtors to creditors and money moves in the opposite direction.

2.2.2 Second period of life (the second part of the first section)

In the first section of period $t + 1$, the bank of generation t receives money from the debtors of generation t by collecting IOUs that expire in period $t + 1$. Subsequently, patient creditors withdraw money from the bank. (They need money to pay the debtors of generation $t + 1$ in the second section.)

First part of the second section. The debtors of generation $t + 1$ visit the patient creditors of generation t and exchange *d*-goods for money.

These economic transactions are summarized in Figure 3.

2.2.3 Structure of the model

We note the following three characteristics with regard to the structure of the model.

First, the agents demanding *c*-goods in each period are young debtors and creditors, with the latter being the initial holders of *c*-goods. They meet in the first section. The agents demanding *d*-goods in each period are young debtors and impatient creditors with old patient creditors with young debtors being the initial holders of *d*-goods. The young creditors and debtors meet in the third section,[10] and the old creditors and young debtors meet in the second section. Therefore, each agent meets, at least on one occasion, the agent who has the good he or she demands. This implies that each agent can physically

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[9] Diamond and Dybvig (1983) considers a bank as a contract to insure the uncertainty about the timing of payments. In our paper, the bank has the same purpose. By establishing banks, it becomes possible to achieve an efficient allocation.

[10] Impatient creditors withdraw money from the bank in the second part of the second section and buy goods from debtors in the third section.

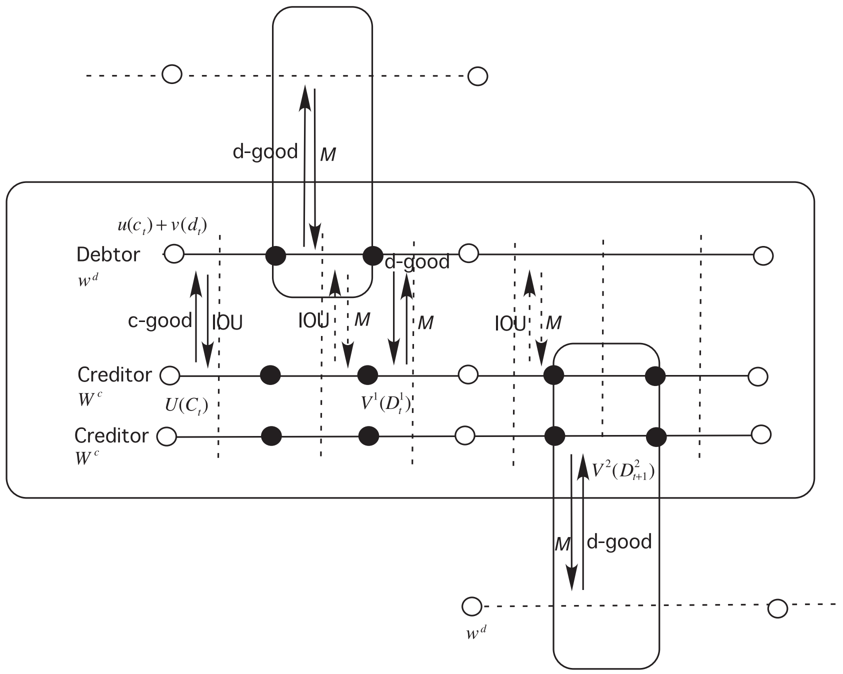

FIGURE 3. Transactions.
Note: M, money

consume all the goods from which he or she derives utility. This setting of the model allows us to analyse the efficiency of allocation naturally.

Second, in the first section, debtors issue IOUs. A part of the IOU is settled with the bank in the second section of this period with the use of money, and the rest of it is settled in the first section of the next period. In this sense, the model in the present paper becomes a payment model in which the privately issued IOU is settled using money.

Third, given the structure in which creditors endogenously establish a bank to pool the risk of trading uncertainty, the model supports a sunspot-type bank run.

Because our model contains the above three characteristics, it can be used to analyze the effect of the LLR function of central banks on resource allocation in a theoretically rigorous manner.

2.3 Allocation

The allocation of debtors in generation t is denoted as (c_t, d_t); that of creditors in generation t, (C_t, D_t^1, D_{t+1}^2); and that of patient creditors of generation 0, D_1^2. The allocation of the economy is defined as

$$(((c_t, d_t), (C_t, D_t^1, D_{t+1}^2))_{t=1}^{\infty}, D_1^2).$$

The allocation becomes feasible if it satisfies $c_t + C_t = W^c$ and $d_t + \alpha D_t^1 + (1 - \alpha)D_t^2 = w^d$. To simplify the analysis, we focus on stationary allocations by assuming that

$c_t = c_{t+1} \equiv c$, $d_t \equiv d$, $C_t = C_{t+1} \equiv C$, $D^1_t = D^1_{t+1} \equiv D^1$, $D^2_t = D^2_{t+1} \equiv D^2$ for all $t \geq 1$. Given this assumption, a stationary allocation is defined by $((c, d), (C, D^1, D^2))$ and the feasibility condition becomes $c + C = W^c$ and $d + \alpha D^1 + (1 - \alpha)D^2 = w^d$.

3. Competitive allocation

Figure 2 shows the flow of goods that is necessary to achieve efficient allocations. We show that this flow is made feasible by economic transactions, including the financial transactions described in Figure 3.

3.1 Economic transactions

3.1.1 Creditor

Let the price of c-goods and d-goods be q^c and q^d, respectively (in monetary terms). Creditors sell $W^c - C$ units of c-goods and receive an IOU. Let B be the amount of IOU and R be the (gross) interest rate. The amount of money needed by impatient creditors is $M^1 = q^d\alpha D^1$ and that needed by patient creditors is $M^2 = q^d(1 - \alpha)D^2$. Then, we obtain $M^1 + M^2 = RB$. Because these IOUs are received when c-goods are sold to debtors, we obtain $q^c(W^c - C) = B$.

Therefore, we have $W^c - C = q^d/Rq^c(\alpha D^1 + (1 - \alpha)D^2)$. Here, we set $p = Rq^c/q^d$. Then, the optimization problem of creditors becomes

$$\max_{c,d} U(C) + \alpha V^1(D^1) + (1 - \alpha)V^2(D^2) \tag{1}$$

subject to

$$C + \frac{\alpha D^1 + (1 - \alpha)D^2}{p} = W^c.$$

3.1.2 Debtor

Debtors issue IOUs, B, in monetary terms. To consume c units of c-goods, $q^c c$ units of IOUs must be issued. Hence, we have $q^c c = B$. Money that is needed to repay the IOUs is obtained by selling d-goods. Let the amount of d-goods sold in the first period of life be d^1 and the price be q^d.[11] Because the amount of money obtained becomes $m^1 = q^d d^1$, we need to obtain $q^d d^1 \geq M^1$. Further more, d-goods are sold to impatient creditors. Let the amount of d-goods sold be d^2. Then, the amount of money obtained becomes $\hat{m}^1 = q^d d^2$. To redeem the IOU for the second period of life, we must have $m^2 = M^2$. Because the amount of money held at the beginning of the second section can be calculated as $m^2 = (m^1 - M^1) + \hat{m}^1$, we have $RB = q^d(d^1 + d^2)$. Therefore, the budget constraint of debtors, $w^d = d + d^1 + d^2$, becomes $Rq^c c = q^d(w^d - d)$. This equation can be rewritten as $pc = w^d - d$, and the debtor's problem becomes

[11] Because we focus on the stationary equilibrium, we find an equilibrium in which the price of d-good remains constant.

$$\max_{c,d} u(c) + v(d) \tag{2}$$

subject to

$$pc + d = w^d.$$

3.2 Behavior of agents

3.2.1 Debtor's optimization

The first-order condition for Equation (2) becomes

$$\frac{1}{p}u'\left(\frac{w^d - d}{p}\right) = v'(d). \tag{3}$$

The demand function of d-good is denoted as $d(p)$.

Proposition 1: *(i) There exists* $d = d(p)(d\colon \Re_{++} \to (0, w^d))$. *(ii)* $\lim_{p\to 0} d(p) = 0$. *(iii)* $d'(p) > 0$.

Proof: See Appendix. ■

Rewriting the first-order condition (Equation (3)), we obtain

$$u'(c) = pv'(w - pc). \tag{4}$$

The demand function of c-good is defined as $c(p)$.

Proposition 2: *(i) There exists* $c = c(p)(d\colon \Re_{++} \to \Re_{+})$. *(ii)* $\lim_{p\to 0} c(p) = \infty$, $\lim_{p\to\infty} c(p) = 0$. *(iii)* $c'(p) < 0$.

Proof: See Appendix. ■

3.2.2 Creditor's optimization

Problem of creditors. Let the solution to problem (1) be $C(p)$, $D^1(p)$, $D^2(p)$. The first-order condition of this problem is calculated as

$$C\colon U'(C) = \lambda, \tag{5}$$

$$D^1\colon \alpha V^{1\prime}(D^1) = \frac{1}{p}\alpha\lambda, \tag{6}$$

$$D^2\colon (1 - \alpha)V^{2\prime}(D^2) = \frac{1}{p}(1 - \alpha)\lambda, \tag{7}$$

where λ is the Lagrange multiplier. From (6) and (7), we obtain

$$\frac{V^{1\prime}(D^1)}{V^{2\prime}(D^2)} = 1. \tag{8}$$

From this equation, we obtain the solution $D^2 = D(D^1)$. Furthermore, $(D: \Re_{++} \to \Re_{++})$ $D' > 0$, $\lim_{x\to 0} D(x) = 0$ and $\lim_{x\to\infty} D(x) = \infty$ hold. In addition, from Equations (5) and (6), we obtain

$$\frac{1}{p}U'\left(W^c - \frac{\alpha D^1 + (1-\alpha)D(D^1)}{p}\right) = V^{1\prime}(D^1). \tag{9}$$

Proposition 3: *(i) There exists* $D^1(p)$*. (ii)* $\lim_{p\to 0} D^1(p) = 0$, $\lim_{p\to\infty} D^1(p) = \infty$. ($\lim_{p\to 0} D^2(p) = 0$ *and* $\lim_{p\to\infty} D^2(p) = \infty$ *also hold.) (iii)* $D^{1\prime} > 0$.

Proof: See Appendix.

Optimal contract. We assume the following to ensure that an optimal contract satisfies the incentive compatibility conditions. Here, we assume that $V^{1\prime}(D) < V^{2\prime}(D)$ for all $D > 0$.

Creditors establish a bank to insure the uncertainty concerning their type. Because the information pertaining to the type of each creditor is private, the bank cannot check the type of agents who withdraw money from the bank. Therefore, it is necessary to execute an optimal contract that allows each creditor to withdraw based on his or her type and of his or her own free will (i.e. patient creditors will not withdraw in the first period of their life). The incentive for this behaviour occurs when $D^1 < D^2$ holds. Considering that Equation (8) holds at the equilibrium, we know that $D^1 < D^2$ holds. Hence, the optimal contract obtained above satisfies the incentive compatibility conditions.

3.3 Existence of competitive equilibrium

The equilibrium conditions are

$$C(p) + c(p) = W^c, \tag{10}$$

$$\alpha D^1(p) + (1-\alpha)D^2(p) + d(p) = w^d. \tag{11}$$

By adding up the budget constraint of creditors and debtors, we obtain

$$pW^c + w^d = \alpha D^1(p) + (1-\alpha)D^2(p) + d(p) + pC(p) + pc(p).$$

Hence, according to Walras's law, either (10) or (11) implies an equilibrium.

Theorem 1: *There exists a competitive equilibrium.*

Proof: See Appendix. ■

This equilibrium is denoted as $((c^*, d^*, C^*, D^{1*}, D^{2*}), p^*)$. In addition, we can show that this competitive equilibrium is efficient.

Theorem 2: *A competitive equilibrium is efficient.*

Proof: See Appendix. ■

3.4 Monetary price. Equilibrium of the monetary market

Let q^c* and q^d* be the equilibrium price of c-good and d-good, respectively. To ensure that there is no difficulty in the repayment of short-term debts in order to focus on the LLR function in the case of a bank run, we assume that $\alpha < 1/2$. First, debtors sell $(1 - \alpha)D^2*$ units of goods to the old creditors and receive $q^d*(1 - \alpha)D^2*$ $(= m_1)$ units of money. Second, creditors demand the repayment of short-term debts. The amount of repayment is $q^d* \alpha D^1*$ at an efficient equilibrium. Therefore, when $(1 - \alpha)D^2* < \alpha D^1*$ holds, there exists a difficulty in repayment of short-term debts in a stationary equilibrium. Because we have assumed that $\alpha < 1/2$, we have $(1 - \alpha)D^2* \geq \alpha D^1*$, which implies that there is no difficulty in repayment. The patient creditors who receive money buy d-goods. In total, $q^d* \alpha D^1*$ units of goods are bought; subsequently, debtors again hold $q^d*(1 - \alpha)D^2*$ $(= m_1)$ units of money and carry it over to the next period.

Then, in each period, $q^d*(1 - \alpha)D^2*$ units of money is carried over. Because the amount of money supplied in the economy is $(1 - \alpha)\bar{M}$, we can calculate the equilibrium monetary price as $q^d* = \bar{M}/D^2*$. With regard to c-goods, for arbitrary R, we have $q^c* = p*q^d*/R$ as an equilibrium price. Here, we set $R = 1$ for convenience; this gives us $q^c* = p*q^d*$. Summarizing the above, we obtain the following theorem.

Theorem 3: *At a competitive equilibrium, we have* $q^d* = \bar{M}/D^2*$ *and* $q^c* = p*q^d*$.

4. Bank run and lender of last resort

4.1 Bank run

Creditors receive q^c*c* in nominal terms in the first period from the debtor at the equilibrium; they deposit this in the bank. The impatient creditors consume D^1* and the amount they withdraw from the bank becomes q^d*D^1*; the patient creditors consume D^2* and the amount they withdraw from the bank becomes q^d*D^2*. This implies that the short-term interest rate is q^d*D^1*/q^c*c* and the long-term interest rate is q^d*D^2*/q^c*c*.

At the equilibrium, we have $D^2* > D^1*$. Hence, the long-term interest rate is greater than the short-term interest rate, and a patient creditor does not withdraw if other creditors also do not withdraw. This then becomes a Nash equilibrium and an efficient allocation is achieved. However, a bank cannot refuse withdrawal to an agent who comes to the bank because the information pertaining to the type of agent is private. The bank copes with the withdrawals of patient creditors by asking debtors to repay all the debts.[12]

[12] Here, we consider the situation wherein the patient creditors can consume d-goods when young and derive the same utility. This implies that if we denote the consumption of d-good when young as D_t^2 and the consumption of d-good when old as D_{t+1}^2, then the utility function of patient creditors can be written as $U(C_t) + V^2(D_t^2 + D_{t+1}^2)$. Although we can develop the model using this form of utility function from the outset, it makes the analysis complicated without altering the result.

The total amount of money that the impatient agents withdraw is $q^{d*}\alpha D^{1*}$, and the total amount that the patient agents try to withdraw is $q^{d*}(1-\alpha)D^{1*}$. This implies that $q^{d*}D^{1*}$ units of money are necessary for the bank. The bank has $q^{d*}\alpha D^{1*}$ units of money and the debtors have $q^{d*}((1-\alpha)D^{2*}-\alpha D^{1*})$ units. Hence, the bank can procure a total of only $q^{d*}(1-\alpha)D^{2*}$ units of money. The total amount that creditors try to withdraw is $q^{d*}D^{1*}$. The amount that the bank can make available is $q^{d*}(1-\alpha)D^{2*}$. Here, we assume that α is significantly large such that at the equilibrium, we have $(1-\alpha)D^{2*} < D^{1*}$. Therefore, a bank run occurs, and an inefficient allocation is achieved.[13]

4.2 Function of the lender of last resort

We consider that the central bank supplies money to banks when patient creditors visit for withdrawal of money. Suppose a percentage of patient creditors $\gamma \in (0, 1)$ try to withdraw. In this case, the shortage of money in a bank amounts to $q^{d*}(1-\alpha)\gamma D^{1*}$. This amount is supplied by the central bank on a temporary basis. Because each creditor can withdraw $q^{d*}D^{1*}$ units of money, a bank run does not occur.

The supply of *d*-goods is αD^{1*}, and the amount of money required to buy them is $(\alpha+(1-\alpha)\gamma)D^{1*}q^{d*}$. Hence, the price of *d*-goods is calculated as

$$q^{d*}\frac{\alpha+(1-\alpha)\gamma}{\alpha} > q^{d*}.$$

Because the price increases, the amount of goods bought for each creditor decreases. The percentage of patient creditors $1-\gamma$ who do not come to the bank in the first period can consume D^{2*} units of *d*-goods, independent of γ. Hence, the utility of creditors becomes higher if they wait and withdraw in the second period of life. This implies that all creditors wait and a bank run does not occur. At this equilibrium, the LLR is not used and the mere existence of the LLR can prevent a bank run. Therefore, the equilibrium becomes efficient with the LLR. We then obtain the following theorem.

Theorem 4: *When the LLR of the central bank is working, an equilibrium will always be efficient.*

5. Economic implications

We have described a bank that has a payments function that offers consumers a deposit contract and have shown that the central bank can prevent the outbreak of a bank run using liquidity supply. The research objectives of previous microeconomic analyses on banking have mainly been related to financial intermediation using a deposit contract, and the models have generally been described in real terms. However, in the present work, we have focused on the payments aspect considering the liquidity supply of the LLR as a part of the payments system. Our model described in the nominal terms has shown that a bank run can be prevented by the supply of outside money. In particular, a bank run is caused

[13] In this paper, the mechanism for realizing an inefficient allocation is the same as that analysed in Diamond and Dybvig (1983).

in a pure exchange economy, irrespective of technology. Therefore, our model suggests that a bank run and its prevention depend on the matter of liquidity.[14]

In our model, monetary transactions are systematically supported by inside money and outside money. In other words, inside money originates from private economic activity and the central bank compensates for the lack of liquidity by means of a discount window. Therefore, outside money is initiated into this process from private economic activity. In this way, our model describes a feature of the Real Bills Doctrine, which advocates that the stock of money should fluctuate to meet the needs of trade. This is the main theme of Freeman's papers (1996a,b). The unique feature of our model is that it has a private bank with a function of risk pooling, and the lack of liquidity causes a bank run as well as inefficient allocations.[15]

6. Appendix

With respect to the utility functions, we assume the following usual boundary conditions.

Assumption 1:

(1) $u : \Re_+ \to \Re$ *and* $v : \Re_+ \to \Re$ *are continuous and twice continuously differentiable in the interior.*

(2) $u' > 0,\ u'' < 0,\ v' > 0,\ v'' < 0.$

(3) $\lim_{x \to 0} u'(x) = \infty,\ \lim_{x \to 0} v'(x) = \infty.$

(4) $U : \Re_+ \to \Re,\ V^1 : \Re_+ \to \Re,$ *and* $V^2 : \Re_+ \to \Re$ *are continuous and twice continuously differentiable in the interior.*

(5) $U' > 0,\ U'' < 0,\ V^{1\prime} > 0,\ V^{1\prime\prime} < 0,\ V^{2\prime} > 0,\ V^{2\prime\prime} < 0.$

(6) $\lim_{x \to 0} U'(x) = \infty,\ \lim_{x \to 0} V^{1\prime}(x) = \infty,\ \lim_{x \to 0} V^{2\prime}(x) = \infty.$

With respect to the utility functions, we also assume the following conditions.

Assumption 2:

(1) $\lim_{x \to \infty} u'(x) \geq \bar{u}$ *where* $\bar{u} > 0.$

(2) $u'(c) + cu''(c) > 0$ *for all* $c > 0.$

Assumption 2(1) is a similar condition that is used to prove the existence of equilibrium in Maeda (1992). In the model with heterogeneous agents, this type of additional boundary condition is required. Assumption 2(2) is satisfied, for example, in the class of CRRA utility functions with an index of constant relative risk aversion being greater than 1.

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[14] Diamond and Dybvig (1983) consider the situation in which a bank run can be prevented by deposit insurance; they also suggest that a bank run can be prevented using a framework like a discount window. However, the rigorous function of a discount window cannot be analysed using a real model like the Diamond and Dybvig model. Our model is a monetary model that can rigorously evaluate the function of a discount window.

[15] We have taken for granted that a bank has both the function of facilitating payments and that of financial intermediation. However, the proposition that the payments function should be separated from financial intermediation has been argued in the trade-off relationship between efficiency and stability. Therefore, our model should provide insight for future researches regarding the narrow bank proposal.

Proof of Proposition 1:

(i) Arbitrarily, fix $p > 0$. Define

$$V(d) \equiv \frac{1}{p}u'\left(\frac{w^d - d}{p}\right) - v'(d). \tag{A1}$$

Then, because $\lim_{d\to 0} V(d) = -\infty$ and $\lim_{d\to w^d} V(d) = \infty$, there exists $\bar{d}$ in $(0, w^d)$, which satisfies $V(\bar{d}) = 0$.

(ii) For arbitrary $d \in (0, w^d)$, the left-hand side of Equation (3) goes to infinity when $p \to 0$ from Assumption 2(1). Therefore, $\lim_{p\to 0} d(p) = 0$ holds.

(iii) By totally differentiating Equation (3), we have

$$\left[\left(-\frac{1}{p^2}\right)u'\left(\frac{w^d - d}{p}\right) + \frac{1}{p}\left(-\frac{w^d - d}{p^2}\right)u''(c)\right]dp = \left[\left(\frac{1}{p}\right)^2 u''(c) + v''(d)\right]dd \tag{A2}$$

$$\left[\left(-\frac{1}{p^2}\right)u'\left(\frac{w^d - d}{p}\right) + \left(-\frac{1}{p^2}\right)\left(\frac{w^d - d}{p}\right)u''(c)\right]dp = \left[\left(\frac{1}{p}\right)^2 u''(c) + v''(d)\right]dd \tag{A3}$$

$$-\frac{1}{p^2}[u'(c) + cu''(c)]\,dp = \left[\left(\frac{1}{p}\right)^2 u''(c) + v''(d)\right]dd \tag{A4}$$

$$\frac{dd}{dp} = -\left(\frac{u'(c) + cu''(c)}{u''(c) + p^2 v''(d)}\right). \tag{A5}$$

From $u'(c) + cu''(c) > 0$ (from Assumption 2(2)), we obtain

$$\frac{dd}{dp} > 0. \ \blacksquare$$

Proof of Proposition 2:

(i) Arbitrarily, fix $p > 0$. Define

$$W(c) \equiv u'(c) - pv'(w - pc).$$

Then, $\lim_{c\to 0} W(c) = \infty$ and $\lim_{c\to w/p} W(c) = -\infty$ hold. Hence, there exists a solution for $W(c) = 0$.

(ii) From the budget constraint, we obtain

$$c(p) = \frac{w^d - d(p)}{p}. \tag{A6}$$

Because $\lim_{p\to 0} d(p) = 0$, from Equation (17), we obtain $\lim_{p\to 0} c(p) = \infty$. When $p \to \infty$, $w^d \geq w^d - d(p) \geq 0$ holds. Then, from Equation (A6), we obtain $\lim_{p\to\infty} c(p) = 0$.

(*iii*) By totally differentiating Equation (4), we obtain

$$(u''(c) + p^2 v''(d))\, dc = (v'(d) - pcv''(d))\, dp \tag{A7}$$

$$\frac{dc}{dp} = \frac{v'(d) - pcv''(d)}{u''(c) + p^2 v''(d)} < 0. \blacksquare \tag{A8}$$

Lemma 1: *For arbitrary $p > 0$, there exists D^1, which satisfies $W^c - [\alpha D^1 + (1-\alpha) D(D^1)]/p = 0$.*

Proof: Define $Z(D^1) \equiv [\alpha D^1 + (1-\alpha)D(D^1)]/p$. For arbitrary $p > 0$, we have $\lim_{D^1 \to 0} Z(D^1) = 0$ and $\lim_{D^1 \to \infty} Z(D^1) = \infty$. Hence, from $Z' > 0$, there uniquely exists $D^1 > 0$ that satisfies $Z(D^1) = W^c$. $\blacksquare$

D^1 in Lemma 1 is denoted as $\bar{D}^1(p)$.

Proof of Proposition 3:

(*i*) Arbitrarily, fix $p > 0$. From Equation (9), defining

$$X(D^1) \equiv \frac{1}{p} U'\left(W^c - \frac{\alpha D^1 + (1-\alpha)D(D^1)}{p} \right) - V^{1\prime}(D^1),$$

we have $\lim_{D^1 \to 0} X(D^1) = -\infty$, $\lim_{D^1 \to \bar{D}(p)} X(D^1) = \infty$. Therefore, there exists a solution for $X(D^1) = 0$.

(*ii*) The left-hand side of Equation (9) goes to infinity when $p \to 0$. Therefore, the right-hand side of Equation (9) must also go to infinity. Hence, we have $\lim_{p \to 0} D^1(p) = 0$. (Then, we also have $\lim_{p \to 0} D^2(p) = 0$.)

Because $p \to \infty$ implies $1/p \to 0$, we suppose that $D^1 \to \infty$; then, the left-hand side of Equation (9) goes to zero. However, the right-hand side of Equation (9) does not reach zero. This is a contradiction. Hence, we have $\lim_{p \to \infty} D^1(p) = \infty$. (Then, we also have $\lim_{p \to \infty} D^2(p) = \infty$.)

(*iii*) Totally differentiating Equation (9), we obtain

$$\left[\left(-\frac{1}{p^2} \right) U'(C) + \frac{1}{p} \left(-\frac{1}{p^2} \right) (-(\alpha D^1 + (1-\alpha)D(D^1)))U''(C) \right] dp =$$

$$\left[\frac{1}{p^2} (\alpha + (1-\alpha)D'(D^1))U''(C) + V^{1\prime\prime}(D^1) \right] dD^1.$$

Then, we have

$$\left(-\frac{1}{p^2} \right) [U'(C) + (-(W - C))U''(C)]\, dp =$$
$$\left[\frac{1}{p^2} (\alpha + (1-\alpha)D'(D^1))U''(C) + V^{1\prime\prime}(D^1) \right] dD, \tag{A9}$$

$$\frac{dD^1}{dp} = \frac{-(U'(C) + (-(W - C))U''(C))}{(\alpha + (1-\alpha)D'(D^1))U''(C) + p^2 V^{1\prime\prime}(D^1)} > 0. \blacksquare \tag{A10}$$

Proof of Theorem 1: From Equation (11), we define

$$Y(p) \equiv \alpha D^1(p) + (1-\alpha)D^2(p) + d(p) - w^d.$$

Because $\lim_{p\to 0} d(p) = 0$, $\lim_{p\to 0} D^1(p) = 0$ and $\lim_{p\to 0} D^2(p) = 0$ hold, we have $\lim_{p\to 0} Y(p) = -w^d$. From $d'(p) > 0$, $D^{1\prime}(p) > 0$ and $D^{2\prime}(p) > 0$, we have $Y'(p) > 0$. From $\lim_{p\to\infty} D^1(p) = \infty$, $\lim_{p\to\infty} D^2(p) = \infty$ and $d(p) > 0$, we also have $\lim_{p\to\infty} Y(p) = \infty$. Hence, there exists a competitive equilibrium. $\blacksquare$

Proof of Theorem 2: Efficient allocations for an arbitrary weight $\mu \in [0, 1]$ can be obtained by solving the following problem:

$$\max \mu(u(c) + v(d)) + \tag{A11}$$

$$(1-\mu)[\alpha(U(C) + V^1(D^1)) + (1-\alpha)(U(C) + V^2(D^2))] \tag{A12}$$

subject to

$$c + C = W^c, \tag{A13}$$

$$d + \alpha D^1 + (1-\alpha)D^2 = w^d. \tag{A14}$$

The first-order conditions become

$$c\colon \mu u'(c) = \lambda^1, \tag{A15}$$

$$d\colon \mu v'(d) = \lambda^2, \tag{A16}$$

$$C\colon (1-\mu)U'(C) = \lambda^1, \tag{A17}$$

$$D^1\ (1-\mu)V^{1\prime}(D^1) = \lambda^2, \tag{A18}$$

$$D^2\ (1-\mu)V^{2\prime}(D^2) = \lambda^2. \tag{A19}$$

We have

$$\mu u'(c) = (1-\mu)U'(C)$$

from Equations (A15) and (A16), and

$$V^{1\prime}(D^1) = V^{2\prime}(D^2). \tag{A20}$$

from (A18) and (A19). We also have

$$\frac{u'(c)}{v'(d)} = \frac{\lambda^1}{\lambda^2} = \frac{U'(C)}{V^{1\prime}(D^1)} \tag{A21}$$

from (A15)–(A18). Hence, the stationary allocation $((c, d), (C, D^1), (C, D^2))$ becomes efficient when (A13)–(A14) and (A20)–(A21) are satisfied.

From Equation (8), at equilibrium, we have

$$V^{1\prime}(D^1*) = V^{2\prime}(D^2*). \tag{A22}$$

Hence, we know that (A20) holds.

Furthermore, from Equation (3), at the equilibrium, we have

$$\frac{1}{p^*}u'(c^*) = v'(d^*). \tag{A23}$$

From this equation, we obtain

$$\frac{u'(c^*)}{v'(d^*)} = p^*. \tag{A24}$$

From Equation (5) and (6),

$$\frac{U'(C^*)}{V^{1\prime}(D^1*)} = p^* \tag{A25}$$

holds. Hence, from these two equations, we obtain

$$\frac{u'(c^*)}{v'(d^*)} = \frac{U'(C^*)}{V^{1\prime}(D^1*)}. \tag{A26}$$

This implies that (A21) holds. Therefore, the theorem is proven. ■

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