A Course on Inductive Game Theory 1

by Mamoru Kaneko, 2009 February 27

General Aim:

Experiential origin/emergence of **belief/knowledge** of a player about the structure of the game.

For this aim, first we should make:

0: the basic assumption that a player has little **knowledge of the structure** of the game.

Under this assumption, we should explore:

1. how can a player learn the structure from experiences?
2. in the first place, **what are experiences?**
3. how does he construct some view from experiences?
4. are his experiences enough to have a view?

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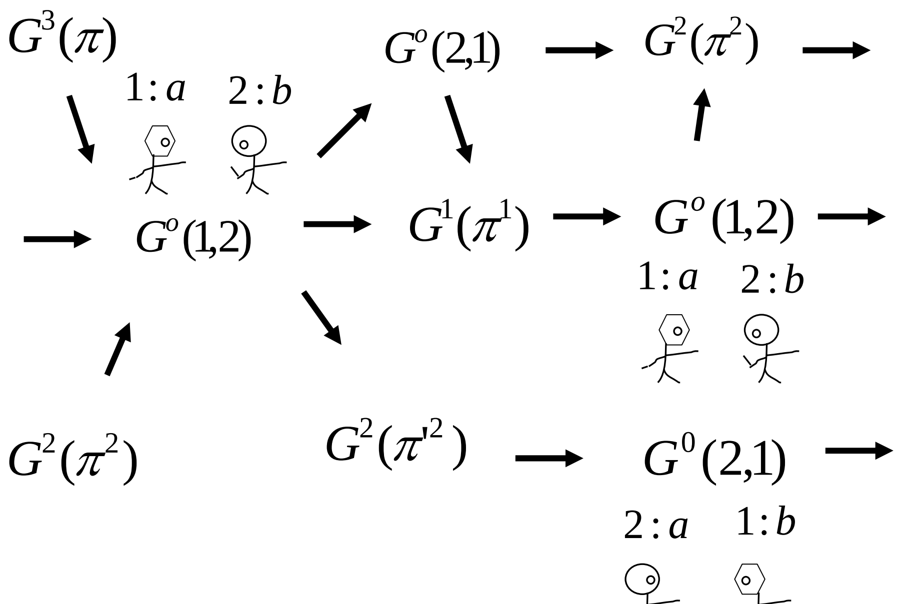

Social Web

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Various Concepts need to be examined.

1. To learn the structure of the target situation, **repetition** of the target situation is required.
   Q1: Why not the **repeated game approach**?
2. The entire situation is formulated as a <u>one-shot game</u> and is considered from the <u>*ex ante*</u> point of view.
   Q2: Is anything wrong with this treatment?
3. A repeated game can be regarded as an extensive game.
   Q3: What is the status of an **extensive game theory**?
4. The theory of extensive games treats the problem of information in the most general manner in game theory.
   Q4: Are there any problems with this treatment of "**information**"?
5. An **information partition** describes **individual memory** in addition to **information transmission**.
   Q5: Is there any difficulty with this?
6. In game theory, the probabilistic behavior (mixed strategy) is assumed widely.
   Q6: Is this a sound assumption? First of all, what is **"probability**"?

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Historical Background of Game Theory

1900~1910: Crises in
Mathematics - - Russell's finding of a contradiction in Cantor's set theory
Physics - - anomalies for relativity theory and quantum mechanics

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D. Hilbert

⇩

J. von Neumann

⇩

Nash, Shapley, Shubik

⇩

Harsanyi, Aumann, Selten

- Minimax Theorem (1928)
- Gödel's Completeness Theorem (1930)
  Incompleteness Theorem (1931)
- Theory of Computation (1934~40)
- Gentzen's Proof Theory (1935)
- Neumann's Balanced Growth Model: Perfect Competition (1937)
- Wald's answer (1937) to von Mises's Frequentist Probability Theory
- Neumann's self-reproducing automata (1953-57)

Reductionism since Harsanyi and Aumann

- Harsanyi (1955): Eliminating such a differences, we have the common universal utility function:

$$u_i(x_i) \mapsto U(x_i, p_i)$$

the common utility function $U(\cdot,\cdot)$

$$u_j(x_j) \mapsto U(x_j, p_j)$$

- Harsanyi (1967): Why people have different prior (probablistic) beliefs?

  Because their experiences are different. There is still the common universal prior beliefs.

- Aumann (1976): each primitive state of the world contains all information.

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Two Opposite Traditions

- Reductionism:

  We look for a more basic structure in a primitive element.

Substructures (Basic Structure) ← Reductionist — Constructivism → Superstructures

- Constructivism:

  We construct a superstructure on a less meaningful substructure (base).

> Inductive Game Theory follows the tradition of constructivism and experientialism.

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Implications from Inductive Game Theory

- Once the experiential source for individual belief/ knowledge becomes explicit, we can discuss various aspects of individual beliefs:

  - falsity
  - limited experiences
  - limited beliefs
  - limited inferential abilities
  - limited interpersonal reciprocities

  ⇨

  - social Institutions
  - social roles
  - education & communication

- We should change the **standard assumption** of game theory/ economics that a player is presumed to know the entire structure of the game.

- By this assumption, we are unable to understand the present “small and narrow” earth through game theory and economics.

- What is the status of perfect competition?
- Various assumptions of omniscience? ➞ Page 18.

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Von Neumann-Morgenstern (1944): 1

- Minimax Theorem for a 2-person zero-sum game;

$G = (\{1,2\}, S_1, S_2, h_1, h_2)$

- $h_1(s_1, s_2) + h_2(s_1, s_2) = 0$ for all $(s_1, s_2) \in S_1 \times S_2$.
- Maximin decision criterion :

  maximize the guarantee level of his payoffs : $\max_{s_1 \in S_1} (\min_{s_2 \in S_2} h_1(s_1, s_2))$

  what is $\min_{s_2 \in S_2} h_1(s_1, s_2)$?
- (s_1^o, s_2^o) is a saddle point for h_1 iff for all $s_1 \in S_1$ and $s_2 \in S_2$

$$h_1(s_1^o, s_2) \le h_1(s_1^o, s_2^o) \le h_1(s_1^o, s_2).$$

Minimax Theorem : In the mixed extension $\hat{G}$ of G,

(1) : $\min_{s_2 \in \hat{S}_2} \max_{s_1 \in \hat{S}_1} \hat{h}_1(s_1, s_2) = \max_{s_1 \in \hat{S}_1} \min_{s_2 \in \hat{S}_2} \hat{h}_1(s_1, s_2)$;

(2) : there is a saddle point for $\hat{h}_1$ in $\hat{G}$.

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Nash (1951)

- Nash equilibrium for an n-person game;

  $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$

  - (s_1^o, s_2^o) is a <u>Nash equilibrium</u> iff for all $i \in N$,

    $h_i(s_i; s_{-i}^o) \leq h_i(s^o)$ for all $s_i \in S_i$.

  - <u>Nash Theorem</u>: The mixed extension of G has a Nash equilibrium.

  - <u>Interchangeability</u>: for all $i \in N$,

    if s and t are NE's, then $(t_i; s_{-i})$ is an NE.

  - Any 0-sum 2-person game has the interchanable equi.set.

<u>The battle of the sexes</u> is not solvable: $\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}$

Nash equilibrium is interpreted as an *Ex Ante* Decision Criterion

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Von Neumann-Morgenstern (1944): 2

- Reduction of an extensive game to a normalized form:

An extensive form game

↓ ← reduction

A normalized form game

↓

A characteristic function game:
cooperative game (N,v)
+
"stable set" + its interpretation.

Kuhn's (1953) perfect recall condition supports this reduction

a strategy: a complete list of contingent actions

- Cooperation is an axiom.
- Almost no substantive relations to the concepts of extensive games
- Almost 4 quarters are devoted to this concept

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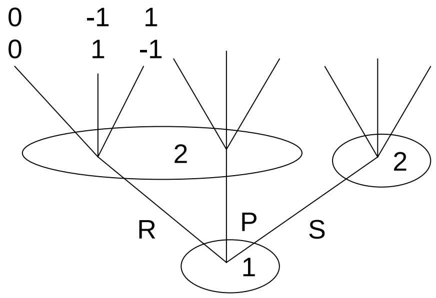
A Game in Extensive Form

1. Information?
2. Memory?
3. Strategy?



- **<u>Merits</u>** of an extensive game
- <u>Faithful</u> representation of the rules of the game (society)
- Interactions between information and actions

- **<u>Demerits</u>** of an extensive game
- The viewpoint is unclear; objective or subjective?
- Does an individual player captures the objective description?
- Treatment of information and memory

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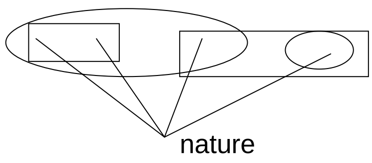
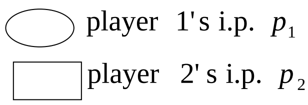
Aumann's Information Partition Model

- Ω : the set of all possible states of the worlds;
- p_i is a partition of Ω for $i = 1,2$, where if $\omega \in \Omega$ occurs as the true state of the world, player i receives information $p_i(\omega)$.

$\omega \mapsto$ information $p_i(\omega)$:
if $p_i(\omega) \subseteq E \subseteq \Omega$, then player i <u>knows</u> event E.



Simple extensive game:



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Difficulties found in Aumann's definition of common knowledge

An event E is <u>common knowledge at the "true state"</u> ω_0 iff

$$p_1(\omega_0) \subseteq E : \text{"1 knows } E\text{"}$$

$$p_2(\omega_0) \subseteq E : \text{"2 knows } E\text{"}$$

$$\forall \omega_1 \in p_1(\omega_0); p_2(\omega_1) \subseteq E : \text{"1 knows that 2 knows } E\text{"}$$

$$\forall \omega_1 \in p_2(\omega_0); p_1(\omega_1) \subseteq E : \text{"2 knows that 1 knows } E\text{"}$$

$$\forall \omega_1 \in p_1(\omega_0); \forall \omega_2 \in p_2(\omega_1); p_1(\omega_2) \subseteq E :$$

$$\text{"1 knows, 2 knows 1knows } E\text{"}$$

• • •

1): the entire structure is known to the players
2): no distinction between "information" and "knowledge"
3): "Event" is the object of "information" or "knowledge".

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- Information partitions are, implicitly, assumed to be "common knowledge".
- Aumann (and his followers) adopts the reductionist view:

  - ω : each primitive state of the world including the description of i.p.structures.

- This attitude comes from and is coherent in Savage's view and, more generally, probability theory.

- What is an alternative view to reductionism?

superstructures
↓
substructures

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What is “information”?

Semantical way:
Information is expressed as a set of possibilities $\{x_1,..., x_k\}$.

Syntactical way
Information is expressed as a proposition p.

Connection?
This requires the back ground universal set universal set expressing possibilities.

$$\{x_1,..., x_k\} = \{x : p(x)\}\ ?$$

* This needs to choose a universal set.
* $\{x_1,..., x_k\} = \{x \in X : p(x)\}$?

* Is the empty information expressed as the entire set?
* Does rich information contains less contents?

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絶句

杜甫

江碧鳥愈々白

山青花欲燃

今春看々又過

何日是帰年

絶句　杜甫
江碧にして　鳥愈々白く
山青くして　花燃えんと欲す
今春　看々又た過ぐ
何れの日か　是れ帰年ならん

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Constructions of Symbolic (mathematical) Logic

0: List of Pure Symbols:
01: propositional variables symbols: $p_0, p_1, \cdots$
02: logical connectives: $\wedge$ (intended meaning, "and"), $\vee$ (or), $\supset$ (implies), $\neg$ (not);
03: auxiliary symbols: (,)

0: Rules to generate formulae (permissible expressions):
01: every propositional variable is a formula;
02: if *A* and *B* are formulae, so are $(A \wedge B), (A \vee B), (A \supset B), (\neg A)$;
03: every formula is generated by a finite number of applications of 01 and 02.

Gentzen-Style Sequent Calculus

0: Initial Sequents and Inference Rules:
01: Axioms: $A \to A$, where $\to$ is an additional symbol.
02: Inference rules, e.g.,

$$\frac{A \to B}{C \wedge A \to B}$$

- Definition of a proof;
- Provability: Existence of a proof.

Semantics in Symbolic Logic

A truth assignment is a function $v : \{p_0, p_1, \ldots\} \to \{t, f\}$.

This is extended to the set of all formulae as follows:

- We define $\Vdash_v$ as follows:

0 : for all propositional variable p, $\Vdash_v p$ if and only if $v(p) = t$;

1 : $\Vdash_v \neg A$ if and only if not $\Vdash_v A$;

2 : $\Vdash_v A \supset B$ if and only if not $\Vdash_v A$ or $\Vdash_v B$;

3 : $\Vdash_v A \wedge B$ if and only if $\Vdash_v A$ and $\Vdash_v B$;

4 : $\Vdash_v A \vee B$ if and only if $\Vdash_v A$ or $\Vdash_v B$.

Validity: We write that $\Vdash A$ **iff** $\Vdash_v A$ **for all truth assignments** ***v*** **.**

Completeness Theorem: Validity is equivalent to Provability.

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Various Hidden Assumptions of Omniscience

- Completeness-Soundness Theorem in logic:
  provability (existence of a proof)
  ⟷ validity (true for all possible models)
- Instantaneous Understanding;
  contents of an information piece can be extracted instantaneously.
- Instantaneous Utility Maximization;
  maximization of utility is instantaneously made.

More serious and similar omniscience assumptions are:
- a player knows the model which he lives in.
- he can make a plan over a long-horizon such as in the repeated game approach.

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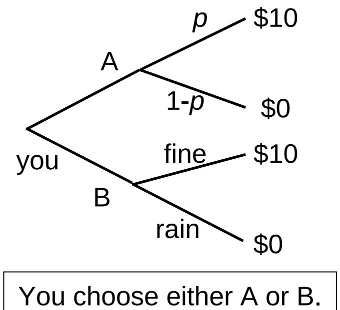
Use of "Subjective Probability"

- What is a subjective probability? (Savage (`54), Aumann-Anscombe (`61))
  Subjective belief? Degree of certainty for the subject?
- Both answers are simply changing the expressions.
- Real questions are: What does "degree of certainty for the subject" mean?

Any meaningful (experiential) definition of "subjective probability?
Now, what is the subjective probability of "tomorrow's fair weather"?



You choose either A or B.

<u>Choice A</u>: $10 is given iff the well-designed random mechanism chooses "up" with probability p
<u>Choice B</u>: $10 is given iff the weather is fine.

Changing p from 0 to 1, we find a p^* so that A(p^*) and B are indifferent.

Mathematically, completeness, continuity and monotonicity with respect p imply $\exists p^*$.

Anything wrong?

von Neumann-Morgenstern (1944)（3）

substructures

(N, v): an n-person game in <u>characteristic function form</u>:

- $N = \{1, ..., n\}$; the set of players;
- $v: 2^N \to R$ with $v(\phi) = 0$; each value $v(S)$ is the total amount of surplus that can be distributed amon the players in S.

superstructures

We say that $x = (x_1, ..., x_n)$ is an <u>imputation</u> iff

- $x_i \geq v(\{i\})$ for all $i \in N$;
- $\sum_{i \in N} x_i = v(N)$.

The set of all imputaitons is denoted by I.

We say that x <u>dominates</u> y <u>via coalition</u> S iff

- $x_i > y_i$ for all $i \in S$;
- $v(S) \geq \sum_{i \in S} x_i$.
- This is denoted by $x \; dom_S \; y$.

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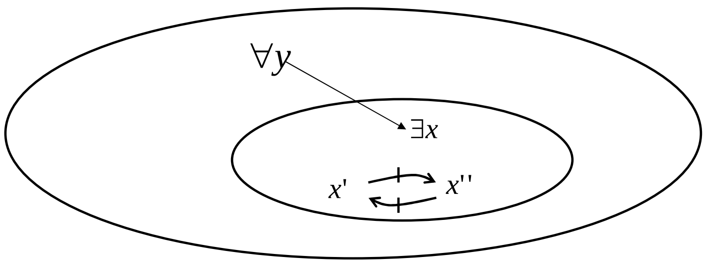
von Neumann-Morgenstern (1944)（3）

We say that a subset V of I is a <u>stable set</u> in game (N, v) iff

- (External Stability): for any $y \in I - V$, there is a $x \in V$ such that $x \text{ dom}_S \, y$ for some $S \in 2^N$;
- (Internal Stability): for any $x, y \in I$ and $S \in 2^N$, it does not holds that $x \text{ dom}_S \, y$.

superstructures



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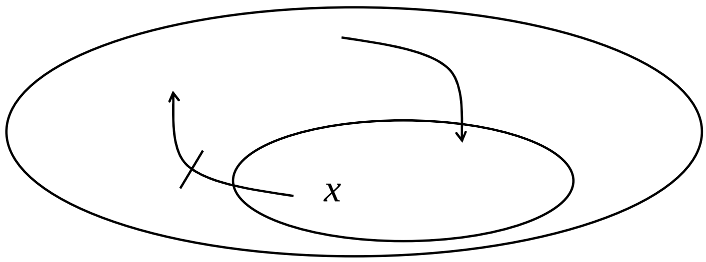
von Neumann-Morgenstern (1944)（3）

Interpretation of a stable set (Standard Behavior) :

- a triple $\langle x;V, dom \rangle$
- $x \in V$ is a stationary state;
- x continues to be a stationary state, which is supported by the internal and external stability conditions.



More basic postulate: the players know the structure of <V, dom>, and then more substructures?

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A Course on Inductive Game Theory：References

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http://infoshako.sk.tsukuba.ac.jp/~kaneko/

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