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On the Rawlsian social ordering over infinite utility streams

Kaname Miyagishima

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Kaname Miyagishima*

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Abstract

In this paper, we characterize a Rawlsian ordering over infinite utility streams in terms of two principles. The first is a fairness principle called *Hammond Equity*. The second is a liberal principle named *Weak Harm Principle*.

Keywords : Infinite utility streams; Rawlsian social ordering

Journal of Economic Literature Classification: D63; D71

1 Introduction

In this paper, we axiomatize a Rawlsian ordering comparing infimum values of infinite utility streams in terms of the following two principles. The first is a fairness principle called *Hammond Equity* proposed by Hammond (1976). The second is a liberal principle named *Weak Harm Principle* proposed by Lombardi and Veneziani (2009).

Lauwers (1997) characterize the Rawlsian ordering by using *Hammond Equity*. The differences between Lauwers' (1997) characterization and ours are as follows: Lauwers (1997) introduces an axiom called *Repetition-approximation Principle*, and proves the characterization through a function representation result shown by Diamond (1965).

*Waseda University. E-mail: Kaname1128@gmail.com

Instead, we use an axiom named *Preference Consistency.* (This axiom is often referred to as *Preference Continuity.*) Moreover, Lauwers (1997) uses completeness of social ranking and *Full Anonymity*, while we do not use these properties.

Lombardi and Veneziani (2009) axiomatize *an order extension* of a Rawlsian ranking comparing *minimum* values of utility streams, using *Weak Harm Principle.* In contrast, we characterize the Rawlsian ordering comparing *infimum* values of utility streams. We use axioms *Completeness*, *Preference Consistency*, and *Sup Continuity* which are stronger than their axioms *Minimal Completeness*, *Weak Independence Continuity*, and *Weak Continuity*, respectively.

2 Notation and definitions

Let $N = \{1, 2, ...\}$ be the set of infinite generations. $\mathbb{R}$ is the set of real numbers. The set of infinite utility streams is denoted by

$$X^N \equiv \left\{ \boldsymbol{u} = (u_1, u_2, ..., u_t, ...) \in \mathbb{R}^\infty | \sup_{t \in \mathbb{N}} |u_t| < \infty \right\},$$

where u_t is the utility level of generation t. For all $\boldsymbol{u} \in X^N$, $\boldsymbol{u}^{-T} = (u_1, ..., u_T)$ and $\boldsymbol{u}^{+(T+1)} = (u_{T+1}, u_{T+2}, ...)$. For all $u \in \mathbb{R}$, $(u)_{con} = (u, u, ..., u, ...)$ denotes a constant stream. Let Π be the set of all finite permutations over N.

A social ranking over utility vectors is denoted by R. For any two utility vectors $\boldsymbol{u}, \boldsymbol{v} \in X^N$, $[\boldsymbol{u} R \boldsymbol{v}]$ is interpreted as "$\boldsymbol{u}$ is socially at least as good as $\boldsymbol{v}$." The symmetric and asymmetric parts of R are denoted by I and P, respectively. A binary relation is a quasi-ordering if it satisfies reflexivity and transitivity. A binary relation is an ordering if it satisfies completeness and transitivity.

We define the Rawlsian social ordering.

Definition: The Rawlsian social ordering R_r is defined as follows:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in X^N, \; \boldsymbol{u} R_r \boldsymbol{v} \Longleftrightarrow \inf_{t \in N} u_t \geq \inf_{t \in N} v_t.$$

This social ordering compares utility streams based on the infimum utility level of each stream.

We introduce some axioms to characterize the Rawlsian social ordering.

Finite Anonymity (FA): For all $\boldsymbol{u} \in X^N$ and each $\pi \in \Pi$, $\boldsymbol{u} I \pi(\boldsymbol{u})$.

Weak Pareto (WP): For all $\boldsymbol{u}, \boldsymbol{v} \in X^N$ and all $\epsilon > 0$, if $u_t \geq v_t + \epsilon$ for all $t \in N$, then $\boldsymbol{u} P \boldsymbol{v}$.

Sup Continuity: For all $\boldsymbol{u} \in X^N$, if a sequence of vectors $\{\boldsymbol{v}^k\}_{k=1}^{\infty}$ converges to $\boldsymbol{v} \in X^N$ in terms of sup norm and $\boldsymbol{u} R \boldsymbol{v}^k$ (resp. $\boldsymbol{v}^k R \boldsymbol{u}$) holds for all $k \in \mathbb{N}$, then $\boldsymbol{u} R \boldsymbol{v}$ (resp. $\boldsymbol{v} R \boldsymbol{u}$).

Preference Consistency (PC): For all $\boldsymbol{u}, \boldsymbol{v} \in X^N$, if there exists $T \geq 1$ such that $(\boldsymbol{u}^{-t}, \boldsymbol{v}^{+(t+1)}) R \boldsymbol{v}$ for all $t \geq T$, then $\boldsymbol{u} R \boldsymbol{v}$.

FA restricts the application of the standard anonymity as a impartiality requirement to situations where utility streams differ in at most a finite number of components. *WP* requires that, if all generations are better off, it should be socially desirable. *SC* requires social orderings to be continuous in terms of sup norm. *PC* is considered, in the literature, as a condition that establishes a link between finite and infinite settings of distributive justice. See Asheim and Tungodden (2004, p. 223).

3 Characterization in terms of fairness

In this section, we characterize R_r in terms of fairness in the sense of *Hammond Equity.*

Hammond Equity: For all $\boldsymbol{u}, \boldsymbol{v} \in X^N$, if $v_t > u_t > u_{t'} > v_{t'}$ for some $t,\ t' \in N$, and $u_k = v_k$ for all $k \in N/\{t, t'\}$, then $\boldsymbol{u} R \boldsymbol{v}$.

This axiom insists that a reduction of inequality in utilities between two generations should be socially accepted.

Theorem 1: A *quasi-ordering* R satisfies *Hammond Equity, Weak Pareto, Preference Consistency,* and *Sup Continuity* if and only if $R = R_r$.

Proof. It is obvious that R_r satisfies the axioms in the Theorem. We show the converse result. Suppose that a quasi-ordering R satisfies the axioms. We first prove that, for any utility vectors $\boldsymbol{u}, \boldsymbol{v} \in X^N$

$$\inf_{t \in N} u_t > \inf_{t \in N} v_t \Longrightarrow \boldsymbol{u} P \boldsymbol{v}. \tag{1}$$

For convenience, we introduce a utility stream. Define

$$\hat{x} = \frac{\inf_{t \in N} u_t + \inf_{t \in N} v_t}{2}.$$

Since $\inf_{t \in N} u_t > \inf_{t \in N} v_t$, there exists T^* such that, for some $t \leq T^*$, $v_t < \inf_{t \in N} u_t$. Define $\boldsymbol{v}' \in X^N$ as follows:

$$v'_t = \hat{x} \text{ for all } t \leq T^*,$$
$$v'_t = v_t \text{ for all } t > T^*.$$

We first show that one can go from $\boldsymbol{v}$ to $\boldsymbol{v}'$ through a sequence of utility streams $\boldsymbol{z}^{(1)}, ..., \boldsymbol{z}^{(H)} \in X^N$ such that $\boldsymbol{z}^{(1)} = \boldsymbol{v}$, $\boldsymbol{z}^{(H)} = \boldsymbol{v}'$, and for all $t = 1, ..., H-1$, either
(Case 1) $z_t^{(h+1)} > z_t^{(h)}$ for all $t \leq T^*$, or
(Case 2) for two generations $t, t' \leq T^*$,

$$z_t^{(h)} > z_t^{(h+1)} > z_{t'}^{(h+1)} > z_{t'}^{(h)},$$

and for all other generations $t'' \leq T^*$, $z_{t''}^{(h+1)} > z_{t''}^{(h)}$.

We construct such a sequence of streams.[1] Let $m \leq T^*$ be a generation such that $v_m \leq v_t$ for all $t \leq T^*$. Define $S = \{t \leq T^* | \ v_t > v_m\}$ and $V(S) = \min_{t \in S}\{v_t\}$. Let $\epsilon > 0$ be such that

$$\epsilon < \frac{1}{T^*}\Big(\min\Big\{V(S),\ \hat{x}\Big\} - v_m\Big).$$

Let $H = |S| + 2$ and $s : \{1, ..., |S|\} \to S$ be a bijection. At every step $h = 1, ..., H-2$, let
(a) $z_t^{(h+1)} = v_m + (h+1)\epsilon$ for $t = s(h) \in S$,
(b) $z_{t''}^{(h+1)} = z(h)_{t''} + \epsilon$ for all $t'' \leq T^*$ such that $t'' \neq t$,

[1]The construction is similar to Fleurbaey (2005, proof of Theorem 3, Step 1). See also Miyagishima (2010).

(c) $\boldsymbol{z}^{(h)[+(T^*+1)]} = \boldsymbol{v}^{+(T^*+1)}$ for all h.

(In particular, $z_m^{(h+1)} = v_m + h\epsilon$.)

For $h = 1, ..., H-2$, the step from $\boldsymbol{z}^{(h)}$ to $\boldsymbol{z}^{(h+1)}$ corresponds to (Case 2) above with $t = s(h)$ and $t' = m$, since

$$z_m^{(h)} < z_m^{(h+1)} = v_m + h\epsilon < v_m + (h+1)\epsilon = z_t^{(h+1)} < z_t^{(h)}, \tag{2}$$

where the last inequality is derived from

$$v_m + (h+1)\epsilon < v_m + \frac{h+1}{T^*}[V(S) - v_m] \leq V(S) \leq z_t^{(h)}.$$

The last step from $\boldsymbol{z}^{(H-1)}$ to $\boldsymbol{z}^{(H)} = \boldsymbol{v}'$ corresponds to (Case 1). This is because, for all $t \leq T^*$,

$$z_t^{(H-1)} \leq v_m + (H-1)\epsilon \leq v_m + T^*\epsilon < \hat{x},$$

where the last inequality follows the assumption regarding ϵ above.

Next, we show $\boldsymbol{v}'R\boldsymbol{v}$. For $h = 1, ..., H-2$, by (2) and *HE*,

$$\big(z_t^{(h+1)}, z_m^{(h+1)}, \boldsymbol{z}_{-tm}^{(h)}\big) R \big(z_t^{(h)}, z_m^{(h)}, \boldsymbol{z}_{-tm}^{(h)}\big),$$

where $\boldsymbol{z}_{-tm}^{(h)}$ is the components of $\boldsymbol{z}^{(h)}$ other than t and m. By *WP* and *SC*,

$$\big(z_t^{(h+1)}, z_m^{(h+1)}, \boldsymbol{z}_{-tm}^{(h+1)}\big) R \big(z_t^{(h+1)}, z_m^{(h+1)}, \boldsymbol{z}_{-tm}^{(h)}\big).$$[2]

By transitivity, $\boldsymbol{z}^{(h+1)}R\boldsymbol{z}^{(h)}$. At the last step, by *WP* and *SC*, $\boldsymbol{z}^{(H)}R\boldsymbol{z}^{(H-1)}$. By transitivity, $\boldsymbol{z}^{(H)}R\boldsymbol{z}^{(1)}$, which means $\boldsymbol{v}'R\boldsymbol{v}$.

We now show (1). By repeating the same argument as above, $\big((\hat{x})_{con}^{-T}, \boldsymbol{v}^{+(T+1)}\big)R(\boldsymbol{v}^{-T}, \boldsymbol{v}^{+(T+1)})$ holds for all $T \geq T^*$. Hence, by *PC*, $(\hat{x})_{con}R\boldsymbol{v}$. On the other hand, by *WP*, $\boldsymbol{u}P(\hat{x})_{con}$. By transitivity, we obtain $\boldsymbol{u}P\boldsymbol{v}$.

From (1) and *SC*, we can easily show that, for any $\boldsymbol{u}, \boldsymbol{v} \in X^N$,

$$\inf_{i\in N} u_i = \inf_{i \in N} v_i \Longrightarrow \boldsymbol{u}I\boldsymbol{v}. \tag{3}$$

By (1) and (3), we have completed the proof. □

[2]When R satisfies *WP* and *SC*, it is easy to show that, for all $\boldsymbol{u}, \boldsymbol{v} \in X^N$, if $u_t \geq v_t$ for all $t \in N$ and $u_{t'} > v_{t'}$ for some $t' \in N$, then $\boldsymbol{u}R\boldsymbol{v}$.

4 Characterization in terms of liberal principle

In this section, we characterize the Rawlsian social ordering using a liberal principle as follows:

Weak Harm Principle (WHP) : For all $\boldsymbol{u}, \boldsymbol{v} \in X^N$, if $\boldsymbol{u}P\boldsymbol{v}$, and $\boldsymbol{u}', \boldsymbol{v}' \in X^N$ are such that, for some $t \in N$,

$$\begin{aligned} &u'_t < u_t, \\ &v'_t < v_t, \\ &u'_j = u_j \text{ for all } j \neq t, \\ &v'_j = v_j \text{ for all } j \neq t, \end{aligned}$$

then not$[\boldsymbol{v}'P\boldsymbol{u}']$ whenever $v'_t < u'_t$.

This axiom is introduced by Lombardi and Veneziani (2009). They argue that the axiom captures a liberal view of noninterference whenever individual choices have no effect on others.

Theorem 2 : An *ordering* R satisfies *Weak Harm Principle, Weak Pareto, Finite Anonymity, Sup Continuity*, and *Preference Consistency* if and only if $R = R_r$.

Proof: It is obvious that R_r satisfies the axioms in the Theorem. We show the necessity part. Suppose that an ordering R satisfies the axioms. We first prove that, for any utility vectors $\boldsymbol{u}, \boldsymbol{v} \in X^N$

$$\inf_{t \in N} u_t > \inf_{t \in N} v_t \Longrightarrow \boldsymbol{u}P\boldsymbol{v}. \tag{4}$$

Define

$$\hat{x} = \frac{\inf_{t \in N} u_t + \inf_{t \in N} v_t}{2}.$$

First, we show that, for all $\boldsymbol{\epsilon} \in \mathbb{R}^{\infty}_{++}$,

$$\exists T \geq 1, \forall t \geq T, \ \big((\hat{x})^{-t}_{con}, \ \boldsymbol{v}^{+(t+1)} + \boldsymbol{\epsilon}\big)R\boldsymbol{v}. \tag{5}$$

The proof is by contradiction. Suppose that, for all $T \geq 1$, there exists $t \geq T$ and $\boldsymbol{\epsilon}$ such that $\boldsymbol{v} P\big((\hat{x})^{-t}_{con}, \boldsymbol{v}^{+(t+1)} + \boldsymbol{\epsilon}\big)$. From $\hat{x} > \inf_{t \in N} v_t$, there exists $T^* < \infty$ such that $\hat{x} > v_{T^*}$. By assumption, for this T^*, there exists $t^* \geq T^*$ such that $\boldsymbol{v} P\big((\hat{x})^{-t^*}_{con}, \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}\big)$. Let $v_{[t]}$ be the t-th lowest utility in $\{v_t | t \leq t^*\}$. Denote

$$z = \min\Big\{\hat{x},\ \min\{v_t | t \leq t^*, v_t > v_{[1]}\}\Big\}.$$

(If $\{v_t | t \leq t^*, v_t > v_{[1]}\} = \phi$, we obtain (5) by *WP*.) Define $\tilde{\boldsymbol{v}} \in X^N$ be such that $\tilde{\boldsymbol{v}}^{+(t^*+1)} = \boldsymbol{v}^{+(t^*+1)}$, and for all $t \leq t^*$,

$$\tilde{v}_t = \frac{[t^* - (t-1)]v_{[1]} + (t-1)z}{t^*}.$$

Moreover, let $\tilde{\boldsymbol{u}} \in X^N$ be such that $\tilde{\boldsymbol{u}}^{+(t^*+1)} = \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}$, and

$$\begin{aligned} &\tilde{u}_1 = \hat{x}, \\ &\tilde{u}_t = \frac{\tilde{v}_{[t-1]} + \tilde{v}_{[t]}}{2} \text{ for } t = 2, ..., t^*. \end{aligned}$$

By *FA*, $(v_{[1]}, ..., v_{[t^*]}, \boldsymbol{v}^{+(t^*+1)}) I (v_1, ..., v_{t^*}, \boldsymbol{v}^{+(t^*+1)})$. Let $(v'_1, v'_2, ..., v'_{t^*}) = (v_{[1]}, v_{[t^*]}, ..., v_{[t^*]})$. By *WP* and *SC*,

$$(v'_1, v'_2, ..., v'_{t^*}, \boldsymbol{v}^{+(t^*+1)}) R (v_{[1]}, v_{[2]}, ..., v_{[t^*]}, \boldsymbol{v}^{+(t^*+1)})$$

Thus, by transitivity,

$$(v'_1, v'_2, ..., v'_{t^*}, \boldsymbol{v}^{+(t^*+1)}) P \big((\hat{x})^{-t^*}_{con}, \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}\big).$$

Moving $(v'_2, ..., v'_{t^*})$ and $(\hat{x}, ..., \hat{x})$ to $(\tilde{v}_2, ..., \tilde{v}_{t^*})$ and $(\tilde{u}_2, ..., \tilde{u}_{t^*})$ respectively, by repeated applications of *WHP* and completeness of R,

$$(\tilde{v}_1, ..., \tilde{v}_{t^*}, \boldsymbol{v}^{+(t^*+1)}) R (\tilde{u}_1, ..., \tilde{u}_{t^*}, \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}). \tag{6}$$

(Note that $\boldsymbol{v}^{+(t^*+1)} = \tilde{\boldsymbol{v}}^{+(t^*+1)}$ and $\tilde{\boldsymbol{u}}^{+(t^*+1)} = \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}$.) Next, consider $\pi \in \Pi$ such that

$$\begin{aligned} &\pi(t^*) = 1 \\ &\pi(t) = t + 1 \text{ for } t = 1, ..., t^* - 1, \\ &\pi(t) = t \text{ for all } t \geq t^* + 1. \end{aligned}$$

Then, since $\tilde{v}_t < \tilde{u}_{\pi(t)}$, from *WP*,

$$(\tilde{u}_{\pi(1)}, ..., \tilde{u}_{\pi(t^*)}, \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}) P (\tilde{v}_1, ..., \tilde{v}_{t^*}, \boldsymbol{v}^{+(t^*+1)}).$$

By *FA* and transitivity,

$$(\tilde{u}_1, ..., \tilde{u}_{t^*}, \boldsymbol{v}^{+(t^*+1)} + \boldsymbol{\epsilon}) P (\tilde{v}_1, ..., \tilde{v}_{t^*}, \boldsymbol{v}^{+(t^*+1)}). \tag{7}$$

From (6) and (7), we have a contradiction. Thus, we have proved (5).

Let $\boldsymbol{\epsilon} \to \mathbf{0}$. Then, by (5) and *SC*,

$$\exists T \geq 1, \forall t \geq T, \ \left((\hat{x})_{con}^{-t}, \ \boldsymbol{v}^{+(t+1)}\right) R \boldsymbol{v}. \tag{8}$$

From *PC*, $(\hat{x})_{con} R \boldsymbol{v}$. By *WP*, $\boldsymbol{u} P (\hat{x})_{con}$. By transitivity of R, $\boldsymbol{u} P \boldsymbol{v}$.

Next, by using the usual argument of From (4) and *Sup Continuity*, we can easily show that, for any $\boldsymbol{u}, \boldsymbol{v} \in X^N$,

$$\inf_{i \in N} u_i = \inf_{i \in N} v_i \Longrightarrow \boldsymbol{u} I \boldsymbol{v}. \tag{9}$$

By (4) and (9), we have completed the proof. □

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