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Characterizing the Boston Mechanism

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Characterizing the Boston Mechanism

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Abstract

This paper characterizes the Boston school choice mechanism using compact axioms along the following lines. First, a mechanism respects preference rankings and is weakly fair if and only if it is the Boston mechanism. Second, weak respect of preference rankings, efficiency, and weak fairness are sufficient to characterize the Boston mechanism. Third, a mechanism satisfies consistency, mutual best, and weak respect of preference rankings if and only if it is the Boston mechanism. Fourth, consistency and respect of top rankings are sufficient to characterize the Boston mechanism. Finally, this mechanism is characterized by double-standard stability, which explains its wide prevalence.

Keywords: Boston mechanism; Respect of preference rankings; Weak stability; Respect of top rankings; Double-standard stability

JEL classification: C78; D78; I21

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1 Introduction

The Boston mechanism is a well-known procedure that is commonly used in both school choice and student placement problems. It is used in many practical situations worldwide. For example, several school districts in the U.S. use this mechanism. In China, a numerically constrained version of the Boston mechanism was used for student placement for more than thirty years after the resumption of Gaokao.[1] In the Boston mechanism, a student's ranking of a school strongly influences his chance of being assigned to that school. Specifically, each school accepts students who rank it as their first choice and it only accepts students who rank it as their second choice when there are seats left. Loosely speaking, the Boston mechanism attempts to assign the maximum possible number of students to their first-choice schools and only when all such assignments have been finished does it consider assigning students to their second-choice schools. If a student is not admitted to his first-choice school, his second-choice school may be filled with students who have listed it as their first choice. This feature may cause strategic behavior when reporting the preference lists; in other words, one problem with the Boston mechanism is that it is not strategy-proof. A graver problem is that it is easy to manipulate, because a student can increase his probability of getting into a school simply by ranking that school higher in his preference order.

Abdulkadiroğlu and Sönmez's (2003) study, which led to renewed interest in the design and study of school choice mechanisms, discusses some serious shortcomings of the Boston mechanism that is used in several school districts in the US. They proposed two alternatives, namely the student-optimal stable mechanism (SOSM) and the top trading cycle mechanism (TTCM). They advocated the SOSM because in addition to deriving the most efficient stable matching, this mechanism is strategy-proof. At first glance, the SOSM seems to solve the problem of the Boston mechanism. The TTCM is strategy-proof and efficient. Therefore, it seems to solve the strategic disadvantage of the Boston mechanism. Thus, these two mechanisms are considered to be promising alternatives to the Boston mechanism.

[1] Gaokao, namely the National Higher Education Entrance Examination, is an academic examination held annually in the mainland of the People's Republic of China.

This paper provides several new axiomatic characterizations of the Boston mechanism, which should help in understanding its advantages and comparing it with other mechanisms such as the SOSM and TTCM. Kojima and Ünver (2010) was the first to characterize the Boston mechanism. Their first result shows that a mechanism is the Boston mechanism induced by some priority profile if and only if the mechanism respects preference rankings and satisfies consistency, resource monotonicity, and rank-respecting Maskin monotonicity. Their second result shows that a mechanism is the Boston mechanism induced by some priority profile if and only if the mechanism respects preference rankings and satisfies individual rationality, population monotonicity, and rank-respecting Maskin monotonicity. While their characterizations consider consistency and respect of preference rankings, the role that fairness plays has been ignored. Their results also do not focus on the dynamic and recursive feature of the Boston mechanism. To further extend and deepen their research, this paper characterizes the Boston mechanism along the following lines.

Our first characterization shows that the Boston mechanism is the only mechanism that respects preference rankings and is weakly fair at the same time. A mechanism respects preference rankings if whenever an acceptable student prefers a school to the school assigned by the mechanism, all the seats of the former are allocated to students who rank it at least as high as the initial student. Second, weak fairness means that if a student likes the assignment of another student more and he puts this school in the same preference ranking, the latter student should have higher priority than the former one. This characterization tells us that the Boston mechanism tries to assign students who value a school as high as possible to this school, and when more students than the quota apply to this school, rejections are made on the basis of priorities, i.e., conflicts are solved fairly. Respect of preference rankings can be decomposed into two axioms: the boundary condition and weak respect of preference rankings. The boundary condition ensures that students have the incentive to participate in the game and have the right to leave at any time. We say that a mechanism weakly respects preference rankings if whenever an acceptable student prefers a school to the school assigned by the mechanism, all the students who are assigned to it rank the school not lower than the initial student. As corollaries of the first theorem, we show that the Boston mechanism is the only mechanism that satisfies the boundary condition and weak respect of preference rankings and that is weakly fair.

Our second result characterizes the Boston mechanism using axioms of weak fairness, efficiency, and weak respect of preference rankings. Efficiency, which is implied by respect of preference rankings, requires that there is no matching that makes every student weakly better off and at least one student strictly better off. It is well known that the TTCM also satisfies efficiency. However, the TTCM does not satisfy weak fairness and does not weakly respect preference rankings, although it is group strategy-proof and weak Maskin-monotonic. Therefore, substituting the Boston mechanism with the TTCM will cause stability and efficiency loss because respect of preference rankings is based on welfare considerations.

Third, we show that a mechanism satisfies consistency, mutual best, and weak respect of preference rankings if and only if it is the Boston mechanism. A mechanism is consistent if whenever we fix the assignment of a student at the mechanism's outcome and reapply the mechanism for the remaining set of students with one less seat at the fixed school of the original student, all remaining students are assigned to the same school with the mechanism's original outcome. A matching satisfies mutual best if when a student likes a school the most and the school also prefers this student to the others, this student and this school are matched together. Mutual best by itself is a very weak property. However, together with consistency, they imply both the boundary condition and weak fairness. Therefore, consistency captures the dynamic and recursive feature of a mechanism and the effect of some weak properties can be extended by it, which is reconfirmed in the next result.

Our fourth result shows that a mechanism satisfies consistency and respect of top rankings if and only if it is the Boston mechanism. Respect of top rankings means that if a student is acceptable to but not assigned to his top choice, this implies that the top school has been fully occupied by students who put this school as their top choice and have higher priority than the original student. Respect of top rankings is one of the main features of the Boston mechanism, which considers that the first choices of students are the most important.

Finally, double-standard stability by itself is sufficient to characterize the Boston mechanism. Double-standard fairness means that if a student likes the assignment of another student more, then the latter student should either put the school in a higher preference ranking or in the same preference ranking, and the latter student has higher priority for the school. Double-standard stability is simply a rearrangement of existing axioms. We mention it here because it is useful for explaining the wide prevalence of the Boston mechanism.

1.1 Literature Review

Many studies have already discussed the advantages and disadvantages of the Boston mechanism. Balinski and Sönmez (1999) was the first to discuss school choice or student placement problems. Abdulkadiroğlu and Sönmez (2003) was the first to recognize the problems in the Boston mechanism. They argued that because the SOSM is both stable and strategy-proof, it is a suitable alternative to the Boston mechanism. Their paper led to renewed interest in the study of school choice problems and the replacement of the Boston mechanism by the SOSM in public schools of the Boston district. Following their study, many studies further investigate the disadvantages of the Boston mechanism. Ergin and Sönmez (2006) found that under the preference revelation game of the Boston mechanism, the set of Nash equilibrium outcomes is equal to the set of stable matchings measured by the true preferences of students. Chen and Sönmez (2006) confirmed the manipulation property of the Boston mechanism in an experimental environment.

Interestingly, recently, many researchers have started to focus on the advantages of the Boston mechanism. Featherstone and Niederle (2008) described some special environments in which truth-telling is an equilibrium under the Boston mechanism and the Boston mechanism can first-order stochastically dominate the SOSM in terms of efficiency, both in theory and in the laboratory. Pathak and Sönmez (2008) showed that the Boston mechanism favors sophisticated students over sincere ones. Haeringer and Klijn (2009) proved that when students face a numerical constraint when reporting their preference lists, Nash equilibrium outcomes are all stable matchings without any restriction on the priority structure under the Boston mechanism, whereas the SOSM and TTCM are not problem-free in this sense. Abdulkadiroğlu, Che and Yasuda (2011) found that under the environment that parents tend to have similar preferences over schools and schools have at best coarse priorities, the Boston mechanism possesses several advantages over the SOSM from the viewpoint of solving conflicts. Zhong and Wu (2011) examined how the matching quality between students and top universities in China is affected by different student placement mechanisms, and their results showed that under the Boston mechanism students with lower scores are selected but they exhibit the same or even better academic performances in colleges than under the SOSM.

The remaining of this paper is organized as follows. Second 2 introduces the basic school choice model and three competing properties for a school choice mechanism. Section 3 presents a description of the Boston mechanism and several revised versions of it. Section 4 presents some axioms. Section 5 characterizes the Boston mechanism. Section 6 presents the conclusions of this study.

2 The model

2.1 Preliminaries

Let I and C denote finite sets of students and schools. Let q be the quota vector associated with each school. To simplify the description, we assume that each school has only one seat, i.e., $q_c = 1$ for any $c \in C$. However, the argument can be easily generalized if schools have more than one seat. For a student, being unmatched is denoted as being matched to the null school $\emptyset_c$. For a school, being unmatched is denoted as being matched to the null student $\emptyset_i$. The quota for null schools and null students is infinite. Each student $i \in I$ has a single unit demand with a strict (complete, transitive, and antisymmetric) preference order P_i over $C \cup \emptyset_c$. Let $\mathcal{P}$ denote the set of all strict orders over $C \cup \emptyset$. The **preference profile** of students denoted by $P = (P_i)_{i \in N} \in \mathcal{P}^{|I|}$ is a vector of linear orders. Let $P_i(c)$ be the ranking of school c at P_i, i.e., if school c is the l^{th} choice of student i under P_i, then $P_i(c) = l$. Then, for any $c, d \in C \cup \{\emptyset_c\}$, $P_i(c) < P_i(d)$ if and only if cP_id.

Each school $c \in C$ has a single unit supply with a strict (complete, transitive, and antisymmetric) priority order $\succeq_c$ over students. A **priority structure** is a vector of strict orders over students $\succeq = (\succeq_c)_{c \in C}$, whereas $i_1 \succeq_c i_2$ means that student i_1 has higher priority than student i_2 at school c. The priority order of the null school is defined as follows: for $\forall i \in I$, $i \succeq_{\emptyset_c} j$ for every $j \in I \backslash \{i\}$. A **matching** is a function $\mu : I \longrightarrow C \cup \{\emptyset_c\}$, with one student consuming at most one school seat and a school seat consumed by at most one student. For all $i \in I$, μ_i denotes the set of schools assigned to student i for all $i \in I$. For all $c \in C$, we stet $\mu_c = \{i \in I : \mu_i = c\}$ for the student assigned to school c. For a set of students $I^{'} \subset I$, $\mu_{I^{'}}$ denotes the set of schools that are assigned to $I^{'}$. Similarly, for a set of schools $C^{'} \subset C$, $\mu_{C^{'}}$ denotes the set of students who are assigned to this set of schools. For a school $c \in C$, I_c denotes the set of students who are acceptable to school c, i.e., for any $i \in I_c$, $i \succeq_c \emptyset_i$.

A school choice **problem** is denoted by $\mathbb{P} = (I, C, q, P, \succeq)$. For simplicity, we occasionally denote a school choice problem as $\mathbb{P}$. A school choice mechanism is a systematic procedure that assigns a matching for each school choice problem. Denote the set of matchings as $\mathcal{M}$ and the set of preference profiles of students as $\mathcal{P}^{|I|}$. A **mechanism** of the school choice problem is a function $\phi : \mathcal{P}^{|I|} \rightarrow \mathcal{M}$ that finds a matching for every school choice problem.

2.2 Three central properties

A mechanism is strategy-proof if no student can benefit by unilaterally misrepresenting his preference. In other words, strategy-proofness means that reporting the true preferences is a dominant strategy for every student in the preference revelation game induced by this mechanism. For any $i \in I$, denote the true preference of student i as P_i. Next, we give the formal definition of strategy-proofness.

Strategy-proofness: A mechanism ϕ is strategy-proof if for any $P_i^{'} \in \mathcal{P}$, $\phi(I, C, q, P_i, P_{-i}, \succeq) P_i \phi(I, C, q, P_i^{'}, P_{-i}, \succeq)$ for any $i \in I$.

The TTCM further satisfies group strategy-proofness. Group strategy-proofness is equivalent to strategy-proofness and nonbossiness. Therefore, if our criterion is strategy-proofness, then the TTCM outperforms the SOSM, and the SOSM outperforms the

Boston mechanism.

It is obvious that the central notion in school choice problems is stability. Stability is a typical feature of the SOSM. Ergin (2002) defined stability using individual rationality and the elimination of a blocking pair. Balinski and Sönmez (1999) and Haeringer and Klijn (2009) defined the stability of a mechanism using three axioms: non-wastefulness, fairness, and individual rationality. Their definitions originated from the presupposition of an acceptant priority structure. Our study assumes the most general priority that does not require acceptance consistent with Ergin (2002). In this study, we follow both methods and give the definition of stability. Note that I_c stands for the set of students who are acceptable to school c.

Stability: A mechanism is stable if it always selects the matching μ that

- is **individually rational**, i.e., $\mu_i R_i \emptyset$ for all $i \in I$.
- is **non-wasteful**, i.e., for any $c \in C$ and $i \in I_c$, $cP_i\mu_i$ implies that $|\mu_c| = q_c$.
- is **fair**, i.e., for any $i, j \in I$ with $\mu_j \in C$, $\mu_j P_i \mu_i$ implies that $j \succeq_{\mu_j} i$.

Note that the definition of non-wastefulness differs slightly from Balinski and Sönmez (1999) and Haeringer and Klijn (2009) because we specify that $i \in I_c$. In fact, our definition is more general. When the priority is acceptant, namely, each school has the right to reject a student unless it has no seat left, $I_c = I$ for any $c \in C$. In this case, non-wastefulness in our study naturally deteriorates into the same as that in Balinski and Sönmez (1999) and Haeringer and Klijn (2009). If a mechanism satisfies individual rationality and non-wastefulness, we say that it satisfies the **boundary condition**. Naturally, a mechanism is stable if and only if it is fair and satisfies the boundary condition. We easily conclude that the SOSM, TTCM, and Boston mechanism all satisfy the boundary condition. The boundary condition ensures that no student is forced to go to a school that causes him negative utility and that school seats are fully occupied unless no acceptable student desires it any longer. Fairness guarantees that there is no justified envy among students. If one student is rejected from a school and the other one is not, then the latter should have higher priority for this school. The SOSM is stable. Later, we prove that the Boston mechanism satisfies a weaker version of stability that is not satisfied by the TTCM. Therefore, if our criterion is stability, the SOSM outperforms the Boston mechanism, and the Boston mechanism outperforms the TTCM.

A matching is efficient if there is no other matching that assigns each student a weakly better school and at least one student a strictly better school and no unacceptable student is assigned to any school. In other words, a matching is efficient if it is impossible to make a student better off without making another student worse off subject to the condition that schools are not forced to accept students whom they find unacceptable.

Efficiency: A matching μ is efficient if there exist no other matchings $\mu^{'} \in \mathcal{M}$ such that $\mu^{'}_i R_i \mu_i$ for all $i \in I$ and $\mu^{'}_j P_j \mu_j$ for some $j \in I$ and $\mu_c \in I_c$ for any $c \in C$.

A mechanism ϕ is efficient if it always selects an efficient matching. Note that the definition of efficiency here is the same as the definition of constrained efficiency in Kojima and Ünver (2010) and different from the usual definition of efficiency because we specify that $i \in I_c$. In fact, our definition is more general. Under acceptant priorities, $I_c = I$ for any $c \in C$ and (constrained) efficiency deteriorates into the usual efficiency. It is easy to check that efficiency implies the boundary condition. The Boston mechanism and the TTCM are both efficient. However, the Boston mechanism satisfies a stronger version of efficiency that is respect of preference rankings. Therefore, if our criterion is efficiency, then the Boston mechanism outperforms the TTCM, and the TTCM outperforms the SOSM.

Stability and efficiency are two desirable properties of a mechanism. Unfortunately, no mechanism is simultaneously stable and efficient in school choice settings (Balinski and Sönmez (1999)). Two competing mechanisms have received much attention in school choice problems. The first is the SOSM, which is strategy-proof and stable. The second is the TTCM, which is strategy-proof and efficient. Neither of them are simultaneously stable and efficient. In other words, there is an unambiguous tradeoff between stability and efficiency. Because both stability and efficiency imply the boundary condition, we can conclude that there is an unambiguous tradeoff between fairness and efficiency.

3 The Boston mechanism

3.1 The Boston mechanism

Given a school choice problem $\mathbb{P}$, the Boston mechanism, denoted by β, determines a matching $\beta(\mathbb{P})$ through the following algorithm:

STEP 1: Only the first choices of the students are considered. For each school, consider the students who listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there is no seat left or there is no student left who has listed it as his first choice and is acceptable to the school.

$\vdots$

STEP k: Consider the remaining school seats and students. Only the k^{th} choices of students are considered. For each school, consider the remaining students who listed it as their k^{th} choice and assign the remaining seats of the school to these students one at a time following their priority order until there is no seat left or there is no student left who has listed it as his k^{th} choice and is acceptable to this school.

The algorithm terminates when all students have been assigned to a seat or rejected by all school seats or all school seats have been removed. All the remaining students remain unassigned.

3.2 The first revised Boston mechanism

Given a school choice problem $\mathbb{P}$, the first revised Boston mechanism, denoted by β^I, determines a matching $\beta^I(\mathbb{P})$ through the following algorithm:

STEP 1: Only the first choices of the students are considered. For each school, consider the students who listed it as their first choice and assign seats of the school to these students. If less students than the quota of a school apply for this school, assign these students to this school. If more students than the quota of a school apply for this school, assign students who have the highest priority among all applicants to this school until there is no seat left or there is no student left who has listed it as his first choice and is acceptable to the school.

The mechanism terminates after the first step. Assign all the remaining students to the null school. The first revised Boston mechanism β^I is in fact the one-step-only Boston mechanism.

3.3 The second revised Boston mechanism

Given a school choice problem $\mathbb{P}$, the second revised Boston mechanism, denoted by β^{II}, determines a matching $\beta^{II}(\mathbb{P})$ through the following algorithm:

STEP 1: Only the first choices of the students are considered. For each school, consider the students who listed it as their first choice and assign seats of the school to these students. If less students than the quota of a school apply for this school, assign these students to this school. If more students than the quota of a school apply for this school, assign students who have **the lowest priority** among all applicants to this school until there is no seat left or there is no student left who has listed it as his first choice and is acceptable to the school.

⋮

STEP k: Consider the remaining school seats and students. Only the k^{th} choices of students are considered. For each school, consider the remaining students who listed it as their k^{th} choice and assign the remaining seats of the school to these students. If less students than the quota of a school apply for this school, assign these students to this school. If more students than the quota of a school apply for this school, assign students who have **the lowest priority** among all applicants to this school until there is no seat left or there is no student left who has listed it as his k^{th} choice and is acceptable to the school.

The algorithm terminates when all students have been assigned to a seat or rejected by all school seats or all school seats have been removed. All the remaining students remain unassigned. The second revised Boston mechanism β^{II} differs from β in solving conflicts. If more students than the quota of a school apply for this school, β solves conflicts fairly, i.e., according to the priority order of this school. However, β^{II} does not solves conflicts fairly.

3.4 The third revised Boston mechanism

Given a school choice problem $\mathbb{P}$, the third revised Boston mechanism, denoted by β^{III}, determines a matching $\beta^{III}(\mathbb{P})$ through two stages. In the first stage, we find all the possible mutual best pairs of students and schools and remove them. Then, we apply the Boston mechanism to the subproblem without the mutual best pairs. The matching is derived through the following algorithm:

STAGE 1: Find all the possible mutual best pairs of students and schools and remove them.

STAGE 2: Apply the Boston mechanism β to the subproblem in which all mutual best pairs of students and their assignment are removed.

The algorithm terminates when all students have been assigned to a seat or rejected by all school seats or all school seats have been removed. All the remaining students remain unassigned.

3.5 A numerical example

The following example might be helpful in demonstrating how the Boston mechanism and its three revised versions work.

EXAMPLE 1 Let $I = \{i_1, i_2, i_3\}$, $C = \{c_1, c_2, c_3\}$, and the quota of each school is one. The preferences of students and the priority profiles are listed below:

P_{i_1}	P_{i_2}	P_{i_3}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$
c_1	c_1	c_3	i_1	i_3	i_1
c_2	c_3	c_2	i_2	i_2	i_2
c_3	c_2	c_1	i_3	i_1	i_3

The algorithm of the Boston mechanism results in the following matching:

$$\beta(I, C, q, P, \succeq) = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

The algorithm of the first revised Boston mechanism results in the following matching:

$$\beta^I(I, C, q, P, \succeq) = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & \emptyset_c & c_3 \end{pmatrix}$$

The algorithm of the second revised Boston mechanism results in the following matching:

$$\beta^{II}(I, C, q, P, \succeq) = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_2 & c_1 & c_3 \end{pmatrix}$$

The algorithm of the third revised Boston mechanism results in the following matching:

$$\beta^{III}(I, C, q, P, \succeq) = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & c_3 & c_2 \end{pmatrix}$$

4 Axioms

4.1 Respect of preference rankings

This section presents some useful axioms to characterize the Boston mechanism. As a main feature of the Boston mechanism, respect of preference rankings was first formalized in Kojima and Ünver (2010). In our study, for a school $c \in C$, I_c is defined as the set of students who are acceptable to school c, i.e., $i \succeq_c \emptyset_i$. The formal definition of respect of preference rankings is given as follows.

Respect of preference rankings: (Kojima and Ünver (2010)) A matching μ respects preference rankings if for all $c \in C \cup \{\emptyset\}$ and $i \in I_c$, $cP_i\mu_i$ implies that $|\mu_c| = q_c$

and $P_j(c) \leq P_i(c)$ for all $j \in \mu_c$.

We say that a mechanism respects preference rankings if whenever an acceptable student prefers a school to the school assigned by the mechanism, all the seats of the former are allocated to students who rank it at least as high as the initial student. Intuitively, the social planner respecting preference rankings tries to assign school seats to students who value them as high as possible. Surprisingly, we can describe it using only two compact axioms: the boundary condition and weak respect of preference rankings.

Boundary condition: A matching μ satisfies the boundary condition if it is individually rational and non-wasteful.

Intuitively speaking, the boundary condition ensures that students have the incentive to participate in the game and have the right to leave at any time. The boundary condition is a very weak condition, and the three well-known school choice mechanisms mentioned in Abdulkadiroğlu and Sönmez (2003), i.e., the SOSM, TTCM, and Boston mechanism, all satisfy it.

Weak respect of preference rankings: A matching μ weakly respects preference rankings if for all $c \in C \cup \{\emptyset\}$ and $i \in I_c$, $cP_i\mu_i$ implies that $P_j(c) \leq P_i(c)$ for all $j \in \mu_c$.

Weak respect of preference rankings is weaker than respect of preference rankings. We say that a mechanism weakly respects preference rankings if whenever an acceptable student prefers a school to the school assigned by the mechanism, all the students who are assigned to the former school rank the school not lower than the initial student. Intuitively, the social planner, who chooses a mechanism that weakly respects preference rankings, tries to assign school seats to students who value them relatively higher. We can easily imagine that null matching trivially satisfies this property. The Boston mechanism weakly respects preference rankings. Neither the SOSM nor TTCM satisfies this property. Therefore, weak respect of preference rankings is one of the main features of the Boston mechanism. The following proposition discusses the relationship between the respect of preference rankings and the weak respect of preference rankings.

Proposition 1. *A mechanism ϕ respects preference rankings if and only if it satisfies the boundary condition and weakly respects preference rankings.*

Proof. The first part of respect of preference rankings shows that if for all $c \in C \cup \{\emptyset\}$ and $i \in I_c$, $cP_i\mu_i$ implies that $|\mu_c| = q_c$. The second part is equivalent to weak respect of preference rankings. We only need to prove that the first part is equivalent to the boundary condition. It is easy to reach this conclusion because first, we assume that the preference profile and priority structure consist of complete orders and second, we assume that the null school and student have infinite supply. □

Because the Boston mechanism respects preference rankings, according to proposition 1, it satisfies the boundary conditions and weakly respects preference rankings. Respect of preference rankings is an axiom related to efficiency. Kojima and Ünver (2010) proved that respect of preference rankings implies efficiency only when the priority profile is acceptant. In their setting, respect of preference rankings implies another (weaker) s-tandard of efficiency, namely constrained efficiency. In our setting, respect of preference ranking implies efficiency because the definition of efficiency in this paper corresponds to that of constrained efficiency in Kojima and Ünver (2010). We refer to the proof of proposition 1 in Kojima and Ünver (2010) for the proof of this conclusion because the process is similar.

Proposition 2. *(Kojima and Ünver (2010))If a mechanism respects preference rankings, then it is efficient.*

Proposition 2 tells us that respect of preference rankings is stronger than efficiency. In other words, respect of preference rankings is a sufficient condition for efficiency. The reverse is not true. The following propositions establish the relationship between efficiency and respect of preference rankings.

Proposition 3. *If a mechanism is efficient, then it satisfies the boundary condition.*

Proposition 4. *Efficiency and weak respect of preference rankings imply respect of preference rankings.*

Proof. Propositions 1, 2, and 3 complete the proof. □

4.2 Weak fairness

Observing the procedure of the Boston mechanism shows that it tries to assign the maximum number of students to their first choice. However, if too many students select one school as their first choice such that some of them will need to be rejected by this school, conflicts arise. How to solve conflicts is another main feature of the Boston mechanism. Similar to the SOSM, rejections are based on the priority rankings of these students. When conflicts arise, the Boston mechanism solves them fairly. However, fairness in this case is weaker than that in the SOSM because fairness is possible only when conflicts arise. Thus, we derive another important property of the Boston mechanism, i.e., weak fairness.

Weak fairness: A matching μ is weakly fair if for any $i, j \in I$ with $\mu_j \in C$, $\mu_j P_i \mu_i$ and $P_i(\mu_j) = P_j(\mu_j)$ imply $j \succeq_{\mu_j} i$.

Weak fairness implies that a mechanism solves a conflict fairly, i.e., according to the priority orders. It means that if a student likes the assignment of another student more and both of them put the school in the same preference ranking, then the latter should have higher priority for this school. Null matching trivially satisfies this property. We can easily check that the TTCM does not satisfy weak fairness. In this sense, weak fairness is an advantage of the Boston mechanism compared with the TTCM. Interestingly, respect of preference rankings and weak fairness are sufficient to characterize the Boston mechanism. As a corollary, weak respect of preference rankings and weak stability are also sufficient to characterize the Boston mechanism.

It is easy to see that fairness implies weak fairness.

Proposition 5. *If a mechanism ϕ is fair, then it is weakly fair.*

Balinski and Sönmez (1999) showed that no mechanism is stable and efficient simultaneously, i.e., stability and efficiency are incompatible in school choice settings. Because both stability and efficiency imply the boundary condition, we can conclude that efficiency and fairness are incompatible. Then, a natural question to ask is whether we can find a weaker version of stability or fairness that is compatible with efficiency. The answer is yes, and we prove that weak fairness and efficiency are compatible and that the Boston mechanism satisfies these two simultaneously.

Proposition 6. *Efficiency and weak fairness are compatible.*

4.3 Consistency and mutual best

We require additional notations to define consistency. For any $i \in I$, denote $\mathcal{P}^{\emptyset_c}$ as the set of preference orders that rank the null school $\emptyset_c$ as the first choice.

Consistency: A mechanism ϕ is consistent if $\phi_j(I, C, q_{\phi_i(I,C,q,P,\succeq)}-1, q_{-\phi_i(I,C,q,P,\succeq)}, P_i^{\emptyset_c}, P_{-i}, \succeq) = \phi_j(I, C, q, P, \succeq)$ for any $j \neq i$.

A mechanism is consistent if whenever a student is removed from the problem with his assignment, the assignment for each remaining student remains the same. As noted in Ergin (2002), consistent mechanisms are coherent in their outcomes for problems involving different groups of students and are robust to nonsimultaneous assignment of the school seats. On the other hand, consistency captures the dynamic and recursive essence of a mechanism. Thus, even a small advantage of this mechanism will be expanded or multiplied to a large extent because the matching of every subproblem should also satisfy the advantageous property and do so recursively.

Mutual best is a really weak condition.

Mutual best: A matching μ satisfies mutual best if for every $i \in I$ and $c \in C \cup \emptyset_c$ such that cP_id for every $d \in C \cup \emptyset_c \backslash \{c\}$ and $i \succ_c j$ for every $j \in I \backslash i$, then $\phi(\mathbb{P})|_i = c$.

Mutual best means that if a student likes a school most and the school also prefers this student to the others, this student and this school should be matched together. A mechanism is mutually best if it always chooses a matching that is mutually best. Mutual best is a very weak condition. It can be considered to be a weaker version of stability and efficiency. The SOSM, TTCM, and Boston mechanism all satisfy it. However, by combining mutual best and consistency, we can derive the boundary condition and weak fairness. This is because consistency captures the dynamic and recursive essence of a mechanism, and even a weak property can be enlarged by it.

Proposition 7. *If a mechanism ϕ satisfies consistency and mutual best, then it satisfies the boundary condition.*

Proof. Suppose that there exists a matching μ that satisfies consistency and mutual best but that is not individually rational, i.e., there exists a student $i \in I$ such that $\emptyset_c P_i \phi_i(\mathbb{P})$. Remove all the other students except for student i and apply ϕ to the remaining students and schools. According to the consistency of ϕ, the assignment of student i will not change. At this time, the null school has infinite supply, $i \succeq_{\emptyset_c} j$ for any $j \in I$, and $\emptyset_c P_i \phi_i(\mathbb{P})$. Thus, we derive a contradiction to the assumption that μ satisfies mutual best.

Suppose that there exists a mechanism ϕ that satisfies consistency and mutual best but that is not non-wasteful, i.e., there exists a student $i \in I_c$ such that $c R_i \phi(\mathbb{P})_i$ and $|\mu_c| < q_c$. Remove the students who have higher priority for school c than student i and their corresponding assignment. Then, school c likes student i most at this time and it has extra quota. Remove schools that student i likes more than school c and their corresponding assignment. Then, student i likes school c most at this time. Because ϕ satisfies consistency, the matching in this step should be the same as in the original problem, namely, $\mu_i \neq c$. This contradicts mutual best of μ. □

Proposition 8. *If a matching μ satisfies consistency and mutual best, then it is weakly fair.*

Proof. Suppose that there exists a matching μ that satisfies consistency and mutual best but that is not weakly fair, i.e., there exist $i, j \in I$ with $\mu_j \in C$, $\mu_j P_i \mu_i$ and $P_i(\mu_j) = P_j(\mu_j)$ but $i \succeq_{\mu_j} j$. Remove students who have higher priority for school μ_j than student i. Remove schools that student i and j like more than school μ_j. Then, we consider the subproblem induced by the previous removal of students and schools. Because μ satisfies consistency, we conclude that μ_j is assigned to student j. However, in this subproblem, school μ_j likes student i most and student i at the same time likes school μ_j most. Assigning school μ_j to student j contradicts the mutual best of μ. □

4.4 Respect of top rankings

Respect of preference rankings is one of the main features of the Boston mechanism and together with weak fairness, it characterizes the Boston mechanism. Someone might ask whether it is possible to discard the property of weak respect of preference rankings and characterize the Boston mechanism completely without it. It seems impossible or difficult at first glance. However, we have found some suitable axioms that can successfully resolve this issue. Coincidentally, the new axioms to be discussed reflect another main feature or motivation of the Boston mechanism, namely, respect of top rankings.

Respect of top rankings: A matching μ respects top rankings if for each $c \in C \cup \{\emptyset_c\}$ and $i \in I_c$ such that $P_i(c) = 1$, $\mu_i \neq c$ implies that

1. $|\mu_c| = q_c$
2. $P_{\mu_c}(c) = 1$
3. $\mu_c \succeq_c i$

Respect of top rankings means that if a student acceptable to his top choice is not assigned his top choice, this implies that the quota of the top school has been fully occupied by students who also put this school as the top choice and have higher priorities than him. Note that respect of top rankings is one of the main motivations of the Boston mechanism. The Boston mechanism considers that the first choice of each student is the most important, and it tries to assign the maximum possible number of students to their top choices and only considers their second choice when it is impossible to do so. This is not satisfied by the SOSM and TTCM. On the other hand, this is not a very strong requirement. For instance, the one-step-only Boston mechanism, which leaves all students who are rejected in the first step unmatched, trivially respects top rankings. Respect of preference rankings puts a restriction on the matching in which the maximum possible number of students are assigned to their top preference rankings. If more students than the quota apply for a school, rejections should be made based on priorities. Respect of top rankings implies mutual best, and the result is formalized in the following proposition.

Proposition 9. *If a mechanism ϕ respects top rankings, then it satisfies mutual best.*

Proposition 10. *If a mechanism ϕ satisfies consistency and respect of top rankings, then it weakly respects preference rankings.*

Proof. Suppose that there exists a matching μ that satisfies consistency and respect of top rankings, but does not weakly respect preference rankings, i.e., there exist $i, j \in I$ with $\mu_j \in C$, $\mu_j P_i \mu_i$ and $P_i(\mu_j) < P_j(\mu_j)$. Consider the students who are rejected by their first $P_i(\mu_j) - 1$ choices and remove schools that have been assigned a student. Now, only the $P_i(\mu_j)^{th}$ choice of the students is considered. At this time, because μ satisfies consistency, we know that the assignment of i and j are now μ_i and μ_j, respectively. However, at this time, student i puts school μ_j in the first place because all other schools have been removed and school μ_j has an extra seat. This contradicts the assumption that μ respects top rankings. □

Proposition 11. *If a mechanism ϕ satisfies consistency and respect of top rankings, then it satisfies the boundary condition.*

Proof. Propositions 7 and 9 complete the proof. □

Proposition 12. *If a mechanism ϕ satisfies consistency and respect of top rankings, then it is weakly fair.*

Proof. Propositions 8 and 9 complete the proof. □

4.5 Double-standard fairness

We combine the definition of weak respect of preference rankings, weak fairness, and the boundary condition to define double-standard stability.

Double-standard fairness: A matching μ is double-standard fair if for any $i, j \in I$ with $\mu_j \in C$, $\mu_j P_i \mu_i$ implies that either of the following two conditions is satisfied:
(i) $P_j(c) < P_i(c)$.
(ii) $P_i(\mu_j) = P_j(\mu_j)$ and $j \succ_{\mu_j} i$.

Double-standard stability: A matching μ is double-standard stable if it is double-standard fair and it satisfies the boundary condition.

Double-standard fairness means that if a student likes the assignment of another student more, then the latter student either puts the school in a higher preference ranking or he puts the school in the same preference ranking and he has higher priority. As mentioned, double-standard fairness and double-standard stability are defined merely by rearranging some statements. This is mentioned here because double-standard stability can at least partially explain the wide prevalence of the Boston mechanism. It is well known that the Boston mechanism is not stable. However, it is applied to many practical problems such as school choice problems in the United States and student placement settings in China. Because of the double-standard stability of this mechanism, no student has the incentive to rebel about the final matching because the result is derived through a fair procedure based on specific and pressured rules such as respect of preference rankings and weak fairness. The Boston mechanism respects preference rankings first, and when conflicts arise, the mechanism allocates students fairly. There is no justified envy among students if we use the standard or criteria of both preference rankings and priority orders. In this double-standard sense, the Boston mechanism is "stable".

5 Characterizing the Boston mechanism

Kojima and Ünver (2010) were the first to characterize the Boston mechanism. Their results are summarized in the following propositions in order to make a clear comparison with our own results.

Proposition 13. *(Kojima and Ünver (2010)) A mechanism is the Boston mechanism induced by some priority profile if and only if the mechanism respects preference rankings and satisfies consistency*[2]*, resource monotonicity*[3]*, and rank-respecting Maskin monotonicity*[4]*.*

[2]A mechanism is consistent if whenever a student is removed from the problem with his assignment, the assignment for each remaining student remains unchanged.

[3]A mechanism is resource monotonic if whenever the supply of non-trivial school seats increases, the matching derived by this mechanism makes each student weakly better off than the original matching.

[4]A mechanism is ranking-respecting Maskin monotonic if the matching derived by it remains unchanged when students promote the rankings of their original assignments, as long as doing so does not

Proposition 14. *(Kojima and Ünver (2010)) A mechanism is the Boston mechanism induced by some priority profile if and only if the mechanism respects preference rankings and satisfies individual rationality, population monotonicity*[5]*, and rank-respecting Maskin monotonicity.*

Following Kojima and Ünver (2010), our paper further characterizes the Boston mechanism in the subsequent theorems.

Theorem 1. *A mechanism ϕ is weakly fair and respects preference rankings if and only if it is the Boston mechanism.*

The proof of theorem 1 appears in the appendix. The SOSM is weakly fair but violates respect of preference rankings. The second revised Boston mechanism β^{II}, stet solves conflicts unfairly, respects preference rankings but violates weak fairness. The previous statement establishes the independence of axioms in theorem 1. Theorem 1 tells us that the Boston mechanism tries to assign the maximum possible number of students to their higher choices and when conflicts arise, a school that has excess demand for seats chooses students according to the priority orders.

Theorem 2. *A mechanism ϕ is weakly fair, satisfies the boundary condition, and weakly respects preference rankings if and only if it is the Boston mechanism.*

Proof. Theorem 1 and proposition 1 complete the proof. □

Null matching trivially weakly respects preference rankings and is weakly fair, but violates only the boundary condition. The SOSM satisfies the boundary condition and is weakly fair, but violates only weak respect of preference rankings. The second revised Boston mechanism β^{II}, stet solves conflicts unfairly, weakly respects preference rankings and satisfies the boundary condition, but violates only weak fairness. The previous statements establish the independence of axioms in the characterization. Decomposing respect of preference rankings into weak respect of preference rankings and the boundary

increase the competition of schools assigned to others.

[5]A mechanism is population monotonic if whenever a student is removed from the problem, the matching derived by this mechanism makes each remaining student weakly better off than the original matching.

condition, which is satisfied by all popular school choice mechanisms, is not meaningless. Weak respect of preference rankings is therefore identified as the unique feature of the Boston mechanism. Theorem 2 tells us that the Boston mechanism, among all mechanisms that satisfy the boundary condition, is the unique one that satisfies weak fairness and weakly respects preference rankings.

Theorem 3. *A mechanism ϕ is weakly fair, efficient, and weakly respects preference rankings if and only if it is the Boston mechanism.*

Proof. Proposition 4 and theorem 2 complete the proof. □

First, we give the mechanism that violates only weak fairness. The second revised Boston mechanism β^{II} satisfies efficiency and weak respect of preference rankings, but violates weak fairness. Second, we give the mechanism that violates only efficiency. Null matching trivially satisfies weak fairness and weak respect of preference rankings, but it is not efficient. Third, we give the mechanism that violates only weak respect of preference rankings. The third revised Boston mechanism β^{III}, which removes mutual best pairs first, satisfies efficiency and weak fairness, but violates weak respect of preference rankings. The previous statements establish the independence of axioms in the characterization. Theorem 3 tells us that among all efficient mechanisms, the Boston mechanism is the only one that satisfies weak fairness and weak respect of preference rankings.

In theorem 3, we characterize the Boston mechanism using both weak fairness and efficiency. It is well known that fairness and efficiency are incompatible in school choice settings, i.e., there is an unambiguous tradeoff between fairness and efficiency. However, theorem 3 tells us that weak fairness and efficiency are compatible and the Boston mechanism satisfies these two simultaneously. Theoretically, we do not know whether weak fairness stands for the strongest axiom on fairness that is compatible with efficiency. Whether it is possible to find conditions stronger than weak fairness that are compatible with efficiency is left as an open question.

Theorem 4. *A mechanism ϕ satisfies consistency, mutual best, and weak respect of preference rankings if and only if it is the Boston mechanism.*

Proof. Propositions 7 and 8 and theorem 2 complete the proof. □

The first revised Boston mechanism β^I satisfies mutual best and weak respect of preference rankings, but violates only consistency. The second revised Boston mechanism β^{II} satisfies consistency and weak respect of preference rankings, but violates only mutual best. The third revised Boston mechanism β^{III} satisfies consistency and mutual best, but violates only weak respect of preference rankings. The previous statement establishes the independence of axioms in this characterization. The SOSM, TTCM, and Boston mechanism all satisfy mutual best. Theorem 4 tells us that among all mechanisms satisfying mutual best, the Boston mechanism is the unique one that also satisfies consistency and weak respect of preference rankings.

Mutual best, like the boundary condition, is a rather weak and common property. However, together with consistency, they imply weak fairness. The property of consistency plays a role like a multiplier or magnifying glass that reinforces the power of some other properties.

Theorem 5. *A mechanism ϕ satisfies consistency and respect of top rankings if and only if it is the Boston mechanism.*

Proof. Propositions 10, 11 and 12 and theorem 2 finish the proof. □

The first revised Boston mechanism β^I respects top rankings, but violates consistency. The simple serial dictatorship where each student chooses their favorite schools according to a predetermined ordering satisfies consistency, but violates respect of top rankings. The previous statement establishes the independence of axioms in this characterization. Theorem 5 tells us that among all consistent mechanisms, the Boston mechanism is the unique one that respects top rankings.

Theorem 6. *A mechanism ϕ is double-standard stable if and only if it is the Boston mechanism.*

Proof. The definition of double-standard stability and theorem 2 complete the proof. □

As noted previously, double-standard stability is simply a concept integrating weak fairness, weak respect of preference rankings, and the boundary condition. However, double-standard stability has important practical implications because it can explain the wide existence and long persistence of the Boston mechanism although it is not desirable theoretically. Double-standard stability guarantees that students have no incentive to rebel about their assignment because the mechanism allocates school seats to them based on, in order the preference rankings and the priority orders. Theorem 4 tells us that the Boston mechanism satisfies this property and that it is the only mechanism to do so.

The SOSM is stable, and the single standard for measuring its stability is the priority order. The Boston mechanism respects preference rankings first, and when conflicts arise, it assigns students fairly. There is no justified envy among students if we use the standard or criterion of both preference rankings and priority orders to measure stability. In this double-standard sense, the Boston mechanism is "stable". Because of the double-standard stability of the Boston mechanism, no student has the incentive to rebel about the final matching because the result is derived through a fair procedure based on specific and pressured rules such as respect of preference rankings and weak fairness. No or few rebelions against the outcome of the Boston mechanism partially explains its widespread use.

6 Conclusions

The following table lists the properties of the three competing school choice mechanisms. We can observe from the table that the Boston mechanism possesses many good properties that the SOSM and TTCM do not. Thus, contrary to the conclusion of Abdulkadiroğlu and Sönmez (2003), which is that the Boston mechanism should be replaced by the SOSM and TTCM, our paper largely argues for the Boston mechanism. Because the SOSM does not respect preference rankings, replacing the Boston mechanism with it will cause efficiency loss. Because the TTCM is not weakly fair, replacing the Boston mechanism with it will cause stability loss. The new characterizations provided in this paper make it easier to compare the Boston mechanism with other school choice mechanisms such as the SOSM and TTCM.

	BOSM	SOSM	TTCM
The Boundary Condition	√	√	√
Mutual Best	√	√	√
Double-standard Fairness	√	×	×
Respect of Top Rankings	√	×	×
Consistency	√	×	×
Weak Respect of Preference Rankings	√	×	×
Resource Monotonicity	√	√	×
Population Monotonicity	√	√	×
Weak Fairness	√	√	×
Fairness	×	√	×
Weak Maskin Monotonicity	×	√	√
Strategy-proofness	×	√	√
Group Strategy-proofness	×	×	√
Efficiency	√	×	√

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Appendix

Proof of theorem 1

Proof. It is easy to see that the Boston mechanism satisfies weak fairness and respects preference rankings. We only need to consider the reverse part, i.e., the Boston mechanism is the unique mechanism that satisfies weak fairness and respect of preference rankings.

Suppose on the contrary that there exists a mechanism ϕ that is weakly fair and respects preference rankings, but is not the Boston mechanism. In other words, for a problem $\mathbb{P} = (I, C, q, P, \succ)$, $\phi(I, C, q, P, \succ) \neq \beta(I, C, q, P, \succ)$. Denote the set of students who are assigned to a real school seat in set C instead of themselves as I_+^β under β and I_+^ϕ under ϕ, respectively. Denote the matching of these students as C_+^β and C_+^ϕ, respectively. We will prove theorem 1 using the following results.

Proof of step 1: We first prove that if ϕ is weakly fair and respects preference rankings, then $C_+^\beta = C_+^\phi$. Suppose that $C_+^\beta \neq C_+^\phi$. There are two cases:

Step 1.1 There exists a school $c_0 \in C$ such that $c_0 \in C_+^\beta$ but $c_0 \notin C_+^\phi$. Then, c_0 is assigned to the null student $\emptyset_i$ under ϕ. Suppose that c_0 is assigned to student i_0 under β. Denote the assignment of student i_0 under ϕ as c_1. Denote the school that is assigned to student i_1 under β as c_1. Denote the school that is assigned to student i_1 under ϕ as c_2 and the student who is assigned to school c_2 under β as i_2, and so on. We generalize the relationship in the following table. For a student i, we write down his assignment under two mechanisms as μ_i^β and μ_i^ϕ, respectively.

i	μ_i^β	i	μ_i^ϕ
i_0	c_0	$\emptyset_i$	c_0
i_1	c_1	i_0	c_1
i_2	c_2	i_1	c_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i_t	c_t	i_{t-1}	c_t
i_{t+1}	c_{t+1}	i_t	c_{t+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The relationships among the preference rankings of these students over their assigned schools are as follows:

(1) $P_{i_0}(c_1) < P_{i_0}(c_0)$

(2) $P_{i_1}(c_1) \leq P_{i_0}(c_1)$

(3) $P_{i_1}(c_2) < P_{i_1}(c_1)$

(4) $P_{i_2}(c_2) \leq P_{i_1}(c_2)$

$\vdots$

(5) $P_{i_t}(c_{t+1}) < P_{i_t}(c_t)$

(6) $P_{i_{t+1}}(c_{t+1}) \leq P_{i_t}(c_{t+1})$

$\vdots$

Suppose that (1) is not correct, i.e., $P_{i_0}(c_1) > P_{i_0}(c_0)$ (Note that it is trivial to consider the case where $P_{i_0}(c_1) = P_{i_0}(c_0)$). Then, under mechanism ϕ, $c_0 P_{i_0} c_1$, but $|\mu_{c_0}| < q_{c_0}$, which contradicts the assumption that ϕ respects preference rankings. Because (1) is correct, we know that student i_0 has been rejected by school c_1 under β. This means that $P_{i_1}(c_1) \leq P_{i_0}(c_1)$ according to the respect of preference rankings of β. Thus, (2) is proved. Suppose that (3) is not correct and student i_1 prefers school c_1 to school c_2, i.e., $P_{i_1}(c_2) > P_{i_1}(c_1)$ (Note that it is trivial to consider the case where $P_{i_1}(c_2) = P_{i_1}(c_1)$. We consider two cases. First, $P_{i_1}(c_1) < P_{i_0}(c_1)$. In this case, under ϕ, $c_1 P_{i_1} c_2 = \mu_{i_1}^\phi$ but $P_{i_1}(c_1) < P_{i_0}(c_1)$, which contradicts the assumption that ϕ respects preference rankings. Second, $P_{i_1}(c_1) = P_{i_0}(c_1) = T$. According to weak fairness of β, $i_1 \succ_{c_1} i_0$. Consider ϕ, $c_1 P_{i_1} c_2 = \mu_{i_1}^\phi$ but $i_1 \succeq_{c_1} i_0$, which contradicts the weak fairness of ϕ. Because (3) is correct, we can conclude that student i_1 has been rejected by school c_2 under β. This means that $P_{i_2}(c_2) \leq P_{i_1}(c_2)$, i.e., condition (4) holds, according to respect of preference rankings of β. Conditions (5) and (6) are proved using the same logic to prove (3) and (4).

(1), (2), (3), (4), (5), and (6) imply that $P_{i_0}(c_0) > P_{i_1}(c_1) > \ldots > P_{i_t}(c_t) > P_{i_{t+1}}(c_{t+1}) > \ldots (t \geq 0)$. Because of the finiteness of the problem, we will finally arrive at a stage where it is impossible to find a student to assign to the undecided school seat caused by the leaving of student i_0 under ϕ. Thus, case 1.1 is impossible.

Step 1.2 There exists a school $c_0 \in C$ such that $c_0 \notin I_+^\beta$ but $c_0 \in I_+^\phi$.

It is easy to imagine that step 1.2 is the symmetric case of step 1.1. Therefore, we refer to the proof of step 1.1 and ignore detailed proof here.

Proof of step 2: We now prove that if ϕ is weakly fair and respects preference rankings, then $I_+^\beta = I_+^\phi$. Suppose that $I_+^\beta \neq I_+^\phi$. There are two cases:

Step 2.1 There exists a student i_0 such that $i_0 \in I_+^\beta$ and $i_0 \notin I_+^\phi$.

i	μ_i^β	i	μ_i^ϕ
i_0	c_0	i_0	$\emptyset_c$
i_1	c_1	i_1	c_0
i_2	c_2	i_2	c_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i_t	c_t	i_t	c_{t-1}
i_{t+1}	c_{t+1}	i_{t+1}	c_t
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The relationships among the preference rankings of these students over their assigned schools are as follows:

(1) $P_{i_1}(c_0) \leq P_{i_0}(c_0)$

(2) $P_{i_1}(c_1) < P_{i_1}(c_0)$

(3) $P_{i_2}(c_1) \leq P_{i_1}(c_1)$

(4) $P_{i_2}(c_2) < P_{i_2}(c_1)$

$\vdots$

(5) $P_{i_{t+1}}(c_t) \leq P_{i_t}(c_t)$

(6) $P_{i_{t+1}}(c_{t+1}) < P_{i_{t+1}}(c_t)$

$\vdots$

Suppose that (1) is not correct, i.e., $P_{i_1}(c_0) > P_{i_0}(c_0)$. Then, under ϕ, $\emptyset_c P_{i_0} c_1$, but $P_{i_1}(c_0) > P_{i_0}(c_0)$, which contradicts the assumption that ϕ respects preference rankings. Because (1) is correct, we know that student i_0 has been rejected by school c_1 under β. This means that $P_{i_1}(c_1) < P_{i_1}(c_0)$ according to respect of preference rankings of ϕ (Note that it is trivial to consider the case where $P_{i_1}(c_1) = P_{i_1}(c_0)$). Thus, (2) is proved. Suppose that (3) is not correct and student i_1 puts school c_1 in a higher preference ranking than student i_2, i.e., $P_{i_2}(c_1) > P_{i_1}(c_1)$. From (2) we know that $P_{i_1}(c_1) < P_{i_1}(c_0)$, i.e., student i_1 prefers school c_1 to school c_0. Thus, under ϕ, $c_1 P_{i_1} c_0$ but $P_{i_2}(c_1) > P_{i_1}(c_1)$, which contradicts the assumption that ϕ respects preference rankings. Next, we prove the correctness of (4). Suppose that (4) is not correct, namely, $P_{i_2}(c_2) > P_{i_2}(c_1)$. (Note that it is trivial to consider the case where $P_{i_2}(c_2) = P_{i_2}(c_1)$. We consider two cases. First, $P_{i_2}(c_1) < P_{i_1}(c_1)$. In this case under β, $c_1 P_{i_2} c_2$ but $P_{i_2}(c_1) < P_{i_1}(c_1)$, which contradicts the assumption that β respects preference rankings. Second, $P_{i_2}(c_1) = P_{i_1}(c_1) = T$. If $P_{i_2}(c_2) > P_{i_2}(c_1)$, we know that student i_2 has been rejected by school c_1 under β. According to weak fairness of β, $i_1 \succ_{c_1} i_2$. Consider the matching of β, $c_1 P_{i_1} c_2$ but $i_1 \succ_{c_1} i_2$, which contradicts weak fairness of β. Conditions (5) and (6) are proved using the same logic as that used to prove (3) and (4).

(1), (2), (3), (4), (5), and (6) imply that $P_{i_0}(c_0) > P_{i_1}(c_1) > \ldots > P_{i_t}(c_t) > P_{i_{t+1}}(c_{t+1}) > \ldots (t \geq 0)$. Because of the finiteness of the problem, we will finally arrive at a stage where it is impossible to find a student to assign to the undecided school seat caused by the leaving of student i_0 under ϕ. Thus, case 2.1 is impossible.

Step 2.2 There exists a student i_0 such that $i_0 \notin I_+^{\beta}$ and $i_0 \in I_+^{\phi}$.

It is easy to imagine that step 2.2 is the symmetric case of step 2.1. Therefore, we refer to the proof of step 2.1 and ignore the detailed proof here. The conclusion is that case 2.2 is impossible.

Before starting step 3, we present a new definition first.

Definition 1. *For a matching μ, a set of students $I_b = \{i_1, \ldots, i_n\}(n \geq 2)$ and their corresponding assignment of school seats contains a* ***strong blocking pair*** *if $\exists i_t, i_{t'} \in I_b$ such that $\mu_{t'} P_{i_t} \mu_t$ and $P_{i_{t'}}(c_{t'}) = P_{i_t}(c_{t'})$, but $i_t \succeq_{c_{t'}} i_{t'}$.*

Proof of step 3: Let μ^ϕ be a matching that is weakly fair and respects preference rankings induced by mechanism ϕ. Then it admits a strong blocking pair.

Denote the set of students who are assigned a different school seat under ϕ than under β as I_d^ϕ. Denote the set of assignment of these students under ϕ as $\mu^\phi_{I_d^\phi}$. Then, according to the proof of step 2, we have $\mu^\beta_{I_d^\phi} = \mu^\phi_{I_d^\phi}$ and, for any $i \in I_d^\phi$, $\mu_i^\beta \neq \mu_i^\phi$. Consider a student $i \in I_d^\phi$ who is assigned the school with the highest preference rankings among I_d^ϕ. Denote the assignment of this student as μ_i^ϕ under ϕ. Then, $\mu_i^\phi P_i d$ for any $d \in \mu^\beta_{I_d^\phi} = \mu^\phi_{I_d^\phi}$. Because student i is assigned a school other than μ_i^ϕ, we conclude that under β, student i is rejected by school μ_i^ϕ. Suppose that school μ_i^ϕ is assigned to student $k \in I_d^\phi$ under the β. Because ϕ respects preference rankings, we know that $P_i(\mu_i^\phi) \leq P_k(\mu_i^\phi)$. Because β respects preference rankings, we know that $P_i(\mu_i^\phi) \geq P_k(\mu_i^\phi)$. Thus, $P_i(\mu_i^\phi) = P_k(\mu_i^\phi)$. According to weak fairness of β, $k \succeq_{\mu_i^\phi} i$. However, under ϕ, now, because $i \in I_d^\phi$ is assigned the school with the highest preference rankings among I_d^ϕ and $P_i(\mu_i^\phi) = P_k(\mu_i^\phi)$, for student k, $P_k(\mu_i^\phi) < P_k(\mu_k^\phi)$, which means that student k envies student i, i.e., $\mu_i^\phi P_k \mu_k^\phi$. Thus, under ϕ, $\mu_i^\phi P_k \mu_k^\phi$ and $P_i(\mu_i^\phi) = P_k(\mu_i^\phi)$ but $k \succeq_{\mu_i^\phi} i$, i.e., ϕ admits a strong blocking pair. This contradicts our assumption that ϕ is weakly fair. □