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Aggregating infinite utility streams with domain restrictions

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Aggregating infinite utility streams with domain restrictions*

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Abstract

We show that anonymous and partially Paretian social welfare functions that satisfy Restricted Non-Substitution exist when each generation's utilities lie in the set of reciprocals of positive and/or negative integers.

1 Introduction

Some recent studies on the problem of aggregating infinite utility streams with a social welfare function (SWF) suppose that the set of feasible utilities is a discrete set because human perception or cognition is not endlessly fine. Furthermore, some impossibility results for the existence of an anonymous and Paretian SWF are reversed on the supposition that the feasible utilities are discrete.

Basu and Mitra (2007) proved that there exists an anonymous and partially Paretian SWFs when each generation's utilities lie in $\mathbb{N} \cup \{0\}$, whereas they proved that there is no anonymous and weakly Paretian SWF if the possible utilities contain $[0, 1]$. Furthermore, Alcantud and García-Sanz (2010) proved that there are anonymous and partially Paretian SWFs satisfying Restricted Non-Substitution when each generation's utilities lie in $\mathbb{N} \cup \{0\}$, while Banerjee (2006) proved that there is no weakly dominant SWF satisfying Hammond Equity for the Future if the feasible utilities are $[0, 1]$.

On the other hand, Dubey and Mitra (2011) provided a complete characterization of domains (of the one period utilities) on which an anonymous and weakly Paretian aggregation is possible.[*1] By way of example, they proved that there exists an anonymous and weakly Paretian SWF when the possible utilities are $\{1/n\}_{n\in\mathbb{N}}$ or $\{-1/n\}_{n\in\mathbb{N}} \cup \{1/n\}_{n\in\mathbb{N}}$, whereas Basu and Mitra (2007) proved that there is no anonymous and weakly Paretian SWF if the feasible utilities contain $[0, 1]$.

The objective of the work is to consider whether we obtain anonymous and partially Paretian SWFs that satify Restricted Non-Substitution exist when each generation's utilities lie in $\{1/n\}_{n\in\mathbb{N}}$ or $\{-1/n\}_{n\in\mathbb{N}} \cup \{1/n\}_{n\in\mathbb{N}}$.

The structure of the article is as follows. Mathematical preliminaries and axioms are introduced in Section 2. In Section 3, we will give alternative proofs of Proposition 1 in Basu and Mitra (2007) and Theorem 1 in Alcantud and García-Sanz (2010), which prove to be useful to show our results. The main results is reported in Sections 4 and 5.

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*1 Sekiguchi (2011) proved that the domain condition is also necessary and sufficient for the possibility of an anonymous and partially Paretian aggregation.

2 Preliminaries

Let $\boldsymbol{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, which represents a domain of utility streams. We adopt the usual notation for such utility streams: $\boldsymbol{x} = (x_1, \ldots, x_n, \ldots) \in \boldsymbol{X}$. By $(y)_{\text{con}}$, we mean the constant sequence $(y, y, \ldots)$, $(x, (y)_{\text{con}})$ holds for $(x, y, y, \ldots)$, and $(x_1, \ldots, x_k, (y)_{\text{con}}) = (x_1, \ldots, x_k, y, y, \ldots)$ means an eventually constant (to y) sequence. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for each $i = 1, 2, \ldots$, and $\boldsymbol{x} \gg \boldsymbol{y}$ if $x_i > y_i$ for each $i = 1, 2, \ldots$. Also, $\boldsymbol{x} > \boldsymbol{y}$ holds for $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$.

An SWF is a function $W : \boldsymbol{X} \to \mathbb{R}$. Alcantud and García-Sanz (2010) introduced the following equity principles.

Hammond Equity for the Future$^+$. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ are such that $\boldsymbol{x} = (x_1, (x)_{\text{con}})$ and $\boldsymbol{y} = (y_1, (y)_{\text{con}})$ $(x_1 > y_1 > y > x)$, then $W(\boldsymbol{y}) > W(\boldsymbol{x})$.[*2]

Restricted Non-Substitution. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ are such that $\boldsymbol{x} = (x_1, \ldots, x_k, (x)_{\text{con}})$ and $\boldsymbol{y} = (y_1, \ldots, y_l, (y)_{\text{con}})$ with $y > x$, then $W(\boldsymbol{y}) > W(\boldsymbol{x})$.[*3]

The next axiom is the classical way to avoid discrimination among generations à la Sidgwick, which-by contrast to HEF$^+$ and RNS does not incorporate any preference for egalitarian distribution of utilities among them.

Anonymity. A finite permutation of a utility stream produces a utility stream with the same social utility.

Useful evaluations must account for some form of efficiency too. The literature usually invokes the following axiom:

Strong Pareto. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\boldsymbol{x} > \boldsymbol{y}$, then $W(\boldsymbol{x}) > W(\boldsymbol{y})$.

However, in the current context, weaker forms of the above postulate seem to be founded on safer ground (v., Basu and Mitra, 2007, Section 5.3). In particular, we are concerned with the next successively weaker forms of Paretianism.

Weak Dominance. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \neq j$, then $W(\boldsymbol{x}) > W(\boldsymbol{y})$.

Weak Pareto. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\boldsymbol{x} \gg \boldsymbol{y}$, then $W(\boldsymbol{x}) > W(\boldsymbol{y})$.

Partial Pareto. The SWF, W, satisfies Weak Dominance and Weak Pareto.

Following Basu and Mitra (2007), we now define a partition and functions as follows. For each $\boldsymbol{x} \in \boldsymbol{X}$, let $E(\boldsymbol{x}) = \{\boldsymbol{y} \in \boldsymbol{X} : \text{there is } N \in \mathbb{N} \text{ such that } y_k = x_k \text{ for all } k \geq N\}$. Let $\Im$ be the collection $\{E : E = E(\boldsymbol{x}) \text{ for some } \boldsymbol{x} \in \boldsymbol{X}\}$. Then $\Im$ is a partition of $\boldsymbol{X}$.[*4] By the axiom of choice, there is a function, $g : \Im \to \boldsymbol{X}$, such that $g(E) \in E$ for each $E \in \Im$. Given any $\boldsymbol{x}, \boldsymbol{y}$ in $E \in \Im$, define $h(\boldsymbol{x}, \boldsymbol{y}) = \lim_{N\to\infty} \sum_{k=1}^{N} (x_k - y_k)$. Notice that h is well-defined, since given any $\boldsymbol{x}, \boldsymbol{y}$ in $E \in \Im$, there is $M \in \mathbb{N}$ such that $\sum_{i=1}^{N} (x_i - y_i)$ is a constant for all $N \geq M$.

[*2] Hammond Equity for the Future condition was introduced in Asheim and Tungodden (2004).
[*3] Non-substitution axiom originates with Lauwers (1997).
[*4] If E and F belong to $\Im$, then either $E = F$, or E is disjoint from F; further, $\cup_{E\in\Im} E = \boldsymbol{X}$.

3 Alternative proofs of the theorems on the non-negative integers

In this section, we give alternative proofs of the following results, which is due to Basu and Mitra (2007) and Alcantud and García-Sanz (2010), by showing a slightly different social welfare functions from theirs, which turn out to be useful.

Proposition 1 (Basu and Mitra, 2007, Proposition 1). *There are SWFs on* $\boldsymbol{X} = Y^{\mathbb{N}}$*, where* $Y \subset \{0, 1, \dots\}$*, that satisfy Anonymity and Partial Pareto.*

Proof. Given any $\boldsymbol{x} \in \boldsymbol{X}$, let $f(\boldsymbol{x}) = -2^{-\min\{x_1, x_2, \dots\}}$.[*5]

Given any $\boldsymbol{x}, \boldsymbol{y}$ in $E \in \Im$, define $H(\boldsymbol{x}, \boldsymbol{y}) = 0.5[3 - h(\boldsymbol{x}, \boldsymbol{y})/[1 + |h(\boldsymbol{x}, \boldsymbol{y})|]]$. Then $H(\boldsymbol{x}, \boldsymbol{y}) \in (1, 2)$.

We now define $W : \boldsymbol{X} \to (-2, 0)$ as follows. Given any $\boldsymbol{x} \in \boldsymbol{X}$, we associate with it its equivalence class, $E(\boldsymbol{x})$. Then, using the function g, we get $g(E(\boldsymbol{x})) \in E(\boldsymbol{x})$. Next, using the functions, h and H, we obtain $h(\boldsymbol{x}, g(E(\boldsymbol{x})))$ and $H(\boldsymbol{x}, g(E(\boldsymbol{x})))$. Finally, define $W(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot H(\boldsymbol{x}, g(E(\boldsymbol{x})))$.

Anonymity can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \neq i, j$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting this common set by E, we see that $h(\boldsymbol{x}, g(E)) = h(\boldsymbol{y}, g(E))$, and so $H(\boldsymbol{x}, g(E)) = H(\boldsymbol{y}, g(E))$. Further, the set $\{x_1, x_2, \dots\}$ is the same as the set $\{y_1, y_2, \dots\}$, so that $f(\boldsymbol{x}) = f(\boldsymbol{y})$. Thus, we obtain: $W(\boldsymbol{x}) = W(\boldsymbol{y})$.

Weak Dominance can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exists $i \in \mathbb{N}$ such that $x_i > y_i$, while $x_k = y_k$ for all $k \neq i$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting the common set by E, we see that $h(\boldsymbol{x}, g(E)) > h(\boldsymbol{y}, g(E))$. This implies $H(\boldsymbol{y}, g(E)) > H(\boldsymbol{x}, g(E)) > 1$. Further, the smallest element of the set $\{x_1, x_2, \dots\}$ is at least as large as the smallest element of the set $\{y_1, y_2, \dots\}$, so that we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x}) < 0$. Thus, we obtain the desired inequality: $W(\boldsymbol{y}) < W(\boldsymbol{x}) < 0$.

Weak Pareto can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\boldsymbol{x} \gg \boldsymbol{y}$, then $E(\boldsymbol{x}) \neq E(\boldsymbol{y})$. Thus, we will not be able to compare $H(\boldsymbol{x}, g(E(\boldsymbol{x})))$ with $H(\boldsymbol{y}, g(E(\boldsymbol{y})))$. However, we do know that $H(\boldsymbol{x}, g(E(\boldsymbol{x}))) > 1$ and $H(\boldsymbol{y}, g(E(\boldsymbol{y}))) < 2$. Further, since $\boldsymbol{x} \gg \boldsymbol{y}$, we have $f(\boldsymbol{y}) \leq 2f(\boldsymbol{x}) < 0$. Thus, we obtain:

$$W(\boldsymbol{y}) = f(\boldsymbol{y}) \cdot H(\boldsymbol{y}, g(E(\boldsymbol{y}))) < f(\boldsymbol{y}) \leq 2f(\boldsymbol{x}) < f(\boldsymbol{x}) \cdot H(\boldsymbol{x}, g(E(\boldsymbol{x}))) = W(\boldsymbol{x}) < 0.$$

□

Remark 1. Basu and Mitra's (2007) SWF W_{BM} and our SWF W_1 are compatible with each other, that is, for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, $W_{\mathrm{BM}}(\boldsymbol{x}) \geq W_{\mathrm{BM}}(\boldsymbol{y})$ if and only if $W_1(\boldsymbol{x}) \geq W_1(\boldsymbol{y})$.

Proposition 1 ensures the existence of a map $W_1 : \boldsymbol{X} \to (-2, 0)$ that satisfies Anonymity and Partial Pareto.

Theorem 1 (Alcantud and García-Sanz, 2010, Theorem 1). *There are SWFs on* $\boldsymbol{X} = Y^{\mathbb{N}}$*, where* $Y \subset \{0, 1, \dots\}$*, that satisfy Anonymity, Partial Pareto, and Restricted Non-Substitution.*

Proof. For any $\boldsymbol{x} = (x_1, x_2, \dots) \in \boldsymbol{X}$, let

$$U(\boldsymbol{x}) = \begin{cases} \frac{-2+W_1(\boldsymbol{x})}{2} \cdot 2^{-\liminf_n (x_n)} & \text{if } \boldsymbol{x} \text{ is bounded} \\ W_1(\boldsymbol{x}) + 2 & \text{otherwise.} \end{cases}$$

This is an anonymous SWF on $\boldsymbol{X}$.

[*5] Notice that $\min\{x_1, x_2, \dots\}$ is well-defined (Munkres, 1975, p. 32).

In order to prove that U satisfies Partial Pareto, suppose that $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and either $\boldsymbol{x} \gg \boldsymbol{y}$ or there is $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. By construction, $W_1(\boldsymbol{y}) < W_1(\boldsymbol{x}) < 0$. If $\boldsymbol{x}$ is bounded, then so is $\boldsymbol{y}$, and because $2^{-\liminf_n(y_n)} \geq 2^{-\liminf_n(x_n)} > 0$, we obtain $U(\boldsymbol{y}) < U(\boldsymbol{x}) < 0$. If $\boldsymbol{x}$ is unbounded but $\boldsymbol{y}$ is bounded, then $U(\boldsymbol{x}) = W_1(\boldsymbol{x}) + 2 > 0 > U(\boldsymbol{y})$. If both $\boldsymbol{x}$ and $\boldsymbol{y}$ are unbounded, then $U(\boldsymbol{x}) = W_1(\boldsymbol{x}) + 2 > W_1(\boldsymbol{y}) + 2 = U(\boldsymbol{y})$.

In order to prove that U satisfies Restricted Non-Substitution, let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ be such that $\boldsymbol{x} = (x_1, \ldots, x_k, (x)_{\text{con}})$ and $\boldsymbol{y} = (y_1, \ldots, y_l, (y)_{\text{con}})$ with $x > y$. We need to ensure $U(\boldsymbol{x}) > U(\boldsymbol{y})$. Because $\liminf_n(x_n) = x \geq 1 + y = 1 + \liminf_n(y_n)$ and $\frac{-2+W_1}{2}$ takes values in $(-2, -1)$, we deduce

$$\frac{-2 + W_1(\boldsymbol{y})}{2} \cdot 2^{-\liminf_n(y_n)} = \frac{-2 + W_1(\boldsymbol{y})}{2} \cdot 2^{-y} < -2^{1-\liminf_n(x_n)} < \frac{-2 + W_1(\boldsymbol{x})}{2} \cdot 2^{-\liminf_n(x_n)} < 0$$

thus $U(\boldsymbol{y}) < U(\boldsymbol{x}) < 0$. □

Remark 2. Alcantud and García-Sanz's (2010) SWF and our SWF are compatible with each other.

4 Reciprocals of positive integers

In this section, we consider the case where $Y = \{1/n\}_{n \in \mathbb{N}}$ (Dubey and Mitra, 2011, Example 2) (which captures such a possibility that the poorer one generation is, the finer the perception is). The domain restriction allows us to establish the existence of an anonymous and partially Paretian social welfare function.[*6]

Proposition 2. *There are SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$, where $Y \subset \{1, 1/2, \ldots\}$, that satisfy Anonymity and Partial Pareto.*

Proof. Given any $\boldsymbol{x} \in \boldsymbol{X}$, let $f(\boldsymbol{x}) = 2^{-1/\max\{x_1, x_2, \ldots\}}$.

Given any $\boldsymbol{x}, \boldsymbol{y}$ in $E \in \Im$, define $H(\boldsymbol{x}, \boldsymbol{y}) = 0.5[3 + h(\boldsymbol{x}, \boldsymbol{y})/[1 + |h(\boldsymbol{x}, \boldsymbol{y})|]]$. Then $H(\boldsymbol{x}, \boldsymbol{y}) \in (1, 2)$.

We now define $W : \boldsymbol{X} \to (0, 1)$ as follows. Given any $\boldsymbol{x} \in \boldsymbol{X}$, we associate with it its equivalence class, $E(\boldsymbol{x})$. Then, using the function g, we get $g(E(\boldsymbol{x})) \in E(\boldsymbol{x})$. Next, using the functions, h and H, we obtain $h(\boldsymbol{x}, g(E(\boldsymbol{x})))$ and $H(\boldsymbol{x}, g(E(\boldsymbol{x})))$. Finally, define $W(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot H(\boldsymbol{x}, g(E(\boldsymbol{x})))$.

Anonymity can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \neq i, j$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting this common set by E, we see that $h(\boldsymbol{x}, g(E)) = h(\boldsymbol{y}, g(E))$, and so $H(\boldsymbol{x}, g(E)) = H(\boldsymbol{y}, g(E))$. Further, the set $\{x_1, x_2, \ldots\}$ is the same as the set $\{y_1, y_2, \ldots\}$, so that $f(\boldsymbol{x}) = f(\boldsymbol{y})$. Thus, we obtain: $W(\boldsymbol{x}) = W(\boldsymbol{y})$.

Weak Dominance can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exists $i \in \mathbb{N}$ such that $x_i > y_i$, while $x_k = y_k$ for all $k \neq i$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting the common set by E, we see that $h(\boldsymbol{x}, g(E)) > h(\boldsymbol{y}, g(E))$. This implies $H(\boldsymbol{x}, g(E)) > H(\boldsymbol{y}, g(E)) > 0$. Further, the largest element of the set $\{x_1, x_2, \ldots\}$ is at least as large as the largest element of the set $\{y_1, y_2, \ldots\}$, so that we have $f(\boldsymbol{x}) \geq f(\boldsymbol{y}) > 0$. Thus, we obtain the desired inequality: $W(\boldsymbol{x}) > W(\boldsymbol{y})$.

Weak Pareto can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\boldsymbol{x} \gg \boldsymbol{y}$, then $E(\boldsymbol{x}) \neq E(\boldsymbol{y})$. Thus, we will not be able to compare $H(\boldsymbol{x}, g(E(\boldsymbol{x})))$ with $H(\boldsymbol{y}, g(E(\boldsymbol{y})))$. However, we do know that $H(\boldsymbol{x}, g(E(\boldsymbol{x}))) > 1$ and $H(\boldsymbol{y}, g(E(\boldsymbol{y}))) < 2$. Further, since $\boldsymbol{x} \gg \boldsymbol{y}$, we have $f(\boldsymbol{x}) \geq 2f(\boldsymbol{y}) > 0$. Thus, we obtain:

$$W(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot H(\boldsymbol{x}, g(E(\boldsymbol{x}))) > f(\boldsymbol{x}) \geq 2f(\boldsymbol{y}) > f(\boldsymbol{y}) \cdot H(\boldsymbol{y}, g(E(\boldsymbol{y}))) = W(\boldsymbol{y}).$$

□

[*6] Basu and Mitra (2007) footnoted this possibility result.

Proposition 2 ensures the existence of a map $W_2 : \boldsymbol{X} \to (0,1)$ satisfying Anonimity and Partial Pareto.

Banerjee (2006) proved that when the feasible utilities are $[0,1]$, axioms Hammond Equity for the Future and Weak Dominance are incompatible under the Basu-Mitra approach. The following theorem proves that in the current setting, the situation is much more favourable.[*7]

Theorem 2. *There are SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$, where $Y \subset \{1, 1/2, \dots\}$, that satisfy Anonymity, Partial Pareto, and Restricted Non-Substitution.*

Proof. For any $\boldsymbol{x} = (x_1, x_2, \dots) \in \boldsymbol{X}$, let

$$U(x) = \begin{cases} 1 + (1 + W_2(\boldsymbol{x})) \cdot 2^{-1/\limsup_n(x_n)} & \text{if } \limsup_n(x_n) \text{ is positive} \\ W_2(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

This is an anonymous SWF on $\boldsymbol{X}$.

In order to prove that U satisfies Partial Pareto, suppose that $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and either $\boldsymbol{x} \gg \boldsymbol{y}$ or there is $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. By construction, $W_2(\boldsymbol{x}) > W_2(\boldsymbol{y}) > 0$. If $\limsup_n(y_n)$ is positive, then so is $\limsup_n(x_n)$, and because $2^{-1/\liminf_n(x_n)} \geq 2^{-1/\liminf_n(y_n)} > 0$, we obtain $U(\boldsymbol{x}) > U(\boldsymbol{y}) > 0$. If $\limsup_n(y_n)$ is not positive but $\limsup_n(x_n)$ is positive, then $U(\boldsymbol{x}) = 1 + (1 + W_2(\boldsymbol{x})) \cdot 2^{-(\limsup_n(x_n))^{-1}} > 1 > U(\boldsymbol{y})$. If neither $\limsup_n(y_n)$ nor $\limsup_n(x_n)$ is positive, then $U(\boldsymbol{x}) = W_2(\boldsymbol{x}) > W_2(\boldsymbol{y}) = U(\boldsymbol{y})$.

In order to prove that U satisfies Restricted Non-Substitution, let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ be such that $\boldsymbol{x} = (x_1, \dots, x_k, (x)_{\mathrm{con}})$ and $\boldsymbol{y} = (y_1, \dots, y_l, (y)_{\mathrm{con}})$ with $x > y$. We need to ensure $U(\boldsymbol{x}) > U(\boldsymbol{y})$. Because $2^{-1/\limsup_n(x_n)} = 2^{-1/x} \geq 2^{1-1/y} = 2 \cdot 2^{-1/\limsup_n(y_n)}$ and $1 + W_2$ takes values in $(1,2)$, we deduce

$$1 + (1 + W_2(\boldsymbol{x})) \cdot 2^{-1/\limsup_n(x_n)} > 1 + 2^{-1/x} \geq 1 + 2 \cdot 2^{-1/\limsup_n(y_n)} > 1 + (1 + W_2(\boldsymbol{y})) \cdot 2^{-1/\limsup_n(y_n)}$$

thus $U(\boldsymbol{x}) > U(\boldsymbol{y})$. □

5 Inverses of positive and negative integers

In this section, we consider the case where $Y = \{-1/n\}_{n \in \mathbb{N}} \cup \{1/n\}_{n \in \mathbb{N}}$ (Dubey and Mitra, 2011, Example 3) (which generalizes the domain restriction in the previous section). The domain restriction also allows us to establish the existence of an anonymous and partially Paretian social welfare function.[*8]

Proposition 3. *There are SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$, where $Y \subset \{-1, -1/2, \dots\} \cup \{1, 1/2, \dots\}$, that satisfy Anonymity and Partial Pareto.*

Proof. For any $\boldsymbol{x} \in \boldsymbol{X}$, let

$$f^1(\boldsymbol{x}) = \begin{cases} \frac{\sup_n(x_n)}{|\sup_n(x_n)|} \cdot 2^{-1/|\sup_n(x_n)|} & \text{if } \sup_n(x_n) \text{ is not zero} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^2(\boldsymbol{x}) = \begin{cases} \frac{\inf_n(x_n)}{|\inf_n(x_n)|} \cdot 2^{-1/|\inf_n(x_n)|} & \text{if } \inf_n(x_n) \text{ is not zero} \\ 0 & \text{otherwise.} \end{cases}$$

[*7] This result is the counterpart of Theorem 1 in Alcantud and García-Sanz (2010).

[*8] Sekiguchi (2011) mentioned this possibility result.

Then, using the function $H_2(\boldsymbol{x},\boldsymbol{y}) = 0.5[3 + h(\boldsymbol{x},\boldsymbol{y})/[1 + |h(\boldsymbol{x},\boldsymbol{y})|]]$, we let

$$W^1(\boldsymbol{x}) = \begin{cases} f^1(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))^{\frac{\sup_n(x_n)}{|\sup_n(x_n)|}} & \text{if } \sup_n(x_n) \text{ is not zero} \\ 0 & \text{otherwise} \end{cases}$$

and

$$W^2(\boldsymbol{x}) = \begin{cases} f^2(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))^{\frac{\inf_n(x_n)}{|\inf_n(x_n)|}} & \text{if } \inf_n(x_n) \text{ is not zero} \\ 0 & \text{otherwise.} \end{cases}$$

We now define $W : \boldsymbol{X} \to (-1, 2)$ as follows. For any $\boldsymbol{x} \in \boldsymbol{X}$, we associate with it its equivalence class, $E(\boldsymbol{x})$. Then, using the function g, we get $g(E(\boldsymbol{x})) \in E(\boldsymbol{x})$. Next, using the functions, h and H_2, we obtain $h(\boldsymbol{x}, g(E(\boldsymbol{x})))$ and $H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))$. Finally, define $W(\boldsymbol{x}) = W^1(\boldsymbol{x}) + W^2(\boldsymbol{x})$.

Anonymity can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \neq i, j$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting this common set by E, we see that $h(\boldsymbol{x}, g(E)) = h(\boldsymbol{y}, g(E))$, and so $H_2(\boldsymbol{x}, g(E)) = H_2(\boldsymbol{y}, g(E))$. Further, the set $\{x_1, x_2, \dots\}$ is the same as the set $\{y_1, y_2, \dots\}$, so that $f^1(\boldsymbol{x}) = f^1(\boldsymbol{y})$ and $f^2(\boldsymbol{x}) = f^2(\boldsymbol{y})$. Thus, we obtain: $W(\boldsymbol{x}) = W(\boldsymbol{y})$.

Weak Dominance can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y}$ are in $\boldsymbol{X}$, and there exists $i \in \mathbb{N}$ such that $x_i > y_i$, while $x_k = y_k$ for all $k \neq i$, then $E(\boldsymbol{x}) = E(\boldsymbol{y})$. Furthermore, denoting the common set by E, we see that $h(\boldsymbol{x}, g(E)) > h(\boldsymbol{y}, g(E))$. This implies $H_2(\boldsymbol{x}, g(E)) > H_2(\boldsymbol{y}, g(E)) > 1$. Further, $\sup_n(x_n) \geq \sup_n(y_n)$ and $\inf_n(x_n) \geq \inf_n(y_n)$, so that we have $f^1(\boldsymbol{x}) \geq f^1(\boldsymbol{y})$ and $f^2(\boldsymbol{x}) \geq f^2(\boldsymbol{y})$. If $\sup_n(y_n)$ is positive, then $W^1(\boldsymbol{x}) = f^1(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E)) > f^1(\boldsymbol{y}) \cdot H_2(\boldsymbol{y}, g(E)) = W^1(\boldsymbol{y})$. If $\sup_n(y_n)$ is negative and $\sup_n(x_n)$ is not zero, then $W^1(\boldsymbol{x}) \geq f^1(\boldsymbol{x})/H_2(\boldsymbol{x}, g(E)) > f^1(\boldsymbol{y})/H_2(\boldsymbol{y}, g(E)) = W^1(\boldsymbol{y})$. If $\sup_n(y_n)$ is negative and $\sup_n(x_n)$ is zero, then $W^1(\boldsymbol{x}) = 0 > W^1(\boldsymbol{y})$. If $\sup_n(y_n)$ is zero, then $W^1(\boldsymbol{x}) \geq 0 = W^1(\boldsymbol{y})$. Furthermore, if $\inf_n(y_n)$ is positive, then $W^2(\boldsymbol{x}) = f^2(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E)) > f^2(\boldsymbol{y}) \cdot H_2(\boldsymbol{y}, g(E)) = W^2(\boldsymbol{y})$. If $\inf_n(y_n)$ is negative and $\inf_n(x_n)$ is not zero, then $W^2(\boldsymbol{x}) \geq f^2(\boldsymbol{x})/H_2(\boldsymbol{x}, g(E)) > f^2(\boldsymbol{y})/H_2(\boldsymbol{y}, g(E)) = W^2(\boldsymbol{y})$. If $\inf_n(y_n)$ is negative and $\inf_n(x_n)$ is zero, then $W^2(\boldsymbol{x}) = 0 > W^2(\boldsymbol{y})$. If $\inf_n(y_n)$ is zero, then $W^2(\boldsymbol{x}) \geq 0 = W^2(\boldsymbol{y})$. Further, if both $\sup_n(y_n)$ and $\inf_n(y_n)$ are zero, then $W^1(\boldsymbol{x}) > 0 = W^1(\boldsymbol{y})$ and $W^2(\boldsymbol{x}) > 0 = W^2(\boldsymbol{y})$, so that we obtain the desired inequality: $W(\boldsymbol{x}) = W^1(\boldsymbol{x}) + W^2(\boldsymbol{x}) > W^1(\boldsymbol{y}) + W^2(\boldsymbol{y}) = W(\boldsymbol{y})$.

Weak Pareto can be verified as follows. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\boldsymbol{x} \gg \boldsymbol{y}$, then $E(\boldsymbol{x}) \neq E(\boldsymbol{y})$. Thus, we will not be able to compare $H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))$ with $H_2(\boldsymbol{y}, g(E(\boldsymbol{y})))$. However, we do know that $H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))$ and $H_2(\boldsymbol{y}, g(E(\boldsymbol{y})))$ take values in $(1, 2)$. Further, since $\boldsymbol{x} \gg \boldsymbol{y}$, we have

$$f^1(\boldsymbol{x}) \geq \begin{cases} 2f^1(\boldsymbol{y}) & \text{if } \sup_n(y_n) \text{ is positive} \\ f^1(\boldsymbol{y})/2 & \text{if } \sup_n(y_n) \text{ is negative} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^2(\boldsymbol{x}) \geq \begin{cases} 2f^2(\boldsymbol{y}) & \text{if } \inf_n(y_n) \text{ is positive} \\ f^2(\boldsymbol{y})/2 & \text{if } \inf_n(y_n) \text{ is negative} \\ 0 & \text{otherwise.} \end{cases}$$

If $\sup_n(y_n)$ is positive, then

$$W^1(\boldsymbol{x}) = f^1(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) \geq 2f^1(\boldsymbol{y}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) > f^1(\boldsymbol{y}) \cdot H_2(\boldsymbol{y}, g(E(\boldsymbol{y}))) = W^1(\boldsymbol{y}).$$

If $\sup_n(y_n)$ is negative and $\sup_n(x_n)$ is not zero, then

$$W^1(\boldsymbol{x}) \geq f^1(\boldsymbol{x})/H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) \geq f^1(\boldsymbol{y})/[2H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))] > f^1(\boldsymbol{y})/H_2(\boldsymbol{y}, g(E(\boldsymbol{y}))) = W^1(\boldsymbol{y}).$$

If $\sup_n(y_n)$ is negative and $\sup_n(x_n)$ is zero, then $W^1(\boldsymbol{x}) = 0 > W^1(\boldsymbol{y})$. If $\sup_n(y_n)$ is zero, then $W^1(\boldsymbol{x}) \geq 0 = W^1(\boldsymbol{y})$. Furthermore, if $\inf_n(y_n)$ is positive, then

$$W^2(\boldsymbol{x}) = f^2(\boldsymbol{x}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) \geq 2f^2(\boldsymbol{y}) \cdot H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) > f^2(\boldsymbol{y}) \cdot H_2(\boldsymbol{y}, g(E(\boldsymbol{y}))) = W^2(\boldsymbol{y}).$$

If $\inf_n(y_n)$ is negative and $\inf_n(x_n)$ is not zero, then

$$W^2(\boldsymbol{x}) \geq f^2(\boldsymbol{x})/H_2(\boldsymbol{x}, g(E(\boldsymbol{x}))) \geq f^2(\boldsymbol{y})/[2H_2(\boldsymbol{x}, g(E(\boldsymbol{x})))] > f^2(\boldsymbol{y})/H_2(\boldsymbol{y}, g(E(\boldsymbol{y}))) = W^2(\boldsymbol{y}).$$

If $\inf_n(y_n)$ is negative and $\inf_n(x_n)$ is zero, then $W^2(\boldsymbol{x}) = 0 > W^2(\boldsymbol{y})$. If $\inf_n(y_n)$ is zero, then $W^2(\boldsymbol{x}) \geq 0 = W^2(\boldsymbol{y})$. Further, if both $\sup_n(y_n)$ and $\inf_n(y_n)$ are zero, then $W^1(\boldsymbol{x}) > 0 = W^1(\boldsymbol{y})$ and $W^2(\boldsymbol{x}) > 0 = W^2(\boldsymbol{y})$, so that we obtain: $W(\boldsymbol{x}) = W^1(\boldsymbol{x}) + W^2(\boldsymbol{x}) > W^1(\boldsymbol{y}) + W^2(\boldsymbol{y}) = W(\boldsymbol{y})$. □

Proposition 3 ensures the existence of a map $W_3 : \boldsymbol{X} \to (-1, 2)$ satisfying Anonymity and Partial Pareto.

Banerjee (2006) proved that when the feasible utilities are $[0, 1]$, axioms Hammond Equity for the Future and Weak Dominance are incompatible under the Basu-Mitra approach. The following theorem proves that in the current setting, the situation is much more favourable.[*9]

Theorem 3. *There are SWFs on* $\boldsymbol{X} = Y^{\mathbb{N}}$, *where* $Y \subset \{-1, -1/2, \dots\} \cup \{1, 1/2, \dots\}$, *that satisfy Anonymity, Partial Pareto, and Restricted Non-Substitution.*

Proof. For any $\boldsymbol{x} = (x_1, x_2, \dots) \in \boldsymbol{X}$, let

$$U(\boldsymbol{x}) = \begin{cases} 2 + \dfrac{4 + W_3(\boldsymbol{x})}{3} \cdot 2^{-1/\limsup_n(x_n)} & \text{if } \limsup_n(x_n) \text{ is positive} \\ -2 + \dfrac{5 - W_3(\boldsymbol{x})}{3} \cdot 2^{1/\limsup_n(x_n)} & \text{if } \limsup_n(x_n) \text{ is negative} \\ W_3(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

This is an anonymous SWF on $\boldsymbol{X}$.

In order to prove that U satisfies Partial Pareto, suppose that $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and either $\boldsymbol{x} \gg \boldsymbol{y}$ or there is $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. By construction, $W_3(\boldsymbol{x}) > W_3(\boldsymbol{y})$. If $\limsup_n(y_n)$ is positive, then so is $\limsup_n(x_n)$, and because $\limsup_n(x_n) \geq \limsup_n(y_n)$, we obtain $U(\boldsymbol{x}) > U(\boldsymbol{y})$. If $\limsup_n(y_n)$ is zero but $\limsup_n(x_n)$ is positive, then $U(\boldsymbol{x}) > 2 > U(\boldsymbol{y})$. If both $\limsup_n(y_n)$ and $\limsup_n(x_n)$ are zero, then $U(\boldsymbol{x}) = W_3(\boldsymbol{x}) > W_3(\boldsymbol{y}) = U(\boldsymbol{y})$. If $\limsup_n(y_n)$ is negative but $\limsup_n(x_n)$ is positive or zero, then $U(\boldsymbol{x}) > -1 > U(\boldsymbol{y})$. If both $\limsup_n(x_n)$ and $\limsup_n(y_n)$ are negative, then $U(\boldsymbol{x}) > U(\boldsymbol{y})$ again. The above cases exhaust all logical possibilities, and therefore our claim is established.

In order to prove that U satisfies Restricted Non-Substitution, let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ be such that $\boldsymbol{x} = (x_1, \dots, x_k, (x)_{\text{con}})$ and $\boldsymbol{y} = (y_1, \dots, y_l, (y)_{\text{con}})$ with $x > y$. We need to ensure $U(\boldsymbol{x}) > U(\boldsymbol{y})$. If $\limsup_n(y_n) = y$ is positive, then $\limsup_n(x_n) = x > y = \limsup_n(y_n) > 0$ and $2^{-1/\limsup_n(x_n)} \geq 2 \cdot 2^{-1/\limsup_n(y_n)}$. Because W_3 takes values in $(-1, 2)$, we deduce

$$\begin{aligned} \frac{4 + W_3(\boldsymbol{x})}{3} \cdot 2^{-1/\limsup_n(x_n)} &\geq 2 \cdot \frac{4 + W_3(\boldsymbol{x})}{3} \cdot 2^{-1/\limsup_n(y_n)} \\ &> 2 \cdot 2^{-1/\limsup_n(y_n)} \\ &> \frac{4 + W_3(\boldsymbol{y})}{3} \cdot 2^{-1/\limsup_n(y_n)} \end{aligned}$$

[*9] This result is the counterpart of Theorem 1 in Alcantud and García-Sanz (2010).

thus $U(\boldsymbol{x}) > U(\boldsymbol{y})$. If $\limsup_n(y_n) = y = 0$ but $\limsup_n(x_n) = x > y = 0$, then $U(\boldsymbol{x}) > 2 > U(\boldsymbol{y})$. If $\limsup_n(y_n) = y < 0$ but $\limsup_n(x_n) = x \geq 0 > y$, then $U(\boldsymbol{x}) > -1 > U(\boldsymbol{y})$. If $\limsup_n(x_n) = x$ is negative, then $\limsup_n(y_n) = y < x = \limsup_n(x_n) < 0$ and $2^{1/\limsup_n(x_n)} \leq 2 \cdot 2^{1/\limsup_n(y_n)}$. Because W_3 takes values in $(-1, 2)$, we deduce

$$\begin{aligned}\frac{5 - W_3(\boldsymbol{x})}{3} \cdot 2^{1/\limsup_n(x_n)} &\geq 2 \cdot \frac{5 - W_3(\boldsymbol{x})}{3} \cdot 2^{1/\limsup_n(y_n)} \\ &> 2 \cdot 2^{1/\limsup_n(y_n)} \\ &> \frac{5 - W_3(\boldsymbol{y})}{3} \cdot 2^{1/\limsup_n(y_n)}\end{aligned}$$

thus $U(\boldsymbol{x}) > U(\boldsymbol{y})$. The above cases exhaust all possibilities, and therefore our claim is established. □

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