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A New Keynesian Model with Endogenous Technological Change

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Abstract

In this study, we introduce endogenous technological change based on R&D into the new Keynesian model, in which nominal wages are suppose to be sticky. The purpose of this paper is to examine how money growth affects long-run economic growth. The economy exhibits the sustained growth based on sustained R&D for sufficiently high money growth rates, and along such a balanced growth path, the faster money growth brings the larger employment and faster economic growth. Further, under some parameter restrictions, no balanced growth path exists for small money growth rates, and the economy is trapped in the steady state without long-run growth. These results suggest that money growth may be an important factor for long-run economic growth.

JEL classification: O11, O42, E12, E31
keywords: endogenous growth, R&D-based growth model, new Keynesian Phillips curve, nominal rigidities, money growth

1 Introduction

Macroeconomists discuss separately *the long-run theory* and *the short-run theory*. The central theory of the former studies is optimal growth theory[1] or endogenous growth theory, which analyzes the supply side of the economy. The central theory of the latter is new Keynesian theory, in which prices or

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[1]See Ramsey (1928), Cass (1965), Koopmans (1965).

nominal wages are supposed to be sticky and the price adjustment process is analyzed.[2]

Such divided framework is justified by the natural rate hypothesis.[3] However, if the price stickiness remains at the steady state of the short-run model, money is not superneutral in the long run and the natural rate hypothesis loses its validity. In this situation, price stickiness must be taken into consideration in the long-run model.

Opinions are divided among macroeconomists on long-run superneutrality of money. In this study, assuming the Rotemberg-type adjustment cost in the labor market, we derive the new Keynesian Phillips curve (NKPC) under which long-run superneutrality of money does not hold.[4][5]

Tsuzuki and Inoue (2010) and Inoue and Tsuzuki (2011) have proposed the Dynamic General Equilibrium (DGE) model with the NKPC and technological change. In their model, the natural rate hypothesis did not hold, and the output gap existed when the money growth rates was lower than the rate of the technological change. However, their analyses assumed exogenous technological change, as the Solow model.[6]

This paper provides the new Keynesian DGE model based on Inoue and Tsuzuki (2011) with endogenous technological progress, instead of exogenous growth, introducing the explicit R&D activities. That is, in this paper, the new Keynesian theory, which represents the short-run theory, is integrated with the endogenous growth theory, which represents the long-run theory. Using such a model, we examine how money growth affects long-run output, employment, and economic growth along the balanced growth path.

The rest of this paper is organized as follows: The next section sets up the model used in our theoretical investigation. Section 3 derives the law of motion and the steady state, which characterize the equilibrium path of the economy. It also investigates the existence and the uniqueness of the steady state. Section 4 examines the local determinacy of the steady state. Section

[2]For details of the new Keynesian theory, see Woodford (2003) and Gali (2008). As for the endogenous growth theory, see e.g., Barro and Sala-i Martin (2004).

[3]For the natural rate hypothesis, see Friedman (1968) and Lucas (1972).

[4]Akerlof et al. (1996) has argued that long-run superneutrality holds for the inflation rates that is higher than 3%, however money is not superneutral for low inflation rates. This paper discuss the latter situation.

[5]In this respect, it can be interpreted that our NKPC is based on the spirit of the traditional Keynesian's Phillips curve. On the contrary, the other type NKPC based on Firedman's expectations-augmented Phillips curve is conceivable. Under such NKPC, money is superneutral in the long run.

[6]See Solow (1956). Tsuzuki and Inoue (2011) have proposed the new Keynesian model in which sustained growth becomes endogenous by human capital accumulations.

5 provides conclusions.

2 Model

We consider a continuous time version dynamic model based on Inoue and Tsuzuki (2011) and Grossman and Helpman (1991, Chap.3). Let us assume economy populated by many infinitely lived households under monopolistic competition in the labor market, and there are rigidities of nominal wage. There is a single final good, which is produced using intermediate goods and supplied competitively. A new variety of intermediate goods is invented by allocating labor for R&D activities, and inventors enjoy an infinitely-lived monopoly power.[7] The available intermediate goods are produced by multiple intermediate firms using labor. Finally, the financial authorities adopt the k-percent rule, and expand money supply at a constant rate.[8]

2.1 Employment agency

The manufacturing sector and R&D sector regard each household's labor service as an imperfect substitute for the other household's labor. To simplify the analysis, we assume that an employment agency combines differentiated labor forces into composite labor forces according to the Dixit-Stiglitz function:[9]

$$\ell = \left[\int_0^1 \ell_j^\beta dj \right]^{\frac{1}{\beta}}, \quad \beta \in (0,1),$$

and supplies composite labor to the intermediate goods sector and the R&D sector. ℓ_j denotes differentiated labor forces supplied by household j, and ℓ is composite labor forces. The number of households is normalized to 1. $\eta = 1/(1-\beta)(> 1)$ is the elasticity of substitution between each pair of differentiated labor.

Cost minimization of the employment agency yields the following demand functions for differentiated labor $j \in [0,1]$:

$$\ell_j = \left(\frac{W_j}{W} \right)^{-\frac{1}{1-\beta}} \ell,$$

[7] The analyses on a "patent length" using variety-expanding framework was presented by Kwan and Lai (2003), Futagami and Iwaisako (2003, 2007) and Furukawa (2007b).

[8] See Friedman (1969).

[9] See Dixit and Stiglitz (1977) and Blanchard and Kiyotaki (1987).

where W_j denotes the nominal wage rate of labor force j, and W denotes the nominal wage rate of composite labor forces, which is given by

$$W = \left[\int_0^1 W_j^{-\frac{\beta}{1-\beta}} dj \right]^{-\frac{1-\beta}{\beta}}.$$

2.2 Final goods sector

We assume that perfect competition prevails in the final goods market. The final goods firm produces the quantity y according to the Dixit-Stiglitz function as follows:

$$y = \left[\int_0^N x_i^{\alpha} d_i \right]^{\frac{1}{\alpha}}, \quad \alpha \in (0,1),$$

where x_i is the amount of intermediate goods indexed by $i \in [0, N]$, and $\phi = 1/(1-\alpha)$ (> 1) represents the elasticity of substitution between every pair of intermediate goods. N is the number of available intermediate goods, and represents the technology level of the economy. The final goods firm faces diminishing returns in each intermediate good, therefore larger N implies higher productivity.

Cost minimization of the final goods producing firm yields the following demand functions for intermediate goods $i \in [0, N_t]$:

$$x_i = \left(\frac{p_i}{p} \right)^{-\frac{1}{1-\alpha}} y, \tag{2.1}$$

where p_i is the price of intermediate goods i, and p is the price of final good or the price level, which is given by

$$p = \left[\int_0^N p_i^{-\frac{\alpha}{1-\alpha}} di \right]^{-\frac{1-\alpha}{\alpha}}.$$

2.3 Intermediate goods sector

Each intermediate good is produced using one unit of composite labor, thus the marginal cost is equal to the nominal wage level, W. Because of infinitely-lived patent, all intermediate goods are supplied monopolistically. Maximization of the monopoly profit, $\Pi_i = (p_i - W)x_i$ subject to the demand function (2.1) yields

$$p_i = p_x \equiv \frac{1}{\alpha} W, \quad x_i = x \equiv \frac{\ell_x}{N}, \quad \forall i \in [0, N]. \tag{2.2}$$

where ℓ_x represents the amount of composite labor allocated to the production of the intermediate goods. All intermediate goods enter symmetrically into the production of final good. Moreover, the maximized monopoly profit is

$$\Pi_i = \Pi = \frac{1-\alpha}{\alpha} W x_i = \frac{1-\alpha}{\alpha} W \frac{\ell_x}{N}, \quad \forall i \in [0, N]. \tag{2.3}$$

From (2.2), the market equilibrium levels of output, y, and the price of final good, p, are obtained as

$$y = N^{\frac{1}{\alpha}} x = N^{\frac{1-\alpha}{\alpha}} \ell_x, \tag{2.4}$$

$$p = N^{-\frac{1-\alpha}{\alpha}} p_x = N^{-\frac{1-\alpha}{\alpha}} \frac{1}{\alpha} W. \tag{2.5}$$

We can rewrite (2.5) as

$$w \equiv \frac{W}{p} = \alpha N^{\frac{1-\alpha}{\alpha}}. \tag{2.6}$$

2.4 R&D sector

The number of intermediate goods, N, expands according to the following equation[10]:

$$\frac{\dot{N}}{N} = \mu \ell_n, \quad N(0) > 0, \tag{2.7}$$

where $\mu (> 0)$ is the parameter that reflects the productivity of R&D. ℓ_n represents the amount of composite labor allocated to R&D, and labor market clearing requires $\ell = \ell_x + \ell_n$.

In equilibrium, the following free-entry condition must be satisfied:

$$V \leq \frac{W}{\mu N}, \quad \text{with an equality whenever } \dot{N} > 0. \tag{2.8}$$

The right-hand side is the nominal unit cost of R&D. V represents the value of the patent, which is given by the the discounted stream of the monopoly profit:

$$V(t) = \int_t^\infty \Pi(\tau) e^{-\int_t^\tau R(\iota) d\iota} d\tau, \tag{2.9}$$

[10]Here, we retain the linear relation between increase in knowledge and stock of knowledge based on the first-generation R&D-based endogenous growth model such that Romer (1990). However, Jones (1995a,b) have argued that this linearity is problematic assumption. A survey of this issue is presented by Jones (1999, 2005) and Li (2000, 2002).

where R is the nominal interest rate. Differentiating (2.9) with respect to time, t, yields the following no-arbitrage condition:

$$R = \frac{\Pi + \dot{V}}{V} \tag{2.10}$$

2.5 Households

Household j possesses nominal money balances, M_j, and stock of the monopoly firms, S_j. The stock S_j yields interest at rate R. Thus, the budget constraint of household j is given by

$$\dot{A}_j = \dot{M}_j + \dot{S}_j = W_j \ell_j + RS_j - pc_j, \quad \forall j \in [0, 1],$$

where A_j is the nominal asset of household j, ℓ_j is labor supplied elastically by household j, and c_j is consumption of household j. The final goods market clears when $y = c \equiv \int_0^1 c_j dj$. We can rewrite the budget constraint in real terms as follows:

$$\dot{a}_j = \frac{W_j}{p}\ell_j + ra_j - Rm_j - c_j,$$

where $r \equiv R - \pi$ is the real interest rate, $m_j \equiv M_j/p$ is real money balances and $a_j \equiv A_j/p$ is the stock of assets in real terms.

Household j obtains utility from consumption, c_j, and real money balances, m_j, and disutility from labor supply, ℓ_j, and wage negotiations. Thus, the instantaneous utility function of household j is[11]

$$u(c_j, m_j, \ell_j, \omega_j) = \ln c_j + \ln m_j - \frac{\ell_j^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_j^2,$$

where $\psi(> 0)$ is the elasticity of the marginal disutility of labor supply. $\gamma(\geq 0)$ denotes the scale of the nominal wage adjustment cost from wage negotiations and $\omega_j \equiv \dot{W}_j/W_j$.[12] If $\gamma = 0$, the nominal wage is flexible, however if $\gamma > 0$, the nominal wage is sticky.

[11] The money-in-utility-function approach was initiated by Sidrauski (1967).

[12] We specify the adjustment cost function as a quadratic expression following Rotemberg (1982).

Summarizing the above, household j faces the following dynamical optimization problem:

$$\max_{c_j, m_j, \omega_j} \int_0^\infty \left[\ln c_j + \ln m_j - \frac{\ell_j^{1+\psi}}{1+\psi} - \frac{\gamma}{2}\omega_j^2 \right] e^{-\rho t} dt,$$

$$\text{subject to} \quad \dot{a}_j = r a_j + \frac{W_j}{p}\ell_j - c_j - R m_j,$$

$$\dot{W}_j = \omega_j W_j,$$

$$\ell_j = \left(\frac{W_j}{W}\right)^{-\frac{1}{1-\beta}} \ell,$$

where $\rho(>0)$ is the subjective discount rate. Since all households behave symmetrically according to the same equations, $W_j = W$, $c_j = c$, $w_j = w$, $\ell_j = \ell$, and $m_j = m$ hold. When $\gamma > 0$, the solution to the above optimization problem is characterized by the Euler equation and the wage version of the New Keynesian Phillips curve (NKPC), as follows:[13]。

$$\frac{\dot{c}}{c} + \rho + \pi = R = \frac{c}{m}, \tag{2.11}$$

$$\frac{\dot{\omega}}{\omega} = \rho + \frac{\beta}{1-\beta}\frac{\ell w}{c\gamma\omega} - \frac{1}{1-\beta}\frac{\ell^{1+\psi}}{\gamma\omega}, \tag{2.12}$$

where $m \equiv \int_0^1 m_j dj$ is real money balances in the whole of the economy. The transversality condition for the households is given by

$$\lim_{t\to\infty} \frac{a(t)}{c(t)} e^{-\rho t} = 0 \tag{2.13}$$

On the other hand, when $\gamma = 0$ the following equation holds instead of the NKPC (2.12):

$$\beta \frac{w}{c} = \ell^\psi \tag{2.14}$$

2.6 Money growth

Financial authorities are assumed to expand money supply, M, at a constant rate θ. That is, the financial policy rule is given by $\dot{M}/M = \theta$. Therefore, the following equation holds:

$$\frac{\dot{m}}{m} = \theta - \pi.$$

[13] Appendix A provides detailed derivations.

3 Equilibrium Path

When the nominal wage is sticky ($\gamma > 0$), and the positive composite labor is allocated to R&D at any time ($\ell_n > 0$), the equilibrium path is characterized by the transversality conditions (2.13) and the following differential equations:[14]

$$\frac{\dot{R}}{R} = R - \theta - \rho, \tag{3.1}$$

$$\frac{\dot{\chi}}{\chi} = R - \rho - \omega, \tag{3.2}$$

$$\frac{\dot{\omega}}{\omega} = \rho + \left(\frac{\ell}{\chi} - \ell^{1+\psi}\right)\frac{\eta}{\gamma\omega}, \tag{3.3}$$

where $\chi \equiv \ell_x/(\alpha\beta)$, and

$$\ell = \ell(R, \chi, \omega) = \frac{\omega - R}{\mu} + \beta\chi. \tag{3.4}$$

When R, χ, and ω are given, we obtain the ℓ_x, ℓ_n, π as follows:

$$\ell_x = \alpha\beta\chi, \tag{3.5}$$

$$\ell_n = \frac{\omega - R}{\mu} + (1-\alpha)\beta\chi, \tag{3.6}$$

$$\pi = \pi(R, \ell_x, \omega) = \omega - \frac{1-\alpha}{\alpha}\mu\ell_n. \tag{3.7}$$

3.1 Steady State

If the law of motion (3.1) – (3.3) has fixed points, these are derived as follows:

$$R^* = \theta + \rho,$$
$$\omega^* = \theta,$$
$$\chi^* \equiv \chi^*(\ell^*), \quad \ell^* > \underline{\ell} \equiv \frac{\alpha}{1-\alpha}\frac{\rho}{\mu},$$

where $\chi^*(\ell^*)$ is the increasing function of ℓ^* defined as

$$\chi^*(\ell^*) = \frac{\ell^*}{\beta} + \frac{\rho}{\beta\mu}. \tag{3.8}$$

[14] The full derivations are given in Appendix B. We can show that the similar differential equations system is derived from the *lab-equipment model* based on Rivera-Batiz and Romer (1991).

When ℓ^* is given, the steady-state value of χ is derived according to (3.8). We obtain the steady-state value of the employment level, ℓ^*, as the root of the following implicit function:[15]

$$\Lambda(\ell^*) \equiv \frac{\gamma\theta\rho}{\eta} + \frac{\ell^*}{\chi^*(\ell^*)} - (\ell^*)^{1+\psi} = 0. \tag{3.9}$$

The steady-state values of ℓ_x and ℓ_n are

$$\ell_x^*(\ell^*) = \alpha\ell^* + \alpha\frac{\rho}{\mu}, \tag{3.10}$$

$$\ell_n^*(\ell^*) = (1-\alpha)\ell^* - \alpha\frac{\rho}{\mu}. \tag{3.11}$$

However, to guarantee that ℓ_n^* is positive, ℓ^* must be larger than $\underline{\ell}$.

If ℓ^* $(> \underline{\ell})$ exists , at this fixed point, y, and N grow at constant rates, i.e., the economy achieves the balanced growth. We shall define this steady state as the *balanced growth path.* From (2.4) and (2.7), the balanced growth rate of output is derived as

$$g_y^*(\ell^*) = \frac{1-\alpha}{\alpha}\mu\ell_n^*(\ell^*).$$

From (3.7), the inflation rate along the balanced growth path is given by the difference between the money growth rate and the long-run growth rate as shown by Siegel (1983), that is,

$$\pi^* = \theta - g_y^*(\ell^*). \tag{3.12}$$

However, the long-run growth rate is exogenous and constant in Siegel (1983).[16]

3.1.1 Existence and uniqueness of the balanced growth path

Case of non-negative money growth If the money growth rate, θ, is non-negative, we can show existence and uniqueness of the balanced growth path in the following way.

[15] See Appendix C.1.

[16] Siegel's equation includes the positive population growth rate, which is supposed to be zero in our model.

Proposition 1 Let $\theta \geq 0$. If and only if $\theta > \theta_1$, the implicit function, $\Lambda(\ell) = 0$, has a unique root, such that $\ell = \ell^* > \underline{\ell}$. On the other hand, if $\theta \leq \theta_1$, $\Lambda(\ell) = 0$ has no root in $(\underline{\ell}, \infty)$. θ_1 is a constant defined as follows:

$$\theta_1 \equiv \frac{\eta}{\gamma\rho} \left\{ \left[\frac{\alpha}{1-\alpha} \frac{\rho}{\mu} \right]^{1+\psi} - \alpha\beta \right\}.$$

proof. See Appendix C.2. □

$\theta > \theta_1$ is a necessary and sufficient condition for $\Lambda(\underline{\ell}) > 0$. When the parameters satisfies

$$\frac{\rho}{\mu} < \Gamma_1 \equiv \frac{1-\alpha}{\alpha} (\alpha\beta)^{\frac{1}{1+\psi}},$$

$\theta_1 < 0$ holds, thus $\theta \geq 0 > \theta_1$ always holds, whereas if $\rho/\mu \geq \Gamma_1$, the existence of the balanced growth path requires that the money growth rate, θ, is sufficiently large. When θ is small and the balanced growth path does not exist, there is only the no-growth steady state mentioned below.

Case allowing negative money growth When we allow the negative value of θ, Proposition 1 is rewritten to a weaker proposition as follows.

Proposition 2 If $\theta > \theta_1$, $\Lambda(\ell) = 0$ has a unique root such that $\ell > \underline{\ell}$.

proof. See Appendix C.3. □

That is, $\theta > \theta_1$ is a necessary but not a sufficient condition for the existence of a unique balanced growth path. To repair the sufficiency, the following parameter restriction is required:

Assumption 1

$$\frac{\rho}{\mu} \geq \Gamma_2 \equiv \frac{1-\alpha}{\alpha} \left[\frac{\alpha\beta(1-\alpha)}{1+\psi} \right]^{\frac{1}{1+\psi}}.$$

$\Gamma_1 > \Gamma_2$ always holds. Assumption 1 is a necessary and sufficient condition for $\Lambda'(\underline{\ell}) \leq 0$.

Proposition 3 Let the parameters satisfy Assumption 1. If and only if $\theta > \theta_1$, the implicit function, $\Lambda(\ell) = 0$, has a unique root, such that $\ell = \ell^* > \underline{\ell}$. In contrast, if $\theta \leq \theta_1$, $\Lambda(\ell) = 0$ has no root in $(\underline{\ell}, \infty)$.

	$\theta > \theta_1$	$\theta = \theta_1$	$\theta < \theta_1$
$\theta \geq 0$	1	0 (–)	0 (–)
$\theta \in (\theta_2, 0)$	1	0 (1)	0 (2)
$\theta = \theta_2$	–	0	0 (1)
$\theta < \theta_2$	–	–	0

Table 1: The number of balanced growth paths.

The number in each cell indicates the number of balanced growth paths. The case that Assumption 1 is not fulfilled corresponds to the number in parenthesis. "–' shows that there does not exist such combinations of parameters.

proof. See Appendix C.4. □

That is, under Assumption 1, $\theta > \theta_1$ is a necessary and sufficient condition for a unique balanced growth path, again. Sufficiently high money growth rates are required to achieve sustained economic growth.

On the other hand, when parameters do not satisfy Assumption 1, it is possible that multiple balanced growth paths exist for negative money growth rates.

Proposition 4 Let $\rho/\mu < \Gamma_2$ hold. There exists the threshold of θ, $\theta_2(< \theta_1)$, and $\Lambda(\ell) = 0$ has two roots, ℓ_1^* and ℓ_2^*, which belong to $(\underline{\ell}, \infty)$ if and only if $\theta_2 < \theta < \theta_1$ holds.[17]

proof. See Appendix C.5. □

Letting $\ell_1^* < \ell_2^*$, we obtain $g_y^*(\ell_n^*(\ell_1^*)) < g_y^*(\ell_n^*(\ell_2^*))$. Therefore, when the money growth rate, θ, is negative and belongs to (θ_2, θ_1), the balanced growth paths with the high growth rate and low growth rate coexist. Since R, χ and ω are jump variables, our model has no mechanism to choose between them, i.e., global indeterminacy arises.[18] The behavior of the economy is determined depending on what the agents expect.

The arguments of Propositions 1–4 are summarized in Table 1.

3.2 Money growth, inflation and economic growth

Let $\theta > \theta_1$ holds and a unique balanced growth path exist. Then, we obtain the following proposition.

[17] Since $\Gamma_1 > \Gamma_2$, θ_1 is negative as long as Assumption 1 is not satisfied.

[18] As for local indeterminacy, 4 provides detailed analyses.

Proposition 5 Let $\theta > \theta_1$ hold. In response to a permanent increase in the money growth rate, θ, the economy experiences larger employment, and faster economic growth along the unique balanced growth path.

This proposition can be easily proved as follows. First, applying the implicit function theorem to (3.9), we show

$$\frac{d\ell^*}{d\theta} = -\frac{\Lambda_\theta}{\Lambda_{\ell^*}} > 0, \tag{3.13}$$

where Λ_X denotes a partial derivative of Λ, with respect to X. Λ_θ is equal to $\gamma\rho/\eta$ and positive, and Λ_{ℓ^*} is negative as shown in Appendix C.2, therefore, $d\ell^*/d\theta$ is positive. Since $(\ell_x^*)'(\ell^*) > 0$ and $(\ell_n^*)'(\ell^*) > 0$, a increase in ℓ^* raises labor allocated each sectors.[19] As a result, since $(g_y^*)'(\ell_n^*) > 0$, the larger value of θ raises g_y^*. That is, the faster money growth is, the faster economic growth is.

The positive relation between θ and g_y^* has been shown in Proposition 5. Therefore, even if the financial authorities add 0.1% to the money growth rate, the rise of long-run inflation rate is smaller than 0.1% because of the rise of the long-run growth rate, g_y^* (See (3.12)). Further, for the high productivity of R&D, which is captured by large μ, it is possible that the inflation rate even decreases.

Proposition 6 Let $\theta > \theta_1$ hold. In response to a permanent increase in the money growth rate, θ, the long-run inflation rate, π^*, decreases for sufficiently large μ.

proof. See Appendix C.6. □

3.3 Output Gap

We shall refer to the output and the employment level in the flexible-price economy (i.e., when $\gamma = 0$) as the *natural output level* and the *natural employment level*, respectively. The *output gap* is the difference between the actual output level and the natural output level.

In the flexible-price economy, the employment level, ℓ, is characterized by (2.14) instead of NKPC(2.12). Then, Substituting (2.6), (2.4), (3.10), and $y = c$ into (2.14), we obtain the natural employment level along the balanced growth path, ℓ^{**}, as the root of the following implicit function:

$$\Lambda|_{\gamma=0}(\ell^{**}) \equiv \frac{\ell^{**}}{\chi^*(\ell^{**})} - (\ell^{**})^{1+\psi} = 0.$$

[19] In addition, ℓ_n/ℓ_x increases.

When $\theta = 0$, $\Lambda(\ell)$ becomes the identical form with $\Lambda_{\gamma=0}(\ell)$. Therefore, when the financial authorities apply the monetary policy with $\theta = 0$, $\ell^* = \ell^{**}$ holds and the output gap caused by price stickiness is eliminated. However, if $\rho/\mu \geq \Gamma_1$, the implicit function, $\Lambda|_{\gamma=0}(\ell) = 0$, has no root that is larger than $\underline{\ell}$.

3.4 No-growth steady state

There exists a different steady state from the balanced growth path, at which no labor is allocated to R&D and long-run growth never occur. We refer to such a steady state as the *no-growth steady state*. At the no-growth steady state, since the free-entry condition (2.8) does not hold with an equality, (3.4), (3.5), and (3.6) are not fulfilled, and $\ell_n = 0$ and $\ell = \ell_x$ hold instead of them.

The value of each variable at this steady state is derived as follows:

$$
\begin{aligned}
R^0 &= \theta + \rho, \\
\pi^0 &= \omega^0 = \theta, \\
\chi^0 &= \frac{l_x^0}{\alpha\beta}, \\
\ell^0 &= \ell_x^0 = \left[\frac{\gamma\theta\rho}{\eta} + \alpha\beta\right]^{\frac{1}{1+\psi}}.
\end{aligned}
$$

Under Assumption 1, the no-growth steady state, (R^0, χ^0, ω^0), is a unique steady state of the economy for $\theta \leq \theta_1$, whereas it coexists with the balanced growth path, (R^*, χ^*, ω^*) for $\theta > \theta_1$. Along the balanced growth path, the economy exhibits sustained growth at the rate, g_y^*, while at the no-growth steady state, exhibits no sustained growth. Further, for $\theta > \theta_1$, since $\Lambda(\ell^0) > 0$ holds, then we can show that $\ell^0 < \ell^*$. That is, the no-growth steady state has lower employment level than the balanced growth path. Since R, χ and ω are jump variables, our model has no mechanism to choose between them. That is, again, global indeterminacy arises.[20]

[20]If two balanced growth paths exist as shown in Proposition 4, there are three steady state in all, and global indeterminacy arises among them.

4 Local determinacy of balanced growth paths

In order to examine the local stability, we linearize the system (3.1)–(3.3) around the fixed point, (R^*, χ^*, ω^*).

$$\begin{bmatrix} \dot{R} \\ \dot{\chi} \\ \dot{\omega} \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} R - R^* \\ \chi - \chi^* \\ \omega - \omega^* \end{bmatrix}, \quad \text{where} \quad \boldsymbol{J} \equiv \begin{bmatrix} \theta + \rho & 0 & 0 \\ \chi^* & 0 & -\chi^* \\ 0 & \frac{\eta\beta}{\gamma}\Lambda'(\ell^*) & \rho \end{bmatrix},$$

where $\Lambda'(\ell)$ has been derived as (C.1). One of three eigenvalues of the Jacobian matrix, $\boldsymbol{J}$, is $\theta + \rho (> 0)$, and other two eigenvalues are equal to the eigenvalues of the following sub matrix:

$$\boldsymbol{J}_1 \equiv \begin{bmatrix} 0 & -\chi^* \\ \frac{\eta\beta}{\gamma}\Lambda'(\ell^*) & \rho \end{bmatrix}.$$

Here, $\operatorname{tr}\boldsymbol{J}_1 = \rho > 0$ and $\det \boldsymbol{J}_1 = \frac{\eta\beta}{\gamma}\chi^*\Lambda'(\ell^*)$ hold.

For the unique balanced growth path First, we study the dynamical property of the unique balanced growth path, which is mentioned in Proposition 1 – 3. Since $\Lambda'(\ell^*) < 0$ holds, $\det \boldsymbol{J}_1$ is negative. Therefore $\boldsymbol{J}_1$ has two real eigenvalue with opposite sign. As a result, Jacobian matrix, $\boldsymbol{J}$, has one negative real root and two positive real roots. Since R, χ and ω are jump variables, the fixed point is locally indeterminate.[21]

For the multiple balanced growth path Next, we analyze the case of the multiple equilibrium, which is argued in Proposition 4. Let ℓ_1^* and ℓ_2^* denote the roots of $\Lambda(\ell) = 0$, and $\ell_1^* < \ell_2^*$. Then, $\Lambda'(\ell_1^*) > 0$ and $\Lambda'(\ell_2^*) < 0$ hold. For ℓ_1^*, $\operatorname{tr}\boldsymbol{J}_1 > 0$ and $\det \boldsymbol{J}_1 > 0$ hold, so that the both two roots of J_1 have positive real parts. Since all eigenvalues of $\boldsymbol{J}$ has positive real parts, this fixed point is locally determinate.[22]

On the other hand, as for ℓ_2^*, since $\det \boldsymbol{J}_1 < 0$, $\boldsymbol{J}_1$ or $\boldsymbol{J}$ has one negative real root. Therefore, the fixed point is locally indeterminate.

[21] Such local indeterminacy can be connected with sunspots or business cycle. A analysis on (both local and global) indeterminacy using the variety-expanding framework was presented by Benhabib and Perli (1994), Evans et al. (1998), Furukawa (2007a,b), Haruyama (2009).

[22] However, there are two balanced growth paths and a no-growth steady state, therefore, global indeterminacy remains.

5 Conclusions

This study has developed a new Keynesian model introducing R&D activities and endogenous technological change. When the money growth rate is sufficiently high, the economy has a unique balanced growth path, and can sustain long-run positive growth based on sustained R&D efforts. Furthermore, the faster money growth brings the larger employment and faster economic growth along a balanced growth path. In contrast, under some parameter restrictions, when the money growth rate is sufficiently small, there does not exist a balanced growth path and the economy is trapped in the no-growth steady state. These results suggest that money growth may be an important factor for long-run economic growth.

Appendix A Dynamical Optimization of Households

Let us define the Hamiltonian function of the optimal problem (2.11) as follows:

$$\begin{aligned}\mathcal{H} = &\ln c_j + \ln m_j - \frac{1}{1+\psi}\left[\left(\frac{W_j}{W}\right)^{-\frac{1}{1-\beta}}\ell\right]^{1+\psi} - \frac{\gamma}{2}\omega_j^2 \\ &+ \xi_1\left[ra_j + \frac{W_j}{p}\left(\frac{W_j}{W}\right)^{-\frac{1}{1-\beta}}\ell - c_j - Rm_j\right] + \xi_2\omega_j W_j,\end{aligned}$$

where ξ_1 and ξ_2 are co-state variables of a_j and W_j, respectively. A set of necessary conditions for optimality can be written as follows:

$$\frac{\partial \mathcal{H}}{\partial c_j} = \frac{1}{c_j} - \xi_1 = 0, \tag{A.1}$$

$$\frac{\partial \mathcal{H}}{\partial m_j} = \frac{1}{m_j} - \xi_1 R = 0, \tag{A.2}$$

$$\frac{\partial \mathcal{H}}{\partial \omega_j} = -\gamma\omega_j + \xi_2 W_j = 0, \tag{A.3}$$

$$\dot{\xi}_1 = \rho\xi_1 - \frac{\partial \mathcal{H}}{\partial a_j} = (\rho - r)\xi_1, \tag{A.4}$$

$$\dot{\xi}_2 = \rho\xi_2 - \frac{\partial \mathcal{H}}{\partial W_j} = \rho\xi_2 - \left[\frac{\ell_j^{1+\psi}}{(1-\beta)W_j} - \frac{\beta}{1-\beta}\xi_1\frac{\ell_j}{p} + \xi_2\omega_j\right]. \tag{A.5}$$

Further, the transversality condition is given by

$$\lim_{t\to\infty} \xi_1(t)a_j(t)e^{-\rho t} = 0.$$

Derivation of (2.11) From (A.1) and (A.2), we get $R = c_j/m_j$. In addition, from (A.2) and (A.4), we get $-\dot{c}_j/c_j = \rho - r$. Substituting $c = c_j, m = m_j, \forall j$, and $r = R - \pi$ into these equations yields (2.11).

Derivation of (2.12) **and** (2.14) When $\gamma > 0$, from (A.3), $\xi_2 > 0$ holds. Therefore, We can divide the both sides of (A.5) by ξ_2 as follows:

$$\frac{\dot{\xi}_2}{\xi_2} = \rho - \left[\frac{\ell_j^{1+\psi}}{(1-\beta)W_j} - \frac{\beta}{1-\beta}\xi_1\frac{\ell_j}{p}\right]\xi_2^{-1} - \omega_j.$$

Substituting $\xi_1 = 1/c_j$ and $\xi_2 = \gamma\omega_j/W_j$ into above equation, we obtain

$$\frac{\dot{\omega}_j}{\omega_j} - \frac{\dot{W}_j}{W_j} = \rho - \left[\frac{\ell_j^{1+\psi}}{(1-\beta)W_j} - \frac{\beta}{1-\beta}\frac{\ell_j}{pc_j}\right]\left(\frac{W_j}{\gamma\omega_j}\right) - \omega_j.$$

Since $W_j = W$, $\omega_j = \omega$, $\ell_j = \ell$, $c_j = c$、 $\forall j$, (2.12) holds.

On the other hand, when $\gamma = 0$, ξ_2 and $\dot{\xi}_2$ are equal to zero from (A.3). Then, from (A.5) and $\xi_1 = 1/c_j$, we obtain

$$\beta\frac{(W_j/p)}{c_j} = \ell_j^{\psi}.$$

Therefore, (2.14) holds.

Appendix B Derivation of the law of motion

Appendix B.1 Derivation of (3.7)

From (2.6),

$$\frac{\dot{w}}{w} = \omega - \pi = \frac{1-\alpha}{\alpha}\frac{\dot{N}}{N},$$

or

$$\pi = \omega - \frac{1-\alpha}{\alpha}\mu\ell_n.$$

Substituting $\ell_n = \alpha\beta\chi$, we obtain (3.7).

Appendix B.2 Derivation of (3.4) and (3.6)

From the free-entry condition (2.8),

$$\frac{\dot{V}}{V} = \omega - \mu \ell_n.$$

From (2.10), (2.3), and (2.8),

$$\frac{\dot{V}}{V} = R - \frac{\Pi}{V} = R - \frac{1-\alpha}{\alpha} \ell_x \mu.$$

Eliminating $\dot{V}/V$ from above two equations, we obtain

$$\ell_n = \frac{\omega - R}{\mu} + \frac{1-\alpha}{\alpha} \ell_x,$$

and substituting $\ell_x = \alpha\beta\chi$, we get (3.6). Moreover, substituting (3.6) and (3.5) into the labor market clearing condition, $\ell = \ell_x + \ell_n$, yields (3.4).

Appendix B.3 Derivation of (3.1)

From (2.11),

$$\begin{aligned} \frac{\dot{R}}{R} = \frac{\dot{c}}{c} - \frac{\dot{m}}{m} &= (R - \rho - \pi) - (\theta - \pi) \\ &= R - \rho - \theta. \end{aligned}$$

Appendix B.4 Derivation of (3.2)

From (2.4),

$$\frac{\dot{y}}{y} = \frac{1-\alpha}{\alpha} \frac{\dot{N}}{N} + \frac{\dot{\ell}_x}{\ell_x}.$$

From the Euler equation (2.11) and the final goods market clearing condition, $y = c$,

$$R - \rho - \pi = \frac{1-\alpha}{\alpha} \mu \ell_n + \frac{\dot{\ell}_x}{\ell_x}.$$

Using (3.7) and $\dot{\chi}/\chi = \dot{l}_x/l_x$, we obtain (3.2).

Appendix B.5 Derivation of (3.3)

From (2.4) and (2.6), $(\ell/c)w = (\ell/y)w = \alpha\ell/\ell_x$ holds, and substituting this equation into (2.12) yields

$$\frac{\dot{\omega}}{\omega} = \rho + \left[\alpha\beta\frac{\ell}{\ell_x} - \ell^{1+\psi}\right]\frac{\eta}{\gamma\omega}.$$

Using $l_x = \alpha\beta\chi$, we get (3.3).

Appendix C Balanced growth path

Appendix C.1 Derivation of ℓ^*

At the steady state, $\omega^* - R^* = -\rho$ holds, then substituting into (3.4) yields (3.8). Moreover, from (3.3), we obtain

$$\frac{\dot{\omega}}{\omega} = \rho + \left(\frac{\ell^*}{\chi^*(\ell^*)} - (\ell^*)^{1+\psi}\right)\frac{\eta}{\gamma\theta} = 0$$

Therefore, we show that $\Lambda(\ell^*) = 0$ holds at the steady state.

Appendix C.2 Proof of Proposition 1

The derivative of $\Lambda(\ell)$ is given by

$$\Lambda'(\ell) = \frac{1}{[\chi^*(\ell)]^2}\frac{\rho}{\beta\mu} - (1+\psi)\ell^{\psi} \tag{C.1}$$

Since $(\chi^*)'(\ell) = 1/\beta > 0$, $\Lambda''(\ell) < 0$. $\Lambda'(0) = \beta\mu/\rho > 0$, then $\Lambda(\ell)$ is concave and a unimodal form for $\ell > 0$. Further, since $\Lambda(0) = \gamma\theta\rho/\eta \geq 0$ and $\lim_{\ell^*\to\infty}\Lambda(\ell^*) = -\infty$ hold for $\theta > 0$, $\Lambda(\ell) = 0$ has a unique positive root.

$\Lambda(\underline{\ell}) > 0$ is satisfied for $\theta > \theta_1$. In this case, $\Lambda(\ell) = 0$ has a unique root, ℓ^*, which is larger than $\underline{\ell}$, and $\Lambda'(\ell^*) < 0$ always holds. In contrast, since $\Lambda(\underline{\ell}) \leq 0$ is satisfied for $\theta \leq \theta_1$, $\Lambda(\ell) = 0$ has no root in $(\underline{\ell}, \infty)$. (See Figure 1.) □

Appendix C.3 Proof of Proposition 2

$\Lambda(\underline{\ell}) > 0$ holds for $\theta > \theta_1$. Since $\lim_{\ell^*\to\infty}\Lambda(\ell^*) = -\infty$, $\Lambda(\ell) = 0$ has a unique root that belongs to $(\underline{\ell}, \infty)$.[23] □

[23] However, when $\theta < 0$, we can not rule out the possibility of the existence of balanced growth paths in spite of $\theta \leq \theta_1$ or $\Lambda(\underline{\ell}) < 0$.

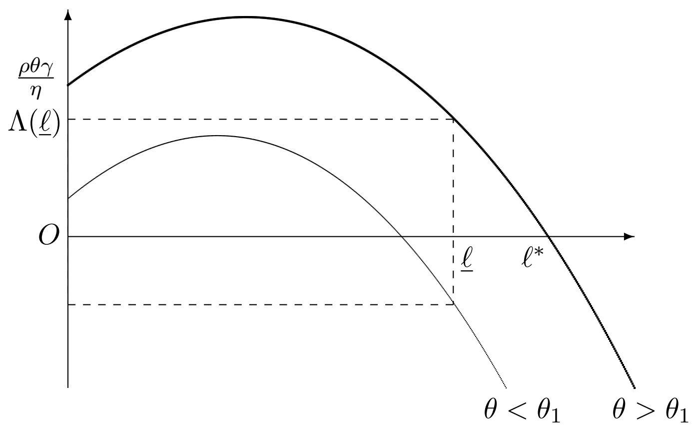

Figure 1: A unique balanced growth path (for $\theta > 0$).

Appendix C.4 Proof of Proposition 3

Some algebra shows that

$$\Lambda'(\underline{\ell}) = \frac{1+\psi}{\underline{\ell}} \left[\frac{1}{1+\psi} \alpha\beta(1-\alpha) - \underline{\ell}^{1+\psi} \right].$$

Therefore, if and only if Assumption 1 is fulfilled, $\Lambda'(\underline{\ell}) \leq 0$ holds. $\Lambda''(\ell) < 0$ guarantees that $\Lambda(\ell)$ is a decreasing function for $\ell > \underline{\ell}$. Since $\Lambda(\underline{\ell}) < 0$ for $\theta \leq \theta_1$, $\Lambda(\ell) = 0$ has no root in $(\underline{\ell}, \infty)$. □

Appendix C.5 Proof of Proposition 4

Since $\Lambda(\ell)$ is concave and unimodal for positive ℓ, there uniquely exists $\ell^{\max} > 0$ such that $\Lambda'(\ell^{\max}) = 0$, and $\Lambda(\ell)$ is maximal at $\ell^{\max}$. $\ell^{\max}$ does not depend on θ, while $\Lambda(\ell^{\max})$ is increasing in θ. Therefore, there uniquely exists a threshold value of θ, θ_2, such that $\Lambda(\ell^{\max}) = 0$ holds for $\theta = \theta_2$ and $\Lambda(\ell^{\max}) > 0$ for $\theta > \theta_2$. Thus, $\Lambda(\ell) = 0$ has two positive root for $\theta > \theta_2$. Since $\Lambda'(\underline{\ell}) > 0$ for $\rho/\mu < \Gamma_2$, both roots are larger than $\underline{\ell}$.(See Figure 2.) □

Appendix C.6 Proof of Proposition 6

Differentiating (3.12) with respect to θ yields

$$\frac{\partial \pi^*}{\partial \theta} = 1 - \frac{\partial g_y^*}{\partial \theta} = 1 - \frac{(1-\alpha)^2}{\alpha} \mu \frac{\partial \ell^*}{\partial \theta}.$$

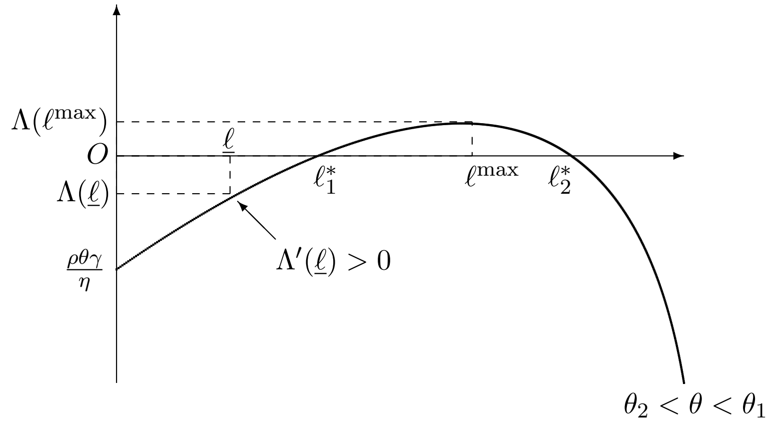

Figure 2: multiple balanced growth paths

Our purpose is to show that this equation becomes negative for large μ. Substituting (3.13) into the above equation, we obtain the following condition:

$$-\frac{(1-\alpha)^2\gamma\rho}{\alpha\eta}\mu < \Lambda'(\ell^*). \tag{C.2}$$

We can calculate ℓ^* as $\mu \to \infty$ as follows:[24][25]

$$\ell^* = \left(\frac{\gamma\theta\rho}{\eta} + \beta\right)^{\frac{1}{1+\psi}}.$$

Then, we obtain

$$\Lambda'(\ell^*) = \frac{\rho\beta}{\ell^*\mu} - (1+\psi)(\ell^*)^{\psi} = -(1+\psi)\left(\frac{\gamma\theta\rho}{\eta} + \beta\right)^{\frac{\psi}{1+\psi}}, \quad \text{as } \mu \to \infty.$$

As $\mu \to \infty$, the right-hand side of (C.2) converges the finite negative value as shown above, whereas, the left-hand side continue to decrease toward $-\infty$. Taking the continuity of both sides into consideration, we can argue that (C.2) holds for sufficiently large μ.

[24] Since $\chi^* = 1/\lambda^*$, we can rewrite the implicit function as

$$\Lambda|_{\mu\to\infty}(\ell^*) = \frac{\gamma\theta\rho}{\eta} + \beta - (\ell^*)^{1+\psi} = 0.$$

[25] Since $\Lambda_\mu > 0$, $\partial\ell^*/\partial\mu = -\Lambda_\mu/\Lambda'(\ell^*) > 0$ holds by applying the implicit function theorem.

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