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Extended anonymity and Paretian relations on infinite utility streams*

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Abstract

This paper examines the range of anonymity that is compatible with a Paretian social welfare relation (SWR) defined on the set of infinite utility streams. We consider three cases of coherence properties for a Paretian SWR: acyclicity, quasi-transitivity, and Suzumura consistency. For each case, we show that the set of permissible permutations of a Paretian SWR, namely a set of permutations with respect to which every utility stream is declared to be indifferent to the permuted stream by a given Paretian SWR, is characterized with a weakening of group structure of a set of permutations. Using the characterization results, we obtain that anonymity defined by the set of all cyclic permutations is the unique maximal anonymity for a Paretian acyclic SWR and a Paretian quasi-transitive SWR. On the other hand, we show that, to define maximal anonymity for a Paretian Suzumura-consistent, a maximal group of cyclic permutations must be used. This paper also characterizes the class of anonymous Paretian SWRs by using an anonymity axiom defined by a set $\mathcal{Q}$ of permutations that satisfies the restrictions specified in the characterizations of permissible permutations. We show that, for each of the three cases of coherence properties, the class of anonymous Paretian SWRs coincides with the class of all SWRs that include the $\mathcal{Q}$-Pareto relation as a subrelation.

Keywords: Intergenerational equity, Pareto-compatible anonymity, Acyclicity, Quasi-transitivity, Suzumura consistency

JEL Classification Numbers: D63, D71

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1 Introduction

In evaluating infinite utility streams which represent the utility levels of infinitely many generations, two basic axioms have been accepted. One is the Pareto axiom, which postulates positive sensitivity of the evaluation to every generation's utility, and the other is an anonymity axiom, which formalizes equal treatment of generations by asserting that two streams related by a permutation are indifferent. In the infinite-horizon setting, whether an anonymity axiom is compatible with a Paretian relation (a binary relation satisfying the Pareto axiom) depends on which set of permutations is considered in the anonymity axiom in question. The anonymity axiom defined by the set of all possible permutations is incompatible with any Paretian relation (Lauwers, 1997a; van Liedekerke, 1995). This incompatibility still holds for some restricted sets of permutations (Fleurbaey and Michel, 2003; Lauwers, 1998; Sakai, 2010b). However, if we consider the set of all finite permutations, there exists a social welfare quasi-ordering (SWQ) that satisfies both the Pareto axiom and the corresponding anonymity axiom (Svensson, 1980).[1] Further, the anonymity axiom defined by the set of all fixed-step permutations, a superset of the set of all finite permutations, is compatible with Paretian SWQs (Lauwers, 1997b).[2]

Recent work of Mitra and Basu (2007) presented a systematic analysis of Pareto-compatible anonymity axioms. They identified the set of *permissible permutations* of a Paretian SWQ, a set of permutations with respect to which every utility stream is declared to be indifferent to the permuted stream by a given Paretian SWQ. They show that the set of permissible permutations of a Paretian SWQ (i) consists only of *cyclic* permutations and (ii) constitutes a *group* with respect to composition of permutations. Furthermore, they establish that the converse implication is true, namely, if a set of permutations is a group of cyclic permutations, then there exists a Paretian SWQ for which the set of permissible permutations coincides with the given set of permutations.

The purpose of this paper is to examine permissible permutations of a Paretian social welfare relation (SWR) that satisfies a coherence property weaker than transitivity.[3] The coherence properties that we consider are *acyclicity*, *quasi-transitivity*, and *Suzumura consistency*. Acyclicity requires that a binary relation has no cycle of strict preferences. Quasi-transitivity postulates that strict preferences are transitive. Suzumura consistency, which is first introduced by Suzumura (1976) under the name consistency, requires that a binary relation has no cycle involving at least one strict preference. That is, Suzumura consistency excludes preference cycles which are known as the phenomenon of money-pump. Both quasi-transitivity and Suzumura consistency imply acyclicity, whereas they are logically independent. Besides being intuitively appealing, these coherence properties are of importance

[1] An SWQ is a reflexive and transitive binary relation.

[2] For the finite anonymous or fixed-step anonymous Paretian SWQs proposed in the literature, see the review article by Asheim (2010). See also the researches we cite in the discussion about Figure 3 in Sect. 5.

[3] An SWR is a reflexive binary relation.

from a theoretical point of view. It is well known that acyclicity is necessary and sufficient for a binary relation to yield maximal elements in a finite set of alternatives. Quasi-transitivity is sufficient and almost necessary for a binary relation to yield social decision making satisfying the Arrow-Platt path independence condition (Blair, Bordes, Kelly, and Suzumura, 1976). Suzumura consistency is necessary and sufficient for the existence of an ordering extension (Suzumura, 1976).[4]

In the literature, it is known that the anonymity axiom defined by the set of all variable-step permutations is compatible with Paretian quasi-transitive SWRs (Fleurbaey and Michel, 2003; Sakai, 2010a).[5] However, there is no systematic analysis that clarifies which sets of permutations yield an anonymity axiom that is compatible with Paretian SWRs when the coherence property required is one of acyclicity, quasi-transitivity, and Suzumura consistency.

In our analysis, we consider four properties of a set of permutations. Property 1 requires that the composition operation is closed in a given set of permutations. Properties 2 and 3, respectively, postulates that a set of permutations must contain the identity permutation and the inverse permutation for each permutations in the set. The conjunction of Properties 1, 2, and 3 is equivalent to requiring a set of permutations to be a group of permutations. The forth property is Property 1*. It requires that the composition of any finite number of permutations in a given set must be a cyclic permutation. When we consider a set of cyclic permutations, this property is a weakening of Property 1. We first establish that cyclicity of permutations and Properties 2 and 3 together are necessary and sufficient for the existence of a Paretian acyclic SWR for which the set of permissible permutations coincides with a given set of permutations. Further, we obtain that even in the case where the acyclicity of a Paretian relation is strengthened to quasi-transitivity, the necessary and sufficient condition is given by the same set of restrictions. On the other hand, we show that if a Paretian SWR is required to be Suzumura-consistent, the necessary and sufficient condition is given by adding Property 1*. In each of the three characterization results, the sufficiency part is proved by showing that the extended Pareto rule, which we call *Q-Pareto relation*, satisfies the requisite conditions.[6]

Using the above-mentioned results, we discuss a maximal anonymity axiom that is compatible with Paretian SWRs for each case of the three coherence properties. We obtain that the set of all cyclic permutations yields the unique maximal anonymity axiom for Paretian acyclic SWRs and Paretian quasi-transitive SWRs. On the other hand, a maximal anonymity axiom that is compatible with Paretian

[4]This characterization result strengthens the variant of Szpilrajn's (1930) lemma due to Arrow (1963) and Hansson (1968) that establishes that any quasi-ordering has an ordering extension.

[5]It should be noted that Campbell (1985), Fleurbaey and Michel (2003), Sakai (2003, 2006), and Shinotsuka (1998) have discussed, without assuming any coherence property of a binary relation, the conflict between the Pareto axiom and the anonymity axiom defined by the set of all finite permutations. However, in their analyses, a certain continuity of a binary relation is assumed.

[6]The Pareto rule has been originally introduced by Sen (1970) and Weymark (1984) in the framework of finite population social choice.

Suzumura-consistent SWRs is a non-constructible object. We establish this by (i) showing that, for any set $\mathcal{Q}$ of cyclic permutations satisfying Properties 1^*, 2, and 3, its closure $\overline{\mathcal{Q}}$ (defined with respect to the composition operation) is a group of cyclic permutations and (ii) combining this with the impossibility of constructing a maximal group of cyclic permutations proved by Lauwers (2010b). In contrast to this non-constructibility result, our result that confirms group structure of $\overline{\mathcal{Q}}$ leads to the following positive result regarding the construction of non-maximal groups of cyclic permutations: given two constructible groups of cyclic permutations, if the union of them satisfies Property 1^*, we can construct another group of cyclic permutations that includes both the two given groups of cyclic permutations. We demonstrate the application of this finding and present a new group of cyclic permutation that includes the set of all fixed-step permutations as a strict subset.

In this paper, we also provide characterizations of classes of anonymous Paretian SWRs. We show that, if an anonymity axiom is defined by a set $\mathcal{Q}$ of cyclic permutations satisfying Properties 2 and 3, the class of all acyclic (resp. quasi-transitive) SWRs that satisfy the Pareto and the corresponding anonymity axioms coincides with the class of all acyclic (resp. quasi-transitive) SWRs that includes the $\mathcal{Q}$-Pareto relation as a subrelation. Further, we obtain that, if the considered set $\mathcal{Q}$ of permutations also satisfies Property 1^*, the class of all Suzumura-consistent SWRs that satisfy the axioms coincides with the class of all Suzumura-consistent SWRs that includes the $\mathcal{Q}$-Pareto relation as a subrelation. This class of Suzumura-consistent SWRs includes a class of SWQs as a subclass. We also provide a characterization of this class of SWQs. We show that the class of all SWQs that satisfy the Pareto and anonymity axioms coincides with the class of all SWQs that include, as a subrelation, the $\overline{\mathcal{Q}}$-*Suppes-Sen* SWQ associated with the closure $\overline{\mathcal{Q}}$ of the given set $\mathcal{Q}$. The similar result has been established by Banerjee (2006) with an anonymity axiom defined by a group $\mathcal{Q}$ of cyclic permutations. The difference between his result and ours is that we weaken the requirement of group structure of $\mathcal{Q}$ to Properties 1^*, 2, and 3.

The next section provides notation and basic definitions. In Sect. 3, the characterizations of permissible permutations of Paretian SWRs are established. In Sect. 4, we present some examples of sets of permutations satisfying the properties considered, and we discuss maximal anonymity axioms. Sect. 5 provides the characterizations of classes of anonymous Paretian SWRs. We conclude in Sect. 6.

2 Preliminaries

2.1 Social welfare relations

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all positive integers. Throughout this paper, we let $X = \mathbb{R}^{\mathbb{N}}$ be the set of all utility streams $\boldsymbol{x} = (x_1, x_2, \dots)$. For all $i \in \mathbb{N}$, x_i is interpreted as the utility level of the ith generation.

Negation of a statement is indicated by the symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, and $\boldsymbol{x} > \boldsymbol{y}$ if $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given two sets A and B, we write $A \subseteq B$ to mean A is a subset of B; $A \subset B$ to mean $A \subseteq B$ and $A \neq B$; and $A \nsubseteq B$ to mean $A \backslash B \neq \varnothing$.

A binary relation $\succsim$ on X is a subset of $X \times X$. For convenience, the fact that $(\boldsymbol{x}, \boldsymbol{y}) \in \succsim$ will be symbolized by $\boldsymbol{x} \succsim \boldsymbol{y}$. The asymmetric part of $\succsim$ is denoted by $\succ$ and the symmetric part by $\sim$, i.e. $\boldsymbol{x} \succ \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\neg(\boldsymbol{y} \succsim \boldsymbol{x})$, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{x}$. A binary relation $\succsim_A$ is said to be a *subrelation* of a relation $\succsim_B$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \sim_A \boldsymbol{y}$ implies $\boldsymbol{x} \sim_B \boldsymbol{y}$ and (ii) $\boldsymbol{x} \succ_A \boldsymbol{y}$ implies $\boldsymbol{x} \succ_B \boldsymbol{y}$. Conversely, we say that a binary relation $\succsim_A$ *extends* a relation $\succsim_B$ if $\succsim_B$ is a subrelation of $\succsim_A$. If an ordering $\succsim_A$ extends a relation $\succsim_B$, we say that $\succsim_A$ is an *ordering extension* of $\succsim_B$.

We consider four coherence properties of a binary relation. A binary relation is said to be (i) *transitive* if and only if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succsim \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$; (ii) *Suzumura-consistent* if and only if, for all $K \in \mathbb{N} \backslash \{1\}$ and all $\boldsymbol{x}^0, \ldots, \boldsymbol{x}^K \in X$, if $\boldsymbol{x}^{k-1} \succsim \boldsymbol{x}^k$ for all $k \in \{1, \ldots K\}$ then $\neg(\boldsymbol{x}^K \succ \boldsymbol{x}^0)$; (iii) *quasi-transitive* if and only if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succ \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{y} \succ \boldsymbol{z}$; (iv) *acyclic* if and only if, for all $K \in \mathbb{N} \backslash \{1\}$ and all $\boldsymbol{x}^0, \ldots, \boldsymbol{x}^K \in X$, $\boldsymbol{x}^{k-1} \succ \boldsymbol{x}^k$ for all $k \in \{1, \ldots K\}$ implies $\neg(\boldsymbol{x}^K \succ \boldsymbol{x}^0)$. Transitivity implies Suzumura consistency and quasi-transitivity, both of which in turn imply acyclicity. Suzumura consistency and quasi-transitivity are logically independent.

A *social welfare relation* (SWR) is a reflexive binary relation, i.e., a binary relation satisfying that, for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \succsim \boldsymbol{x}$. A *social welfare quasi-ordering* (SWQ) is a transitive SWR.

2.2 Permutations

We represent any permutation on the set $\mathbb{N}$ by a permutation matrix.[7] A permutation matrix is an infinite matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ such that (i) for all $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$; and (ii) for all $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$. Let $\mathcal{P}$ be the set of all permutation matrices. Note that, for all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{P}$, the product $\boldsymbol{P}\boldsymbol{x} = (Px_1, Px_2, \ldots)$ belongs to X, where $Px_i = \sum_{k \in \mathbb{N}} p_{ik} x_k$ for all $i \in \mathbb{N}$. For all $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ and all $n \in \mathbb{N}$, $\boldsymbol{P}(n)$ denotes the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\ldots,n\}}$. Let $\boldsymbol{I}$ denote the infinite identity matrix. For any $\boldsymbol{P} \in \mathcal{P}$, let $\boldsymbol{P}'$ be the inverse of $\boldsymbol{P}$ satisfying $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$.[8] A permutation $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ is said to be a *finite permutation* if there exists $n \in \mathbb{N}$ such that $p_{ii} = 1$ for all $i > n$. The set of all finite permutations is denoted by $\mathcal{F}$.

A permutation $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ is said to be *cyclic* if, for any $i \in \mathbb{N}$, there exist $n(i) \in \mathbb{N}$ and $n(i)$-dimensional vector $(i_1, \ldots, i_{n(i)})$ of positive integers such

[7]Hereafter, we will use the terms "permutation" and "permutation matrix" interchangeably. Further, we use the term "product of permutation matrices" for "composition of permutations".

[8]For any $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{P}$, the product $\boldsymbol{P}\boldsymbol{Q}$ is defined by $(r_{ij})_{i,j \in \mathbb{N}}$ with $r_{ij} = \sum_{k \in \mathbb{N}} p_{ik} q_{kj}$.

that $i_1 = i$ and $p_{i_2 i_1} = \cdots = p_{i_{n(i)} i_{n(i)-1}} = p_{i_1 i_{n(i)}} = 1$. The set of all cyclic permutations is denoted by $\mathcal{C}$.

The following characterization of a cyclic permutation has been provided by Lauwers (2010b) and Mitra and Basu (2007), which will prove useful in establishing our results.[9]

Lemma 1 (Lauwers 2010b, Lemma 1; Mitra and Basu 2007, Lemma 1)**.** *A permutation $\boldsymbol{P} \in \mathcal{P}$ is cyclic if and only if there exists no $\boldsymbol{x} \in X$ satisfying $\boldsymbol{P}\boldsymbol{x} > \boldsymbol{x}$.*

A set $\mathcal{Q}$ of permutations is a *group* (together with matrix multiplication) if and only if it satisfies Property 1 (closure), Property 2 (existence of the unit element), and Property 3 (existence of the inverse element).[10]

Property 1: For all $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$, $\boldsymbol{P}\boldsymbol{Q} \in \mathcal{Q}$.

Property 2: For all $\boldsymbol{P} \in \mathcal{Q}$, there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{P} = \boldsymbol{P}$.

Property 3: For all $\boldsymbol{P} \in \mathcal{Q}$, there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{P} = \boldsymbol{I}$.

Since $\mathcal{Q}$ is a set of permutations, the permutation $\boldsymbol{Q}$ in Property 2 must be $\boldsymbol{I}$ and the permutation $\boldsymbol{Q}$ in Property 3 must be $\boldsymbol{P'}$.

We also consider the following variant of Property 1. It requires that any product of a finite number of elements in a set $\mathcal{Q}$ of permutations must be cyclic.

Property 1*: For any finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ with $K > 1$, the product $\boldsymbol{P}^1 \cdots \boldsymbol{P}^K$ belongs to $\mathcal{C}$.

Note that, for any set of cyclic permutations, Property 1 implies Property 1*.

2.3 Permissible permutations and Pareto and anonymity axioms

For a binary relation $\succsim$ on X, the set $\Pi(\succsim)$ of *permissible permutations* associated with $\succsim$ is defined by

$$\Pi(\succsim) = \{\boldsymbol{P} \in \mathcal{P} : \boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x} \text{ for all } \boldsymbol{x} \in X\}. \tag{1}$$

$\Pi(\succsim)$ collects all permutations with respect to which every utility stream $\boldsymbol{x}$ is declared to be indifferent to the permuted stream $\boldsymbol{P}\boldsymbol{x}$ by the given relation $\succsim$. The concept of permissible permutations has been introduced by Mitra and Basu (2007).

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[9]In Mitra and Basu (2007), the original version of this lemma is established for the domain $X = [0, 1]^{\mathbb{N}}$. Lauwers (2010b) strengthens their result by showing that it holds for any domain X satisfying $\{0, 1\}^{\mathbb{N}} \subseteq X$.

[10]In algebra, a set of objects is said to be a group if it satisfies *associativity* in addition to Properties 1, 2, and 3. Given a set $\mathcal{O}$ of objects, $\mathcal{O}$ together with an operation $\circ$ satisfy associativity if, for all $A, B, C \in \mathcal{O}$, $A \circ (B \circ C) = (A \circ B) \circ C$. Since any set of permutations is associative, we omit associativity in the definition of a group of permutations.

For a set $\mathcal{Q}$ of permutations, the anonymity axiom associated with $\mathcal{Q}$ is defined as follows.

$\mathcal{Q}$-Anonymity: For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{Q}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.

Different sets $\mathcal{Q}$ of permutations yield different anonymity axioms. If $\mathcal{Q} = \mathcal{F}$, we obtain $\mathcal{F}$-Anonymity, which is also called finite anonymity in the literature. Some other examples of $\mathcal{Q}$-Anonymity will be presented in Sect. 4. Note that an SWR $\succsim$ satisfies $\mathcal{Q}$-Anonymity if and only if $\mathcal{Q} \subseteq \Pi(\succsim)$. Hence, $\Pi(\succsim)$-Anonymity is the strongest anonymity that can be satisfied by the given SWR.

We consider SWRs satisfying the following Pareto axiom.

Pareto: For all $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} > \boldsymbol{y}$, $\boldsymbol{x} \succ \boldsymbol{y}$.

Throughout this paper, an SWR satisfying the Pareto axiom is referred to as a *Paretian* SWR.

3 Permissible permutations of Paretian SWRs

3.1 Acyclicity and quasi-transitivity

In this section, we provide necessary and sufficient conditions on a set $\mathcal{Q}$ of permutations for the existence of a Paretian SWR $\succsim$ for which the set of permissible permutations $\Pi(\succsim)$ coincides with $\mathcal{Q}$. We consider three cases of the coherence property of an SWR, namely, acyclicity, quasi-transitivity, and Suzumura consistency. We first show that cyclicity of permutations and Properties 2 and 3 together are necessary and sufficient for the existence of a Paretian acyclic SWR $\succsim$ that satisfies $\Pi(\succsim) = \mathcal{Q}$. Further, we will see that even if the acyclicity of an SWR is strengthened to quasi-transitivity, the necessary and sufficient condition is given by the same restrictions. In Sect. 3.2, we show that if the acyclicity of an SWR is strengthened to Suzumura consistency, the necessary and sufficient condition is given by adding Property 1^* to cyclicity of permutations and Properties 2 and 3.

We begin with the following proposition due to Mitra and Basu (2007). It shows that if a Paretian SWR $\succsim$ is acyclic, the set $\Pi(\succsim)$ of permissible permutations must be a set of cyclic permutations satisfying Properties 2 and 3.[11]

Proposition 1 (Mitra and Basu 2007)**.** *Let $\succsim$ be a Paretian acyclic SWR on X.*
(i) $\Pi(\succsim) \subseteq \mathcal{C}$.
(ii) $\Pi(\succsim)$ *satisfies Properties 2 and 3.*

[11] Part (i) of the lemma corresponds to Proposition 1 in Mitra and Basu (2007). The original version of this result is established for a Paretian SWQ. However, since it follows from Lemma 1, the same result holds for a Paretian acyclic SWR. Part (ii) follows from Mitra and Basu's (2007) argument in the proof of their Proposition 2.

By Proposition 1, cyclicity of permutations and Properties 2 and 3 are necessary conditions for the existence of a Paretian acyclic SWR $\succsim$ for which the set of permissible permutations $\Pi(\succsim)$ coincides with a given set $\mathcal{Q}$. We show that these restrictions together are sufficient for the existence of such an SWR. We establish this sufficiency result by constructing an SWR that satisfies the requisite conditions.

For an arbitrary set $\mathcal{Q}$ of cyclic permutations satisfying Properties 2 and 3, we define the binary relation $\succsim_{\mathcal{Q}}$ associated with $\mathcal{Q}$ as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\begin{cases} \boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y} \text{ iff } \boldsymbol{x} > \boldsymbol{y}; & \text{(2a)} \\ \boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y} \text{ iff there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{P}\boldsymbol{x} = \boldsymbol{y}. & \text{(2b)} \end{cases}$$

The asymmetric part of $\succsim_{\mathcal{Q}}$ is an infinite-horizon version of the Pareto rule due to Sen (1970) and Weymark (1984). The symmetric part of it corresponds to $\mathcal{Q}$-Anonymity associated with the given set $\mathcal{Q}$. We will call this extended Pareto rule *$\mathcal{Q}$-Pareto* relation. As Lemma 2 demonstrates, the $\mathcal{Q}$-Pareto relation is well-defined as a binary relation.

Lemma 2. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 2 and 3. $\succsim_{\mathcal{Q}}$ is well-deifned as a binary relation on X.*

Proof. Let $\succsim = \succ_{\mathcal{Q}} \cup \sim_{\mathcal{Q}}$. We show that $\sim_{\mathcal{Q}} = \sim$ and $\succ_{\mathcal{Q}} = \succ$. We first verify $\sim_{\mathcal{Q}} = \sim$. Assume $\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y}$. Note that, by (2b) and Property 3,

$$\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y} \text{ if and only if } \boldsymbol{y} \sim_{\mathcal{Q}} \boldsymbol{x}. \tag{3}$$

Thus, by the definition of $\succsim$, $\boldsymbol{x} \sim \boldsymbol{y}$.

Next, assume $\boldsymbol{x} \sim \boldsymbol{y}$. By the definition of $\succsim$, we obtain

$$\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y} \text{ or } \boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y},$$

and

$$\boldsymbol{y} \succ_{\mathcal{Q}} \boldsymbol{x} \text{ or } \boldsymbol{y} \sim_{\mathcal{Q}} \boldsymbol{x}.$$

Suppose $\neg(\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y})$. By (3), $\neg(\boldsymbol{y} \sim_{\mathcal{Q}} \boldsymbol{x})$. Thus, $\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{\mathcal{Q}} \boldsymbol{x}$ hold. By (2a), we obtain $\boldsymbol{x} > \boldsymbol{y}$ and $\boldsymbol{y} > \boldsymbol{x}$, a contradiction. Thus, $\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y}$ holds.

Next, we show $\succ_{\mathcal{Q}} = \succ$. Assume $\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y}$. By the definition of $\succsim$, $\boldsymbol{x} \succsim \boldsymbol{y}$ holds. We verify that $\boldsymbol{x} \succ \boldsymbol{y}$ holds by showing $\neg(\boldsymbol{x} \sim \boldsymbol{y})$. By (2a), $\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y}$ implies $\boldsymbol{x} > \boldsymbol{y}$. Since $\mathcal{Q} \subseteq \mathcal{C}$, it follows from Lemma 1 that there is no $\boldsymbol{P} \in \mathcal{Q}$ satisfying $\boldsymbol{P}\boldsymbol{y} > \boldsymbol{y}$. Thus, $\boldsymbol{P}\boldsymbol{y} \neq \boldsymbol{x}$ for all $\boldsymbol{P} \in \mathcal{Q}$. By (2b), $\neg(\boldsymbol{y} \sim_{\mathcal{Q}} \boldsymbol{x})$. Since $\sim_{\mathcal{Q}} = \sim$, we obtain $\neg(\boldsymbol{y} \sim \boldsymbol{x})$, or equivalently, $\neg(\boldsymbol{x} \sim \boldsymbol{y})$. Next, assume $\boldsymbol{x} \succ \boldsymbol{y}$. By the definition of $\succsim$, $\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y}$ or $\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y}$. Since $\sim_{\mathcal{Q}} = \sim$, if $\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y}$ holds, then we have a contradiction to $\boldsymbol{x} \succ \boldsymbol{y}$. Thus, $\boldsymbol{x} \succ_{\mathcal{Q}} \boldsymbol{y}$. ■

The next proposition establishes the sufficiency result. It shows that the $\mathcal{Q}$-Pareto relation associated with a set $\mathcal{Q}$ of cyclic permutations satisfying Properties 2 and 3 is a Paretian quasi-transitive SWR, and furthermore, the set $\Pi(\succsim_{\mathcal{Q}})$ of permissible permutations coincides with $\mathcal{Q}$.

Proposition 2. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 2 and 3.*

(i) $\succsim_{\mathcal{Q}}$ *is a Paretian quasi-transitive SWR.*

(ii) $\Pi(\succsim_{\mathcal{Q}}) = \mathcal{Q}$.

Proof. (i) By (2a), $\succsim_{\mathcal{Q}}$ is Paretian and quasi-transitive. Since $\mathcal{Q}$ satisfies Property 2, $\boldsymbol{I} \in \mathcal{Q}$. Thus, by (2b), $\succsim_{\mathcal{Q}}$ is reflexive.

(ii) We first show $\mathcal{Q} \subseteq \Pi(\succsim_{\mathcal{Q}})$. Let $\boldsymbol{P} \in \mathcal{Q}$ and $\boldsymbol{x} \in X$. By (2b), $\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{P}\boldsymbol{x}$. Thus, $\boldsymbol{P} \in \Pi(\succsim_{\mathcal{Q}})$. Next, we show $\Pi(\succsim_{\mathcal{Q}}) \subseteq \mathcal{Q}$. Let $\boldsymbol{P} \in \Pi(\succsim_{\mathcal{Q}})$. Consider $\boldsymbol{x} \in X$ such that $x_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$. By (1), $\boldsymbol{P}\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{x}$. By (2b), there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Q}(\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{x}$. Since $x_i \neq x_j$ for all $i, j \in \mathbb{N}$ with $i \neq j$, it must be that $\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{I}$, i.e., $\boldsymbol{P} = \boldsymbol{Q}'$. Since $\boldsymbol{Q} \in \mathcal{Q}$ and $\mathcal{Q}$ satisfies Property 3, $\boldsymbol{P} \in \mathcal{Q}$. ∎

Recall that quasi-transitivity implies acyclicity. Hence, combining Propositions 1 and 2, we obtain that cyclicity of permutations and Properties 2 and 3 together are necessary and sufficient for the existence of a Paretian acyclic SWR $\succsim$ for which the set $\Pi(\succsim)$ of permissible permutations coincides with a given set $\mathcal{Q}$ of permutations. Further, these restrictions are necessary and sufficient for the existence of a Paretian *quasi-transitive* SWR that satisfies $\Pi(\succsim) = \mathcal{Q}$. This coincidence of the results for an acyclic SWR and a quasi-transitive SWR can be explained as follows. The set of permissible permutations of a relation is defined in terms of the symmetric part of the relation. On the other hand, acyclicity and quasi-transitivity are properties concerned with the asymmetric part of a relation. Hence, which of these coherence properties we impose on an SWR does not affect the structure of the set of permissible permutations of the SWR as long as axioms we impose on the SWR are stated for the asymmetric part of the SWR (e.g., Pareto).

We compare our results with the result of Mitra and Basu (2007) obtained for the case of a Paretian SWQ. Mitra and Basu (2007, Propositions 2 and 3) show that cyclicity of permutations and group structure together are necessary and sufficient for the existence of a Paretian SWQ $\succsim$ satisfying $\Pi(\succsim) = \mathcal{Q}$. Recall that a group of permutations is a set of permutations satisfying Properties 1, 2, and 3. Hence, the class of sets of permissible permutations associated with a Paretian acyclic SWR or a Paretian quasi-transitive SWR is larger as compared to the case where a Paretian SWR is required to be transitive. Some examples of sets of cyclic permutations that satisfy Properties 2 and 3 but violate Property 1 will be provided in Sect. 4.

3.2 Suzumura consistency

We now consider the case where a Paretian SWR is required to be Suzumura-consistent. Recall that Suzumura consistency implies acyclicity. Thus, from Proposition 1, it follows that the set $\Pi(\succsim)$ of permissible permutations of a Paretian Suzumura-consistent SWR $\succsim$ must be a set of cyclic permutations satisfying Properties 2 and 3. The following proposition shows that $\Pi(\succsim)$ additionally satisfies Property 1*.

Proposition 3. *If $\succsim$ is a Paretian Suzumura-consistent SWR on X, then $\Pi(\succsim)$ satisfies Property 1*.*

Proof. Let $\{\boldsymbol{P}^k\}_{k=1}^K$ be a finite sequence in $\Pi(\succsim)$ with $K > 1$. We will show that the product $\boldsymbol{P}^1 \cdots \boldsymbol{P}^K$ is cyclic. The proof is done by contradiction. Let $\{\boldsymbol{Q}^k\}_{k=0}^{K-1}$ be the finite sequence in $\mathcal{P}$ defined by

$$\boldsymbol{Q}^0 = \boldsymbol{P}^K \text{ and } \boldsymbol{Q}^k = \boldsymbol{P}^{K-k}\boldsymbol{Q}^{k-1} \text{ for all } k \in \{1, \dots, K-1\}.$$

By definition, $\boldsymbol{Q}^{K-1} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. Suppose $\boldsymbol{Q}^{K-1} \notin \mathcal{C}$. By Lemma 1, there exists $\boldsymbol{x} \in X$ such that $\boldsymbol{Q}^{K-1}\boldsymbol{x} > \boldsymbol{x}$. Since $\succsim$ is Paretian,

$$\boldsymbol{Q}^{K-1}\boldsymbol{x} \succ \boldsymbol{x}. \tag{4}$$

Since $\{\boldsymbol{P}^k\}_{k=1}^K$ is a sequence in $\Pi(\succsim)$, it follows from the definition of $\{\boldsymbol{Q}^k\}_{k=0}^{K-1}$ that

$$\boldsymbol{x} \sim \boldsymbol{Q}^0\boldsymbol{x} \text{ and } \boldsymbol{Q}^{k-1}\boldsymbol{x} \sim \boldsymbol{Q}^k\boldsymbol{x} \text{ for all } k \in \{1, \dots, K-1\}. \tag{5}$$

From (4) and (5), we obtain a contradiction to that $\succsim$ is Suzumura-consistent. Thus, $\boldsymbol{Q}^{K-1} \in \mathcal{C}$, i.e., the product $\boldsymbol{P}^1 \cdots \boldsymbol{P}^K$ is cyclic. ∎

We now examine whether cyclicity of permutations and Properties 1*, 2, and 3 together are sufficient for the existence of a Paretian Suzumura-consistent SWR $\succsim$ for which the set $\Pi(\succsim)$ of permissible permutations coincides with a given set $\mathcal{Q}$ of permutations. In Proposition 2, we obtained that the $\mathcal{Q}$-Pareto relation associated with a set $\mathcal{Q}$ of cyclic permutations satisfying Properties 2 and 3 is a Paretian SWR and $\Pi(\succsim_{\mathcal{Q}}) = \mathcal{Q}$ holds. As the following proposition shows, if $\mathcal{Q}$ also satisfies Property 1*, it is Suzumura-consistent. Hence, the answer to the above question is in the afirmative.

Proposition 4. *If $\mathcal{Q}$ is a set of cyclic permutations satisfying Properties 1*, 2, and 3, then $\succsim_{\mathcal{Q}}$ is Suzumura-consistent.*

Proof. Let $K \in \mathbb{N}\backslash\{1\}$ and $\boldsymbol{x}^0, \dots, \boldsymbol{x}^K \in X$ and suppose that $\boldsymbol{x}^{k-1} \succsim_{\mathcal{Q}} \boldsymbol{x}^k$ for all $k \in \{1, \dots, K\}$. We show, by contradiction, that $\neg(\boldsymbol{x}^K \succ_{\mathcal{Q}} \boldsymbol{x}^0)$. Suppose $\boldsymbol{x}^K \succ_{\mathcal{Q}} \boldsymbol{x}^0$. By (2a),

$$\boldsymbol{x}^K > \boldsymbol{x}^0. \tag{6}$$

By (2b), for all $k \in \{0, \dots, K-1\}$, if $\boldsymbol{x}^k \sim_{\mathcal{Q}} \boldsymbol{x}^{k+1}$ holds, then there exists $\boldsymbol{P}^k \in \mathcal{Q}$ such that $\boldsymbol{P}^k\boldsymbol{x}^k = \boldsymbol{x}^{k+1}$. Further, by (2a), for all $k \in \{0, 1, \dots, K-1\}$, if $\boldsymbol{x}^k \succ_{\mathcal{Q}} \boldsymbol{x}^{k+1}$, then $\boldsymbol{x}^k > \boldsymbol{x}^{k+1}$. Let

$$\boldsymbol{Q} = \boldsymbol{P}^{K-1} \cdots \boldsymbol{P}^0,$$

where $\boldsymbol{P}^k = \boldsymbol{I}$ for all $k \in \{0, \ldots, K-1\}$ such that $\boldsymbol{x}^k \succ_{\mathcal{Q}} \boldsymbol{x}^{k+1}$. Then, we obtain

$$\begin{aligned}
\boldsymbol{x}^K &\leq \boldsymbol{P}^{K-1}\boldsymbol{x}^{K-1} \\
&\leq \boldsymbol{P}^{K-1}\boldsymbol{P}^{K-2}\boldsymbol{x}^{K-2} \\
&\vdots \\
&\leq \boldsymbol{P}^{K-1}\cdots\boldsymbol{P}^{0}\boldsymbol{x}^{0} \\
&= \boldsymbol{Q}\boldsymbol{x}^0. \qquad (7)
\end{aligned}$$

Since $\mathcal{Q}$ satisfies Property 2, $\boldsymbol{I} \in \mathcal{Q}$. Thus, $\{\boldsymbol{P}^k\}_{k=0}^{K-1}$ is the finite sequence in $\mathcal{Q}$. Since $\mathcal{Q}$ satisfies Property 1*, $\boldsymbol{Q}$ must be cyclic. However, by (6) and (7), we obtain $\boldsymbol{Q}\boldsymbol{x}^0 > \boldsymbol{x}^0$. By Lemma 1, this is a contradiction. Thus, $\neg(\boldsymbol{x}^K \succ_{\mathcal{Q}} \boldsymbol{x}^0)$. ■

By Propositions 1, 2, 3, and 4, cyclicity of permutations and Properties 1*, 2, and 3 together are necessary and sufficient for the existence of a Paretian Suzumura-consistent SWR for which the set $\Pi(\succsim)$ of permissible permutations coincides with a given set $\mathcal{Q}$ of permutations. Hence, the class of sets of permissible permutations associated with a Paretian Suzumura-consistent SWR is smaller as compared to the cases of a Paretian acyclic SWR and of a Paretian quasi-transitive SWR. However, since Property 1* is weaker than Property 1 for all sets of cyclic permutations, it is larger than the class of sets of permissible permutations of a Paretian SWQ (recall the result of Mitra and Basu (2007) we mentioned after Proposition 2). An example of a set of cyclic permutations that satisfies Properties 1*, 2, and 3 but violates Property 1 will be provided in the next section.

4 Examples of sets of cyclic permutations and maximal anonymity

In this section, we present some examples of sets of cyclic permutations that satisfy Properties 2 and 3. Further, we examine which sets of permutations are maximal (with respect to set inclusion) in the class of all sets of cyclic permutations satisfying Properties 2 and 3 and the class of all sets of cyclic permutations satisfying Properties 1*, 2, and 3.

All of the sets of permutations we present below are strict supersets of the set $\mathcal{F}$ of all finite permutations. Thus, $\mathcal{Q}$-Anonymity axioms associated with them realize extended anonymity stronger than $\mathcal{F}$-Anonymity. We begin with two particular sets of cyclic permutations that satisfy Properties 2 and 3 but violate Properties 1 and 1*.

Example 1. The set $\mathcal{C}$ of all cyclic permutations is, of course, a set of cyclic permutations. It satisfies Properties 2 and 3. Next, consider the set $\mathcal{V}$ defined by

$$\mathcal{V} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{c} \text{for all } n \in \mathbb{N}, \text{there exists } n' \in \mathbb{N} \text{ with } n' \geq n \text{ such that} \\ \boldsymbol{P}(n') \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

$\mathcal{V}$ is the set of all *variable-step permutations*, which has been proposed by Fleurbaey and Michel (2003). $\mathcal{V}$ is a strict subset of $\mathcal{C}$ and, thus, it is a set of cyclic permutations. It also satisfies Properties 2 and 3. Neither $\mathcal{C}$ nor $\mathcal{V}$ satisfy Properties 1 and 1* (on this, see, e.g., Example 2 in Mitra and Basu (2007), p. 93).

$\mathcal{C}$-Anonymity is stronger than $\mathcal{V}$-Anonymity since $\mathcal{V} \subset \mathcal{C}$. From Proposition 2, it follows that $\mathcal{C}$-Anonymity and $\mathcal{V}$-Anonymity are compatible with the existence of a Paretian quasi-transitive SWR. However, by Proposition 3, neither $\mathcal{C}$-Anonymity nor $\mathcal{V}$-Anonymity are compatible with a Paretian Suzumura-consistent SWR.

Next, we present sets of cyclic permutations that satisfy Properties 1, 2, and 3, i.e., groups of cyclic permutations. One of the most common examples of groups of cyclic permutations is the set $\mathcal{F}$ of all finite permutations. Two groups of cyclic permutations presented in the next example include $\mathcal{F}$ as a strict subset.

Example 2. Let $\mathcal{S}$ and $\mathcal{V}_p$ be the sets of permutations defined as follows:

$$\mathcal{S} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{c} \text{there exists } s \in \mathbb{N} \text{ such that, for all } n \in \mathbb{N}, \\ \boldsymbol{P}(ns) \text{ is a finite-dimensional permutation matrix} \end{array} \right\},$$
$$\mathcal{V}_p = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{c} \text{there exists } m \in \mathbb{N} \text{ such that, for all } n \in \mathbb{N}\backslash\{1\}, \\ \boldsymbol{P}(n^m) \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

$\mathcal{S}$ is the set of all *fixed-step permutations*, which has been introduced by Lauwers (1997b). $\mathcal{V}_p$ is newly introduced in this paper. It collects particular variable-step permutations that reshuffle contiguous generations taken by a power function. They satisfy

$$\mathcal{S} \subset \mathcal{V} \text{ and } \mathcal{V}_p \subset \mathcal{V}.$$

Thus, they are sets of cyclic permutations. Since $\mathcal{V}_p$ contains permutations in which the size of contiguous generations reshuffled is strictly increasing,

$$\mathcal{V}_p \not\subseteq \mathcal{S}.$$

Further, since 3^m is odd for all $m \in \mathbb{N}$,

$$\mathcal{S} \not\subseteq \mathcal{V}_p.$$

$\mathcal{S}$ is a group of cyclic permutations (Mitra and Basu, 2007, pp. 96-97). Thus, it satisfies Properties 1, 2, and 3. It is straightforward that $\mathcal{V}_p$ satisfies Properties 2 and 3. Further, it satisfies Property 1. This can be verified as follows. Let $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{V}_p$. Then, there exist $m, m' \in \mathbb{N}$ such that, for all $n \in \mathbb{N}\backslash\{1\}$, $\boldsymbol{P}(n^m)$ and $\boldsymbol{Q}(n^{m'})$ are finite-dimensional permutation matrices. Define $\boldsymbol{R} = \boldsymbol{PQ}$ and $\bar{m} = mm'$. Then, for all $n \in \mathbb{N}\backslash\{1\}$, $\boldsymbol{R}(n^{\bar{m}})$ is a finite-dimensional permutation matrix. Thus, $\boldsymbol{R} \in \mathcal{V}_p$.

Since Property 1 implies Property 1* for any set of cyclic permutations, $\mathcal{S}$ and $\mathcal{V}_p$ also satisfy Property 1*.

Since $\mathcal{S} \subset \mathcal{V}$ and $\mathcal{V}_p \subset \mathcal{V}$, both $\mathcal{S}$-Anonymity and $\mathcal{V}_p$-Anonymity are weaker than $\mathcal{V}$-Anonymity. By Propositions 2 and 4, they are compatible with the existence of a Paretian Suzumura-consistent SWR. Further, by the result of Mitra and Basu (2007) we mentioned after Proposition 2, they are compatible with the existence of a Paretian SWQ.

Next, we present an example of a set of cyclic permutations that satisfies Properties 1*, 2, and 3 but violates Property 1.

Example 3. Consider the union $\mathcal{S} \cup \mathcal{V}_p$. For convenience, let $\mathcal{Q}$ denote $\mathcal{S} \cup \mathcal{V}_p$ in the discussion below. By Example 2, $\mathcal{Q}$ is a set of cyclic permutations and it satisfies Properties 2 and 3. We demonstrate that it satisfies Property 1* but violates Property 1. First, we show that it satisfies Property 1*. Consider any finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ with $K > 1$. Let $\boldsymbol{Q} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. For all $k \in \{1, \ldots, K\}$, if $\boldsymbol{P}^k \in \mathcal{S}$, then there exists $s_k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{P}^k(ns_k)$ is a finite-dimensional permutation matrix, and if $\boldsymbol{P}^k \in \mathcal{V}_\mathrm{p}$, then there exists $m_k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}\backslash\{1\}$, $\boldsymbol{P}^k(n^{m_k})$ is a finite-dimensional permutation matrix. Let $s = \Pi_{k=1}^K s_k$ and $m = \Pi_{k=1}^K m_k$ where, for all $k \in \{1, \ldots, K\}$, $s_k = 1$ if $\boldsymbol{P}^k \in \mathcal{V}_\mathrm{p}$ and $m_k = 1$ if $\boldsymbol{P}^k \in \mathcal{S}$. Then, for all $n \in \mathbb{N}\backslash\{1\}$, $\boldsymbol{Q}((ns)^m)$ is a finite-dimensional permutation matrix. This implies $\boldsymbol{Q} \in \mathcal{V}$. Since $\mathcal{V} \subset \mathcal{C}$, $\mathcal{Q} \in \mathcal{C}$.

Next, we verify that $\mathcal{Q}$ violates Property 1. Let $\boldsymbol{P} = (p_{ij})_{i,j\in\mathbb{N}}$ be the permutation defined by

$$p_{ij} = p_{ji} = 1 \text{ if } i \text{ is odd and } j = i + 1.$$

Further, let $\boldsymbol{Q} = (q_{ij})_{i,j\in\mathbb{N}}$ be the permutation defined by

$$\begin{cases} p_{ij} = p_{ji} = 1 \text{ if } (i, j) = (n^2 + 1, (n + 1)^2) \text{ for some } n \in \mathbb{N} \\ p_{ii} = 1 \text{ if } i \neq n^2 + 1 \text{ and } i \neq (n + 1)^2 \text{ for all } n \in \mathbb{N}. \end{cases}$$

Note that $\boldsymbol{P} \in \mathcal{S}$ and $\boldsymbol{Q} \in \mathcal{V}_p$. Thus, $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$. Let $\boldsymbol{R} = \boldsymbol{PQ}$. Note that if $\boldsymbol{R}(n)$ is a finite-dimensional permutation matrix, then n is even. Since 3^m is odd for all $m \in \mathbb{N}$, there is no $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}\backslash\{1\}$, $\boldsymbol{R}(n^m)$ is a finite-dimensional permutation matrix. This implies $\boldsymbol{R} \notin \mathcal{V}_p$. Further, since the size of contiguous generations reshuffled in $\boldsymbol{R}$ is strictly increasing, there is no $s \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{R}(ns)$ is a finite-dimensional permutation matrix. This implies $\boldsymbol{R} \notin \mathcal{S}$. Thus, $\boldsymbol{R} \notin \mathcal{Q}$.

Since $\mathcal{S} \cup \mathcal{V}_p$ includes $\mathcal{S}$ and $\mathcal{V}_p$, $\mathcal{Q}$-Anonymity defined by $\mathcal{Q} = \mathcal{S} \cup \mathcal{V}_p$ is stronger than $\mathcal{S}$-Anonymity and $\mathcal{V}_p$-Anonymity. However, since $\mathcal{S} \cup \mathcal{V}_p \subset \mathcal{V}$, it is weaker than $\mathcal{V}$-Anonymity. By Propositions 2 and 4, $\mathcal{Q}$-Anonymity defined by $\mathcal{Q} = \mathcal{S} \cup \mathcal{V}_p$ is compatible with the existence of a Paretian quasi-transitive SWR and of a Paretian Suzumura-consistent SWR.

We now examine maximal sets of permutations in the class of all sets of cyclic permutations satisfying Properties 2 and 3 and the class of all sets of cyclic permutations satisfying Properties 1*, 2, and 3. We first consider the class of all sets of cyclic permutations satisfying Properties 2 and 3. From Example 1, it is straightforward that the set $\mathcal{C}$ of all cyclic permutations is the unique maximal of the class. By Propositions 1 and 2, this means that $\mathcal{C}$-Anonymity is the strongest anonymity that is compatible with the existence of a Paretian acyclic SWR and of a Paretian quasi-transitive SWR.

Next, we consider the class of all sets of cyclic permutations satisfying Properties 1*, 2, and 3. We show that a maximal set of permutations in this class is a group of cyclic permutations. To establish this result, we define the closure of a set of permutations.

For a set $\mathcal{Q}$ of permutations, the closure $\bar{\mathcal{Q}}$ of $\mathcal{Q}$ is defined as follows:

$$\bar{\mathcal{Q}} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{l} \text{there exists a finite sequence } \{\boldsymbol{P}^k\}_{k=1}^K \text{ in } \mathcal{Q} \text{ satisfying} \\ \boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K \end{array} \right\}. \tag{8}$$

Note that $\mathcal{Q} \subseteq \bar{\mathcal{Q}}$ for all $\mathcal{Q} \subseteq \mathcal{P}$.

The next proposition shows that, for any set $\mathcal{Q}$ of cyclic permutations satisfying Properties 1*, 2, and 3, the closure $\bar{\mathcal{Q}}$ is the smallest group of cyclic permutations that includes $\mathcal{Q}$.

Proposition 5. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 1*, 2, and 3.*

(i) *$\bar{\mathcal{Q}}$ is a group of cyclic permutations.*

(ii) *$\bar{\mathcal{Q}} \subseteq \tilde{\mathcal{Q}}$ holds for any group $\tilde{\mathcal{Q}}$ of cyclic permutations with $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$.*

Proof. (i) First, we prove $\bar{\mathcal{Q}} \subseteq \mathcal{C}$. Let $\boldsymbol{P} \in \bar{\mathcal{Q}}$. By (8), there exists a finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ satisfying $\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. If $K > 1$, then $\boldsymbol{P} \in \mathcal{C}$ since $\mathcal{Q}$ satisfies Property 1*. If $K = 1$, it is straightforward that $\boldsymbol{P} \in \mathcal{C}$ since $\boldsymbol{P} \in \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{C}$. Next, we prove that $\bar{\mathcal{Q}}$ satisfies Properties 1, 2, and 3. We begin with the proof of Property 1. Let $\boldsymbol{P}, \boldsymbol{Q} \in \bar{\mathcal{Q}}$. By (8), there exist finite sequences $\{\boldsymbol{P}^k\}_{k=1}^{K_1}$ and $\{\boldsymbol{Q}^k\}_{k=1}^{K_2}$ in $\mathcal{Q}$ satisfying that $\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^{K_1}$ and $\boldsymbol{Q} = \boldsymbol{Q}^1 \cdots \boldsymbol{Q}^{K_2}$. Let $\boldsymbol{R} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^{K_1} \boldsymbol{Q}^1 \cdots \boldsymbol{Q}^{K_2}$. By (8), $\boldsymbol{R} \in \bar{\mathcal{Q}}$. Note that $\boldsymbol{R} = \boldsymbol{P}\boldsymbol{Q}$. Thus, $\bar{\mathcal{Q}}$ satisfies Property 1.

It is straightforward that $\bar{\mathcal{Q}}$ satisfies Property 2 since $\boldsymbol{I} \in \mathcal{Q}$ and $\mathcal{Q} \subseteq \bar{\mathcal{Q}}$.

Finally, we show that $\bar{\mathcal{Q}}$ satisfies Property 3. Let $\boldsymbol{P} \in \bar{\mathcal{Q}}$. By (8), there exists a finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ satisfying $\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. Since $\mathcal{Q}$ satisfies Property 3, $\mathcal{Q}$ contains the inverse $\boldsymbol{P}^{k\prime}$ for all $k \in \{1, \ldots, K\}$. Let $\boldsymbol{Q} = \boldsymbol{P}^{K\prime} \cdots \boldsymbol{P}^{1\prime}$. By (8), $\boldsymbol{Q} \in \bar{\mathcal{Q}}$. Since $\boldsymbol{P}^{k\prime}\boldsymbol{P}^k = \boldsymbol{P}^k\boldsymbol{P}^{k\prime} = \boldsymbol{I}$ for all $k \in \{1, \ldots, K\}$, $\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{P} = \boldsymbol{I}$ holds. Thus, $\bar{\mathcal{Q}}$ satisfies Property 3.

(ii) Let $\tilde{\mathcal{Q}}$ be a group of cyclic permutations with $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$. We show that $\bar{\mathcal{Q}} \subseteq \tilde{\mathcal{Q}}$. Let $\boldsymbol{P} \in \bar{\mathcal{Q}}$. By (8), there exists a finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ satisfying $\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. Since $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$, $\boldsymbol{P}^k \in \tilde{\mathcal{Q}}$ holds for all $k \in \{1, \ldots, K\}$. If $K = 1$, it is straightforward that $\boldsymbol{P} \in \tilde{\mathcal{Q}}$. Next, we consider the case of $K > 1$. Let $\{\boldsymbol{Q}^k\}_{k=1}^K$ be the finite sequence in $\mathcal{P}$ defined by

$$\boldsymbol{Q}^1 = \boldsymbol{P}^1 \text{ and } \boldsymbol{Q}^k = \boldsymbol{Q}^{k-1}\boldsymbol{P}^k \text{ for all } k \in \{2, \ldots, K\}.$$

$$
\begin{array}{ccccccccc}
\mathcal{F} & \subset & \mathcal{S} & & & & & & \\
\cap & & \cap & & & & & & \\
\mathcal{V}_p & \subset & \mathcal{S} \cup \mathcal{V}_p & \subset & \overline{\mathcal{S} \cup \mathcal{V}_p} & \subset & \mathcal{V} & \subset & \mathcal{C}
\end{array}
$$

Figure 1: Set inclusion among sets of cyclic permutations

Since $\tilde{\mathcal{Q}}$ satisfies Property 1, $\boldsymbol{Q}^k \in \tilde{\mathcal{Q}}$ for all $k \in \{2, \ldots, K\}$. By definition, $\boldsymbol{Q}^K = \boldsymbol{P}$. Thus, $\boldsymbol{P} \in \tilde{\mathcal{Q}}$. ■

From Propositions 5, it follows that a maximal set in the class of all sets of cyclic permutations satisfying Properties 1*, 2, and 3 is a group of cyclic permutations. Thus, by Propositions 1, 2, 3, and 4 and the result of Mitra and Basu (2007) we mentioned after Proposition 2, maximal anonymity that is compatible with the existence of a Paretian Suzumura-consistent SWR coincides with maximal anonymity for a Paretian SWQ. Lauwers (2010b) has shown that a maximal group of cyclic permutations is a non-constructible object since it entails the use of a non-constructive mathematics such as the existence of free ultrafilter. Hence, we cannot obtain an explicit definition of maximal anonymity for a Paretian Suzumura-consisitent SWR.

We should note that Proposition 5 has a positive implication regarding the construction of a non-maximal group of cyclic permutations. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be constructible groups of cyclic permutations (e.g., $\mathcal{S}$ and $\mathcal{V}_p$). Proposition 5 tells that it is possible to construct another group of cyclic permutations that includes both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ if the union $\mathcal{Q}_1 \cup \mathcal{Q}_2$ satisfies Property 1*. This can be verified as follows. Since $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are groups of cyclic permutations, $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is a set of cyclic permutations. Further, it satisfies Properties 2 and 3. Hence, by Proposition 5, the closure $\overline{\mathcal{Q}_1 \cup \mathcal{Q}_2}$ is a group of cyclic permutations if $\mathcal{Q}_1 \cup \mathcal{Q}_2$ satisfies Property 1*. This finding can be applied to the groups $\mathcal{S}$ and $\mathcal{V}_p$ of cyclic permutations. By Example 3, the union $\mathcal{S} \cup \mathcal{V}_p$ satisfies Property 1*. Thus, $\overline{\mathcal{S} \cup \mathcal{V}_p}$ is a group of cyclic permutations.[12]

Figure 1 clarifies the relationship (in terms of set inclusion) between the sets of cyclic permutations that we have discussed in this section. Further, the properties satisfied by them are summarized in Table 1. For each row in Table 1, properties satisfied (resp. violated) by the set in the first column are indicated by + (resp. –).

5 Characterizations of $\mathcal{Q}$-anonymous Paretian SWRs

In Sect. 3, we have established that if $\mathcal{Q}$ is a set of cyclic permutations satisfying Properties 2 and 3, there exists a $\mathcal{Q}$-anonymous and Paretian quasi-transitive

[12] An example of a group $\mathcal{Q}$ of cyclic permutations satisfying $\overline{\mathcal{S} \cup \mathcal{V}_p} \subset \mathcal{Q} \subset \mathcal{V}$ is presented by Demichelis, Mitra, and Sorger (2010). On this, see their Lemma 3.

Table 1: Properties of sets of cyclic permutations

Set of cyclic permutations	Properties 1	1*	2	3
$\mathcal{C}$	−	−	+	+
$\mathcal{V}$	−	−	+	+
$\overline{\mathcal{S} \cup \mathcal{V}_p}$	+	+	+	+
$\mathcal{S} \cup \mathcal{V}_p$	−	+	+	+
$\mathcal{V}_p$	+	+	+	+
$\mathcal{S}$	+	+	+	+
$\mathcal{F}$	+	+	+	+

(and, thus, acyclic) SWR, and if it additionally satisfies Property 1*, there exists a $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWR. In this section, following these findings, we consider a set $\mathcal{Q}$ of cyclic permutations satisfying Properties 2 and 3 and provide characterizations of the classes of all $\mathcal{Q}$-anonymous and Paretian acyclic SWRs and of all $\mathcal{Q}$-anonymous and Paretian quasi-transitive SWRs. Further, adding the restriction that $\mathcal{Q}$ satisfies Property 1*, we characterize the class of all $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWRs. We also provide a characterization of the class of all $\mathcal{Q}$-anonymous Paretian SWQs.

Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 2 and 3. We first establish a characterization of the class of all $\mathcal{Q}$-anonymous and Paretian acyclic SWRs. By (2a) and (2b), the $\mathcal{Q}$-Pareto relation is defined by combining Pareto with $\mathcal{Q}$-Anonymity. Hence, every acyclic SWR that satisfies Pareto and $\mathcal{Q}$-Anonymity includes the $\mathcal{Q}$-Pareto relation as a subrelation. Further, conversely, any acyclic SWR that includes the $\mathcal{Q}$-Pareto relation as a subrelation satisfies Pareto and $\mathcal{Q}$-Anonymity. From this observation, we obtain the following proposition. We omit the detailed proof for the sake of brevity.

Proposition 6. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 2 and 3. An acyclic SWR $\succsim$ on X satisfies Pareto and $\mathcal{Q}$-Anonymity if and only if $\succsim_{\mathcal{Q}}$ is a subrelation of $\succsim$.*

Proposition 6 means that the class of all $\mathcal{Q}$-anonymous and Paretian acyclic SWRs coincides with the class of all acyclic SWRs that includes the $\mathcal{Q}$-Pareto relation as a subrelation. Hence, the $\mathcal{Q}$-Pareto relation is the least element with respect to set inclusion in the class of all $\mathcal{Q}$-anonymous and Paretian acyclic SWRs. For a formal discussion about this, see Banerjee (2006, Theorem 1) and Basu and Mitra (2007, p. 355).

Next, we characterize the class of all $\mathcal{Q}$-anonymous and Paretian quasi-transitive SWRs and of all $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWRs. Since quasi-transitivity and Suzumura consistency, respectively, imply acyclicity, these two classes of SWRs are subclasses of the class of SWRs we characterized in

Proposition 6. Note that, by Proposition 2 (i), the $\mathcal{Q}$-Pareto relation is quasi-transitive, and by Proposition 4, it is Suzumura-consistent if the given set $\mathcal{Q}$ satisfies Property 1^*. Hence, as a corollary of Proposition 6, we obtain the following characterization results.

Corollary 1. (i) *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 2 and 3. A quasi-transitive SWR $\succsim$ on X satisfies Pareto and $\mathcal{Q}$-Anonymity if and only if $\succsim_{\mathcal{Q}}$ is a subrelation of $\succsim$.*

(ii) *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 1*, 2, and 3. A Suzumura-consistent SWR $\succsim$ on X satisfies Pareto and $\mathcal{Q}$-Anonymity if and only if $\succsim_{\mathcal{Q}}$ is a subrelation of $\succsim$.*

We should note that the if-and-only-if statement in Corolalry 1 (ii) is true without the assumption of Property 1^*. However, in this case, the class of $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWR is empty. This result follows from Proposition 3.

We now examine the class of all $\mathcal{Q}$-anonymous Paretian SWQs. This class is a subclass of the class of all $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWRs. Thus, if the set $\mathcal{Q}$ of permutations violates Property 1^*, it is empty by the same reason we noted above. However, it is not empty if $\mathcal{Q}$ satisfies Property 1^*. This can be verified by using *transitive closure*. The transitive closure of a binary relation $\succsim$, denoted by $\overline{\succsim}$, is defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \mathrel{\overline{\succsim}} \boldsymbol{y} \text{ iff } \begin{cases} \text{there exists a finite sequence } \{\boldsymbol{x}^k\}_{k=1}^K \text{ in } X \text{ such that} \\ \boldsymbol{x} = \boldsymbol{x}^1,\, \boldsymbol{y} = \boldsymbol{x}^K, \text{ and } \boldsymbol{x}^k \succsim \boldsymbol{x}^{k+1} \text{ for all } k \in \{1, \ldots, K-1\}. \end{cases} \tag{9}$$

We collect the properties of transitive closure. Let $\succsim$ and $\succsim'$ be binary relations on X. The transitive closures of them satisfy the following properties (see, e.g., Bossert and Suzumura (2010, p. 34)):

(T1): $\succsim \subseteq \overline{\succsim}$;

(T2): $\succsim$ is transitive iff $\succsim = \overline{\succsim}$;

(T3): $\overline{\succsim}$ is the smallest transitive relation containing $\succsim$;

(T4): $\succsim \subseteq \succsim'$ implies $\overline{\succsim} \subseteq \overline{\succsim'}$.

Further, the following property is satified:

(T5): If $\succsim$ is Suzumura-consistent, then $\overline{\succsim}$ extends $\succsim$.

This can be verified as follows. By T1, if $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{x} \mathrel{\overline{\succsim}} \boldsymbol{y}$ holds. Now suppose $\boldsymbol{x} \succ \boldsymbol{y}$. By T1, $\boldsymbol{x} \mathrel{\overline{\succsim}} \boldsymbol{y}$. If $\boldsymbol{y} \mathrel{\overline{\succsim}} \boldsymbol{x}$ also holds, then by (9), there exists a finite sequence $\{\boldsymbol{x}^k\}_{k=1}^K$ in X such that $\boldsymbol{x}^1 = \boldsymbol{y}$, $\boldsymbol{x}^K = \boldsymbol{x}$, and $\boldsymbol{x}^{k-1} \succsim \boldsymbol{x}^k$ for all $k \in \{1, \ldots, K\}$. In the case of $K = 1$, we have a contradiction since $\boldsymbol{x} \succ \boldsymbol{y}$. Further, in the case of $K > 1$, it leads to a contradiction since $\boldsymbol{x} \succ \boldsymbol{y}$ and $\succsim$ is Suzumura-consistent. Thus, $\boldsymbol{x} \succ \boldsymbol{y}$ implies $\boldsymbol{x} \mathrel{\overline{\succsim}} \boldsymbol{y}$ and $\neg(\boldsymbol{y} \mathrel{\overline{\succsim}} \boldsymbol{x})$.

From T3 and T5, it follows that the class of all $\mathcal{Q}$-anonymou Paretian SWQs coincides with the class of all the transitive closures of the $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWRs. By Corollary 1 (ii), this class contains the transitive closure $\overline{\succsim_{\mathcal{Q}}}$ of the $\mathcal{Q}$-Pareto relation. Hence, it is nonempty.

To present a characterization of the class of all $\mathcal{Q}$-anonymous Paretian SWQs, we define the $\mathcal{Q}$-*Suppes-Sen* relation $\succsim_{\mathcal{Q}S}$ associated with a set $\mathcal{Q}$ of permutations as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{\mathcal{Q}S} \boldsymbol{y} \text{ iff there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{P}\boldsymbol{x} \geq \boldsymbol{y}. \tag{10}$$

The $\mathcal{Q}$-Suppes-Sen relation has been introduced by Svensson (1980) for the case of $\mathcal{Q} = \mathcal{F}$. Mitra and Basu (2007) extended it to an arbitrary group $\mathcal{Q}$ of cyclic permutations. If $\mathcal{Q}$ is a group of cyclic permutations, $\succsim_{\mathcal{Q}S}$ is an SWQ (Mitra and Basu, 2007, Proposition 3).

The following lemma shows that if $\mathcal{Q}$ is a set of cyclic permutations satisfying Properties 1*, 2, and 3, then the transitive closure $\overline{\succsim_{\mathcal{Q}}}$ of the $\mathcal{Q}$-Pareto relation coincides with the $\bar{\mathcal{Q}}$-Suppes-Sen SWQ $\succsim_{\bar{\mathcal{Q}}S}$ associated with the closure $\bar{\mathcal{Q}}$ of $\mathcal{Q}$.

Lemma 3. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 1*, 2, and 3. Then,*

$$\overline{\succsim_{\mathcal{Q}}} = \succsim_{\bar{\mathcal{Q}}S} .$$

Proof. First, we show that $\overline{\succsim_{\mathcal{Q}}} \subseteq \succsim_{\bar{\mathcal{Q}}S}$. By (2a), (2b), and (8), we obtain $\succsim_{\mathcal{Q}} \subseteq \succsim_{\bar{\mathcal{Q}}S}$. By T4,

$$\overline{\succsim_{\mathcal{Q}}} \subseteq \overline{\succsim_{\bar{\mathcal{Q}}S}}. \tag{11}$$

By Proposition 5 (i), $\bar{\mathcal{Q}}$ is a group of permutations. Thus, $\succsim_{\bar{\mathcal{Q}}S}$ is transitive. By T2,

$$\overline{\succsim_{\bar{\mathcal{Q}}S}} = \succsim_{\bar{\mathcal{Q}}S} . \tag{12}$$

By (11) and (12), we obtain $\overline{\succsim_{\mathcal{Q}}} \subseteq \succsim_{\bar{\mathcal{Q}}S}$.

Next, we show that $\succsim_{\bar{\mathcal{Q}}S} \subseteq \overline{\succsim_{\mathcal{Q}}}$. Suppose $\boldsymbol{x} \succsim_{\bar{\mathcal{Q}}S} \boldsymbol{y}$. By (10), there exists $\boldsymbol{P} \in \bar{\mathcal{Q}}$ such that $\boldsymbol{P}\boldsymbol{x} \geq \boldsymbol{y}$. By (8), there exists a finite sequence $\{\boldsymbol{P}^k\}_{k=1}^K$ in $\mathcal{Q}$ satisfying $\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^K$. Define the finite sequence $\{\boldsymbol{y}^k\}_{k=1}^K$ in X by

$$\boldsymbol{y}^1 = \boldsymbol{P}^K\boldsymbol{x}, \text{ and } \boldsymbol{y}^k = \boldsymbol{P}^{K-k+1}\boldsymbol{y}^{k-1} \text{ for all } k \in \{2, \ldots, K\}.$$

Note that $\boldsymbol{y}^K = \boldsymbol{P}\boldsymbol{x}$. By (2b), we obtain

$$\boldsymbol{x} \sim_{\mathcal{Q}} \boldsymbol{y}^1 \tag{13}$$

and, for all $k \in \{2, \ldots, K\}$,

$$\boldsymbol{y}^{k-1} \sim_{\mathcal{Q}} \boldsymbol{y}^k \tag{14}$$

Since $\boldsymbol{Px} \geq \boldsymbol{y}$, $\boldsymbol{y}^K \geq \boldsymbol{y}$ holds. Further, since $\mathcal{Q}$ satisfies Property 2, $\boldsymbol{I} \in \mathcal{Q}$ holds. Thus, by (2a, 2b),

$$\boldsymbol{y}^K \succsim_{\mathcal{Q}} \boldsymbol{y}. \tag{15}$$

By (13), (14), and (15), we obtain $\boldsymbol{x} \,\overline{\succsim_{\mathcal{Q}}}\, \boldsymbol{y}$. ■

In the next proposition, we provide a characterization of the class of all $\mathcal{Q}$-anonymous Paretian SWQs.

Proposition 7. *Let $\mathcal{Q}$ be a set of cyclic permutations satisfying Properties 1*, 2, and 3. An SWQ $\succsim$ on X satisfies Pareto and $\mathcal{Q}$-Anonymity if and only if $\succsim_{\bar{\mathcal{Q}}S}$ is a subrelation of $\succsim$.*

Proof. The proof of the if-part is straightforward and can be omitted. We prove the only-if-part. First, we show that $\boldsymbol{x} \succsim_{\bar{\mathcal{Q}}S} \boldsymbol{y}$ implies $\boldsymbol{x} \succsim \boldsymbol{y}$, or equivalently, $\succsim_{\bar{\mathcal{Q}}S} \subseteq \succsim$. By Corollary 1 (ii), $\succsim_{\mathcal{Q}}$ is a subrelation of $\succsim$. Thus, $\succsim_{\mathcal{Q}} \subseteq \succsim$. By T4, $\overline{\succsim_{\mathcal{Q}}} \subseteq \overline{\succsim}$. Since $\succsim$ is transitive, we obtain, by T2, $\overline{\succsim} = \succsim$. Thus, $\overline{\succsim_{\mathcal{Q}}} \subseteq \succsim$. By Lemma 3, $\succsim_{\bar{\mathcal{Q}}S} \subseteq \succsim$.

Next, we show that $\boldsymbol{x} \succ_{\bar{\mathcal{Q}}S} \boldsymbol{y}$ implies $\boldsymbol{x} \succ \boldsymbol{x}$. Assume $\boldsymbol{x} \succ_{\bar{\mathcal{Q}}S} \boldsymbol{y}$. By Lemma 3, we obtain $\boldsymbol{x} \,\overline{\succsim_{\mathcal{Q}}}\, \boldsymbol{y}$ and $\neg(\boldsymbol{y} \,\overline{\succsim_{\mathcal{Q}}}\, \boldsymbol{x})$. By (9), $\boldsymbol{x} \,\overline{\succsim_{\mathcal{Q}}}\, \boldsymbol{y}$ implies that there exists a finite sequence $\{\boldsymbol{x}^k\}_{k=1}^K$ in X such that $\boldsymbol{x}^1 = \boldsymbol{x}$, $\boldsymbol{x}^K = \boldsymbol{y}$, and $\boldsymbol{x}^k \succsim_{\mathcal{Q}} \boldsymbol{x}^{k+1}$ for all $k \in \{1, \dots, K-1\}$. Further, $\neg(\boldsymbol{y} \,\overline{\succsim_{\mathcal{Q}}}\, \boldsymbol{x})$ implies that there exists $h \in \{1, \dots, K-1\}$, $\boldsymbol{x}^h \succ_{\mathcal{Q}} \boldsymbol{x}^{h+1}$. Since $\succsim_{\mathcal{Q}}$ is a subrelation of $\succsim$, we obtain that, for all $k \in \{1, \dots, K-1\}$, $\boldsymbol{x}^k \succsim \boldsymbol{x}^{k+1}$ and $\boldsymbol{x}^h \succ \boldsymbol{x}^{h+1}$. Since $\succsim$ is transitive, we obtain $\boldsymbol{x}^1 \succ \boldsymbol{x}^K$, i.e., $\boldsymbol{x} \succ \boldsymbol{y}$. ■

By Proposition 7, $\succsim_{\bar{\mathcal{Q}}S}$ is the least element of the class of all $\mathcal{Q}$-anonymous Paretian SWQs. The result similar to Proposition 7 is obtained by Banerjee (2006). He shows that if $\mathcal{Q}$ is a group of cyclic permutations, then the class of all $\mathcal{Q}$-anonymous Paretian SWQs coincides with the class of all SWQs that include $\succsim_{\mathcal{Q}S}$ as a subrelation (Banerjee 2006, Proposition 2). Proposition 7 strengthens his characterization result by weakening the requirement of group structure to Properties 1*, 2, and 3.

In Figures 2 and 3, we summarize the relationship (in terms of set inclusion) between the four classes of SWRs we studied in this section: Classes I, II, III, and IV, respectively, correspond to the classes of (i) acyclic, (ii) quasi-transitive, (iii) Suzumura consistent, and (iv) transitive SWRs that satisfy Pareto and $\mathcal{Q}$-Anonymity. In the figures, we also present the least element of Classes I, II, and III: namely $\succsim_{\mathcal{Q}}$ and that of Class IV: $\succsim_{\bar{\mathcal{Q}}S}$. Figure 2 illustrates the case where a set $\mathcal{Q}$ of cyclic permutations satisfies Properties 2 and 3 but violates Property 1*. In this case, there is no $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWR. Thus, Classes III and IV are empty. In Figure 3, we illustrate the case where a set $\mathcal{Q}$ of cyclic permutations satisfies Properties 1*, 2, and 3. Since Suzumura consistency is necessary and sufficient for the existence of an ordering extension, Class IV contains an ordering

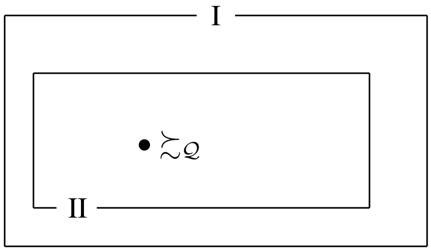

Figure 2: The case of $\mathcal{Q}$ that violates Property 1*

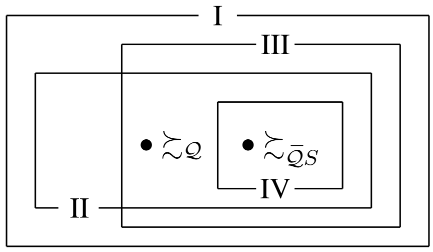

Figure 3: The case of $\mathcal{Q}$ that satisfies Property 1*

extension of the $\mathcal{Q}$-Pareto relation. It is known, however, that any ordering on X that satisfies Pareto and $\mathcal{F}$-Anonymity must involve the use of non-constructive mathematics (Lauwers, 2010a; Zame, 2007).[13] Thus, the ordering extension of the $\mathcal{Q}$-Pareto relation cannot be explicitly described if $\mathcal{Q}$ includes $\mathcal{F}$.

Finally, we briefly note earlier contributions in connection with the four classes of SWRs in Figures 2 and 3. First, Fleurbaey and Michel (2003) and Sakai (2010a) propose complete, quasi-transitive SWRs that satisfy Pareto and $\mathcal{V}$-Anonymity (SWRs called Type 3 and Type 6 in Fleurbaey and Michel (2003) and the future agreement and the future domination extensions in Sakai (2010a)). When we consider $\mathcal{Q} = \mathcal{V}$, their SWRs belong to Class II in Figure 2. Second, for an arbitrary group $\mathcal{Q}$ of cyclic permutations, Banerjee (2006) and Kamaga and Kojima (2009) propose $\mathcal{Q}$-anonymous Paretian SWQs that extends the $\mathcal{Q}$-Suppes-Sen SWQ (the $\mathcal{Q}$-utilitarian SWQ in Banerjee (2006) and the $\mathcal{Q}$-leximin and the $\mathcal{Q}$-generalized Lorenz SWQs in Kamaga and Kojima (2009)). Their SWQs belong to Class IV in Figure 3 when we fix $\mathcal{Q}$ as a group of cyclic permutations. Third, various extensions of the leximin and utilitarian overtaking SWQs that satisfies $\mathcal{S}$-Anonymity have been proposed in the literature (Asheim and Banerjee (2010), Fleurbaey and Michel (2003), Kamaga and Kojima (2010), and Lauwers (1997b)). Those SWQs belong to Class IV when we consider the case of $\mathcal{Q} = \mathcal{S}$. Lastly, in the case of $\mathcal{Q} = \mathcal{F}$, Class IV contains numerous SWQs proposed in the literature (e.g., the utilitarian SWR formulated by Basu and Mitra (2007) and the leximin SWR considered by Bosseet, Sprumont, and Suzumura (2007)).

6 Conclusion

We have characterized the set of permissible permutations of a Paretian SWR for three cases of coherence properties: acyclicity, quasi-transitivity, and Suzumura consistency. Together with the characterization result of Mitra and Basu (2007) obtained for a Paretian SWQ, our results complete the analysis of Pareto-compatible anonymity for various cases of coherence properties of SWRs considered in the

[13]This impossibility result has been conjectured by Fleurbaey and Michel (2003).

literature. Based on our characterization results, we obtained that, in the cases where Paretian acyclic SWRs and Paretian quasi-transitive SWRs are considered, anonymity defined by the set $\mathcal{C}$ of all cyclic permutations is the strongest Pareto-compatible anonymity, which cannot be realized by a Paretian SWQ. On the other hand, maximal anonymity for a Paretian Suzumura-consistent SWR cannot be stronger than those compatible with a Paretian SWQ. This is a negative result. However, we emphasize that we should not conclude that weakening transitivity to Suzumura consistency is of no use in exploring stronger anonymity. As we have demonstrated by using the constructible groups $\mathcal{S}$ and $\mathcal{V}_p$ of cyclic permutations, the analysis of anonymity that is compatible with a Paretian Suzumura-consistent SWR plays an important role in exploring an explicit construction of stronger anonymity axioms that are compatible with a Paretian SWQ.

In this paper, we have also characterized the classes of $\mathcal{Q}$-anonymous and Paretian acyclic SWRs, $\mathcal{Q}$-anonymous and Paretian quasi-transitive SWRs, and $\mathcal{Q}$-anonymous and Paretian Suzumura-consistent SWRs. Further, we established a new characterization of the class of $\mathcal{Q}$-anonymous Paretian SWQs. The least elements of these four classes, namely the $\mathcal{Q}$-Pareto relation and the $\overline{\mathcal{Q}}$-Suppes-Sen SWQ, are very weak in the sense that many pairs of utility streams are declared to be non-comparable. In contrast to the large literature that analyzes SWQs that extends the $\overline{\mathcal{Q}}$-Suppes-Sen SWQ, there have been only a few attempts to analyze anonymous Paretian SWRs satisfying weaker coherence properties like acyclicity, quasi-transitivity, and Suzumura consistency. An issue to be addressed in future work is to analyze acyclic, quasi-transitive, or Suzumura-consistent SWRs that make it possible to compare more paris of utility streams than the $\mathcal{Q}$-Pareto relation.

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