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Comparative Statics and Welfare Variations

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Abstract

By employing the order theoretical comparative statics, this study shows the equivalence of the following three concepts: the normality of consumer demand, the monotonicity of the welfare variations, and the complementarity of an indirect utility function. It is shown that the normality of the demand for a specific commodity is equivalent with Quah's C_i-quasisupermodularity of an indirect utility function, which is satisfied if an indirect utility is concave in income and supermodular in income and the consumption level of that commodity. In addition, this condition is the necessary and sufficient condition for the monotonicity of the welfare variations in income. The order theoretical comparative statics enables us to allow non-quasiconcave and non-differentiable utility functions. In addition, the existence of indivisible commodities and nonlinear price system can also be allowed.

JEL classification: D11, C61.

Keywords: Monotone comparative statics; Consumer theory; Welfare variations; Complementarity; Value order; Indirect utility.

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1 Introduction

Subsequent to a lot of successful applications for game theory, the order theoretical comparative statics has become an important technique in the consumer theory. Although the applicability of lattice programming for consumer theory was already pointed out in the celebrated paper by Milgrom and Shannon (1994), the comprehensive treatment is established in the relatively recent studies by Antoniadou (2007), Quah (2007), and Mirman and Ruble (2008). They show the condition for the monotonicity of the consumer's demand, in particular, the normality of the demand without imposing the classical regularity conditions such as the differentiability and strict quasiconcavity of a utility function. In addition to these studies, Weber (2010) relates Quah's theory to the monotone comparative statics of welfare variations (the compensating variation and equivalent variation) in income level.

Following these literature, this study provides a link between the normality of the demand and the monotonicity of welfare variations from the view point of order theoretical comparative statics. Under the strict quasiconcavity and smoothness of a utility function, the equivalence of these properties is known, for instance, by Hanemann (1980). Hanemann also shows that the monotonicity of welfare variations ensured, if an indirect utility function is concave in income level and supermodular in income and state change. In fact, the result in Weber (2010) can be seen as the generalization of the latter, in that Weber shows the same result without imposing the smoothness and strict quasiconcavity on a direct utility function. As a counterpart of Weber's study, this study generalizes the Hanemann's former result to the non-differentiable and non-quasiconcave environments. In addition, we show the characterization of the monotone welfare variations in terms of the property of an indirect utility function, which is, as in Weber (2010), related to Quah (2007). Similar to the joint assumption of concavity and supermodularity, our condition on an indirect utility function can be seen as a kind of complementarity between income and state change. As a result, the three way connection is provided: the normality of the demand, the "complementarity" of an indirect utility function, and the monotonicity of welfare variations.

More precisely, we take the following three steps: First, we characterize the normality of the demand through the order theoretical comparative statics, that is, the necessary and sufficient condition for the normal demand on a utility function is shown. Second, the monotonicity of welfare variations are characterized by the complementarity of an indirect utility function, and finally, we show the equivelence of the necessary and sufficient condition on a direct utility function and the complementarity of an indirect utility function. The overview of each step is as follows.

On the first step, the normality of the demand is characterized basically by the "value order" method proposed by Antoniadou (2007), and Mirman and Ruble (2008). This theory intends to apply the lattice theoretical comparative statics by Milgrom and Shannon (1994) and Li Calzi and Veinott (2005) for the consumer theory. As it can be easily confirmed, even if the consumption set is a sublattice of $\mathbb{R}^n$, lattice programming cannot be directly applied for the natural vector order. Indeed, for every two income levels, the budget sets in the utility maximization problem are not strong set comparable with respect to the vector order. To fend off this difficulty, Antoniadou, Mirman and Ruble define an "artificial" partial order under which the budget sets are strong set comparable. Actually, this artificial partial order is refered to as the value order, since it is related to the values of consumption vectors. Then, the comparative statics is performed by employing lattice programming, and hence, the quasisupermodularity of a utility function with respect to the value order constitutes the necessary and sufficient condition for a specific type of normal demand. However, if there are more than two commodities, constructing the appropriate partial order is not an easy task. In such a case, the most natural value order, the "direct value order" is a preorder, and hence, the standard lattice programming cannot be applied, even if the value order is defined[1]. We overcome this difficulty by applying the theory of monotone comparative statics on preordered sets by Shirai (2008, 2009). That is, in this paper, the normality of the demand is characterized by a generalized value order method. By employing this approach, to a large extent, we can allow the existence of indivisible commodities.

With regard to the second step, we should refer to the definitions of the welfare variations and an indirect utility function in this study. In fact, this study focuses on the normality of the demand for a specific commodity, say, commodity i. Thus, the welfare variations in this paper are those for the change of the consumption level of commodity i. For instance, given a price vector of other commodities, p_{-i}, two consumption level of commodity i, $x'_i > x_i$, and an income level $\tilde{w}_{-i}$, that can be used for the consumption of other commodities, the compensating variation $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x'_i, x_i)$ is defined as the willingness to pay for the increase of the consumption level of commodity i from x_i to x'_i. On the other hand, we define an indirect utility function as the value function of the utility maximization problem in which the consumption level of commodity i is fixed. Hence, an indirect utility function takes the form of $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$. As stated, Weber (2010) shows that if $\tilde{V}$ is concave in $\tilde{w}_{-i}$ and supermodular in $(\tilde{w}_{-i}, x_i)$, then the welfare variations, the compensating variation $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x'_i, x_i)$ and the equivalent variation are nondecreasing in $\tilde{w}_{-i}$. While the Weber's condition is the joint assumption of two cardinal conditions (concavity and supermodularity), by extracting the ordinal implication of it, we present the necessary and sufficient condition for the monotonicity of the welfare variations.

In fact, our necessary and sufficient condition on an indirect utility function, to which we refer as the "ordinal complementarity," is equivalent with the "C_i-quasisupermodularity," which is proposed by Quah (2007). Quah shows that if a *direct* utility function satisfies the C_i-quasisupermodularity, then the demand for commodity i is normal. That is, this notion is initially introduced as the condition for a direct utility function, which is an alternative to the value order methods. Interestingly, it is shown that the necessary and sufficient condition on a utility function derived from the value order method is equivalent with the C_i-quasisupermodularity of an *indirect* utility function. Thus, the equivalence of the normality of the demand and the monotonicity of the welfare variations is ensured.

The rest of this study is constructed as follows. In section 2, we prepare some mathematical resutls, in particular, the theories of comparative statics by Shirai (2008, 2009) and Quah (2007). The main results of this paper are stated in section 3. In section 3.1, the normality of the demand is characterized, and, in section 3.2, the equivalence relations are shown. In section 4, we show that the essentially same equivalency holds even in a nonlinear price model. The construction of section 4 is similar to section 3.

[1]Although Mirman and Ruble (2008) define a (partial) value order that can be used for n-commodity model, it derives the quite strong sufficient condition for the normality, which is not suitable in this paper, since we seek to characterize the normality.

2 Mathematical Preliminary

In this section, we introduce some mathematical concepts and results that are emplyed in the subsequent parts of this study. First, we state the generalized comparative statics by Shirai (2008, 2009), and then, a few mathematical concepts defined by Quah (2007) is explained. The former is employed to characterize the normality of the demand, while the latter is related to the property of an indirect utility function.

In light of the mathematical structure of the comparative statics of consumer's problem, it seems natural to focus on comparative statics of the solution set of the maximization problem with respect to the changes of feasible sets. Formally, the problem that we address can be written as follows. Let X be the domain of the objective function $f : X \to \mathbb{R}$, that is, the whole set of alternatives, and let $S \subset X$ be a feasible set. Now, consider the maximization problem

$$\max_{x \in S \subset X} f(x).$$

Let $M(S)$ denote the solution set of the above problem when the feasible set is S. We intend to analyze the effects of the change of the feasible set from S to S' on $M(\cdot)$. Specifically, in this study, monotone comparative statics forms the core of the analysis in this paper. Throughout this paper, we assume the nonemptyness of the solution set of every optimization problem.

To perform the comparative statics, we employ the theory of Shirai (2008, 2009), which can be regarded as the generalization of the lattice theoretical comparative statics by Milgrom and Shannon (1994) and Li Calzi and Veinott (2005) in that it is equivalent to the conventional results when lattice programming can be applied. More concretely, it is the theory of comparative statics on preordered sets with "lattice-like" properties, which is called a "preordered lattice structure."

Definition 1: Let $(X, \preccurlyeq_X)$ be a preordered set[2]. We say that X is a *preordered lattice structure* if $T_{x,y} \neq \emptyset$ and $A_{x,y} \neq \emptyset$, where

$$T_{x,y} = \sup\{z \mid z \preccurlyeq_X x \;\&\; z \preccurlyeq_X y\},$$

and

$$A_{x,y} = \inf\{z \mid x \preccurlyeq_X z \;\&\; y \preccurlyeq_X z\}.$$

That is, $T_{x,y}$ and $A_{x,y}$ is the set of the greatest lower bounds and that of the least upper bounds in X respectively.

Note that if $(X, \preccurlyeq_X)$ is a partially ordered set, then a preordered lattice structure is a lattice. In this sense, it is an extension of the concept of a lattice. Similar to the above case, we extend the notions of the strong set order and quasisupermodularity to preordered lattice structures. If $\preccurlyeq_X$ is a partial order, then each of the extended notions is equivalent with the corresponding notion in the standard lattice programming: the strong set order and quasisupermodularity.

[2] A binary relation is a preorder, if it is reflexive and transitive. If a preorder satisfies antisymmetry, then it is called a partial order.

Definition 2: Let $(X, \preccurlyeq_X)$ be a preordered lattice structure and $S, S' \subset X$. We say that S is lower than S' in terms of the *w-strong set order* if $T_{x,y} \cap S \neq \emptyset$ and $A_{x,y} \cap S' \neq \emptyset$ for every $x \in S$ and $y \in S'$. We denote this as $S \leq_{wa} S'$. We say that S is lower than S' in terms of the *s-strong set order* if $T_{x,y} \subset S$ and $A_{x,y} \subset S'$ for every $x \in S$ and $y \in S'$.

Definition 3: Let $(X, \preccurlyeq_X)$ be a preordered lattice structure and $f : X \to \mathbb{R}$. We say that f is *w-quasisupermodular* if

$$\forall t \in T_{x,y}; f(x) \geq (>) f(t) \Rightarrow \exists a \in A_{x,y}; f(a) \geq (>) f(y)$$

for every $x, y \in X$.

Then, we can extend Milgrom and Shannon (1994) as follows: if X is partially ordered, the theorem below is equal to the monotone comparative statics through lattice programming.

Theorem 4: *Let $(X, \preccurlyeq)$ be a preordered lattice structure and $f : X \to \mathbb{R}$. Let $M(S)$ be the solution set of the maximization problem*

$$\max_{x \in S \subset X} f(x).$$

For every $S \leq_{sa} S'$, $M(S) \leq_{wa} M(S')$ if and only if f is w-quasisupermodular.

Proof First, we demonstrate the sufficiency. Let $x \in M(S)$ and $y \in M(S')$. Since $S \leq_{sa} S'$, $T_{x,y} \subset S$ and $A_{x,y} \subset S'$. By definition, $f(x) \geq f(t)$ for all $t \in T_{x,y}$. Hence, by the w-quasisupermodularity of f, there exists $a \in A_{x,y}$ such that $f(a) \geq f(y)$; this implies that $A_{x,y} \cap M(S') \neq \emptyset$. Similarly, $T_{x,y} \cap M(S) \neq \emptyset$. Otherwise, by the w-quasisupermodularity of f, there exists $a \in A_{x,y}$ such that $f(a) > f(y)$; this contradicts $y \in M(S')$.

Then, we show the necessity. Fix $x, y \in X$ and let $S = \{x\} \cup T_{x,y}$ and $S' = \{y\} \cup A_{x,y}$ for $x, y \in X$. Obviously, $S \leq_{sa} S'$. Suppose that $f(x) \geq f(t)$ for all $t \in T_{x,y}$. Then, $x \in M(S)$. Since $M(S) \leq_{wa} M(S')$, there exists $a \in A_{x,y}$ such that $a \in M(S')$; this implies that $f(a) \geq f(y)$. Suppose that $f(x) > f(t)$ for every $T_{x,y}$ and $y \in M(S')$. Then, there exists $t \in T_{x,y}$ such that $t \in M(S)$; this contradicts the assumption. Hence, there exists $a \in A_{x,y}$ such that $f(a) > f(y)$. *[Q.E.D.]*

Then, we turn to the "C-flexible" method by Quah (2007). Although, by using the following concepts, Quah provides the theory of monotone comparative statics which is tractable and easy to interpret, in this study, it suffices to refer to some fundamental definitions and those implications. In the rest of this section, we assume that X is a convex sublattice of $\mathbb{R}^n$. For every $i = 1, 2, ..., n$ and $\lambda \in [0, 1]$, we define the two operations Δ_i^λ and ∇_i^λ on X such that

$$\begin{aligned} x \Delta_i^\lambda y &= x && if\ x_i \leq y_i \\ &= \lambda y + (1 - \lambda)(x \wedge_E y) && if\ x_i > y_i \\ x \nabla_i^\lambda y &= y && if\ x_i \leq y_i \\ &= \lambda x + (1 - \lambda)(x \vee_E y) && if\ x_i > y_i, \end{aligned}$$

where $\wedge_E$ and $\vee_E$ denote the supremum and infimum with respect to the vector order, respectively.

Definition 5: A function $f : X \to \mathbb{R}$ is C_i^λ-quasisupermodular, if for every $x, y \in X$,

$$f(x) \geq (>) f(x \Delta_i^\lambda y) \Rightarrow f(x \nabla_i^\lambda y) \geq (>) U(y).$$

A function f is C_i^λ-supermodular, if for every $x, y \in X$,

$$f(x) + f(y) \leq f(x \Delta_i^\lambda y) + f(x \nabla_i^\lambda y).$$

If f is C_i^λ-quasisupermodular (C_i^λ-supermodular) for every $\lambda \in [0, 1]$, it is C_i-quasisupermodular (C_i-supermodular). In addition, if f is C_i-quasisupermodular (C_i-supermodular) for every $i = 1, 2, ..., n$, it is C-quasisupermodular (C-supermodular).

Note that C_i-quasisupermodularity (C_i-supermodularity) is stronger than quasisupermodularity (supermodularity), and hence, it can be regarded as a kind of complementarity between variables. Figure 1 below, which is borrowed from Quah (2007), provides us with the intuition for these properties. In addition, it is known that C_i-quasisupermodularity, more specifically, C_i-supermodularity is

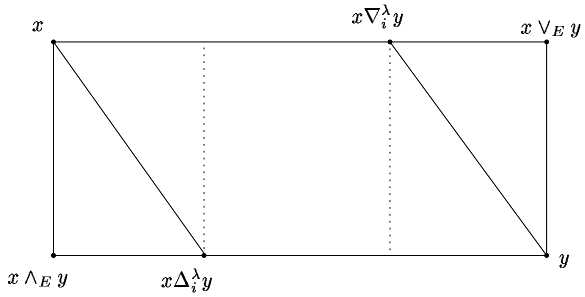

Figure 1: C_i-quasisupermodularity

ensured, if a function f is supermodular with respect to the vector order and *i-concave*. A function f is i-concave, if the function $\phi_v : s \mapsto f(x + sv)$ is concave for every $x \in X$ and $v \in \mathbb{R}_+^n$ such that $v_i = 0$, where the domain of ϕ_v is the set of nonnegative reals s such that $x + sv \in X$. In particular, if f is concave in $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, it is i-concave. It is also worth noting that the definition of C_i-quasisupermodularity is valid, as long as the set $X_{-i}(x_i) = \{x_{-i} \mid (x_{-i}, x_i) \in X\}$ is convex for every x_i. In case of $n = 2$, we can characterize the C_i-quasisupermodularity as follows.

Proposition 6: *Suppose that X is a convex sublattice of $\mathbb{R}^2$. Then, a function $f : X \to \mathbb{R}$ is C_i-quasisupermodular, if and only if*

$$f(x_{-i} - c; x'_i) \geq (>) f(x_{-i}; x_i) \Rightarrow f(x'_{-i} - c; x'_i) \geq (>) f(x'_{-i}; x_i)$$

for every $x'_{-i} > x_{-i}$, $x'_i > x_i$, and $c \geq 0$.

Proof Fix $c \geq 0$ and assume that f is C_i-quasisupermodular. Let $a = (x_{-i}, x'_i)$ and $b = (x'_{-i}, x_i)$. We show that there exists $\lambda^* \in [0,1]$ such that $a\Delta_i^{\lambda^*} b = (x_{-i}, x_i)$ and $a\nabla_i^{\lambda^*} b = (x'_{-i} - c, x'_i)$. Since, for every $\lambda \in [0,1]$, $a\Delta_i^{\lambda} b = (\lambda x'_{-i} + (1-\lambda)(x_{-i} - c), x_i)$ and $x_{-i} \in [x_{-i} - c, x'_{-i}]$, there exists a desirable λ^*. It is straightforward that, for this λ^*, $a\nabla_i^{\lambda^*} b = (x'_{-i} - c, x'_i)$. Thus, the latter condition follows.

Conversely, assume that f satisfies the latter condition, and that $a \parallel b$ and $a_i > b_i$. We can write $a = (x_{-i}, x'_i)$ and $b = (x'_{-i}, x_i)$. Then, for every $\lambda \in [0,1]$, $a\Delta_i^{\lambda} b = ((1-\lambda)a_{-i} + \lambda b_{-i}, b_i)$, and hence, we have $a = (x_{-i} - c, x'_i)$, where $c = \lambda(b_{-i} - a_{-i})$. Similarly, for this c, it follows that $a\nabla_i^{\lambda} b = ((1-\lambda)b_{-i} + \lambda a_{-i}, a_i) = (x'_{-i} - c, x'_i)$. Then, the C_i-quasisupermodularity follows. *[Q.E.D.]*

3 Comparative Statics and Welfare Variations

3.1 Characterization of Normality

In this subsection, the formal definition and characterization of the normality of demand is provided. Our idea of monotone comparative statics is based on the "value-order" method developed by Antoniadou (2007) and Mirman and Ruble (2008)[3]. In the standard linear price and divisible commodities model, our condition is satisfied by every concave and supermodular utility function (See also Chipman (1977) and Quah (2007)). In addition, our comparative statics is still valid, even if the existence of the indivisible commodities is allowed.

Let $X \subset \mathbb{R}^n_+$ be a consumption set, which is a sublattice of $\mathbb{R}^n_+$ with respect to the vector order on $\mathbb{R}^n_+$. In general, we do not assume the convexity of X. In particular, we allow the existence of indivisible commodities. Let $U : X \to \mathbb{R}$ be a utility function. Given a price vector $p \in \mathbb{R}^n_+$ and income level $w > 0$, define the budget set of the consumer as $B(p,w) = \{x \in X \mid p \cdot x \leq w\}$. Then, the utility maximization problem of the consumer is,

$$\max_{x \in B(p,w)} U(x).$$

Let $D(p,w)$ be the solution set of this problem. In this paper, we are interested in the condition under which the demand for i-th commodity is normal in the following sense.

Definition 7: The demand $D(p,w)$ is *strongly i-normal* for $w < w'$, if for every $x \in D(p,w)$ and $y \in D(p,w')$, there exist $t \in D(p,w)$ and $a \in D(p,w')$ such that $t_i = \min\{x_i, y_i\}$ and $a_i = \max\{x_i, y_i\}$.

Although, by definition, the consumption set X is a lattice under the standard vector order, it can be easily verified that for every $w < w'$, the budget set $B(p,w)$ and $B(p,w')$ are not strong set comparable. Hence, in general, the vector order cannot be directly applied for this comparative statics[4]. To employ the lattice theoretical comparative statics, Antoniadou (2007), and Mirman and Ruble (2008) develop the "value order" method. The main idea of their approach is to construct

[3]For further applications of the value order method, see Antoniadou et.al (2009).

[4]Milgrom and Shannon (1994) address this problem in the two commodity framework, by reducing the maximization problem to the univariate maximization problem. However, this does not work in the higher dimension.

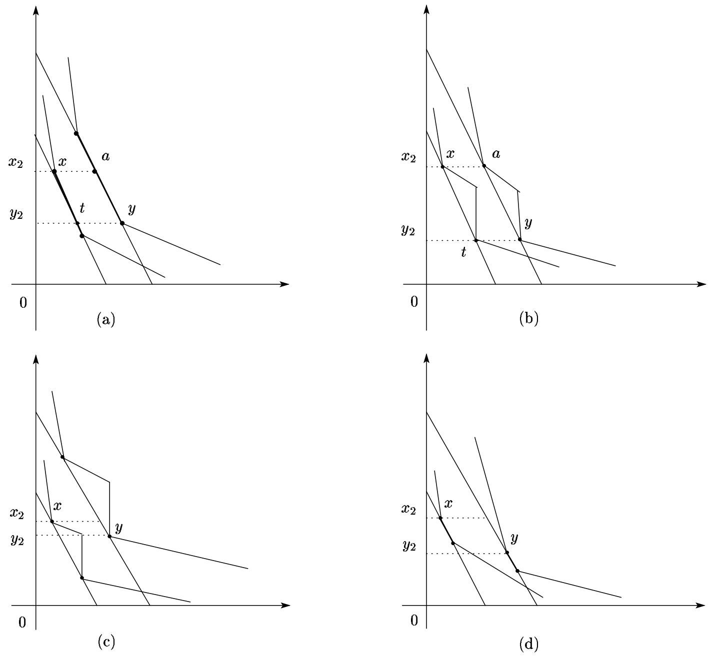

Figure 2: (a), (b): Strongly i-normal. (c), (d): Not strongly i-normal

the binary relation such that the budget sets are strong set comparable with respect to that "artifitial" order, and the strong set comparability of the demand sets implies the strong i-normality. Concretely, the value-order in this model is defined as follows.

Definition 8: Given a price vector p, define the binary relation $\preccurlyeq_{dv(p,i)}$ on X, such that $x \preccurlyeq_{dv(p,i)} y$ if $x_i \leq y_i$ and $p \cdot x \leq p \cdot y$. We refer to this binary relation as the direct (p,i)-value order.

As shown by Antoniadou (2007), the direct (p,i)-value order $\preccurlyeq_{dv(p,i)}$ is a preorder. In particular, if $n = 2$, it is a partial order. By its definition, the set of the greatest lower bounds with respect to this order is $T_{x,y} = \{z \in X \mid z_i = \min\{x_i, y_i\},\, p \cdot z = \min\{p \cdot x, p \cdot y\}\}$. Similarly, the set of the least upper bounds with repect to this order is $A_{x,y} = \{z \in X \mid z_i = \max\{x_i, y_i\},\, p \cdot z = \max\{p \cdot x, p \cdot y\}\}$. If there exists at least one divisible commodity other than i, both sets are nonempty. The proofs of this fact and propositions in this subsection are provided in Section 4.1, by using more general setting.

Lemma 9: *If there exists at least one divisible commodity other than i, the consumption set X is a preordered lattice structure with respect to the direct (p,i)-value order $\preccurlyeq_{dv(p,i)}$.*

Note that, in the two commodities model, the direct (p,i)-value order $\preccurlyeq_{dv(p,i)}$ is a partial order, and X is a lattice with respect to this order. Antoniadou (2007), and Mirman and Ruble (2008) mainly deal with the two commodities models, since they employ the standard lattice programming[5]. The following two lemmas imply the plausibility of employing this order to perform the comparative statics of the consumers demand. The proofs are provided in Section 4.1 using more general model.

Lemma 10: *Suppose that there exists at least one divisible commodity other than commodity i. Given a price vector p, the budget sets are s-strong set comparable with respect to $\preccurlyeq_{dv(p,i)}$. That is, $B(p,w) \leq_{sa} B(p,w')$ for every $w < w'$.*

Lemma 11: *Suppose that there exists at least one divisible commodity other than commodity i. Given a price vector p, the demand $D(p,w)$ is strongly i-normal, if and only if $D(p,w) \leq_{wa} D(p,w')$ with respect to $\preccurlyeq_{dv(p,i)}$ for every $w < w'$.*

Thus, by recalling Theorem 4 in the previous section, the necessary and sufficient condition for strong i-normality is as follows.

Proposition 12: *Suppose that there exists at least one divisible commodity other than commodity i. Given a price vector p, $D(p,w)$ is strongly i-normal, if and only if a utility function U is w-quasisupermodular with respect to $\preccurlyeq_{dv(p,i)}$.*

Remark: The strong i-normality is, in fact, stronger than the most standard concept of normality in this literature, which is introduced by Antoniadou (2007) and Quah (2007): the demand $D(p,w)$ is *i-normal* for $w < w'$, if for every $x \in D(p,w)$, there exists $y \in D(p,w')$ such that $x_i \leq y_i$, and for every $y \in D(p,w')$, there exists $x \in D(p,w)$ such that $x_i \leq y_i$. If $D(p,w)$ is i-normal for every $w < w'$, we say that the demand $D(p,w)$ is i-normal.

It is obvious that if the demand is strongly i-normal, then it is i-normal. However, in general, the converse does not follows, which is depicted in Figure 2(c), where the demand is i-normal, but not strongly i-normal. Hence, the w-quasisupermodularity of a utility function with respect to the direct (p,i)-value order is sufficient condition for the i-normality, but not necessary. Nevertheless, when we assume the convexity of X, and the local non-satiation and quasiconcavity of a utility function, this condition is also the necessary condition, which is depicted in Figure 2(a).

As the closely related results, under the convexity of X, Quah (2007) shows the sufficient condition for the strong i-normality by employing the C_i-flexible approach, which is stated in the second half of the last section. Although Quah refers to only the i-normality, as shown below, his sufficient condition ensures the strong i-normality.

[5] In Mirman and Ruble (2008), as the generalization of the direct value order, the iterated value order is defined as follows. Let $X = \mathbb{R}^n_+$ and $p(x) = p_1x_1 + p_2x_2 + ... + p_nx_n$, where every $p_k \geq 0$. The iterated value order $\leq_{(p,iv)}$ is defined such that for every $x, x' \in X$, $x \leq_{(p,iv)} x'$ if $p_1x_1 \leq p_1x'_1$, $p_1x_1 + p_2x_2 \leq p_1x'_1 + p_2x'_2$,..., and $\sum_{k=1}^n p_kx_k \leq \sum_{k=1}^n p_kx'_k$. It is shown that $(X, \leq_{(p,iv)})$ is a lattice and if a utility function U satisfies quasisupermodularity with respect to this order, the demand for x_1 is pathwisely normal. However, this sufficient condition depends on the numbering of coordinates and more restrictive than w-quasisupermodularity with respect to $\preccurlyeq_{(p,\leq_1)}$. Indeed, it can be confirmed from the fact that for every $x, y \in X$, the infimum and the supremum with respect to $\leq_{(p,iv)}$ are elements of $T_{x,y}$ and $A_{x,y}$ with respect to $\preccurlyeq_{(p,\leq_1)}$ respectively.

Proposition 13: *Suppose that X is convex. If a utility function U satisfies the C_i-quasisupermodularity, then it satisfies the w-quasisupermodularity with respect to $\preccurlyeq_{dv(p,i)}$ for every $p \in \mathbb{R}^n_+$.*

Proof Fix a price vector $p \in \mathbb{R}^n_+$. Let $x, y \in \mathbb{R}^n_+$. Without loss of generality, we can assume that $x_i > y_i$ and $p \cdot x < p \cdot y$. Note that there exists $\lambda^* \in [0,1]$ such that $(x\Delta_i^{\lambda^*}y)_i = y_i$, and for this λ^*, it follows that $p \cdot (x\Delta_i^{\lambda^*}y)$, that is, $(x\Delta_i^{\lambda^*}y) \in T_{x,y}$. Thus, if $U(x) \geq (>)U(t)$ for all $t \in T_{x,y}$, then $U(x) \geq (>)U(x\Delta_i^{\lambda^*}y)$. By C_i-quasisupermodularity, $U(x\nabla_i^{\lambda^*}y) \geq (>)U(y)$. Since $(x\nabla_i^{\lambda^*}y)_i = x_i$ and $p \cdot (x\nabla_i^{\lambda^*}y) = p \cdot y$, it follows that $(x\nabla_i^{\lambda^*}y) \in A_{x,y}$. Hence, U is w-quasisupermodular with respect to the direct (p,i)-value order. Since p is arbitrary, our claim follows. *[Q.E.D.]*

Corollary 14: *Suppose that X is convex. If a utility function U satisfies the C_i-quasisupermodularity, then the demand $D(p,w)$ is strongly i-normal for every $p \in \mathbb{R}^n_+$.*

Corollary 15: *Suppose that X is convex. If a utility function U is i-concave and supermodular with respect to the vector order, then the demand $D(p,i)$ is strongly i-normal for every p. If a utility function U is concave and supermodular with respect to the vector order, then the demand $D(p,w)$ is strongly i-normal for every $i = 1, 2, ..., n$.*

If $n = 2$, the C_i-quasisupermodularity and the w-quasisupermodularity with respect to $\preccurlyeq_{dv(p,i)}$ for every p is equivalent. Since the direct (p,i)-value order is a partial order, for every $p \in \mathbb{R}^n_+$, each of $T_{x,y}$ and $A_{x,y}$ is singleton, that is, the former is equal to $(x\Delta_i^{\lambda^*}y)$ and the latter is equal to $(x\nabla_i^{\lambda^*}y)$. Hence, the C_i-quasisupermodularity implies the w-quasisupermodularity with respect to $\preccurlyeq_{dv(p,i)}$ for every p.

As the preceding corollary indicates, under the convexity of the consumption set, Quah's approach provides us with the tractable sufficient condition for the normality of demand. In particular, both the concavity and supermodularity are familiar concepts in economic theory. On the other hand, our order theoretical approach has advantages when we deal with the models with indivisible commodities. In addition, as shown in Section 4, the latter can also deal with the model with nonlinear prices. However, most important aspect of the order theoretical approach is that, the necessary and sufficient condition for the normal demand is equivalent with the monotonicity of the welfare variations, which is addressed in the next subsection.

3.2 Welfare Variations, Indirect Utility, and Normality

Here, our main results in this paper is shown: the equivalence of the w-quasisupermodularity of a direct utility function, the monotonicity of the welfare variations, and the complementarity of an indirect utility function. As a result, three characterizations of the strong i-normality of the demand are provided. Here, the "complementarity" of an indirect utility function is defined as the C_i-quasisupermodularity of it. That is, as the property of an indirect utility function, C_i-quasisupermodularity is the necessary and sufficient condition for the normality of the demand. Thus, there is another link between our order theoretical comparative statics and Quah's C-flexible approach. Note that, in this subsection, we allow the existence of indivisible commodities, since the normality of the demand here is characterized by the w-quasisupermodularity of a utility function, which does not depend on the convexity of the consumption set.

Let us start from defining an indirect utility function and the welfare variations. As known, with regard to the welfare variations, there are two types of welfare variations: the compensating variation and the equivalent variation. Recall that, for every fixed x_i, $X_{-i}(x_i)$ stands for the consumption set for other commodities when the consumption level of i-th commodity is x_i, that we write the consumption vector of commodities other than i as x_{-i}, and that the price vector of commodities other than i is represented by p_{-i}. The disposable income for other commodities is represented by $\tilde{w}_{-i} > 0$, to distinguish from the income level in the utility maximization problem in the previous subsection. Then, for every fixed x_i, consider the following maximization problem:

$$\max_{x_{-i}\in B_{-i}(p,\tilde{w}_{-i})} U(x_{-i}; x_i),$$

where $B_{-i}(p, \tilde{w}_{-i}) = \{x_{-i} \in X_{-i}(x_i) \mid p_{-i} \cdot x_{-i} \leq \tilde{w}_{-i}\}$. That is, $\tilde{w}_{-i}$ denotes the income that can be used for the consumption of commodities other than i. Let $\tilde{V}(p, \tilde{w}_{-i}; x_i)$ be the value function of this maximization problem, and we refer to it as an indirect utility function. The two welfare variations are defined as follows.

Definition 16: Let $x_i' > x_i$. Given a price vector p_{-i} and income level $\tilde{w}_{-i}$, the compensating variation $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ is defined such that

$$\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) = \max\{c \mid \tilde{V}(p, \tilde{w}_{-i} - c; x_i') = \tilde{V}(p, \tilde{w}_{-i}; x_i)\}.$$

Similarly, the equivalent variation $\tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ is defined such that

$$\tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) = \min\{e \mid \tilde{V}(p, \tilde{w}_{-i}; x_i') = \tilde{V}(p, \tilde{w}_{-i} + e; x_i)\}.$$

In the following, we are interested in the monotonicity of the welfare variations in the income level $\tilde{w}_{-i}$. First, we state the following proposition by Weber (2010).

Proposition 17: *Suppose that the preference is locally non-satiated, that is, $\tilde{V}(p, \tilde{w}_{-i}; x_i)$ is increasing in $\tilde{w}_{-i}$. Then, the following statements are equivalent:*

1. *$\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ is nondecreasing in $\tilde{w}_{-i}$.*
2. *$\tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ is nondecreasing in $\tilde{w}_{-i}$.*
3. *$\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) \leq \tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$, in particular,*

$$\tilde{C}(p_{-i}, \tilde{w}_{-i} + \tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i), x_i', x_i) = \tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i).$$

This is closely related to the $WTP < WTA$ problem in the literature of the valuation for non-market good. For this issue, see Randall and Stoll (1980), Hanemann (1991), Weber (2003), and Plott and Zeiler (2005, 2007). From the preceding proposition, as long as the comparative statics matters, we can concentrate on the property of the compensating variation, and we do so when it is convenient. Weber (2010) points out the close relationship between the monotonicity of welfare variations and the comparative statics in Quah (2007). Indeed, Weber shows the following:

Proposition 18: *Suppose that the preference is locally non-satiated, that is, $\tilde{V}(p, \tilde{w}_{-i}; x_i)$ is increasing in $\tilde{w}_{-i}$. In addition, assume that the commodity i is desireble. Then, given a price vector p_{-i}, the welfare variations $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ and $\tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i', x_i)$ is nondecreasing in $\tilde{w}_{-i}$, if $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ is concave in $\tilde{w}_{-i}$ and supermodular in $(\tilde{w}_{-i}; x_i)$ with respect to the vector order in $\mathbb{R}^2_+$.*

Weber refers to the assumption of supermodularity in $\tilde{w}_{-i}$ and x_i as the *complementarity* of $\tilde{V}$. Then, slightly abusing the terminology, we can deduce that the C_i-supermodularity of the indirect utility function ensures the monotonicity of the welfare variations. This deduction is correct, and moreover, we can show that C_i-quasisupermodularity of the indirect utility function is the necessary and sufficient condition for the monotonicity of the welfare variations. Formally, an indirect utility function $\tilde{V}(p, \tilde{w}_{-i}; x_i)$ is C_i-quasisupermodular, if $\tilde{V}(p_{-i}, \tilde{w}_{-i} - c; x_i') \geq (>) \tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ implies $\tilde{V}(p_{-i}, \tilde{w}'_{-i} - c; x_i') \geq (>) \tilde{V}(p_{-i}, \tilde{w}'_{-i}; x_i)$ for every $\tilde{w}_{-i} < \tilde{w}'_{-i}$, and $c \geq 0$. We refer to this property as the *ordinal complementarity* of an indirect utility function.

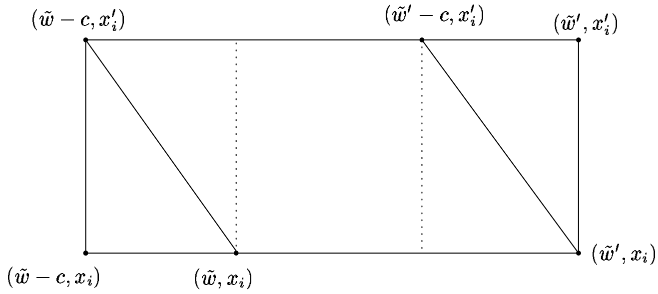

Figure 3: Ordinal complementarity of $\tilde{V}$

Proposition 19: *Suppose that the preference is locally non-satiated, that is, $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ is increasing in $\tilde{w}_{-i}$. In addition, assume that the i-th commodity is desireble. Then, the welfare variations are nondecreasing in $\tilde{w}_{-i}$, if and only if $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ satisfies the ordinal complementarity.*

Proof First, we show the monotonicity of the compensating variation. Suppose that V satisfies the C_i-quasisupermodularity. Then, $\tilde{V}(p_{-i}, \tilde{w}''_{-i} - \tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i); x_i') = \tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$, and hence, $\tilde{V}(p_{-i}, \tilde{w}'_{-i} - \tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i); x_i') \geq \tilde{V}(p_{-i}, \tilde{w}'_{-i}; x_i)$. This implies that $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) \leq \tilde{C}(p_{-i}, \tilde{w}'_{-i}, x_i', x_i)$.

To show the converse, suppose that $\tilde{V}$ does not satisfy the condition in the statement. Then, there exist some $\tilde{w}_{-i} < \tilde{w}'_{-i}$ and $x_i < x_i'$ such that $\tilde{V}(p_{-i}, \tilde{w}_{-i} - c; x_i') \geq (>) \tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ and $\tilde{V}(p_{-i}, \tilde{w}_{-i} - c; x_i') < (\leq) \tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ for some $c \geq 0$. This implies that $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) \geq (>) c$ and $\tilde{C}(p_{-i}, \tilde{w}'_{-i}, x_i', x_i) < (\leq) c$, and hence, we have $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i', x_i) > \tilde{C}(p_{-i}, \tilde{w}'_{-i}, x_i', x_i)$. *[Q.E.D.]*

Thus, we have proved the equivalence between the monotocity of the welfare variations and the ordinal complementarity of the indirect utility function. Finally, we add the third equivalent statement: the strong i-normality of the demand. Since, by Proposition 12, we know the equivalence between the strong i-normality of the demand and the w-quasisupermodularity with respect to the direct (p, i)-value order, the following proposition suffices. The proof is provided in the next section by proving more general claim.

Proposition 20: *Suppose that the preference is locally non-satiated, and the i-th commodity is desirable. In addition, assume that there exists at least one divisible commodity $j \neq i$. Then, given a price vector p_{-i}, a utility function U is w-quasisupermodular with respect to the direct (p, i)-value order for every $p = (p_i, p_{-i}) \in \mathbb{R}^n_+$, if and only if an indirect utility function $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ satisfies the ordinal complementarity.*

Corollary 21: *Suppose that the preference is locally non-satiated, and the i-th commodity is desirable. In addition, assume that $X_{-i}(x_i)$ is convex for every x_i. If a utility function satisfies the C_i-quasisupermodularity, then, for every p_{-i}, an indirect utility function $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ satisfies the ordinal complementarity (the C_i-quasisupermodularity).*

Corollary 22: *Suppose that the preference is locally non-satiated, and the i-th commodity is desirable. In addition, assume that there exists at least one divisible commodity $j \neq i$. Then, the following statements are equivalent:*

1. *Given a price vector p_{-i}, the demand $D(p, w)$ is strongly i-normal for every price $p = (p_i, p_{-i})$.*
2. *The welfare variations are nondecreasing in $\tilde{w}_{-i}$.*
3. *An indirect utility function $\tilde{V}(p_{-i}, \tilde{w}_{-i}; x_i)$ satisfies the ordinal complementarity.*
4. *A utility function U is w-quasisupermodular with respect to the direct (p, i)-value order for every $p = (p_i, p_{-i})$.*

Note that, in the above proposition and its corollary, the strong i-normality does not depend on the price of the i-th commodity. That is, given a price vector of other commodities, p_{-i}, the monotonicity of the welfare variations implies the strong i-normality of the demand whatever the price of the i-th commodity is.

4 Nonlinear Price System

4.1 The Value Order for Nonlinear Price System

In fact, even if the price system is nonlinear, the similar equivalence holds. In the next section, we prove it. In a nonlinear price model, we need slight modifications of the definition of the welfare variations, and an indirect utility function. The notions of monotonicity and complementarity are also adjusted to the general model.

Similar to the previous section, we start from the characterization of the normality of the demand. Let X be the consumption set, which is a sublattice of the vector lattice $\mathbb{R}^n_+$. Suppose that the preference of the consumer is represented by a utility function $U : X \to \mathbb{R}$, and let $p : X \to \mathbb{R}_+$ be a continuous and increasing price system. Then, given an income level $w > 0$, the budget set of the consumer is $B(p, w) = \{x \in X \mid p(x) \leq w\}$, and the utility maximization problem is defined in a similar fashion to the previous section. Let $D(p, w)$ denote the demand. We are interested in the necessary and sufficient condition for the strong i-normality of the demand. In fact, the idea of the direct (p, i)-value order is still valid in this setting.

Definition 23: Given a price system $p(\cdot)$, define a binary relation $\preccurlyeq_{dv(p,i)}$ such that

$$x \preccurlyeq_{dv(p,i)} y \iff x_i \leq y_i, \textit{ and } p(x) \leq p(y).$$

Obviously, this is a preorder on X, and the sets of greatest lower bounds and least upper bounds are defined as $T_{x,y} = \{z \in X \mid z_i = \min\{x_i, y_i\}, p(z) = \min\{p(x), p(y)\}\}$, and $A_{x,y} = \{z \in X \mid z_i = \max\{x_i, y_i\}, p(z) = \max\{p(x), p(y)\}\}$. We show that, under the assumption below, (1) the consumption set is a preordered lattice structure with respect to this order, and (2) the budget sets are strong set comparable, each of which is respectively the generalization of Lemmas 9, 10, and 11.

Assumption 24: There exists at least one commodity $j \neq i$, which is divisible and, for every x_{-j}, $\lim_{x_j \to +\infty} p(x_{-j}, x_j) = +\infty$.

Proposition 25: *Under Assumption 24, given a price system $p(\cdot)$, $(X, \preccurlyeq_{dv(p,i)})$ is a preordered lattice structure.*

Proof Let x and y be two incomparable points in X, with $x_i > y_i$ and $p(x) < p(y)$. Then, define x_i^t such that $x_i^t = y_i$. Since $p(\cdot)$ is increasing, $p(x_i^t, x_{-i}) < p(x)$. Then, by Assumption 24, there exists x_{-i}^t such that $p(x_i^t, x_{-i}^t) = p(x)$. This (x_i^t, x_{-i}^t) is obviously an element of $T_{x,y}$. Similarly, define x_i^a such that $x_i^a = x_i$. Since $p(x) = p(x_i^a, x_{-i}) < p(y_i, y_{-i}) = p(y)$, there exists x_{-i}^a such that $p(x_i^a, x_{-i}^a) = p(y_i, y_{-i})$, by Assumption 24. This $x^a = (x_i^a, x_{-i}^a)$ is an element of $A_{x,y}$. *[Q.E.D.]*

Proposition 26: *Under Assumption 24, given a price system $p(\cdot)$, the budget sets are s-strong set comparable for every $w < w'$. That is, $B(p, w) \leq_{sa} B(p, w')$.*

Proof Let $x \in B(p, w)$ and $y \in B(p, w')$. Suppose that $x_i > y_i$ and $p(x) < p(y)$. However, for every element $x^t \in T_{x,y}$, we have $p(x^t) = p(x) \leq w$, which implies that $T_{x,y} \subset B(p, w)$. Similarly, $A_{x,y} \subset B(p, w')$. Let $x, y \in B(p, w)$ with $x_i > y_i$ and $p(x) < p(y)$. However, for every $x^t \in T_{x,y}$, $p(x^t) = p(x) \leq w$, and for every $x^a \in A_{x,y}$, $p(x^a) = p(y) \leq w$. *[Q.E.D.]*

Proposition 27: *Under Assumption 24, given a price system $p(\cdot)$, the demand $D(p, w)$ is strongly i-normal, if and only if $D(p, w) \leq_{wa} D(p, w')$ with respect to $\preccurlyeq_{dv(p,i)}$ for every $w < w'$.*

Proof Suppose that the demand is strongly i-normal. Let $x \in D(p, w)$ and $y \in D(p, w')$, where $w < w'$. Then, there exist $x^t \in D(p, w)$ and $x^a \in D(p, w')$, which satisfies $x_i^t = \min\{x_i, y_i\}$, $p(x^t) = \min\{p(x), p(y)\}$, $x_i^a = \max\{x_i, y_i\}$, and $p(x^a) = \max\{p(x), p(y)\}$. It is obvious that $x^t \in T_{x,y}$ and $x^a \in A_{x,y}$.

Conversely, suppose that $D(p,w) \leq_{wa} D(p,w')$. Then, there exist $x^t \in D(p,w) \cap T_{x,y}$ and $x^a \in D(p,w') \cap A_{x,y}$. By definition, $x_i^t = \min\{x_i, y_i\}$ and $p(x^t) = \min\{p(x), p(y)\}$. Similarly, $x_i^a = \max\{x_i, y_i\}$ and $p(x^a) = \max\{p(x), p(y)\}$. This implies the strong i-normality of the demand. *[Q.E.D.]*

Then, by applying Theorem 4, we have the following characterization of the strong i-normality of the demand.

Proposition 28: *Under Assumption 24, given a price system $p(\cdot)$, the demand $D(p,w)$ is strongly i-normal, if only if a utility function satisfies the w-quasisupermodularity with respect to the direct (p,i)-value order.*

4.2 Welfare Variations under Nonlinear Price System

In this subsection, we shows the equivalence of the w-quasisupermodularity of a utility function, (and hence, the strong i-normality of the demand) the weak sense of monotonicity of the welfare variations (the single-crossing property), and the single-crossing property of an indirect utility function (it can be seen as the ordinal complementarity) in the setting of a nonlinear price system. As stated in the head of this section, we modify the definition of the welfare variations and an indirect utility function. It is shown that the results in section 3.2 can be seen as the special case of those in this subsection.

Suppose that $x_i', x_i \in X_j$ and $x_i < x_i'$. Consider the maximization problem

$$\max_{x_{-i} \in B_{-i}(p,w;x_i)} U(x_{-i}; x_i),$$

where $B_{-i}(p,w;x_i) = \{x_{-i} \in X_{-i} \mid p(x_{-i}, x_i) \leq w\}$. Let $V(p,w;x_i)$ denote the value function of this problem. As before, we refer to this as an indirect utility function. Note that the domain of $V(p,\cdot;x_i)$ is the set of w such that $B_{-i}(p,w;x_i) \neq \emptyset$.

Definition 29: Let $x_i < x_i'$ and fix a price system $p(\cdot)$ and an income level $w > 0$. The compensating variation for the change from x_i to x_i' is defined as

$$C(p,w,x_i',x_i) = \max\{c \mid V(p,w-c;x_i') = V(p,w;x_i)\}.$$

Similarly, the equivalent variation for the change from x_i to x_i' is defined as

$$E(p,w,x_i',x_i) = \min\{e \mid V(p,w;x_i') = V(p,w+e;x_i)\}.$$

Since the price system is increasing, the domains of $C(p,\cdot,x_i',x_i)$ and $E(p,\cdot,x_i',x_i)$ are equal to the domain of $V(p,\cdot;x_i')$.

Note that, even if the i-th commodity is desirable, the welfare variations in Definition 28 may be negative, since the higher consumption level of the i-th commodity straightens the budget set $B_{-i}(p,w;x_i)$, that is,

$$\{x_{-i} \in X_{-i} \mid p(x_{-i}, x_i') \leq w\} \subset \{x_{-i} \in X_{-i} \mid p(x_{-i}, x_i) \leq w\}$$

for every $x_i < x_i'$. The reason we modify the definition of the welfare variations is as follows. Since we are interested in the comparative statics of the welfare variations with respect to income levels, the price system that affects the decision in the reduced utility maximization problem need to be fixed. Indeed, the welfare variations in the previous section are defined for a given price vector for other commodities, p_{-i}. It was possible since the price system for other commodities is not affected by the consumption level of the i-th commodity. However, if the price system takes the general form $p(x_i, x_{-i})$, it is not the case. Hence, we have redefined the welfare variations in such a way that the entire price system is fixed. As shown below, if the price of the i-th commodity is separable, then the welfare variations in Definition 29 can be translated into those in Definition 23.

For the welfare variations and an indirect utility function in this section, we can show the similar results with the previous section. That is, the monotonicity of the welfare variations is equivalent with the complementarity of the indirect utility function, both of which are equivalent with the strong i-normality of the demand. We define the weaker form of monotonicity of the welfare variations, the single-crossing property, as follows.

Definition 30: Fix a price system $p(\cdot)$. The compensating variation $C(p, w, x_i', x_i)$ satisfies the *single-crossing property* in $(x_i; w)$ if

$$C(p, w, x_i', x_i) \geq (>) 0 \Rightarrow C(p, w', x_i', x_i) \geq (>) 0$$

for every $w < w'$ and $x_i < x_i'$ satisfying $B_{-i}(p, w, x_i') \neq \emptyset$. The single-crossing property of the equivalent variation is similarly defined.

The term "single-crossing property" stems from the fact that if the welfare variation satisfies it, then it crosses 0 at most once from below. In fact, the single-crossing property of the welfare variations is equivalent with the single-crossing property of the indirect utility function $V(p, w; x_i)$. Obviously, the single-crossing property is strictly weaker than the standard monotonicity, since every monotonic function trivially satisfies the single-crossing property. In addition, if the sign of a function is constant, then it satisfies the single-crossing property. The following proposition implies that the weak monotonicity (the single-crossing property) of the welfare variations is equivalent with that the indirect utility function satisfies the complementarity between income levels and the consumption level of the i-th commodity.

Proposition 31: *Fix a price system $p(\cdot)$. The compensating variation satisfies the single-crossing property in $(x_i; w)$ if and only if V satisfies the single-crossing property in $(x_i; w)$, that is, for every $w < w'$ and $x_i < x_i'$ satisfying $B_{-i}(p, w, x_i') \neq \emptyset$,*

$$V(p, w; x_i') \geq (>) V(p, w; x_i) \Rightarrow V(p, w'; x_i') \geq (>) V(p, w'; x_i).$$

Similarly, the equivalent variation satisfies the single-crossing property if and only if V satisfies the single-crossing property in $(x_i; w)$.

Proof First, we show the case with the compensating variation. Suppose that V satisfies the single-crossing property in $(x_i; w)$. Let $w < w'$ and $C(p, w, x_i', x_i) \geq (>) 0$. This implies that $V(p, w; x_i') \geq$

$(>)V(p,w;x_i)$. By the single-crossing property of V, we have $V(p,w';x_i') \geq (>)V(p,w;x_i)$. This implies that $C(p,w',x_i',x_i) \geq (>)0$.

To show the converse, suppose that V does not satisfy the single-crossing property. Then, there exist some $w < w'$ and $x_i < x_i'$ such that $V(p,w;x_i') \geq (>)V(p,w;x_i)$ and $V(p,w';x_i') < (\leq)V(p,w';x_i)$. The former inequation $V(p,w;x_i') \geq (>)V(p,w;x_i)$ implies that $C(p,w,x_i',x_i) \geq (>)0$, while the latter inequation implies that $C(p,w',x_i',x_i) < (\leq)0$.

The proof for the case with the equivalent variation is omitted because it is the same as that for the above argument, except that each of $C(p,w,x_i',x_i)$ and $C(p,w',x_i',x_i)$ are replaced with $E(p,w,x_i',x_i)$ and $E(p,w',x_i',x_i)$ respectively. *[Q.E.D.]*

Under the local non-satiation of the preference, we can characterize the w-quasisupermodularity of a utility function with respect to the direct (p,i)-value order by the single-crossing property of the welfare variations. That is, the equivalent relation in the preceding proposition is characterized by the strong i-normality of the demand.

Proposition 32: *Suppose that the preference is locally non-satiated. Then, under Assumption 24, given a price system $p(\cdot)$ a utility function U satisfies w-quasisupermodularity with respect to the direct (p,i)-value order if and only if $V(p,w;x_i)$ satisfies the single-crossing property in $(x_i;w)$.*

Proof First, we show the "if" part. Let $x,x' \in X$. Suppose that $x_i' > x_i$ and $p(x) < p(x')$. Define x^{t^*} and x^{a^*} such that

$$x_{-i}^{t^*} \in \underset{z_{-i} \in B_{-i}(p,p(x);x_i')}{\operatorname{argmax}} U(z_{-i};x_i'),$$

$$x_{-i}^{a^*} \in \underset{z_{-i} \in B_{-i}(p,p(x');x_i)}{\operatorname{argmax}} U(z_{-i};x_i),$$

with $x_i^{t^*} = x_i'$ and $x_i^{a^*} = x_i$. Since the preference is locally non-satiated, $p(x^{t^*}) = p(x)$ and $p(x^{a^*}) = p(x')$. It is obvious that $x^{t^*} \in T_{x,x'}$ and $x^{a^*} \in A_{x,x'}$. In the following, we prove that

$$U(x) \geq (>)U(x^{t^*}) \Rightarrow U(x^{a^*}) \geq (>)U(x').$$

By the analogy of x^{t^*} and x^{a^*}, we define x^* and $x^{'*}$ such that

$$x_{-i}^{*} \in \underset{z_{-i} \in B_{-i}(p,p(x);x_i)}{\operatorname{argmax}} U(z_{-i};x_i),$$

$$x_{-i}^{'*} \in \underset{z_{-i} \in B_{-i}(p,p(x');x_i')}{\operatorname{argmax}} U(z_{-i};x_i'),$$

with $x_i^* = x_i$ and $x_i^{'*} = x_i'$. Since $U(x^{t^*}) = V(p,p(x);x_i')$, $U(x^{a^*}) = V(p,p(x');x_i)$, $U(x^*) = V(p,p(x);x_i)$, and $U(x^{'*}) = V(p,p(x');x_i')$, by the single-crossing property, we have

$$\begin{aligned} U(x^*) \geq (>)U(x^{t^*}) &\iff V(p,p(x);x_i) \geq (>)V(p,p(x);x_i') \\ &\Rightarrow V(p,p(x');x_i) \geq (>)V(p,p(x');x_i') \\ &\iff U(x^{a^*}) \geq (>)U(x^{'*}). \end{aligned}$$

By definition, if $U(x) \geq (>)U(t)$ for all $t \in T_{x,x'}$, then $U(x^*) \geq (>)U(x^{t^*})$. The above inequations imply that $U(x^{a^*}) \geq (>)U(x')$ in such cases.

To show the converse, suppose that V does not satisfy the single-crossing property. Without loss of generality, we can assume $V(p, w; x_i') \geq (>)V(p, w; x_i)$ and $V(p, w'; x_i') < (\leq)V(p, w'; x_i)$. Then, the inequations at the end of the previous paragraph imply that $U(x^*) \geq (>)U(x^{t^*})$ and $U(x^{a^*}) < (\leq)U(x^{'*})$. This implies that U does not satisfy w-quasisupermodularity. *[Q.E.D.]*

Corollary 33: *Fix a price system $p(\cdot)$. Then, under the local non-satiation of the preference, and Assumption 24, the following statements are equivalent.*

1. *An indirect utility function V satisfies the single-crossing property in $(x_i; w)$.*
2. *The welfare variations satisfies the single-crossing property in $(x_i; w)$.*
3. *A utility function satisfies w-quasisupermodularity with respect to the direct (p, i)-value order.*
4. *The demand $D(p, w)$ is strongly i-normal.*

Finally, we relate the above results with the propositions in the previous section. In fact, if the price system $p(x_{-i}, x_i)$ is separable with respect to the i-th commodity, that is, $p(x_{-i}, x_i) = p_{-i}(x_{-i}) + p_i(x_i)$, the results in this section can be translated into those of the previous section. Note that, for a given price system $p(x)$ and the consumption level of commodity i, x_i, the income that can be used for the consumption of other commodities is equal to $\tilde{w}_{-i} = w - p_i(x_i)$. Hence, $\tilde{V}(p_{-i}, \tilde{w}_{-i}, x_i) = V(p, w, x_i)$, if and only if $\tilde{w}_{-i} = w - p_i(x_i)$. From this observation, it is easily confirmed that $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i, x_i') = C(p, w, x_i, x_i') + p_i(x_i') - p_i(x_i)$. Then, we can prove the following. Combining it with Propositions 31 and 32, the proof of Proposition 20 is straightforward.

Proposition 34: *Fix a price system $p(\cdot)$, which is separable with respect to commodity i. Suppose that a utility function is locally non-satiated and increasing in x_i. Then, under Assumption 24, the compensating (equivalent) variation in the previous section, $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i, x_i')$ ($\tilde{E}(p_{-i}, \tilde{w}_{-i}, x_i, x_i')$), is nondecreasing in $\tilde{w}_{-i}$, if and only if $C(p', w, x_i, x_i')$ ($E(p', w, x_i, x_i')$) satisfies the single-crossing property in $(x_i; w)$ for every $p'_{-i}(x_{-i}) = p_{-i}(x_{-i})$.*

Proof The "if" part is shown as follows. Suppose that $\tilde{C}(p, \tilde{w}_{-i}, x_i, x_i') > \tilde{C}(p, \tilde{w}'_{-i}, x_i, x_i')$ for some $w_{-i} < w'_{-i}$. Since a utility function is increasing in x_i, $\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i, x_i') > 0$. Define $p_i'(\cdot)$ such that

$$\tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i, x_i') > p_i'(x_i') - p_i'(x_i)$$
$$\tilde{C}(p_{-i}, \tilde{w}'_{-i}, x_i, x_i') < p_i'(x_i') - p_i'(x_i).$$

Then, by letting $w = \tilde{w}_{-i} + p_i'(x_i)$ and $w' = \tilde{w}'_{-i} + p_i'(x_i)$, $C(p', w, x_i, x_i') > 0$ and $C(p', w', x_i, x_i') < 0$; this violates the single-crossing property in $(x_i; w)$.

The "only if" part can be shown by defining the above $p_i'(\cdot)$ as $p_i(\cdot)$ and following the preceding conversely. That is, if $C(p', w, x_i, x_i') > 0$ and $C(p', w', x_i, x_i') < 0$, then,

$$\begin{aligned}\tilde{C}(p_{-i}, \tilde{w}'_{-i}, x_i, x_i') &< p_i(x_i') - p_i(x_i)\\ &< \tilde{C}(p_{-i}, \tilde{w}_{-i}, x_i, x_i')\end{aligned}$$

for $\tilde{w}_{-i} = w - p_i(x_i)$ and $\tilde{w}'_{-i} = w' - p(x_i)$. *[Q.E.D.]*

References

[1] Antoniadou, E. (2007): Comparative statics for the consumer problems. *Economic Theory*, 31(1), 189-203.

[2] Antoniadou, E., Mirman, L., Ruble, R. (2009): Lattices and lotteries. Mimeo.

[3] Chipman, J. (1977): An empirical implication of Auspitz-Lieben-Edgeworth-Pareto complementarity. *Journal of Economic Theory*, 14(1), 228-231.

[4] Hanemann, W.M. (1980): Quality changes, consumer's surplus, and hedonic price indices. Mimeo.

[5] Hanemann, W.M. (1991): Willingness to pay and willingness to accept: How much can they differ? *American Economic Review* 81(3), 635-647.

[6] Milgrom, P., Shannon, C. (1994): Monotone comparative statics. *Econometrica*, 62(1), 157-180.

[7] Mirman, L., Ruble, R. (2008): Lattice theory and the consumer's problem. *Mathematics of Operations Research*, 33, 301-314.

[8] Li Calzi, M., Veinott, A. (2005): Subextremal functions and lattice programming. Mimeo.

[9] Plott, C.R., Zeiler, K. (2005): The willingness to pay-willingness to accept gap, the "endowment effect," subject misconceptions, and experimental procedures for eliciting valuations. *American Economic Review* 95(3), 530-545.

[10] Plott, C.R., Zeiler, K. (2007): Exchange asymmetries incorrectly interpreted as evidence of endowment effect theory and prospect theory? Subject misconceptions, and experimental procedures for eliciting valuations. *American Economic Review* 97(4), 1449- 1466.

[11] Quah, J.K-H. (2007): The comparative statics of constrained optimization problems. *Econometrica*, 75(2), 401-431.

[12] Randall, A., Stoll, J.R., (1980): Consumer's surplus in the commodity space. *American Economic Review* 70(3), 449-455.

[13] Shirai, K. (2008): A generalization of monotone comparative statics. *Economics Bulletin*, 3(39), 1-9

[14] Shirai, K. (2009): A generalization of monotone comparative statics: A correction. *Economics Bulletin*, 29(1), 116-121.

[15] Weber, T.A. (2003): An exact relation between willingness to pay and willingness to accept. *Economics Letters* 80(3), 311-315.

[16] Weber, T. (2010): Hicksian welfare measures and the normative endowment effect. *American Economic Journal: Microeconomics*, forthcoming.