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On the Existence of a Submodular Utility Function

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Abstract

This paper shows that, on lattices in the nonnegative integer lattices, if an weakly increasing, and quasisubmodular binary relation has a representation, then it also has a submodular representation. This contrasts with the known result that every weakly increasing and quasisupermodular binary relation admits a supermodular representation on finite lattices, as long as it has a representation. In addition, we show that, if the binary relation is increasing then not only submodularity, but the discrete version of the law of diminishing marginal utility is also non-refutable. By applying this result, we can show the discrete version of Afriat's theorem: the decreasing increment of a utility function cannot be refuted from the finite price-consumption data.

JEL Classification: D11

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1 Introduction

Since the notion of quasisupermodularity (quasisubmodularity) was introduced by Milgrom and Shannon (1994), the condition for supermodularizability (submodularizability) of such functions has been considered. Although quasisupermodularity is the ordinal property that is satisfied by every supermodular function, there exist quasisupermodular functions that can not be supermodularized by any strictly increasing transformation. That is, in general, supermodularity contains ordinal implications more than quasisupermodularity. Nevertheless, by the recent study by Chambers and Echenique (2009), it is shown that if an weakly increasing and quasisupermodular binary relation has a representation, it must have a supermodular representation. The affinity of supermodularity for monotonicity is also supported by Li Calzi (1990), who shows that a twice continuously differentiable function defined a compact rectangle in $\mathbb{R}^n$ is supermodular if every partial derivative of it is positive. These results imply that, under weak monotonicity and some additional conditions, supermodularity has no ordinal implication except quasisupermodularity. Equivalently, it can

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be said that, under weak monotone decreasingness, submodularity has no ordinal implication other than quasisubmodularity.

From these observations, one may wonder what if weak increasingness is combined with submodularity. The aim of this note is to answer this question. Our answer is negative since it asserts that, for every binary relations on lattices in $\mathbb{Z}_+^L$, the representability and the submodular representability is equivalent if it is weakly increasing and quasisubmodular[1]. That is, under monotonicity, submodularity has no ordinal implication more than quasisubmodularity. Note that this implies that every increasing binary relation on a finite lattice has both a supermodular representation and a submodular representation as long as it has a representation. Moreover, under increasingness, there exists a submodular representation that satisfies the additional property—the discrete version of the law of diminishing marginal utility. From the mathematical perspective, this property is the dual of the "ultramodularity" defined by Marinacci and Montrucchio (2005), to which we refer as "dual ultramodularity." These results are derived by employing the well-known functional form that is used in the classical existence theorem for a utility function by Rader (1963).

We apply the dual ultramodular, and hence, submodular representability for the same topic with Chambers and Echenique (2009): the rationalizability of finite price-consumption data by a utility function. They prove that, as long as the consumption set is a finite lattice, the supermodularity of a utility function cannot be refuted from finite data sets. On the other hand, we can conclude that, on a lattice in $\mathbb{Z}_+^L$, the dual ultramodular representability, in particular, the submodular representability cannot be refuted. Although both are the counterparts of Afriat's theorem, which shows the non-refutability of the concave representability on $\mathbb{R}_+^L$, ours seems more natural discretization in the sense that it ensures the discrete version of the law of diminishing marginal utility.

2 Results

Consider a lattice in the nonnegative integer lattice $X \subset \mathbb{Z}_+^L$ and a binary relation $\mathcal{R}$ on it. Suppose that $\mathcal{R}$ is weakly increasing, and quasisubmodular. Formally, the weak increasingness of $\mathcal{R}$ requires that $x \geq y$ implies $x\mathcal{R}y$. Quasisubmodularity requires that $(x \wedge y)\mathcal{R}(\mathcal{P})x$ implies $y\mathcal{R}(\mathcal{P})(x \vee y)$, where $\mathcal{P}$ denotes the irreflexible part of $\mathcal{R}$. Note that, under weak increasingness, quasisubmodularity is equivalent with the property that $(x \wedge y)\mathcal{I}x$ implies $y\mathcal{I}(x \vee y)$, where $\mathcal{I}$ is the symmetric part of $\mathcal{R}$. It should also be noted that if $\mathcal{R}$ satisfies increasingness, then it is weakly increasing and quasisubmodular.

We say that $f : X \subset \mathbb{Z}_+^L \to \mathbb{R}$ is a representation of $\mathcal{R}$ when $f(x) \geq f(y) \iff x\mathcal{R}y$. A representation f is submodular (supermodular) if $f(x) + f(y) \geq (\leq) f(x \wedge y) + f(x \vee y)$. Suppose that $\mathcal{R}$ has a representation, say, U. The set $U(X) \subset \mathbb{R}$ is an well-ordered set with respect to the standard order[2]. This can be shown as follows. Let $V \subset U(X)$. Obviously, this is a totally ordered set. We show the existence of the minimum element. Consider $U^{-1}(V) \subset X \subset \mathbb{Z}_+^V$. Let $L \subset U^{-1}(V)$ be the set of minimal elements of $U^{-1}(V)$, that is, $L = \{x \in U^{-1}(V) \mid \nexists y \in U^{-1}(V); y < x\}$. Since this is a finite set, the set of minimizer on L is nonempty. In addition, the monotonicity of U ensures that every minimizer on L is the minimizer on $U^{-1}(V)$. Thus, we can represent $U(X) = \{u_1, u_2, ..., u_k, ...\}$ such that $u_k < u_{k+1}$ for every $k \in \mathbb{N}$. Define the set $A_k = U^{-1}(u_k) \subset X \subset \mathbb{Z}_+^L$. By analogy of Rader (1963), we can construct a representation as follows.

[1]The ordinal implication of submodularity is also studied by Chambers and Echenique (2008) from the different standpoint.

[2]On this point, I'm grateful to Christopher Chambers and Federico Echenique, who pointed out the inaccuracy in the first version.

Proposition 1: *Given a representation $U : X \subset \mathbb{Z}_+^L \to \mathbb{R}$, define a function $U^R : X \subset \mathbb{Z}_+^L \to \mathbb{R}$ such that*

$$U^R(x) = \sum_{i \in N(x)} \frac{1}{2^i}$$

where $N(x) = \{i \in \mathbb{N} \mid x\mathcal{R}a_i,\ a_i \in A_i\}$. Then, this U^R is also a representation of $\mathcal{R}$.

Proof We show that $x\mathcal{P}y$ implies $U^R(x) > U^R(y)$. It suffices to prove that $N(x) \supsetneq N(y)$. Since $X = \cup_k A_k$, there exist $i(x)$ and $i(y)$ such that $x \in A_{i(x)}$ and $y \in A_{i(y)}$. Our assumption implies that $i(x) > i(y)$. Since $N(z) = \{i \in \mathbb{N} \mid i \leq i(z)\}$, our claim follows. *[Q.E.D.]*

Based on its functional form, we refer to U^R as Rader's utility function. According to the preceding proposition, the existence of a submodular representation follows if Rader's utility function satisfies submodularity. We can prove it as follows.

Theorem 2: *Suppose that a binary relation $\mathcal{R}$ on a lattice $X \subset \mathbb{Z}_+^L$ satisfies weak increasingness and quasisubmodularity. Then it has a submodular representation as long as it has a representation.*

Proof We show that Rader's utility function satisfies submodularity. Let $x, y \in X$. Without loss of generality, we can assume that $x \parallel y$. By the weak increasingness of the binary relation, it follows that $U^R(x \wedge y) \leq U^R(y)$ and $U^R(x) \leq U^R(x \vee y)$. We can restrict our interest to the cases with strict inequalities. Indeed, if the former follows by equality, then the quasisubmodularity of the binary relation implies that $U^R(x) = U^R(x \vee y)$ and it follows that $U^R(x) + U^R(y) = U^R(x \wedge y) + U^R(x \vee y)$. If the latter follows by equality, it is trivial that $U^R(x) + U^R(y) \geq U^R(x \wedge y) + U^R(x \vee y)$. In the following, we assume that $U^R(y) \leq U^R(x)$, $U^R(x \wedge y) < U^R(y)$ and that $U^R(x) < U^R(x \vee y)$. Since $i(x \wedge y) < i(y) \leq i(x) < i(x \vee y)$, it follows that

$$\begin{aligned} U^R(y) - U^R(x \wedge y) &= \sum_{i=i(x \wedge y)+1}^{i(y)} \frac{1}{2^i} \\ &\geq \frac{1}{2^{i(x \wedge y)+1}} > \sum_{i=i(x)+1}^{i(x \vee y)} \frac{1}{2^i} \\ &= U^R(x \vee y) - U^R(x) \end{aligned}$$

This implies that $U^R(\cdot)$ satisfies submodularity. *[Q.E.D.]*

Intuitively, the submodularity of a utility function implies that the marginal utility of every commodity is nonincreasing in the consumption level of other commodities—the monotonicity of *cross* effect. In fact, if $\mathcal{R}$ satisfies increasingness, Rader's utility function also satisfies the discrete version of the law of diminishing marginal utility, that is, the marginal utility of every commodity is nonincreasing in the consumption level of it—the monotonicity of *own* effect. This is the dual of *ultramodularity* defined by Marinacci and Montrucchio (2005). Formally, a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is ultramodular if $f(y + h) - f(y) \leq f(x + h) - f(x)$ for every $x \geq y$ and $h \geq 0$. If A is a lattice, then every ultramodular function satisfies supermodularity by definition. We say that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is *dual ultramodular* if $-f$ is an ultramodular function. Similar to submodularity (supermodularity) and quasisubmodularity (quasisupermodularity), we can consider the ordinal version of this property. We say that a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is *quasiultramodular* if $f(y+h) \geq (>) f(y) \Rightarrow f(x+h) \geq (>) f(x)$ for every $x \geq y$ and $h \geq 0$. If $-f$ is quasiultramodular, the function f is said to be *dual quasiultramodular*. Every dual quasiultramodular function also satisfies quasisubmodularity. Note that under the weak monotonicity, a binary relation $\mathcal{R}$ satisfies dual quasiultramodular if $y + h\mathcal{I}y$ implies $x + h\mathcal{I}x$ for every $x \geq y$ and $h \geq 0$.

Theorem 3: *Suppose that a binary relation $\mathcal{R}$ on a lattice $X \subset \mathbb{Z}^L_+$ satisfies weak increasingness and dual quasiultramodularity. Then it has a dual ultramodular representation as long as it has a representation.*

Proof We show that Rader's utility function satisfies dual ultramodularity. Let $x \geq y$ and $h \geq 0$[3]. By the weak increasingness of $\mathcal{R}$, $U^R(x) \leq U^R(x+h)$ and $U^R(y) \leq U^R(y+h)$. Similar to the proof of Theorem 2, without loss of generality, we can suppose that $U^R(x) < U^R(x+h)$ and $U^R(y) < U^R(y+h)$ and $U^R(y+h) \leq U^R(x)$. Then, it follows that $i(y) < i(y+h) \leq i(x) < i(x+h)$. Hence,

$$
\begin{aligned}
U^R(x+h) - U^R(x) &= \sum_{i=i(x)+1}^{i(x+h)} \frac{1}{2^i} \\
&\leq \frac{1}{2^{i(y)+1}} < \sum_{i=i(y)+1}^{i(y+h)} \frac{1}{2^i} \\
&= U^R(y+h) - U^R(y),
\end{aligned}
$$

which implies the dual ultramodularity of U^R. *[Q.E.D.]*

Corollary 4: *Suppose that $\mathcal{R}$ on a lattice $X \subset \mathbb{Z}^L_+$ satisfies increasingness. Then it has a dual ultramodular representation as long as it has a representation. In particular, it is a submodular representation.*

By combining our results with those in Chambers and Echenique (2009), we have the following two corollaries. The first corollary implies that, if the domain of a preordering is a finite lattice, supermodular representability and submodular representability are compatible.

Corollary 5: *Let $(X, \leq)$ be a finite lattice. Then, every increasing binary relation on X has both a supermodular representation and a submodular representation, as long as it has a representation. In particular, both a supermodular representation and a submodular representaion exist.*

Proof Follows from our Theorem 2 and Theorem 1 of Chambers and Echenique (2009), where it is shown that every binary relation that has a quasisupermodular and weakly increasing representation admits a supermodular representation. *[Q.E.D.]*

On the supermodular representability, Shmaya (2010) shows the existence of an increasing, qusisupermodular, and continuous function on $[0,1] \times [0,1]$ that admits no supermodular representation. However, this

[3]Here, h is an element of $\mathbb{Z}^L_+$ such that $x+h, y+h \in X$.

does not mean the infiniteness of the domain solely demolishes the ordinal equivalency between quasisupermodularity and supermodularity under weak increasingness. Indeed, suppose that $X \subset \mathbb{Z}_-$ be a lattice (may be infinite) and consider $\mathcal{R}$ be an weakly increasing and quasisupermodular binary relation that has a representation $U : X \to \mathbb{R}$. Note that, under weak increasingness, quasisupermodularity requires that $x\mathcal{P}(x \wedge y)$ implies $(x \vee y)\mathcal{P}y$. Then, the set $U(X)$ can be written as $U(X) = \{u_1, u_2, ..., u_k, ...\}$ such that $u_k > u_{k+1}$ for every $k \in \mathbb{N}$. Then, we can define $A_k = U^{-1}(u_k)$ and Rader's Utility function $U^R(x) = \sum_{i=i(x)}^{\infty} \frac{1}{2^i}$, where $i(x) = \{i \in \mathbb{N} \mid x \in A_i\}$. This U^R is also a representation and satisfies supermodularity. Similarly, if an weakly increasing binary relation $\mathcal{R}$ satisfies quasiultramodularity and has a representation, then it has an ultramodular representation. The proofs of these are similar to those of Theorems 2 and 3. For instance, if $\mathcal{R}$ is weakly increasing, quasisupermodular, and $U(x \wedge y) < U(y) \leq U(x) < U(x \vee y)$, then

$$\begin{aligned}
U^R(x \vee y) - U^R(x) &= \sum_{i(x \vee y)}^{i(x)-1} \frac{1}{2^i} \\
&\geq \frac{1}{2^{i(x \vee y)}} > \sum_{i(y)}^{i(x \wedge y)-1} \frac{1}{2^i} \\
&= U(y) - U(x \wedge y).
\end{aligned}$$

Thus, we can say that what is important for ordinal equivalency between quasisubmodularity (quasisupermodularity) and submodularity (supermodularity) is the (reversed) well-orderedness of the images of the representations. This relationship is similar to the case with quasiultramodularity (dual quasiultramodularity) and ultramodularity (dual ultramodularity). Then, the preceding corollary can be generalized as follows.

Corollary 5': *Let $(X, \leq)$ be a finite lattice. Then, every increasing binary relation on X has both an ultramodular representation and a dual ultramodular representation, as long as it has a representation. In particular, both a supermodular representation and a submodular representaion exist.*

Finally, we state the second corollary, which is the counterpart of Proposition 7 of Chambers and Echenique (2009): the non-refutability of a dual ultramodular utility function by finite price-consumption data. By Afriat (1967), it is well-known that finite observations of pairs of price vectors and consumption choices can be rationalized by a utility function if and only if it can be rationalized by a concave utility function. Although Afriat works in the standard linear price setting, Matzkin (1991) and Forges and Minelli (2009) deal with this topic without assuming the linear budget constraint. The latter proves that, under the general budget set, the rationalizability by a utility function does not necessarily implies the rationalizability by a concave utility function. Chambers and Echenique (2009) prove the discrete version of Afriat's theorem by restricting the consumption set to finite lattices. As a result, they show that the rationalizablity by a utility function is equivalent with that by a supermodular utility function even under the general budget set. By almost the same argument with Chambers and Echenique, we can confirm that the rationalizability by a utility function is equivalent with that by a dual ultramodular utility function, whether the price system is linear or not. Our result is more general in the sense that we can allow the infinite consumption set like $\mathbb{Z}_+^L$. In addition, dual ultramodularity seems more natural counterpart of concavity, because it exhibits the law of diminishing marginal utility.

Before stating our result, let us provide the formal definition of the rationalizability along the line of

Chambers and Echenique (2009). Let $X \subset \mathbb{Z}_+^L$ be a lattice. For all $k = 1, 2, ..., K$, let $S^k \in 2^X \setminus \{\emptyset\}$ and let $x^k \in S^k$. The former denotes the feasible set and the latter denotes the choice of the consumer when the feasible set is S^k. Thus, the pairs $\{(x^k, S^k)\}_{k=1}^K$ constitute a data set. The following three conditions are imposed on the data set.

1. For all $k \in \{1, 2, ..., K\}$, there is no $y^k \in S^k$ such that $x^k < y^k$.
2. For all $k \in \{1, 2, ..., K\}$, if $x \in S^k$ and $y \leq x$ then $y \in S^k$.
3. For all $k, k' \in \{1, 2, ..., K\}$, if $x^{k'} \in S^k$ and $x^{k'} \neq x^k$, then $x^k \notin S^k$.

The first condition corresponds to a non-satiation. The second is the free disposal assumption. The last requirement is a variant of the weak axiom of revealed preference. As usual, we can define the revealed preference $\mathcal{R}^*$ such that $x\mathcal{R}^*y$ if there exists $k \in \{1, 2, ..., K\}$ such that $x = x^k$ and $y \in S^k$. The third condition on the data set ensures the antisymmetry of $\mathcal{R}^*$, that is, if $x\mathcal{R}^*y$ and $x \neq y$ then $x\mathcal{P}^*y$. A function $U : X \to \mathbb{R}$ rationalizes the data $\{(x^k, S^k)\}_{k=1}^K$ if it represents $\mathcal{R}^*$. We say that the data $\{(x^k, S^k)\}_{k=1}^K$ is rationalizable if there exists a function U that rationalizes $\mathcal{R}^*$.

Corollary 6: *Assume that $\{(x^k, S^k)\}_{k=1}^K$ satisfies conditions 1, 2, and 3. Then the data is rationalizable if and only if it is rationalized by a dual ultramodular function.*

The formal proof is omitted because it is exactly the same with the proof of Proposition 7 of Chambers and Echenique (2009), except that we apply Theorem 3 above instead of Theorem 1 of their paper. The main idea is as follows. Define the binary relation $\mathcal{B}$ such that $x\mathcal{B}y$ if $x\mathcal{R}^*y$ or $x \geq y$. By condition 3, it can be verified that if $x > y$ then $x\mathcal{P}_{\mathcal{B}}y$. Thus, if $\mathcal{B}$ has a representation, it must be increasing. Chambers and Echenique (2009) show that every representation of $\mathcal{B}$ also represents $\mathcal{R}^*$ and that if $\mathcal{R}^*$ has a representation, then $\mathcal{B}$ also has a representation. As a result, by applying Theorem 3, $\mathcal{B}$ has a dual ultramodular representation as long as $\{(x^k, S^k)\}_{k=1}^K$ is rationalizable. Hence, the rationalizability of the data and the rationalizability by a dual ultramodular function is equivalent[4].

References

[1] Afriat, S.N. (1967): The construction of utility functions from expenditure data. *International Economic Review*, 8(1), 67-77.

[2] Chambers, C.P., Echenique, F. (2008): Ordinal notions of submodularity. *Journal of Mathematical Economics*, 44(11), 1243-1245.

[3] Chambers, C.P., Echenique, F. (2009): Supermodularity and preferences. *Journal of Economic Theory*, 144(3), 1004-1014.

[4] Forges, F., Minelli, E. (2009): Afriat's theorem for general budget sets. *Journal of Economic Theory*, 144(1), 135-145.

[5] Li Calzi, M. (1990): Generalized symmetric supermodular functions. Mimeo.

[4] It should be noted that the well known result by Richter (1966) plays an important role in the proof of Chambers and Echenique (2009).

[6] Matzkin, R.L. (1991): Axioms of revealed preference for nonlinear choice sets. *Econometrica*, 59(6), 1779-1786.

[7] Milgrom, P., Shannon, C. (1994): Monotone comparative statics. *Econometrica*, 62(1), 157-180.

[8] Marinacci, M., Montrucchio, L. (2005): Ultramodular functions. *Mathematics of Operations Research*, 30(2), 311-332.

[9] Rader, T. (1963): The existence of a utlity function to represent preferences. *Review of Economic Studies*, 30(3), 229-232.

[10] Richter, M.K. (1966): Revealed preference theory. *Econometrica*, 34(3), 635-645.

[11] Shmaya, E. (2010): A quasisupermodular function without supermodular equivalent. Mimeo.