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Existence of competitive equilibrium in unbounded exchange economies with satiation

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Existence of competitive equilibrium in unbounded exchange economies with satiation☆

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Abstract

In this paper, we extend the results of Sato [Sato, N., 2009. Nonsatiation and existence of competitive equilibrium, G-COE GLOPE II Working Paper Series No.15] to economies with unbounded-from-below choice sets: we prove the existence of a competitive equilibrium (to be exact, a quasi-equilibrium) by assuming the "boundary satiation" condition introduced in Sato (2009) and the "strong compactness of individually rational utility set" introduced in Martins-da-Rocha and Monteiro [Martins-da-Rocha, V. F., Monteiro, C. K., 2009. Unbounded exchange economies with satiation: how far can we go? Journal of Mathematical Economics 45, 465–478]. As a result, we obtain a new equilibrium existence theorem that can be applied to the case in which choice sets are unbounded from below and satiation occurs only inside the set of individually rational feasible choices.

JEL classification: C62; D50

Key words: Satiation, Unbounded-from-below choice sets, Individually rational utility set, Competitive equilibrium.

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1. Introduction

This paper is a sequel to Sato (2009), [1] in which we introduced a new condition concerning the satiation property of preferences and established the existence of a competitive equilibrium under it. The new condition, which we here call *boundary satiation* (BS), asserts that if satiation occurs inside the individually rational feasible consumption set, then satiation should also occur on a "boundary" of the set. The condition BS generalizes the standard nonsatiation assumption and Allouch and Le Van's (2008, 2009) *weak nonsatiation.* [2] In particular, of these conditions, only BS allows the set of satiation points to be a subset of the individually rational feasible consumption set.

Unfortunately, Sato's (2009) existence results under BS depend on the boundedness of consumption sets (or that of individually rational feasible allocation set), and therefore, cannot be applied to securities markets with short-selling, in which choice sets are unbounded from below.

For the case of standard nonsatiation, Dana et al. (1999) show that the *compactness of the individually rational utility set* (CU) is sufficient for the existence of a competitive equilibrium. This condition CU allows choice sets (and the individually rational feasible allocation set) to be unbounded. Moreover, CU is implied by various types of *no-arbitrage conditions*, which are intended to bound the economies with unbounded-from-below choice sets endogenously by limiting arbitrage opportunities. [3]

On the other hand, Martins-da-Rocha and Monteiro (2009) show that under Allouch and Le Van's (2008, 2009) weak nonsatiation, CU is not sufficient for the existence of a competitive equilibrium. However, they establish the existence of a competitive equilibrium under weak nonsatiation by introducing the notion of a *strong compactness of the individually rational utility set* (SCU). The condition SCU is stronger than CU in general, but it allows the individually rational feasible allocation set to be unbounded. Moreover, CU is equivalent to SCU under standard nonsatiation, and therefore Martins-da-Rocha and Monteiro's (2009) results unify the existence results of Dana

[1] A revised version is attached to this paper.

[2] Weak nonsatiation allows satiation inside the individually rational feasible consumption set, provided that satiation also occurs outside the set.

[3] For details about the no-arbitrage conditions, in addition to Dana et al. (1999), see Hart (1974), Hammond (1983), Werner (1987), Page (1987), Nielsen (1989), Page and Wooders (1996), Allouch et al. (2002) and Allouch et al. (2006).

et al. (1999) and Allouch and Le Van (2009). [4]

The purpose of this paper is, based on the results of Martins-da-Rocha and Monteiro (2009), to extend the results of Sato (2009) to economies with unbounded-from-below choice sets. Since BS contains weak nonsatiation as a special case, CU is not sufficient for the existence of a competitive equilibrium under BS. Therefore, we prove the existence of a competitive equilibrium (to be exact, a quasi-equilibrium) under BS and SCU. In the proof, we require no additional conditions other than the standard assumptions, as a result of which, we also generalize the results of Martins-da-Rocha and Monteiro (2009).

Nielsen (1990), Allingham (1991) and Won et al. (2008) investigate the existence of a competitive equilibrium in the context of the CAPM without a riskless asset. [5] In a more general setting, Won and Yannelis (2006) provide existence results that contain these results as a special case. In fact, the results of Won and Yannelis (2006) can be applied to the case in which neither BS nor SCU holds. [6] However, as shown in this paper, there are cases in which our existence result can be applied, while those of Won and Yannelis (2006) cannot. Therefore, this paper provides one of the weakest sets of conditions in the literature for the existence of a competitive equilibrium.

This paper is organized as follows: In Section 2, we describe the model and list the assumptions. In Section 3, we first provide a preliminary existence result (Section 3.1), and then prove our main existence theorem (Section 3.2). Finally, in Section 4, we provide an example of an economy in which both BS and SCU hold, while Won and Yannelis's (2006) condition concerning satiation does not hold.

[4] Martins-da-Rocha and Monteiro's (2009) results also generalize several existence results based on the no-arbitrage conditions, including the results of Hart (1974), Werner (1987), Allouch et al. (2006).

[5] In the CAPM without a riskless asset, satiation is rather a rule than an exception. See, for example, Nielsen (1987) and Won et al. (2008).

[6] See also, Won and Yannelis (2008).

2. Model and Assumptions

2.1. Model

We consider a pure exchange economy $\mathcal{E}$ with ℓ commodities and n agents ($\ell, n \in \mathbb{N}$). [7] For convenience, let I be the set of all agents, that is, $I = \{1, \cdots, n\}$. Each agent $i \in I$ is characterized by a choice set $X_i \subset \mathbb{R}^\ell$, a utility function $u_i : X_i \to \mathbb{R}$, and an initial endowment $e_i \in \mathbb{R}^\ell$. Let $X = \prod_{i \in I} X_i$ with a generic element $x = (x_i)_{i \in I}$, and put $e = (e_i)_{i \in I} \in \mathbb{R}^{\ell n}$.

The pure exchange economy $\mathcal{E}$ is thus summarized by the list

$$\mathcal{E} = \Big(\mathbb{R}^\ell, (X_i, u_i, e_i)_{i \in I}\Big).$$

An allocation $x \in X$ is *feasible* if $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. Note that we do not allow free disposal. We denote the set of all feasible allocations by F.

An allocation $x \in X$ is *individually rational feasible* if $x \in F$ and $u_i(x_i) \geq u_i(e_i)$ for all $i \in I$. We denote the set of all individually rational feasible allocations by A. Let A_i be the projection of A onto X_i, and refer to it as the *individually rational feasible choice set of agent* $i \in I$. In addition, let $R_i = \{x_i \in X_i : u_i(x_i) \geq u_i(e_i)\}$ for each $i \in I$.

We define the *individually rational utility set* $\mathcal{U}$ as follows:

$$\mathcal{U} = \{(t_i) \in \mathbb{R}^n : \exists x \in A \text{ s.t. } u_i(e_i) \leq t_i \leq u_i(x_i)\ \forall i \in I\}.$$

The utility function u_i is *satiated at* $s_i \in X_i$ if s_i maximizes u_i over X_i, and s_i is called a *satiation point of* u_i. Let S_i denote the set of all satiation points of u_i, that is,

$$S_i = \{s_i \in X_i : u_i(s_i) \geq u_i(x_i) \text{ for all } x_i \in X_i\}.$$

Put $S = \prod_{i \in I} S_i$.

We adopt the following standard definitions of competitive equilibrium and quasi-equilibrium.

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[7] We use the following mathematical notations. The symbols $\mathbb{N}$, $\mathbb{R}^\ell$, and $\mathbb{R}^\ell_+$ denote the set of natural numbers, ℓ-dimensional Euclidean space, and the nonnegative orthant of $\mathbb{R}^\ell$, respectively. For $x, y \in \mathbb{R}^\ell$, we denote by $x \cdot y = \sum_{j=1}^{\ell} x_j y_j$ the inner product, and by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm. Let $B(x_0, r) = \{x \in \mathbb{R}^\ell : \|x - x_0\| < r\}$ denote the open ball centered at x_0 with radius r. For $a \in \mathbb{R} = \mathbb{R}^1$, we denote by $|a|$ the absolute value of a. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote by (a, b) and $[a, b]$, the open interval and closed interval between a and b, respectively. For a set $A \subset \mathbb{R}^\ell$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{bd} A$, the interior, closure, and boundary of A in $\mathbb{R}^\ell$, respectively.

Definition 1. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *competitive equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot e_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then $\overline{p} \cdot x_i > \overline{p} \cdot e_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} e_i$.

Definition 2. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *quasi-equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot e_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then $\overline{p} \cdot x_i \geq \overline{p} \cdot e_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} e_i$.

We can divide the existence proof of a competitive equilibrium of $\mathcal{E}$ into two parts: (i) proving that there exists a quasi-equilibrium of $\mathcal{E}$, and (ii) proving that the quasi-equilibrium is also a competitive equilibrium. Since the analysis concerning part (ii) is well established (or can be done independent of (i)), [8] in this paper we focus on the existence of a quasi-equilibrium of $\mathcal{E}$.

2.2. Assumptions

In this subsection, we present the assumptions used in this paper.

The following two sets of assumptions are quite standard in the literature.

Assumption 1. For each $i \in I$,

(a) X_i is closed, (b) X_i is convex, (c) $e_i \in X_i$.

Assumption 2. For each $i \in I$,

(a) u_i is upper semicontinuous on X_i, [9]

[8] There are several known sets of assumptions under which every quasi-equilibrium is also a competitive equilibrium. The following is the simplest: (a) $e_i \in \operatorname{int} X_i$ and (b) u_i is continuous on X_i for each $i \in I$. For further details, see Geistdoerfer-Florenzano (1982).

[9] A function $f : X \to \mathbb{R}$ is upper semicontinuous on $X \subset \mathbb{R}^\ell$ if and only if for all $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \geq \alpha\}$ is closed in X.

(b) u_i is strictly quasi-concave. [10]

For each $i \in I$, let $\mathrm{int}_{R_i} A_i$ denote the interior of A_i in the relative topology on $R_i \subset \mathbb{R}^\ell$, that is, for $x_i \in R_i$, we have $x_i \in \mathrm{int}_{R_i} A_i$ if and only if there exists an open ball $B(x_i, r)$ centered at x_i with radius r, such that $B(x_i, r) \cap R_i \subset A_i$. Let A_i^c and $(\mathrm{int}_{R_i} A_i)^c$ denote the complements of A_i and $\mathrm{int}_{R_i} A_i$ in X_i, that is, $A_i^c = X_i \setminus A_i$ and $(\mathrm{int}_{R_i} A_i)^c = X_i \setminus \mathrm{int}_{R_i} A_i$.

Concerning the satiation property of preferences, we use the following assumption introduced in Sato (2009).

Assumption 3 (BS). For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap (\mathrm{int}_{R_i} A_i)^c \neq \varnothing$.

Assumption 3 is a generalization of the standard nonsatiation assumption (that is, $S_i \cap A_i = \varnothing$ for all $i \in I$) and *weak nonsatiation* introduced in Allouch and Le Van (2008, 2009). [11] We refer to Assumption 3 as *boundary satiation* (BS) based on the fact that if $S_i \subset A_i$, then Assumption 3 implies that $S_i \cap (A_i \setminus \mathrm{int}_{R_i} A_i) \neq \varnothing$, and the set $A_i \setminus \mathrm{int}_{R_i} A_i$ coincides with the boundary of A_i in the relative topology on R_i under Assumptions 1 and 2.

We also use the following assumption that is stronger than BS.

Assumption 4 (INS). For each $i \in I$, (a) $S_i \cap \mathrm{int}_{R_i} A_i = \varnothing$, and (b) if $S_i \cap (A_i \setminus \mathrm{int}_{R_i} A_i) \neq \varnothing$, the set is a singleton.

Under Assumption 4, every u_i must be nonsatiated on $\mathrm{int}_{R_i} A_i$, the interior of A_i (in the relative topology of R_i), and therefore, we refer to the assumption as *interior nonsatiation* (INS). Note that INS is weaker than the standard nonsatiation assumption by (b).

Dana et al. (1999) show that under Assumptions 1 and 2 and the standard nonsatiation, the following condition is sufficient for the existence of a quasi-equilibrium.

Assumption 5 (CU). The individually rational utility set $\mathcal{U}$ is compact.

The compactness of $\mathcal{U}$ is a weaker property than the compactness of A. In particular, it allows A to be unbounded.

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[10] A function $f : X \to \mathbb{R}$ is strictly quasi-concave if and only if for all $x, y \in X$ with $f(x) > f(y)$, and for all $\lambda \in (0,1)$, we have $f(\lambda x + (1-\lambda)y) > f(y)$.

[11] Weak nonsatiation asserts that $S_i \cap A_i^c \neq \varnothing$ for all $i \in I$.

On the other hand, Martins-da-Rocha and Monteiro (2009) show that CU is not sufficient for the existence of a quasi-equilibrium under weak nonsatiation. [12] Instead, the authors establish the existence of a quasi-equilibrium under weak nonsatiation by introducing the notion of the *strong compactness of* $\mathcal{U}$.

Definition 3. The individually rational utility set $\mathcal{U}$ is *strongly compact* if for every sequence $(x^\nu)_{\nu\in\mathbb{N}}$ in A, there exist a feasible allocation $y \in F$ and a subsequence $(x^{\nu_k})_{k\in\mathbb{N}}$ satisfying

$$u_i(y_i) \geq \lim_{k\to\infty} u_i(x_i^{\nu_k}) \quad \text{for all} \quad i \in I,$$

together with

$$\lim_{k\to\infty} \frac{\mathbf{1}_{S_i}(x_i^{\nu_k})}{1+\|x_i^{\nu_k}\|^2}(y_i - x_i^{\nu_k}) = 0 \quad \text{for all} \quad i \in I.^{13} \tag{1}$$

The strong compactness of $\mathcal{U}$ is stronger than the compactness of $\mathcal{U}$ in general, but weaker than the compactness of A. In addition, it is readily verified that both the compactness and the strong compactness of the individually rational utility set are preserved under any upper semicontinuous and strictly increasing transformation of the utility functions.

Since BS contains weak nonsatiation as a special case, CU is not sufficient for the existence of a quasi-equilibrium under BS. Therefore, we assume the strong compactness of $\mathcal{U}$, instead of CU, when investigating equilibrium existence under BS.

Assumption 6 (SCU). The individually rational utility set $\mathcal{U}$ is strongly compact.

3. Results

3.1. Existence under interior nonsatiation

The main purpose of this paper is to prove the existence of a quasi-equilibrium under Assumptions 1, 2, BS, and SCU (Theorem 1 in Section

[12] See Martins-da-Rocha and Monteiro (2009, Section 5).

[13] We denote by $\mathbf{1}_{S_i} : X_i \to \{0,1\}$ the indicator function of S_i: $\mathbf{1}_{S_i}(x_i) = 1$ if $x_i \in S_i$, and $\mathbf{1}_{S_i}(x_i) = 0$ otherwise. In Martins-da-Rocha and Monteiro (2009), the space $\mathbb{R}^\ell$ is endowed with the L^1 norm, that is, $\|x\| = \sum_{j=1}^{\ell} |x_j|$ for all $x \in \mathbb{R}^\ell$, while we consider $\mathbb{R}^\ell$ with the Euclidean norm. Obviously, this difference does not affect the definition of the strong compactness of $\mathcal{U}$.

3.2). To this end, we first prove the existence of a quasi-equilibrium by assuming INS instead of BS.

The following proposition extends the result of Dana et al. (1999, Theorem 3) and is used to prove our main existence theorem. Although in the proposition we impose not SCU but CU, as we will see later (Remark 1 in Section 3.2), CU implies SCU under INS. Proposition 1 is therefore a special case of Theorem 1.

Proposition 1. *Under Assumptions 1, 2, INS, and CU, there exists a quasi-equilibrium* $(\overline{x}, \overline{p}) \in A \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$.

Proof. First by CU, it is readily verified that there exists $\xi_i \in \operatorname{argmax}\{u_i(x_i) : x_i \in A_i\}$ for each $i \in I$. Then, INS implies that there exists $\zeta_i \in (\operatorname{int}_{R_i} A_i)^c$ with $u_i(\zeta_i) \geq u_i(\xi_i)$.

For $\nu \in \mathbb{N}$, let $\mathcal{E}^\nu$ be the truncated economy obtained by replacing X_i with $X_i^\nu = X_i \cap \operatorname{cl} B(0, \nu)$ for each $i \in I$. Let $\overline{\nu} \in \mathbb{N}$ be such that $e_i \in X_i^\nu$ and $\zeta_i \in X_i \cap B(0, \nu) \subset X_i^\nu$ for all $\nu \geq \overline{\nu}$ and $i \in I$.

Let $R_i^\nu = \{x_i \in X_i^\nu : u_i(x_i) \geq u_i(e_i)\}$, and let A_i^ν denote the individually rational feasible choice set of $i \in I$ in $\mathcal{E}^\nu$. Note that $R_i^\nu = R_i \cap \operatorname{cl} B(0, \nu)$ and $\zeta_i \in R_i^\nu$.

For each $\nu \geq \overline{\nu}$, the economy $\mathcal{E}^\nu$ satisfies all the assumptions of the existence theorem in Sato (2009, Theorem 2). In particular, $\mathcal{E}^\nu$ satisfies BS. To observe this we first prove the following claim.

Claim 1. $\zeta_i \in (\operatorname{int}_{R_i^\nu} A_i^\nu)^c$.

Proof of Claim 1. Suppose that $\zeta_i \in \operatorname{int}_{R_i^\nu} A_i^\nu$.

By the definition of $\operatorname{int}_{R_i^\nu} A_i^\nu$, there exists $r_1 > 0$ such that

$$B(\zeta_i, r_1) \cap R_i^\nu \subset A_i^\nu.$$

On the other hand, since $\zeta_i \in B(0, \nu)$, there exists $r_2 > 0$ such that $B(\zeta_i, r_2) \subset B(0, \nu)$.

Let $r = \min\{r_1, r_2\}$. Then,

$$B(\zeta_i, r) \cap R_i \subset B(\zeta_i, r_1) \cap \Big(B(\zeta_i, r_2) \cap R_i\Big) \subset B(\zeta_i, r_1) \cap R_i^\nu \subset A_i^\nu \subset A_i.$$

Therefore, $\zeta_i \in \operatorname{int}_{R_i} A_i$, which contradicts our choice of ζ_i. □

Let S_i^ν denote the set of all satiation points of agent $i \in I$ on X_i^ν, that is, $S_i^\nu = \{s_i \in X_i^\nu : u_i(s_i) \geq u_i(x_i) \text{ for all } x_i \in X_i^\nu\}$. Note that $S_i^\nu \neq \varnothing$ by the compactness of X_i^ν and upper semicontinuity of u_i. We now prove that $\mathcal{E}^\nu$ satisfies BS.

Claim 2. $S_i^\nu \cap (\operatorname{int}_{R_i^\nu} A_i^\nu)^c \neq \varnothing$.

Proof of Claim 2. If $\zeta_i \in S_i^\nu$, we have $\zeta_i \in S_i^\nu \cap (\operatorname{int}_{R_i^\nu} A_i^\nu)^c$ by Claim 1. Suppose that $\zeta_i \notin S_i^\nu$, then, for any element s_i of $S_i^\nu \neq \varnothing$, we have $u_i(s_i) > u_i(\zeta_i)$. Moreover, by the definition of ζ_i, we have $u_i(s_i) > u_i(\zeta_i) \geq u_i(x_i)$ for all $x_i \in A_i^\nu \subset A_i$, which implies that $s_i \notin A_i^\nu$. Therefore, we have $s_i \in S_i^\nu \cap (\operatorname{int}_{R_i^\nu} A_i^\nu)^c$. □

For each $\nu \geq \overline{\nu}$, applying the existence theorem of Sato (2009), [14] we obtain a quasi-equilibrium $(\overline{x}^\nu, \overline{p}^\nu) \in X^\nu \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}^\nu$ with $u_i(\overline{x}_i^\nu) \geq u_i(e_i)$ for all $i \in I$. In view of the definition of the quasi-equilibrium, we may assume that $\overline{p}^\nu \in S = \{p \in \mathbb{R}^\ell : \|p\| = 1\}$. Put $\overline{u}_i^\nu = u_i(\overline{x}_i^\nu)$ for each $i \in I$ and $\overline{u}^\nu = (\overline{u}_i^\nu)_{i \in I} \in \mathbb{R}^n$. Clearly, $\overline{u}^\nu \in \mathcal{U}$.

Consider the sequence $((\overline{u}^\nu, \overline{p}^\nu))_{\nu \geq \overline{\nu}} \subset \mathcal{U} \times S$. Since $\mathcal{U} \times S$ is compact, we may assume that the sequence has a limit point $(\overline{u}, \overline{p}) \in \mathcal{U} \times S$. Then, by the definition of $\mathcal{U}$, there exists $\overline{x} \in A$ such that

$$u_i(\overline{x}_i) \geq \overline{u}_i \quad \text{for all} \quad i \in I.$$

We prove that the element $(\overline{x}, \overline{p}) \in A \times \mathbb{R}^\ell \setminus \{0\}$ is a quasi-equilibrium of the original economy $\mathcal{E}$. Since $\overline{x} \in A$, it suffices to show that $(\overline{x}, \overline{p})$ satisfies (i) and (ii) of Definition 2.

We first prove that $\overline{p} \cdot \overline{x}_i \geq \overline{p} \cdot e_i$ for each $i \in I$.

Claim 3. $\overline{p} \cdot \overline{x}_i \geq \overline{p} \cdot e_i$ *for each* $i \in I$

Proof of Claim 3. Fix $i \in I$. We divide the proof into two cases.

Case 1. $\overline{x}_i \notin S_i$.

In this case, there exists $y_i \in X_i$ such that $u_i(y_i) > u_i(\overline{x}_i)$.

[14]See also Remark 2 of the revised version of Sato (2009) which is attached to this paper.

For a fixed $\lambda \in (0,1)$, let $y_i(\lambda) = \lambda y_i + (1-\lambda)\overline{x}_i$. By Assumption 2 (b), we have $u_i(y_i(\lambda)) > u_i(\overline{x}_i)$. Since $u_i(\overline{x}_i^\nu) \to \overline{u}_i \leq u_i(\overline{x}_i) < u_i(y_i(\lambda))$, there exists $\overline{\nu}' \in \mathbb{N}$ such that $u_i(y_i(\lambda)) > u_i(\overline{x}_i^\nu)$ for all $\nu \geq \overline{\nu}'$. In view of the definition of X_i^ν, we may assume without loss of generality that $y_i(\lambda) \in X^\nu$ for all $\nu \geq \overline{\nu}'$. Then, for each $\nu \geq \overline{\nu}'$, since $(\overline{x}^\nu, \overline{p}^\nu) \in X^\nu \times S$ is a quasi-equilibrium of $\mathcal{E}^\nu$, we have

$$\overline{p}^\nu \cdot y_i(\lambda) \geq \overline{p}^\nu \cdot e_i.$$

Taking the limit as $\nu \to \infty$, we obtain

$$\overline{p} \cdot y_i(\lambda) \geq \overline{p} \cdot e_i.$$

The above inequality holds for an arbitrary $\lambda \in (0,1)$. Therefore, taking the limit as $\lambda \to 0$, we finally obtain

$$\overline{p} \cdot \overline{x}_i \geq \overline{p} \cdot e_i.$$

Case 2. $\overline{x}_i \in S_i$.

Then, by INS, $\overline{x}_i$ is the unique element of $S_i \cap A_i$. Let $\overline{\nu}'' \geq \overline{\nu}$ be a natural number such that $\overline{x}_i \in X_i^\nu$ for all $\nu \geq \overline{\nu}''$. We first prove that $\overline{p}^\nu \cdot \overline{x}_i \geq \overline{p}^\nu \cdot e_i$ for all $\nu \geq \overline{\nu}''$.

For each $\nu \geq \overline{\nu}''$, since $\overline{x}_i \in S_i$, either of the following two cases holds:

(a) $u_i(\overline{x}_i^\nu) = u_i(\overline{x}_i)$.
(b) $u_i(\overline{x}_i^\nu) < u_i(\overline{x}_i)$.

In case (a), we have $\overline{x}_i^\nu = \overline{x}_i$ because $\overline{x}_i^\nu \in S_i \cap A_i = \{\overline{x}_i\}$. Since $(\overline{x}^\nu, \overline{p}^\nu) \in X^\nu \times S$ is a quasi-equilibrium of $\mathcal{E}^\nu$, we have $\overline{p}^\nu \cdot \overline{x}_i = \overline{p}^\nu \cdot \overline{x}_i^\nu = \overline{p}^\nu \cdot e_i$.

In case (b), since $\overline{x}_i \in X_i^\nu$ with $u_i(\overline{x}_i) > u_i(\overline{x}_i^\nu)$, we must have $\overline{p}^\nu \cdot \overline{x}_i \geq \overline{p}^\nu \cdot e_i$.

Since $\overline{p}^\nu \cdot \overline{x}_i \geq \overline{p}^\nu \cdot e_i$ for all $\nu \geq \overline{\nu}''$ by the above argument, we obtain by taking the limit as $\nu \to \infty$,

$$\overline{p} \cdot \overline{x}_i \geq \overline{p} \cdot e_i,$$

which completes the proof of Claim 3. □

Since $\overline{x} \in F$, Claim 3 implies that $\overline{p} \cdot \overline{x}_i = \overline{p} \cdot e_i$ for all $i \in I$. Therefore, $(\overline{x}, \overline{p})$ satisfies (i) of Definition 2.

We next prove that (ii) of Definition 2 holds. Suppose that for some $i \in I$, there exists $y_i \in X_i$ such that $u_i(y_i) > u_i(\overline{x}_i)$ and $\overline{p} \cdot y_i < \overline{p} \cdot e_i$. Since $u_i(y_i) > u_i(\overline{x}_i) \geq \overline{u}_i$ and $u_i(\overline{x}_i^\nu) \to \overline{u}_i$, we have $u_i(y_i) > u_i(\overline{x}_i^\nu)$ for sufficiently large ν. Moreover, we may assume without loss of generality that $y_i \in X_i^\nu$ and $\overline{p}^\nu \cdot y_i < \overline{p}^\nu \cdot e_i$ for this ν. However, this contradicts the fact that $(\overline{x}^\nu, \overline{p}^\nu) \in X^\nu \times S$ is a quasi-equilibrium of $\mathcal{E}^\nu$. □

3.2. Existence under boundary nonsatiation

We now state and prove the existence of a quasi-equilibrium under BS and SCU:

Theorem 1. *Under Assumptions 1, 2, BS, and SCU, there exists a quasi-equilibrium* $(\overline{x}, \overline{p}) \in A \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$.

Note that Theorem 1 generalizes the results of Martins-da-Rocha and Monteiro (2009, Theorem 6.1) and Sato (2009, Theorem 2).

Proof of Theorem 1. In this proof, we assume that $e_i \notin S_i$ for all $i \in I$. This assumption is not a real restriction as far as the existence of a quasi-equilibrium is concerned. To observe this, let $I_s = \{i \in I : e_i \in S_i\}$, and consider the economy obtained by removing agents who belong to I_s. [15] If we can prove the existence of a quasi-equilibrium $(\overline{x}, \overline{p}) \in \prod_{i \in I \setminus I_s} X_i \times \mathbb{R}^\ell \setminus \{0\}$ in the modified economy, then together with $\overline{p}$, the allocation $\overline{x}' \in \prod_{i \in I} X_i$, defined by $\overline{x}_i' = \overline{x}_i$ for $i \in I \setminus I_s$ and $\overline{x}_i = e_i$ for $i \in I_s$, clearly constitutes a quasi-equilibrium of the original economy.

Let $\xi_i \in \operatorname{argmax}\{u_i(x_i) : x_i \in A_i\}$ for each $i \in I$, as in the proof of Theorem 1. Then, BS implies that there there exists $\zeta_i \in (\operatorname{int}_{R_i} A_i)^c$ such that $u_i(\zeta_i) \geq u_i(\xi_i)$. In particular, if $S_i \neq \varnothing$, we can find such ζ_i in $S_i \cap (\operatorname{int}_{R_i} A_i)^c$.

Following Martins-da-Rocha and Monteiro (2009), for each $i \in I$, we define a function $v_i : X_i \to \mathbb{R}$ by

$$v_i(x_i) = u_i(x_i) + \mathbf{1}_{S_i}(x_i) \exp(-\|x_i - \zeta_i\|),$$

where $\mathbf{1}_{S_i} : X_i \to \{0, 1\}$ is the indicator function of S_i. Note that $v_i(x_i) = u_i(x_i)$ for $x_i \notin S_i$, especially $v_i(e_i) = u_i(e_i)$.

[15] If $I_s = I$, for an arbitrary $p \in \mathbb{R}^\ell$, the element $(e, p) \in X \times \mathbb{R}^\ell$ clearly constitutes a quasi-equilibrium of $\mathcal{E}$.

We consider the auxiliary economy $\mathcal{E}(v)$ obtained by replacing u_i with v_i for all $i \in I$ in $\mathcal{E}$, that is, $\mathcal{E}(v) = \left(\mathbb{R}^\ell, (X_i, v_i, e_i)_{i\in I}\right)$. Note that since $v_i(e_i) = u_i(e_i)$ for all $i \in I$, we have $R_i(v) = R_i$ and $A_i(v) = A_i$, where $R_i(v) = \{x_i \in X_i : v_i(x_i) \geq v(e_i)\}$, and $A_i(v)$ denotes the individually rational feasible choice set of $i \in I$ in $\mathcal{E}(v)$.

We prove that $\mathcal{E}(v)$ satisfies all the assumptions of Proposition 1.

The following claim can be shown in the same way as in the proofs of Martins-da-Rocha and Monteiro (2009, Claims 6.1–6.3).

Claim 4. *The economy $\mathcal{E}(v)$ satisfies Assumptions 1, 2, and CU.*

Proof of Claim 4. See Martins-da-Rocha and Monteiro (2009, Claims 6.1–6.3). ☐

Let $S_i(v)$ be the set of all satiation points of $i \in I$ in $\mathcal{E}(v)$. Note that if $S_i(v) \neq \varnothing$, then $S_i(v) = \{\zeta_i\}$. We now prove that $\mathcal{E}(v)$ satisfies INS.

Claim 5. *For each $i \in I$, we have (a) $S_i(v) \cap \operatorname{int}_{R_i(v)} A_i(v) = \varnothing$, and (b) if $S_i(v) \cap A_i(v) \neq \varnothing$, the set is a singleton.*

Proof of Claim 5. Suppose that $S_i(v) \cap \operatorname{int}_{R_i(v)} A_i(v) \neq \varnothing$ for some $i \in I$. Since $S_i(v) = \{\zeta_i\}$, we have $\zeta_i \in \operatorname{int}_{R_i(v)} A_i(v)$. However, since $R_i(v) = R_i$ and $A_i(v) = A_i$, we have $\zeta_i \in \operatorname{int}_{R_i(v)} A_i(v) = \operatorname{int}_{R_i} A_i$, which contradicts our choice of ζ_i. Therefore, we have $S_i(v) \cap \operatorname{int}_{R_i(v)} A_i(v) = \varnothing$ for all $i \in I$.

The second part of the claim immediately follows from the fact that $S_i(v) = \{\zeta_i\}$ if $S_i(v) \neq \varnothing$. ☐

Applying Proposition 1, we obtain a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}(v)$. In fact, $(\overline{x}, \overline{p})$ is a quasi-equilibrium of the original economy $\mathcal{E}$. To observe this, it suffices to show that (ii) of Definition 2 holds.

For $i \in I$, let $y_i \in X_i$ with $u_i(y_i) > u_i(\overline{x}_i)$. Then, $\overline{x}_i \notin S_i$, and thus $v_i(\overline{x}_i) = u_i(\overline{x}_i)$. Since $v_i(y_i) \geq u_i(y_i) > v_i(\overline{x}_i)$ and $(\overline{x}, \overline{p})$ is a quasi-equilibrium of $\mathcal{E}(v)$, we have $\overline{p} \cdot y_i \geq \overline{p} \cdot e_i$, which is the desired conclusion. ☐

Remark 1. As noted before, under INS, the assumption CU implies SCU.

To observe this, suppose that CU and INS hold. Let $(x^\nu)_{\nu\in\mathbb{N}}$ be a sequence in A. Then, CU implies that there exist a feasible allocation $y \in A$, and a subsequence $(x^{\nu_k})_{k\in\mathbb{N}}$ satisfying

$$u_i(y_i) \geq \lim_{k\to\infty} u_i(x_i^{\nu_k}) \quad \text{for all} \quad i \in I. \tag{2}$$

Let

$$I_{ns} = \{i \in I : \exists k(i) \in \mathbb{N} \text{ s.t. } x_i^{\nu_k} \notin S_i \ \forall\, k \geq k(i)\}.$$

Then, equation (1) clearly holds for all $i \in I_{ns}$.

Suppose that $I \setminus I_{ns} \neq \varnothing$. Passing to a subsequence if necessary, we may assume that $x_i^{\nu_k} \in S_i$ for all k and $i \in I \setminus I_{ns}$. Then, (2) implies that $y_i \in S_i$ for all $i \in I \setminus I_{ns}$. Since $S_i \cap A_i$ is a singleton by INS, we must have $y_i = x_i^{\nu_k}$ for all k and $i \in I \setminus I_{ns}$. It is now clear that (1) holds for all $i \in I$.

4. Example

In this section, we provide an example of an economy in which both BS and SCU hold. In the economy, the individually rational feasible allocation set is unbounded, and neither standard nonsatiation nor weak nonsatiation holds. Moreover, the economy does not satisfy Won and Yannelis's (2006) condition concerning satiation.

Let $\mathcal{E}$ be an exchange economy with two commodities and two agents defined as follows. Agents' choice sets are

$$X_1 = \{x_1 \in \mathbb{R}^2 : x_{12} \leq x_{11} + 3\}.$$
$$X_2 = \mathbb{R} \times [0, \infty),$$

where x_{ij} denotes the quantity of j-th commodity chosen by agent $i = \{1, 2\}$. Agents' utility functions are

$$u_1(x_1) = -|x_{11} - 1| + 1,$$
$$u_2(x_2) = x_{21}.$$

Note that $S_1 = \{x_1 \in X_1 : x_{11} = 1\}$ and $S_2 = \varnothing$. Finally, the initial endowments are $e_1 = e_2 = (2, 2)$.

It is then easy to check that $\mathcal{E}$ satisfies Assumptions 1 and 2. We next prove that $\mathcal{E}$ satisfies CU.

First, we prove that $\mathcal{U}$ is bounded. Let $\lambda \in \mathcal{U}$, then there exists $x \in A$ such that

$$u_i(e_i) \leq \lambda_i \leq u_i(x_i) \quad \text{for all} \quad i \in I.$$

Clearly, $0 \leq \lambda_1 \leq 1$. We also have $(2 \leq)\lambda_2 \leq 4$. To observe this, suppose that $\lambda_2 > 4$. Then, $x_{21} = u_2(x_2) \geq \lambda_2 > 4$ and $x_{11} < 0$. However, this implies that $u_1(x_1) < 0 = u_1(e_1)$, which is a contradiction. Therefore,

$$\mathcal{U} \subset [0, 1] \times [2, 4],$$

in other words, $\mathcal{U}$ is bounded.

We next prove that $\mathcal{U}$ is closed. Let $(\lambda^\nu)_{\nu\in\mathbb{N}}$ be a sequence on $\mathcal{U}$ converging to some $\overline{\lambda} \in \mathbb{R}^2$. Then, for each $\nu \in \mathbb{N}$, there exists $x^\nu \in A$ such that

$$\lambda_i^\nu \leq u_i(x_i^\nu) \quad \text{for all} \quad i \in I.$$

Suppose first that $\overline{\lambda}_2 \leq 3$, and let $x = (x_1, x_2)$ be the allocation with $x_1 = (1, 1)$ and $x_2 = (3, 1)$. Clearly, $x \in A$. Since $x_1 \in S_1$, we have $\lambda_1^\nu \leq u_1(x_1^\nu) \leq u_1(x_1)$ for all $\nu \in \mathbb{N}$, which implies that $\overline{\lambda}_1 \leq u_i(x_1)$. Since $\overline{\lambda}_2 \leq 3$, we also have $\overline{\lambda}_2 \leq u_2(x_2)$. Therefore, $\overline{\lambda} \in \mathcal{U}$.

Suppose then that $\overline{\lambda}_2 > 3$. For all $\nu \in \mathbb{N}$, since $u_1(x_1^\nu) \geq u_1(e_1)$, we have $x_{11}^\nu \in [0, 2]$. Moreover, since $x_{21}^\nu = 4 - x_{11}^\nu \in [2, 4]$ for all ν, the sequence $(x_{11}^\nu, x_{21}^\nu)_{\nu\in\mathbb{N}}$ is bounded. Therefore, passing to a subsequence if necessary, we may suppose that the sequence converges to a vector $(\overline{x}_{11}, \overline{x}_{21})$ in $\mathbb{R}^2$, which satisfies $\overline{x}_{11} + \overline{x}_{21} = 4$.

Since $\overline{\lambda}_2 > 3$, we have $\lambda_2^\nu > 3$ for sufficiently large ν. Then, $x_{21}^\nu = u_2(x_2^\nu) \geq \lambda_2^\nu > 3$. Moreover, since $x^\nu \in A$, we must have $x_{11}^\nu < 1$, which implies that $u_1(x_1^\nu) = x_{11}^\nu$. Therefore, we have $\lambda_i^\nu \leq x_{i1}^\nu$ for all $i \in I$, and for all sufficiently large ν. Taking the limit as $\nu \to \infty$ for each $i \in I$, we have

$$u_i(e_i) \leq \overline{\lambda}_i \leq \overline{x}_{i1}.$$

Note that we also have $\overline{x}_{11} \leq 1$. Let $y_i = (\overline{x}_{i1}, 2)$ for each $i \in I$. Then, $y = (y_1, y_2) \in A$ and

$$u_i(e_i) \leq \overline{\lambda}_i \leq \overline{x}_{i1} = u_i(y_i),$$

which implies that $\overline{\lambda} \in \mathcal{U}$. Therefore, $\mathcal{U}$ is compact.

According to Martins-da-Rocha and Monteiro (2009, Proposition 7.1), in an economy with at most two agents CU is equivalent to SCU. Therefore, $\mathcal{E}$ satisfies SCU. Note that A_i is unbounded for each $i \in I$, which implies the unboundedness of A.

We prove that $\mathcal{E}$ satisfies BS. Since $S_2 = \varnothing$, it suffices to check that agent 1 satisfies BS. Let $s_1 = (1, 4)$. Then, it is clear that $s_1 \in S_1 \cap A_1$. We prove that $s_1 \in (A_1 \setminus \operatorname{int}_{R_1} A_1)$. Let $(x_1^\nu)_{\nu\in\mathbb{N}}$ be the sequence defined by

$$x_1^\nu = s_1 + \frac{1}{\nu}(1, 1) \quad \text{for all} \quad \nu \in \mathbb{N}.$$

Then, it is clear that $x_1^\nu \in R_1 \cap A_1^{\mathrm{c}}$ for each $\nu \in \mathbb{N}$ and $x_1^\nu \to s_1$ as $\nu \to \infty$. Therefore, $s_1 \in (A_1 \setminus \operatorname{int}_{R_1} A_1)$, and thus, $\mathcal{E}$ satisfies BS. Note that since $S_1 \subset A_1$, agent 1 satisfies neither standard nonsatiation nor weak nonsatiation.

Finally, we prove that $\mathcal{E}$ does not satisfy Won and Yannelis's (2006) condition concerning satiation.

Let $I_s = \{i \in I : S_i \subset A_i\}$, and for an allocation $x \in X$, let $I_{ns}(x) = \{i \in I : x_i \notin S_i\}$. For each $i \in I$ and $x_i \in X_i$, let $A(x_i) = \{y \in A : y_i = x_i\}$. Moreover, for $x_i \in X_i$, let $P_i(x_i) = \{y_i \in X_i : u_i(y_i) > u_i(x_i)\}$.

In our framework, the condition of Won and Yannelis (2006, Assumption S5') can be expressed as follows:

Condition 1. There exists $(s_i)_{i \in I_s} \in \prod_{i \in I_s} S_i$ such that for each $x \in \cup_{i \in I_s} A(s_i)$, and for any $p \in \mathbb{R}^\ell \setminus \{0\}$ that satisfies $p \cdot P_k(x_k) > p \cdot x_k$ for all $k \in I_{ns}(x)$, [16] we have $p \cdot x_i \geq p \cdot e_i$ for all $i \in I \setminus I_{ns}(x)$.

Note first that $I_s = \{1\}$ in our economy. Let $x \in A(s_1)$ for an arbitrary chosen $s_1 \in S_1$. Then, there exists $x_2 \in X_2$ such that $x = (s_1, x_2) \in A$. We also have $I_{ns}(x) = \{2\}$ and $I \setminus I_{ns}(x) = \{1\}$. Since $s_1 \in S_1$, we have $s_{11} = 1$. Then, $x_{21} = 3$ and $P_2(x_2) = \{y_2 \in X_2 : y_{21} > 3\}$.

Let $p = (1, 0)$. Then, for agent 2 we have

$$p \cdot P_2(x_2) > 3 = p \cdot x_2.$$

However, for agent 1 we have

$$p \cdot s_1 = 1 < 2 = p \cdot e_1.$$

Therefore, Condition 1 does not hold in $\mathcal{E}$.

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[16] By "$p \cdot P_k(x_k) \geq p \cdot x_k$", we mean $p \cdot y_k \geq p \cdot x_k$ for all $y_k \in P_k(x_k)$.

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Satiation and existence of competitive equilibrium☆

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Abstract

In this paper, we introduce a new assumption concerning the (non)satiation property of preferences and establish the existence of a competitive equilibrium under it. The assumption is weaker than the standard nonsatiation assumption and "weak nonsatiation" introduced in Allouch and Le Van [Allouch, N., Le Van, C., 2008. Walras and dividends equilibrium with possibly satiated consumers. Journal of Mathematical Economics 44, 907–918]. In particular, the new assumption allows, under certain conditions, preferences to be satiated only inside the set of individually rational feasible consumptions. Moreover, just like the two nonsatiation assumptions, our assumption depends solely on the characteristics of consumers.

JEL classification: C62; D50

Key words: Satiation, Competitive equilibrium, Individually rational feasible consumption.

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1. Introduction

In classical general equilibrium theory (Arrow and Debreu 1954, Debreu 1959, among others), consumers' preferences are assumed to be "nonsatiated," that is, assumed to have no consumption bundle that is preferred to all other. However, in some cases, we observe that consumption sets are

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naturally compact (see Mas-Colell 1992) and every continuous preference has therefore at least one satiation point. As is well known in the literature, a simple way to avoid this inconsistency without affecting the existence of a competitive equilibrium is to assume that when a preference has satiation points, they are always outside the set of individually rational feasible consumptions. [1] This modified nonsatiation assumption allows preferences to be satiated, but excludes the case in which satiation occurs inside the individually rational feasible consumption sets.

It is well known that a competitive equilibrium may fail to exist when satiation occurs inside the individually rational feasible consumption set. Recently, however, Allouch and Le Van (2008, 2009) have shown that even if there exists a consumer whose preference reaches satiation in his or her individually rational feasible consumption set, one can still obtain the existence of a competitive equilibrium by assuming that satiation also occurs outside the set. This assumption is a generalization of the standard nonsatiation assumption (including the modified one) and called "weak nonsatiation." [2]

Won and Yannelis (2006) introduce a different assumption that allows for satiation inside the individually rational feasible consumption sets. In fact, their assumption applies to the case in which satiation occurs *only* inside the individually rational feasible consumption sets and contains Allouch and Le Van's weak nonsatiation as a special case. Moreover, in their existence proofs, consumers' preferences are allowed to be non-ordered and individually rational feasible consumption sets do not need to be bounded. These advantages make their results applicable to securities markets with unlimited short-selling, in which choice sets are unbounded from below, and the capital asset pricing model (CAPM) without a riskless asset, in which satiation is rather a rule than an exception. However, Won and Yannelis's assumption contains a restriction on the price system, while weak nonsatiation, just like standard nonsatiation, depends solely on the characteristics of consumers.

The main contribution of this paper is to establish the existence of a competitive equilibrium under a new assumption that is weaker than Allouch

[1]A consumption bundle is said to be individually rational feasible if it can be achieved by a trade in which every consumer involved attains at least the same utility as that gained from his or her initial endowment. For the existence proof under this assumption, see for example, Bergstrom (1976), Dana et al. (1999).

[2]Allouch and Le Van (2008, 2009) also prove that in securities markets with short-selling, Werner's (1987) nonsatiation assumption implies weak nonsatiation.

and Le Van's weak nonsatiation, and therefore, the standard nonsatiation assumption. Our assumption allows each consumer's preference to be satiated only inside the individually rational feasible consumption set, provided that at least one satiation point lies on a "boundary" of the set. One's individually rational consumption bundle belongs to the "boundary" if every neighborhood of the bundle contains at least one point that is at least as good as his or her initial endowment but is not individually rational feasible. Although our existence results rely on the ordered preferences and bounded individually rational feasible consumption sets, the new assumption does not imply Won and Yannelis's (2006) assumption. Moreover, just like standard nonsatiation and weak nonsatiation, our assumption depends solely on the characteristics of consumers.

This paper is organized as follows. In Section 2, we describe the model and introduce the new assumption. In Section 3, we provide our main existence results. In Section 4, we consider an alternative to the new assumption and provide some related results. As a concluding remark, in Section 5, we compare our assumption with the assumption introduced by Won and Yannelis (2006). Finally, the proofs of some propositions are provided in Appendix.

2. Model and Assumptions

2.1. Model

We consider a pure exchange economy $\mathcal{E}$ with ℓ commodities and n consumers ($\ell, n \in \mathbb{N}$). [3] For convenience, let I be the set of all consumers, that is, $I = \{1, \cdots, n\}$. Each consumer $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^\ell$, an initial endowment $\omega_i \in \mathbb{R}^\ell$, and a utility function $u_i : X_i \to \mathbb{R}$. Let $X = \prod_{i \in I} X_i$ with a generic element $x = (x_i)_{i \in I}$, and put $\omega = (\omega_i)_{i \in I} \in \mathbb{R}^{\ell n}$.

[3] We use the following mathematical notations. The symbols $\mathbb{N}$, $\mathbb{R}^\ell$ and $\mathbb{R}^\ell_+$ denote the set of natural numbers, ℓ-dimensional Euclidean space and nonnegative orthant of $\mathbb{R}^\ell$, respectively. For $x, y \in \mathbb{R}^\ell$, we denote by $x \cdot y = \sum_{j=1}^{\ell} x_j y_j$ the inner product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm. Let $B(x_0, r) = \{x \in \mathbb{R}^\ell : \|x - x_0\| < r\}$ denote the open ball centered at x_0 with radius r. For $a \in \mathbb{R} = \mathbb{R}^1$, we denote by $|a|$ the absolute value of a. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote by (a, b) and $[a, b]$, the open interval and closed interval between a and b, respectively. For a set $A \subset \mathbb{R}^\ell$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$, the interior, closure and boundary of A in $\mathbb{R}^\ell$, respectively.

The pure exchange economy $\mathcal{E}$ is thus summarized by the list

$$\mathcal{E} = \Big(\mathbb{R}^\ell, (X_i, u_i, \omega_i)_{i\in I}\Big).$$

An allocation $x \in X$ is *feasible* if $\sum_{i\in I} x_i = \sum_{i\in I} \omega_i$. Note that we do not allow free disposal. We denote the set of all feasible allocations by F. Moreover, let F_i be the *individually feasible consumption set of consumer* $i \in I$ that is defined as the projection of F onto X_i. In other words, for $x_i \in X_i$, we have $x_i \in F_i$ if and only if there exists $(x_j)_{j\neq i} \in \prod_{j\neq i} X_j$ such that $x = (x_k)_{k\in I} \in F$. Then, it is easy to check that $F_i = X_i \cap (-\sum_{j\neq i} X_j + \sum_{k\in I}\{\omega_k\})$ for all $i \in I$.

An allocation $x \in X$ is *individually rational feasible* if $x \in F$ and $u_i(x_i) \geq u_i(\omega_i)$ for all $i \in I$. We denote the set of all individually rational feasible allocations by A. Let A_i be the projection of A onto X_i, and call it *individually rational feasible consumption set of consumer* $i \in I$.

Let $R_i = \{x_i \in X_i : u_i(x_i) \geq u_i(\omega_i)\}$ for each $i \in I$. Then, it is easy to check that

$$A_i = R_i \cap \left(-\sum_{j\neq i} R_j + \sum_{k\in I}\{\omega_k\}\right) \quad \text{for all} \quad i \in I.$$

For simplicity of notation, we put

$$G_i = -\sum_{j\neq i} R_j + \sum_{k\in I}\{\omega_k\} \quad \text{for each} \quad i \in I.$$

Note that $A_i = R_i \cap G_i \subset R_i$ for all $i \in I$.

The utility function u_i is *satiated at* $s_i \in X_i$ if s_i maximizes u_i over X_i, and we call s_i a *satiation point of* u_i. Let S_i denote the set of all satiation points of u_i, that is,

$$S_i = \{s_i \in X_i : u_i(s_i) \geq u_i(x_i) \text{ for all } x_i \in X_i\}.$$

Put $S = \prod_{i\in I} S_i$.

We adopt the following standard definitions of competitive equilibrium and quasi-equilibrium.

Definition 1. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *competitive equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot \omega_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then, $\overline{p} \cdot x_i > \overline{p} \cdot \omega_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} \omega_i$.

Definition 2. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *quasi-equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot \omega_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then, $\overline{p} \cdot x_i \geq \overline{p} \cdot \omega_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} \omega_i$.

2.2. Assumptions

We first make the following two sets of assumptions on the economy $\mathcal{E}$.

Assumption 1. For each $i \in I$,

(a) X_i is closed and convex, (b) X_i is bounded, (c) $\omega_i \in X_i$.

Assumption 2. For each $i \in I$,

(a) u_i is upper semicontinuous on X_i, [4]

(b) u_i is strictly quasi-concave. [5]

As shown in Section 3.1, the existence of a quasi-equilibrium is ensured under Assumptions 1 and 2 and our new assumption on preference satiation introduced below. To prove the existence of a competitive equilibrium, however, we need some additional assumptions (see Section 3.2). It is worth noting that in our main existence theorems (Theorems 2 and 3), Assumption 1 (b) can be weakened to the boundedness of A by the standard truncation technique.

It is easy to check that under Assumptions 1 and 2, we have the following facts.

[4]A function $f : X \to \mathbb{R}$ is upper semicontinuous on $X \subset \mathbb{R}^\ell$ if and only if for all $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \geq \alpha\}$ is closed in X.

[5]A function $f : X \to \mathbb{R}$ is strictly quasi-concave if and only if for all $x, y \in X$ with $f(x) > f(y)$ and for all $\lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) > f(y)$.

Fact 1. $S_i \neq \varnothing$ for each $i \in I$.

Fact 2. R_i, G_i and A_i are nonempty, compact and convex in $\mathbb{R}^\ell$ for each $i \in I$.

In particular, the convexity of R_i in Fact 2 follows from the quasi-concavity of u_i, [6] which is implied by Assumption 2 (b) under Assumptions 1 (a) and 2 (a).

Before introducing our new assumption on satiation property of preferences, we first define some additional notations.

For each $i \in I$, let $\operatorname{int}_{R_i} A_i$ denote the interior of A_i in the relative topology on $R_i \subset \mathbb{R}^\ell$, that is, for $x_i \in R_i$, we have $x_i \in \operatorname{int}_{R_i} A_i$ if and only if there exists an open ball $B(x_i, r)$ centered at x_i with radius r such that $B(x_i, r) \cap R_i \subset A_i$.

Roughly speaking, if $x_i \in \operatorname{int}_{R_i} A_i$, when consumer $i \in I$ slightly changes his or her consumption plan from x_i in such a way that the resulting consumption bundle x'_i is within R_i, the bundle x'_i will also lie on A_i. In contrast, if $x_i \in A_i \setminus \operatorname{int}_{R_i} A_i$, the resulting consumption bundle $x'_i \in R_i$ may not lie on A_i, no matter how small the change is.

Let A_i^c and $(\operatorname{int}_{R_i} A_i)^c$ denote the complements of A_i and $\operatorname{int}_{R_i} A_i$ in X_i, that is, $A_i^c = X_i \setminus A_i$ and $(\operatorname{int}_{R_i} A_i)^c = X_i \setminus \operatorname{int}_{R_i} A_i$.

We now introduce the following assumption.

Assumption 3. For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap (\operatorname{int}_{R_i} A_i)^c \neq \varnothing$.

Since $(\operatorname{int}_{R_i} A_i)^c = A_i^c \cup (A_i \setminus \operatorname{int}_{R_i} A_i)$, this assumption allows consumer's satiation area to be a subset of the individually rational feasible consumption set, provided that it touches the complement of $\operatorname{int}_{R_i} A_i$ in A_i. In other words, under Assumption 3, we must have $S_i \cap (A_i \setminus \operatorname{int}_{R_i} A_i) \neq \varnothing$ if $S_i \subset A_i$. Note that under Assumptions 1 and 2, the set $A_i \setminus \operatorname{int}_{R_i} A_i$ coincides with the boundary of A_i in the relative topology on R_i.

Assumption 3 generalizes the following two assumptions.

[Nonsatiation] For each $i \in I$, we have $S_i \cap A_i = \varnothing$.

[Weak nonsatiation] For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap A_i^c \neq \varnothing$.

[6] A function $f : X \to \mathbb{R}$ is quasi-concave if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.

[Nonsatiation] is a standard assumption in the classical general equilibrium theory. It excludes the case in which satiation occurs inside the individually rational feasible consumption sets.

[Weak nonsatiation], introduced by Allouch and Le Van (2008, 2009), is a generalization of [Nonsatiation]. This assumption allows consumer's satiation points to be inside the individually rational feasible consumption set, provided that at least one satiation point lies outside A_i. However, unlike Assumption 3, it does not apply to the case in which S_i is a subset of A_i.

Note that [Weak nonsatiation] coincides with [Nonsatiation] when S_i is a singleton for all $i \in I$ with $S_i \neq \varnothing$. Note also that Assumption 3 coincides with [Weak nonsatiation] if $R_i = A_i$ for all $i \in I$ because we have $\operatorname{int}_{R_i} A_i = A_i$ in such a case.

In the following example, only Assumption 3 holds among the above three assumptions concerning satiation.

Example 1. Consider an exchange economy $\mathcal{E}$ with two commodities and two consumers. Let $X_1 = X_2 = \mathbb{R}^2_+$ and $\omega_1 = \omega_2 = (4, 4)$. Consumers' utility functions are as follows.

$$u_1(x_1) = -\|(x_{11}, x_{12}) - (4, 6)\|^2 \quad \text{and} \quad u_2(x_2) = x_{21}.$$

Note that u_1 has a unique satiation point $s_1 = (4, 6)$, while u_2 is never satiated on X_2 (see Figure 1).

Let $y_2 = (4, 2) \in X_2$. Then, the allocation $(s_1, y_2) \in X_1 \times X_2$ is feasible. Moreover, since

$$u_2(y_2) = 4 = u_2(\omega_2),$$

we have $s_1 \in A_1$. Therefore, neither [Nonsatiation] nor [Weak nonsatiation] holds.

We prove that Assumption 3 holds. Let $\varepsilon_1 = (1, 0) \in \mathbb{R}^2$, and for each $t \in (0, 1]$, let

$$\begin{aligned} z_1(t) &= s_1 + t\varepsilon_1 = (4 + t, 6) \in X_1, \\ z_2(t) &= y_2 - t\varepsilon_1 = (4 - t, 2) \in X_2. \end{aligned}$$

We claim that $z_1(t) \in R_1 \setminus A_1$ for all $t \in (0, 1]$. To see this, note first that we have $z_1(t) + z_2(t) = \sum_{i \in I} \omega_i$ for all $t \in (0, 1]$. Next, since $u_1(z_1(t)) = -t^2 > -4 = u_1(\omega_1)$, we have $z_1(t) \in R_1$ for all $t \in (0, 1]$. Moreover, for all $t \in (0, 1]$, since

$$u_2(z_2(t)) = 4 - t < 4 = u_2(\omega_2),$$

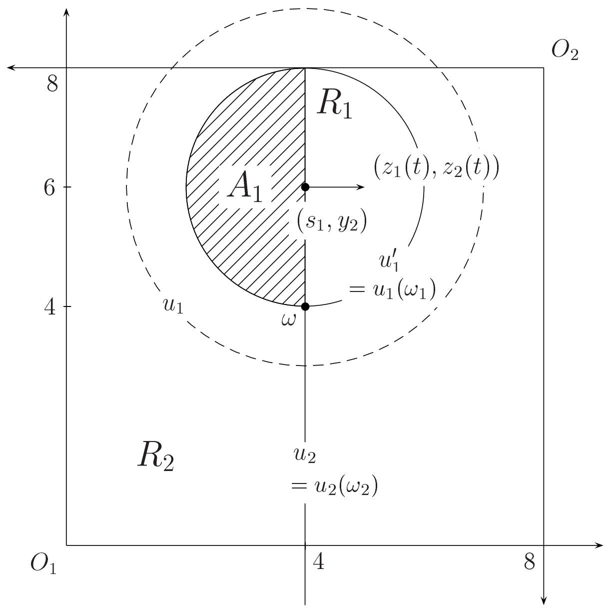

Figure 1: Example in which Assumption 3 holds but [Nonsatiation] and [Weak nonsatiation] do not.

we have $z_2(t) \notin R_2$. Therefore, $z_1(t) \notin A_1$ for all $t \in (0, 1]$.

Since $z_1(t) \in R_1 \setminus A_1$ for all $t \in (0, 1]$ and $z_1(t) \to s_1$ as $t \to 0$, we obtain $s_1 \in A_1 \setminus \mathrm{int}_{R_1} A_1$. Therefore, Assumption 3 holds.

It is easy to check that the allocation $\overline{x} = (s_1, y_2)$ together with the price $\overline{p} = (1, 0) \in \mathbb{R}^2$ is a competitive equilibrium of $\mathcal{E}$.

Note that this example also shows that Assumption 3 does not coincide with [Nonsatiation] even if S_i is a singleton for all $i \in I$ with $S_i \neq \varnothing$.

3. Main Results

3.1. Existence of quasi-equilibrium

The purpose of this section is to demonstrate the existence of a quasi-equilibrium of $\mathcal{E}$ under Assumptions 1 – 3 (Theorem 2). In the proof, we use the following existence result by Allouch and Le Van (2009).

Theorem 1. *(Allouch and Le Van, 2009)*

Under Assumptions 1, 2 and [Weak nonsatiation], there exists a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell$ of $\mathcal{E}$. [7]

Remark 1. A careful reading of the original proof of Theorem 1 by Allouch and Le Van (2009), which is based on the result of Dana et al. (1999, Theorem 1), shows that the stated quasi-equilibrium allocation $\overline{x}$ is, in fact, an element of A, that is, $u_i(\overline{x}_i) \geq u_i(\omega_i)$ for all $i \in I$.

The strategy of our existence proof is as follows.

First, under our assumptions, we can choose $s = (s_i)_{i \in I} \in S$ so that

$$s_i \in (\operatorname{int}_{R_i} A_i)^c \quad \text{for all} \quad i \in I. \tag{1}$$

Next, for this s, we construct a sequence $\{\omega^\nu\}_{\nu \in \mathbb{N}} = \{(\omega_i^\nu)_{i \in I}\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^{\ell n}$ that satisfies the following properties:

(a) $\omega^\nu \to \omega$ as $\nu \to \infty$,

(b) there exists $\overline{\nu} \in \mathbb{N}$ such that for all $\nu \geq \overline{\nu}$,

(b-1) $\omega_i^\nu \in X_i$ for all $i \in I$, and

(b-2) $s_i \notin R_i^\nu \cap (-\sum_{j \neq i} R_j^\nu + \sum_{k \in I}\{\omega_k^\nu\})$ for all $i \in I$, where

$$R_k^\nu = \{x_k \in X_k : u_k(x_k) \geq u_k(\omega_k^\nu)\} \quad \text{for each} \quad k \in I.$$ [8]

We then define an auxiliary economy $\mathcal{E}^\nu$ for each $\nu \geq \overline{\nu}$ by $\mathcal{E}^\nu = (\mathbb{R}^\ell, (X_i, u_i, \omega_i^\nu)_{i \in I})$ (the economy $\mathcal{E}^\nu$ differs from the initial economy only in its initial endowments). By the definition, each $\mathcal{E}^\nu$ satisfies all the assumptions in Theorem 1. In particular, [Weak nonsatiation] holds by the property (b-2) of $\{\omega^\nu\}_{\nu \in \mathbb{N}}$.

Therefore, by Theorem 1, we obtain a sequence $\{(\overline{x}^\nu, \overline{p}^\nu)\}_{\nu \geq \overline{\nu}} \subset X \times \mathbb{R}^\ell$ in which each term is a quasi-equilibrium of $\mathcal{E}^\nu$. Under our assumptions, we may assume that the sequence has a limit point, and we can prove that the

[7] In the original version of Theorem 1 (Allouch and Le Van 2009, Theorem 2), the boundedness of A is assumed instead of Assumption 1 (b).

[8] To be precise, we cannot always find a sequence $\{\omega^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^{\ell n}$ that satisfies all the properties stated above. However, as will be shown later, we may assume without loss of generality that there exists a sequence that satisfies the properties (a) and (b) as far as the existence of a quasi-equilibrium matters.

point is a quasi-equilibrium of the original economy.

The next lemma shows that for a fixed $s \in S$ that satisfies (1), we can find a sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ with certain properties. It will be used in our main existence theorem to construct the sequence $\{\omega^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^{\ell n}$ described above.

Lemma 1. *Suppose that Assumptions 1 and 2 hold, and suppose that there exists $s = (s_i)_{i\in I} \in S$ that satisfies*

$$s_i \in (\operatorname{int}_{R_i} A_i)^c \quad \textit{for all} \quad i \in I.$$

Then, there exist $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ and $\overline{\nu} \in \mathbb{N}$ such that $\nu\varepsilon^\nu \to 0$ as $\nu \to \infty$, and for every $\nu \geq \overline{\nu}$ and $i \in I$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}).$$

Proof. Let $s = (s_i)_{i\in I} \in S$ be the element that satisfies $s_i \in (\operatorname{int}_{R_i} A_i)^c$ for all $i \in I$. Put $I_{bd} = \{i \in I : s_i \in A_i \setminus \operatorname{int}_{R_i} A_i\}$ and $I_{out} = I \setminus I_{bd} = \{i \in I : s_i \notin A_i\}$.

In the following, we divide the proof into several cases, in each of which we construct a sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ that satisfies the properties in the statement of the lemma.

Case 1. $I_{bd} = \varnothing$ (equivalently, $I_{out} = I$).

It is clear that the sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ defined by $\varepsilon^\nu = 0$ for all $\nu \in \mathbb{N}$ satisfies the desired properties.

Case 2. $I_{bd} \neq \varnothing$.

We first consider the sequence $\{\varepsilon^\nu_{out}\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ defined by

$$\varepsilon^\nu_{out} = \frac{1}{\nu^2} \sum_{h\in I_{bd}} (s_h - \omega_h) \quad \text{for each} \quad \nu \in \mathbb{N}.$$

It is clear that $\varepsilon^\nu_{out} \to 0$ and $\nu\varepsilon^\nu_{out} \to 0$ as $\nu \to \infty$.

Then, for $i \in I_{out}$, we have the following claim.

Claim 1. *For each $i \in I_{out}$, there exists $\overline{\nu}_i \in \mathbb{N}$ such that*

$$s_i \notin R_i \cap (G_i - \{\varepsilon^{\nu}_{out}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}_i.$$

Proof of Claim 1. First, for each $i \in I_{out}$, since $s_i \notin A_i = R_i \cap G_i$ and $s_i \in R_i$, we must have $s_i \notin G_i$. Then, since G_i is closed in $\mathbb{R}^{\ell}$ (by Fact 2), there exists a positive real number $r_i > 0$ such that $B(s_i, r_i) \cap G_i = \varnothing$. Since $\varepsilon^{\nu}_{out} \to 0$ as $\nu \to \infty$, there exists $\overline{\nu}_i$ such that $s_i + \varepsilon^{\nu}_{out} \in B(s_i, r_i)$ for all $\nu \geq \overline{\nu}_i$, which implies that $s_i \notin G_i - \{\varepsilon^{\nu}_{out}\}$ for all $\nu \geq \overline{\nu}_i$. □

With respect to $i \in I_{bd}$, we have the following claim.

Claim 2. *For each $i \in I_{bd}$, either (a) there exists $\overline{\nu}_i \in \mathbb{N}$ such that*

$$s_i \notin R_i \cap (G_i - \{\varepsilon^{\nu}_{out}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}_i$$

or (b) there exists $\overline{\nu}'_i \in \mathbb{N}$ such that

$$s_i \in R_i \cap (G_i - \{\varepsilon^{\nu}_{out}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}'_i.$$

The proof of Claim 2 is provided in Appendix.

Let I_{in} be the subset of I_{bd} such that $i \in I_{in}$ if and only if there exists $\overline{\nu}_i \in \mathbb{N}$ that satisfies

$$s_i \in R_i \cap (G_i - \{\varepsilon^{\nu}_{out}\}) \quad \text{for all} \quad \nu \geq \overline{\nu}_i.$$

Note that if $I \setminus I_{in} \neq \varnothing$, by the definition of I_{in} and Claims 1 and 2, there exists $\overline{\nu}_{out}$ such that for every $\nu \geq \overline{\nu}_{out}$ and $i \in I \setminus I_{in}$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^{\nu}_{out}\}).$$

Let $\overline{\nu}_{out} = 1$ if $I \setminus I_{in} = \varnothing$.

Then, we have two cases.

Case 2-A. $I_{in} = \varnothing$.

It is clear that the sequence $\{\varepsilon^{\nu}\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^{\ell}$ defined by $\varepsilon^{\nu} = \varepsilon^{\nu}_{out}$ for each $\nu \in \mathbb{N}$ satisfies all the properties in the statement of the lemma.

Case 2-B. $I_{in} \neq \varnothing$.

For simplicity of notation, we assume without loss of generality that $I_{in} = \{1, 2, \cdots, M\}$, where $M = |I_{in}| \leq n$.

In view of the definition of I_{in}, there exists $\overline{\nu}_{in}$ such that for all $i \in I_{in}$ and $\nu \geq \overline{\nu}_{in}$,

$$s_i \in R_i \cap (G_i - \{\varepsilon^\nu_{out}\}).$$

Put $\overline{\nu} = \max\{\overline{\nu}_{out}, \overline{\nu}_{in}\}$.

We construct the sequence $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}}$ in several steps.

First, for each fixed $\nu \geq \overline{\nu}$, we successively define M vectors $\varepsilon^\nu_1, \cdots, \varepsilon^\nu_M \in \mathbb{R}^\ell$ that satisfy the following two properties for each $m \in I_{in}$:

(i) $s_m + \varepsilon^\nu_m \in R_m \setminus A_m$ and $\|\varepsilon^\nu_m\| < 1/2^m \nu^{2m+2}$, and

(ii) for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m} 2^q \nu^{2q} \{\varepsilon^\nu_q\} \right).$$

We first define ε^ν_1 as follows.

Suppose first that $I \setminus I_{in} \neq \varnothing$. Since $\nu \geq \overline{\nu}$ $(\geq \overline{\nu}_{out})$ and the set $G_i - \{\varepsilon^\nu_{out}\}$ is closed in $\mathbb{R}^\ell$ for each $i \in I \setminus I_{in}$, there exists a positive real number r^ν_1 such that

$$B(s_i, r^\nu_1) \cap (G_i - \{\varepsilon^\nu_{out}\}) = \varnothing \quad \text{for all} \quad i \in I \setminus I_{in}.$$

Then, since $s_1 \in A_1 \setminus \operatorname{int}_{R_1} A_1$, there exists $\varepsilon^\nu_1 \in \mathbb{R}^\ell \setminus \{0\}$ such that

$$s_1 + \varepsilon^\nu_1 \in R_1 \setminus A_1 \quad \text{and} \quad \|\varepsilon^\nu_1\| < \min\left\{ \frac{r^\nu_1}{2\nu^2}, \frac{1}{2\nu^4} \right\}.$$[9]

Since $2\nu^2 \|\varepsilon^\nu_1\| < r^\nu_1$, we have

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu_{out}\} - 2\nu^2 \{\varepsilon^\nu_1\}) \quad \text{for all} \quad i \in I \setminus I_{in}. \tag{2}$$

We need to show that (2) also holds for $i = m = 1$.

[9] Recall that for $x_i \in A_i$, we have $x_i \in A_i \setminus \operatorname{int}_{R_i} A_i$ if and only if $B(x_i, r) \cap R_i \not\subset A_i$ for any positive real number r.

Claim 3.

$$s_1 \notin R_1 \cap (G_1 - \{\varepsilon^\nu_{out}\} - 2\nu^2\{\varepsilon^\nu_1\}).$$

The proof of Claim 3 is provided in Appendix.

If $I \setminus I_{in} = \varnothing$, we may define ε^ν_1 as the vector that satisfies

$$s_1 + \varepsilon^\nu_1 \in R_1 \setminus A_1 \quad \text{and} \quad \|\varepsilon^\nu_1\| < \frac{1}{2\nu^4}.$$

The proof of Claim 3 is not affected by this change.

Let $m \in I_{in}$ with $m \geq 2$, and suppose that $\varepsilon^\nu_1, \cdots, \varepsilon^\nu_{m-1}$ are the vectors that satisfy properties (i) and (ii) for each $q \in \{1, \cdots, m-1\}$. We define ε^ν_m as follows.

First, by property (ii) with respect to $m-1$, we have, for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m-1} 2^q \nu^{2q}\{\varepsilon^\nu_q\}\right). \tag{3}$$

Note that for all $q \in \{1, \cdots, m-1\}$, by the first part of property (i) and the convexity of R_q, we have $s_q + \lambda\varepsilon^\nu_q \in R_q$ for all $\lambda \in [0,1]$.

By (3), there exists a positive real number r^ν_m such that for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$B(s_i, r^\nu_m) \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m-1} 2^q \nu^{2q}\{\varepsilon^\nu_q\}\right) = \varnothing.$$

Since $s_m \in A_m \setminus \operatorname{int}_{R_m} A_m$, we can choose $\varepsilon^\nu_m \in \mathbb{R}^\ell$ so that

$$s_m + \varepsilon^\nu_m \in R_m \setminus A_m \quad \text{and} \quad \|\varepsilon^\nu_m\| < \min\left\{\frac{r^\nu_m}{2^m\nu^{2m}}, \frac{1}{2^m\nu^{2m+2}}\right\}.$$

Since $2^m\nu^{2m}\|\varepsilon^\nu_m\| < r^\nu_m$, we have, for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m} 2^q \nu^{2q}\{\varepsilon^\nu_q\}\right). \tag{4}$$

We claim that (4) also holds for $i = m$.

Claim 4.
$$s_m \notin R_m \cap \left(G_m - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m} 2^q \nu^{2q} \{\varepsilon^\nu_q\} \right).$$

The proof of Claim 4 is provided in Appendix.

Therefore, we conclude that for each $\nu \geq \overline{\nu}$, there exist M vectors $\varepsilon^\nu_1, \cdots, \varepsilon^\nu_M \in \mathbb{R}^\ell$ that satisfy properties (i) and (ii) for each $m \in I_{in}$. Note that by property (ii) with respect to $m = M$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{m \in I_{in}} 2^m \nu^{2m} \{\varepsilon^\nu_m\} \right) \quad \text{for all} \quad i \in I. \tag{5}$$

We now define a sequence $\{\varepsilon^\nu_{in}\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^\ell$ by

$$\varepsilon^\nu_{in} = \begin{cases} 1 & \text{for} \quad \nu < \overline{\nu} \\ \sum_{m \in I_{in}} 2^m \nu^{2m} \varepsilon^\nu_m & \text{for} \quad \nu \geq \overline{\nu}. \end{cases}$$

Since
$$\|\varepsilon^\nu_{in}\| \leq \sum_{m \in I_{in}} 2^m \nu^{2m} \|\varepsilon^\nu_m\| < \frac{M}{\nu^2} \quad \text{for all} \quad \nu \geq \overline{\nu},$$
we have $\nu \varepsilon^\nu_{in} \to 0$ as $\nu \to \infty$ (recall that for all $\nu \geq \overline{\nu}$ and $m \in I_{in}$, by the second part of property (i), we have $2^m \nu^{2m} \|\varepsilon^\nu_m\| < 1/\nu^2$).

Finally, define a sequence $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^\ell$ by

$$\varepsilon^\nu = \varepsilon^\nu_{out} + \varepsilon^\nu_{in}.$$

Then, from the definition, we have $\nu \varepsilon^\nu \to 0$ as $\nu \to \infty$. Moreover, by (5), we have, for every $\nu \geq \overline{\nu}$ and $i \in I$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}),$$

which completes the proof. □

Let $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}}$ be the sequence that satisfies the properties stated in Lemma 1. The next Lemma shows that we may assume that $\sum_{i \in I} \omega_i - \nu \varepsilon^\nu \in \sum_{i \in I} R_i$ for all sufficiently large $\nu \in \mathbb{N}$ as far as the existence of a quasi-equilibrium matters.

Lemma 2. *Suppose Assumptions 1 and 2 hold. Suppose that there exists a sequence* $\{\delta^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ *such that* $\delta^\nu \to 0$ *as* $\nu \to \infty$*, and*

$$\sum_{i\in I}\omega_i - \delta^\nu \notin \sum_{i\in I} R_i \quad \textit{for all} \quad \nu \in \mathbb{N}.$$

Then, there exists a ℓ*-dimensional vector* $\overline{p} \neq 0$ *such that*

$$\overline{p}\cdot R_i \geq \overline{p}\cdot\omega_i \quad \textit{for all} \quad i \in I.$$ [10]

Therefore, $(\omega, \overline{p}) \in X \times \mathbb{R}^\ell$ *is a quasi-equilibrium of* $\mathcal{E}$*.*

Proof. Since $\sum_{i\in I}\omega_i \in \sum_{i\in I} R_i$ by Assumption 1 (c), if there exists a sequence $\{\delta^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ that satisfies the properties in the statement of the lemma, we must have $\sum_{i\in I}\omega_i \in \mathrm{bd}(\sum_{i\in I} R_i)$. Since $\sum_{i\in I} R_i$ is convex, by applying the support theorem (see Florenzano and Le Van 2001, Corollary 2.1.1), we obtain $\overline{p} \neq 0$ such that

$$\overline{p}\cdot z \geq \overline{p}\cdot\sum_{i\in I}\omega_i \quad \text{for all} \quad z \in \sum_{i\in I} R_i.$$

Take arbitrary $i \in I$ and $x_i \in R_i$. Since $x_i + \sum_{j\neq i}\omega_j \in \sum_{k\in I} R_k$, we have

$$\overline{p}\cdot x_i + \overline{p}\cdot\sum_{j\neq i}\omega_j \geq \overline{p}\cdot\omega_i + \overline{p}\cdot\sum_{j\neq i}\omega_j,$$

and thus,

$$\overline{p}\cdot x_i \geq \overline{p}\cdot\omega_i.$$

Therefore, we obtain

$$\overline{p}\cdot R_i \geq \overline{p}\cdot\omega_i \quad \text{for all} \quad i \in I,$$

which completes the proof. □

We now state and prove the existence of a quasi-equilibrium of $\mathcal{E}$.

Theorem 2. *Under Assumptions 1 – 3, there exists a quasi-equilibrium* $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$*.*

[10] By "$\overline{p}\cdot R_i \geq \overline{p}\cdot\omega_i$," we mean $\overline{p}\cdot x_i \geq \overline{p}\cdot\omega_i$ for all $x_i \in R_i$.

Proof. By Assumptions 1 (a), 1 (b), 2 (a) and 3, there exists $s = (s_i)_{i \in I} \in S$ such that $s_i \in (\operatorname{int}_{R_i} A_i)^c$ for all $i \in I$.

Then, by Lemma 1, there exist a sequence $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^\ell$ and a natural number $\overline{\nu}_1 \in \mathbb{N}$ such that $\nu\varepsilon^\nu \to 0$ as $\nu \to \infty$, and for every $\nu \geq \overline{\nu}_1$ and $i \in I$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}). \tag{6}$$

Suppose that the sequence $\{\nu\varepsilon^\nu\}_{\nu \in \mathbb{N}}$ contains a subsequence $\{\nu_\mu \varepsilon^{\nu_\mu}\}_{\mu \in \mathbb{N}}$ that satisfies

$$\sum_{i \in I} \omega_i - \nu_\mu \varepsilon^{\nu_\mu} \notin \sum_{i \in I} R_i \quad \text{for all} \quad \mu \in \mathbb{N}.$$

Then, by Lemma 2, there exists $\overline{p} \in \mathbb{R}^\ell \setminus \{0\}$ such that the element $(\omega, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ constitutes a quasi-equilibrium of $\mathcal{E}$.

Therefore, we may suppose without loss of generality that there exists $\overline{\nu}_2 \in \mathbb{N}$ such that for all $\nu \geq \overline{\nu}_2$,

$$\sum_{i \in I} \omega_i - \nu\varepsilon^\nu \in \sum_{i \in I} R_i.$$

By this relation, for each $\nu \geq \overline{\nu} = \max\{\overline{\nu}_1, \overline{\nu}_2\}$, there exists $x^\nu = (x_i^\nu)_{i \in I} \in \prod_{i \in I} R_i$ such that $\sum_{i \in I} x_i^\nu = \sum_{i \in I} \omega_i - \nu\varepsilon^\nu$. Note that since R_i is compact, the sequence $\{x_i^\nu\}_{\nu \geq \overline{\nu}} \subset R_i$ is bounded for each $i \in I$.

For each $\nu \geq \overline{\nu}$ and $i \in I$, let

$$\omega_i^\nu = \left(1 - \frac{1}{\nu}\right)\omega_i + \frac{1}{\nu} x_i^\nu.$$

Then, we have $\omega_i^\nu \in R_i \subset X_i$ by the convexity of R_i, and

$$\sum_{i \in I} \omega_i^\nu = \left(1 - \frac{1}{\nu}\right) \sum_{i \in I} \omega_i + \frac{1}{\nu} \sum_{i \in I} x_i^\nu = \sum_{i \in I} \omega_i - \varepsilon^\nu. \tag{7}$$

Moreover, $\omega_i^\nu \to \omega_i$ as $\nu \to \infty$ for each $i \in I$. Indeed, since $\{x_i^\nu\}_{\nu \geq \overline{\nu}}$ is bounded,

$$\|\omega_i - \omega_i^\nu\| \leq \frac{1}{\nu}\|\omega_i\| + \frac{1}{\nu}\|x_i^\nu\| \to 0 \quad \text{as} \quad \nu \to \infty.$$

We now define, for each $\nu \geq \overline{\nu}$, an auxiliary economy $\mathcal{E}^\nu$ by

$$\mathcal{E}^\nu = (\mathbb{R}^\ell, (X_i, u_i, \omega_i^\nu)_{i \in I}).$$

Note that $\mathcal{E}^\nu$ differs from $\mathcal{E}$ only in its initial endowments.

By the definition, each $\mathcal{E}^\nu$ satisfies all the assumptions in Theorem 1. In particular, each $\mathcal{E}^\nu$ satisfies [Weak nonsatiation]. To see this, note first that for each $i \in I$, by (6) and (7),

$$\begin{aligned} s_i &\notin G_i - \{\varepsilon^\nu\} \\ &= -\sum_{j\neq i} R_j + \sum_{k\in I}\{\omega_k\} - \{\varepsilon^\nu\} \\ &= -\sum_{j\neq i} R_j + \sum_{k\in I}\{\omega_k^\nu\}. \end{aligned} \tag{8}$$

For each $i \in I$, let

$$R_i^\nu = \{x_i \in X_i : u_i(x_i) \geq u_i(\omega_i^\nu)\}.$$

Then, since $\omega_i^\nu \in R_i$, we have $R_i^\nu \subset R_i$. Therefore, for each $i \in I$,

$$-\sum_{j\neq i} R_j^\nu + \sum_{k\in I}\{\omega_k^\nu\} \subset -\sum_{j\neq i} R_j + \sum_{k\in I}\{\omega_k^\nu\}.$$

Finally, by this relation and (8),

$$s_i \notin R_i^\nu \cap \left(-\sum_{j\neq i} R_j^\nu + \sum_{k\in I}\{\omega_k^\nu\}\right) = A_i^\nu,$$

where A_i^ν denotes the individually rational feasible consumption set of consumer $i \in I$ in $\mathcal{E}^\nu$.

Therefore, by Theorem 1, each $\mathcal{E}^\nu$ ($\nu \geq \overline{\nu}$) admits a quasi-equilibrium $(\overline{x}^\nu, \overline{p}^\nu) \in X \times \mathbb{R}^\ell \setminus \{0\}$. In view of Definition 2, we may assume without loss of generality that $\overline{p}^\nu \in S(0,1) = \{p \in \mathbb{R}^\ell : \|p\| = 1\}$ for all $\nu \geq \overline{\nu}$.

We now obtain a sequence $\{(\overline{x}^\nu, \overline{p}^\nu)\}_{\nu \geq \overline{\nu}} \subset X \times S(0,1)$ in which each term $(\overline{x}^\nu, \overline{p}^\nu)$ is a quasi-equilibrium of $\mathcal{E}^\nu$. Since $X \times S(0,1)$ is compact, we may assume without loss of generality that the sequence has a limit point $(\overline{x}, \overline{p}) \in X \times S(0,1)$. We prove that $(\overline{x}, \overline{p})$ is a quasi-equilibrium of the original economy.

We first show that (a-2) of Definition 2 holds. Suppose that for some $i \in I$, there exists $x_i \in X_i$ with

$$u_i(x_i) > u_i(\overline{x}_i) \quad \text{and} \quad \overline{p} \cdot x_i < \overline{p} \cdot \omega_i.$$

Then, since $(\overline{x}^\nu, \overline{p}^\nu) \to (\overline{x}, \overline{p})$ and $\omega_i^\nu \to \omega_i$ as $\nu \to \infty$, and u_i is upper semicontinuous, we have

$$u_i(x_i) > u_i(\overline{x}_i^\nu) \quad \text{and} \quad \overline{p}^\nu \cdot x_i < \overline{p}^\nu \cdot \omega_i^\nu$$

for sufficiently large ν. However, this contradicts the fact that $(\overline{x}^\nu, \overline{p}^\nu)$ is a quasi-equilibrium of $\mathcal{E}^\nu$. Thus, (a-2) of Definition 2 holds. It is easy to check that $(\overline{x}, \overline{p})$ satisfies (a-1) and (b) of Definition 2.

Therefore, we conclude that $(\overline{x}, \overline{p}) \in X \times S(0,1)$ is a quasi-equilibrium of the original economy $\mathcal{E}$. $\square$

Remark 2. In fact, in the proof of Theorem 2, we can find a quasi-equilibrium allocation $\overline{x}$ in the individually rational feasible allocation set A. To see this, let $\{(\overline{x}^\nu, \overline{p}^\nu)\}_{\nu \geq \overline{\nu}} \subset X \times S(0,1)$ be the sequence obtained in the proof. In view of Remark 1, we may assume that $\overline{x}^\nu \in A^\nu$ for each $\nu \geq \overline{\nu}$, where A^ν denotes the set of all individually rational feasible allocations in $\mathcal{E}^\nu$. Note that $u_i(\overline{x}_i^\nu) \geq u_i(\omega_i^\nu) \geq u_i(\omega_i)$ for each $\nu \geq \overline{\nu}$ and $i \in I$. Then, since $\overline{x}_i^\nu \to \overline{x}_i$ and u_i is upper semicontinuous, we have $u_i(\overline{x}_i) \geq u_i(\omega_i)$ for all $i \in I$, which implies that $\overline{x} \in A$.

3.2. Existence of competitive equilibrium

There are several known sets of assumptions under which a quasi equilibrium is a competitive equilibrium (see, for example, Geistdoerfer-Florenzano 1982). In this paper, we employ the simplest one:

Assumption 4. For all $i \in I$,

(a) $\omega_i \in \operatorname{int} X_i$, and

(b) u_i is continuous on X_i.

We now establish the existence of a competitive equilibrium under Assumption 3.

Theorem 3. *Under Assumptions 1 – 4, there exists a competitive equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}$.*

Proof. By Theorem 2, there exists a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}$. We prove that $(\overline{x}, \overline{p})$ is a competitive equilibrium of $\mathcal{E}$.

It is clear that $(\overline{x}, \overline{p})$ satisfies (a-1) and (b) of Definition 1. Suppose that (a-2) of Definition 1 does not hold for some $i \in I$. Then, there exists $x_i \in X_i$ such that

$$u_i(x_i) > u_i(\overline{x}_i) \quad \text{and} \quad \overline{p} \cdot x_i = \overline{p} \cdot \omega_i.$$

Since $\overline{p} \neq 0$ and $\omega_i \in \operatorname{int} X_i$, there exists $y_i \in X_i$ such that $\overline{p} \cdot y_i < \overline{p} \cdot \omega_i$. Let $x_i(t) = tx_i + (1-t)y_i$ for each $t \in (0,1)$. It is clear that for all $t \in (0,1)$,

$$x_i(t) \in X_i \quad \text{and} \quad \overline{p} \cdot x_i(t) < \overline{p} \cdot \omega_i.$$

Moreover, since $u_i(x_i) > u_i(\overline{x}_i)$ and u_i is continuous on X_i, we have $u_i(x_i(t)) > u_i(\overline{x}_i)$ for t sufficiently close to 1. However, this contradicts the fact that $(\overline{x}, \overline{p})$ is a quasi-equilibrium of $\mathcal{E}$.

Therefore, we conclude that $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a competitive equilibrium of $\mathcal{E}$. □

4. Alternative assumption

In this section, we introduce an alternative to Assumption 3 and provide some related results.

Consider the following assumption.

Assumption 5. For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$.

The symbol $\operatorname{int}_{X_i} F_i$ denotes the interior of F_i in the relative topology on X_i that is, for $x_i \in X_i$, we have $x_i \in \operatorname{int}_{X_i} F_i$ if and only if there exists a positive real number $r > 0$ such that $B(x_i, r) \cap X_i \subset F_i$. Assumption 5 allows S_i to be a subset of the individually feasible consumption set F_i, provided that S_i touches the complement of $\operatorname{int}_{X_i} F_i$ in X_i. This assumption is a generalization of Sato's (2008) nonsatiation assumption, which asserts that $S_i \cap \operatorname{int}_{X_i} F_i = \varnothing$ for each $i \in I$.

By replacing A_i by F_i and R_i by X_i for all $i \in I$ in the statements and proofs of all the propositions provided in 3.1 (including Theorem 1), we obtain the existence of a quasi-equilibrium under Assumption 5.

Corollary 1. *Under Assumptions 1, 2 and 5, there exists a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}$.* [11]

Assumptions 3 and 5 do not imply each other in general. Indeed, Assumption 5 does not hold in Example 1 in 2.2 (in which s_1 lies on $\operatorname{int}_{X_1} F_1$). In contrast, in Example 2 below, we observe that only Assumption 5 holds.

[11] In this corollary, Assumption 1 (b) can be weakened to the boundedness of F by the standard truncation technique.

Example 2. Consider an exchange economy $\mathcal{E}$ with two commodities and two consumers. Let $X_1 = \{x_1 \in \mathbb{R}^2 : 0 \leq x_{11} \leq 10 \text{ and } x_{12} \geq 0\}$ and $X_2 = \mathbb{R}^2_+$. Let $\omega_1 = (0, 10)$ and $\omega_2 = (10, 0)$. Consumers' utility functions are as follows.

$$u_1(x_1) = \begin{cases} -|10 - x_{12}| & \text{if} \quad x_{12} \neq 10 \\ -|5 - x_{11}| + 5 & \text{if} \quad x_{12} = 10, \end{cases}$$

$$u_2(x_2) = x_{22}.$$

Note that u_1 is satiated at $s_1 = (5, 10)$, while u_2 is never satiated on X_2 (see Figure 2).

By the above definitions, it is easy to check that

$$A_1 = \{x_1 \in X_1 : 0 \leq x_{11} \leq 10 \text{ and } x_{12} = 10\} = R_1$$

and $s_1 \in A_1$. Since $A_i \setminus \operatorname{int}_{R_i} A_i = \varnothing$ when $A_i = R_i$, Assumption 3 does not hold. However, since $s_1 \in F_1 \setminus \operatorname{int}_{X_1} F_1$ (note that s_1 requires the total amount of the second good in the economy while consumer $i \in I$ can consume more of it), Assumption 5 holds.

However, Assumption 5 implies Assumption 3 if consumers' utility functions are continuous and not satiated at the initial endowment.

Proposition 1. *Suppose that u_i is continuous on X_i and $\omega_i \notin S_i$ for all $i \in I$, then Assumption 5 implies Assumption 3.* [12]

Proof. It suffices to show that for all $i \in I$, if $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$, then, $S_i \cap (\operatorname{int}_{R_i} A_i)^c \neq \varnothing$. Suppose that $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$ and $S_i \cap (\operatorname{int}_{R_i} A_i)^c = \varnothing$ for some $i \in I$. Let $s_i \in S_i \cap (\operatorname{int}_{X_i} F_i)^c$. Then, by the supposition, we have $s_i \in \operatorname{int}_{R_i} A_i$.

Since $s_i \in \operatorname{int}_{R_i} A_i$, there exists a positive real number r_1 such that

$$B(s_i, r_1) \cap R_i \subset A_i.$$

Moreover, since u_i is continuous on X_i and $u_i(s_i) > u_i(\omega_i)$, there exists a positive real number r_2 such that

$$B(s_i, r_2) \cap X_i \subset R_i.$$

[12] Example 1 in 2.2 shows that the converse of the statement is not true.

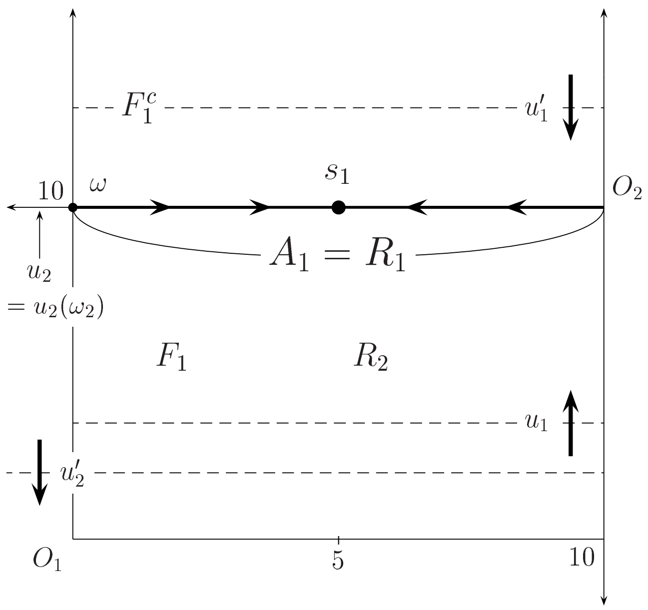

Figure 2: Example in which Assumption 5 holds but Assumption 3 does not.

Let $r = \min\{r_1, r_2\}$. Then, from the above two relations,

$$B(s_i, r) \cap X_i \subset B(s_i, r_1) \cap R_i \subset A_i \subset F_i.$$

Therefore, $s_i \in \operatorname{int}_{X_i} F_i$, which is a contradiction. $\square$

5. Concluding Remark

As a concluding remark, we compare Assumption 3 with the assumption introduced by Won and Yannelis (2006).

Won and Yannelis (2006) establish the existence of a competitive equilibrium with satiation in more general settings than ours. For example, in their analysis, consumers' preferences are allowed to be non-ordered and individually rational feasible consumption sets do not need to be bounded. Moreover, their assumption concerning satiation allows each consumer's satiation area S_i to be a subset of $\operatorname{int}_{R_i} A_i$, and therefore, does not imply Assumption 3. These advantages make their results applicable to securities markets with unlimited short-selling, in which the set A may be unbounded, and to the capital asset pricing model (CAPM) without a riskless asset, in which satiation inside $\mathrm{int}_{R_i} A_i$ is likely to occur. [13]

Nevertheless, as shown below, our assumption does not imply Won and Yannelis's assumption either.

To simplify the arguments, we consider the case in which if $S_i \neq \varnothing$ for some $i \in I$, it consists of a unique element $s_i \in X_i$.

For an allocation $x \in X$, let $I_s(x) = \{i \in I : x_i \in S_i\}$ and $I_{ns}(x) = I \setminus I_s(x)$. For a consumption bundle $x_i \in X_i$, let $P_i(x_i) = \{y_i \in X_i : u_i(y_i) > u_i(x_i)\}$. Then, Won and Yannelis's (2006) assumption reduces to the following form:

> Let $x = (x_i)_{i\in I} \in A$ with $I_s(x) \neq \varnothing$ and $I_{ns}(x) \neq \varnothing$. Then, for each $p \in \mathbb{R}^\ell \setminus \{0\}$ that satisfies $p \cdot P_j(x_j) > p \cdot x_j$ for all $j \in I_{ns}(x)$, we have $p \cdot s_i \geq p \cdot \omega_i$ for all $i \in I_s(x)$. [14]

Note that since s_i is a unique element of S_i if the set is nonempty, we have $x_i = s_i$ for $i \in I_s(x)$.

We now provide the following example.

Example 3. Consider an exchange economy $\mathcal{E}$ with two commodities and three consumers. Let $X_i = \mathbb{R}^2_+$ for all $i \in I = \{1, 2, 3\}$. Let $\omega_1 = \omega_2 = (2, 2)$ and $\omega_3 = (4, 4)$. Consumers' utility functions are as follows.

$$u_1(x_1) = -\|(x_{11}, x_{12}) - (8, 0)\|^2,$$

$$u_2(x_2) = -\|(x_{21}, x_{22}) - (0, 8)\|^2,$$

$$u_3(x_3) = x_{32} - x_{31}.$$

Note that $s_1 = (8, 0)$ ($\neq \omega_1$) and $s_2 = (0, 8)$ ($\neq \omega_2$) are the unique satiation points of consumers 1 and 2, while u_3 is never satiated on X_3 (see Figure 3). Note also that u_i is continuous on X_i for each $i \in I$.

Consider an allocation $x = (s_1, s_2, y_3) \in X$, where $y_3 = (0, 0) \in X_3$. It is clear that x is individually rational feasible. Note that $I_s(x) = \{1, 2\}$ and $I_{ns} = \{3\}$.

[13] For satiation in CAPM without a riskless asset, see, for example, Nielsen (1987, 1990) and Won et al. (2008).

[14] Won and Yannelis 2006, Assumption **S5**.

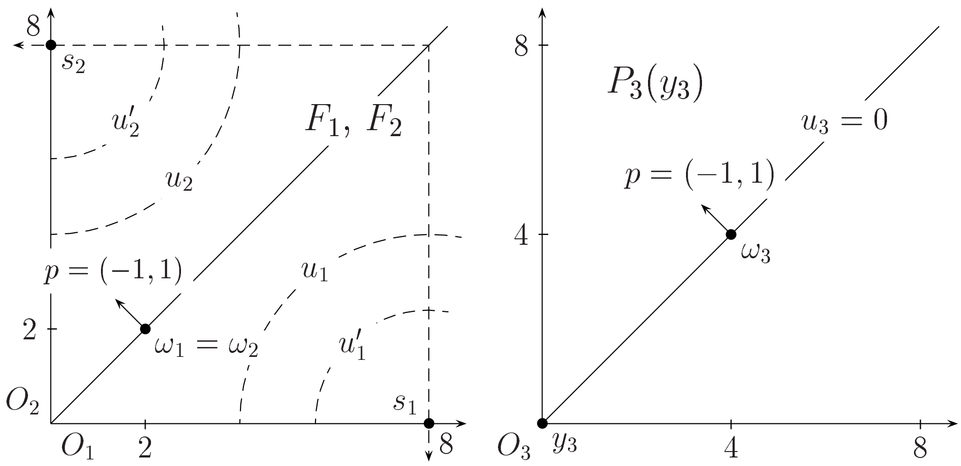

Figure 3: Example in which Assumption 3 holds but Won and Yannelis's (2006) assumption does not.

By the definition of u_3, we have $P_3(y_3) = \{x_3 \in X_3 : x_{32} > x_{31}\}$. Therefore, for a price $p = (-1, 1) \in \mathbb{R}^2$, we have

$$p \cdot P_3(y_3) > 0 = p \cdot y_3.$$

However, since $p \cdot s_1 = -8 < 0 = p \cdot \omega_1$ for consumer 1, Won and Yannelis's assumption does not hold.

To prove that Assumption 3 holds, we first observe that Assumption 5 holds. Indeed, we have $s_1 \in F_1 \setminus \mathrm{int}_{X_1} F_1$ because s_1 requires the total amount of the first good in the economy while consumer 1 can consume more of it. Likewise, we have $s_2 \in F_2 \setminus \mathrm{int}_{X_2} F_2$. Therefore, this economy satisfies Assumption 5. Then, by Proposition 1, we conclude that Assumption 3 holds.

Appendix

Proof of Claim 2. First, if $\sum_{h \in I_{bd}} (s_h - \omega_h) = 0$ (that is, $\varepsilon^{\nu}_{out} = 0$ for all $\nu \in \mathbb{N}$), case (b) clearly holds. Thus, in the following, we suppose that $\sum_{h \in I_{bd}} (s_h - \omega_h) \neq 0$.

Suppose that the assertion of the lemma is not true. Then, for an arbitrarily chosen $\nu \in \mathbb{N}$, there exist $\nu', \nu'' \geq \nu$ such that

$$s_i \notin R_i \cap \left(G_i - \frac{1}{(\nu')^2} \sum_{h \in I_{bd}} \{s_h - \omega_h\} \right) \tag{9}$$

and

$$s_i \in R_i \cap \left(G_i - \frac{1}{(\nu'')^2} \sum_{h \in I_{bd}} \{s_h - \omega_h\} \right). \tag{10}$$

Without loss of generality, we may assume that $\nu' > \nu''$.

Then, by (10), there exists $(x_j)_{j \neq i} \in \prod_{j \neq i} R_j$ such that

$$s_i = -\sum_{j \neq i} x_j + \sum_{k \in I} \omega_k - \frac{1}{(\nu'')^2} \sum_{h \in I_{bd}} (s_h - \omega_h). \tag{11}$$

Since $s_i \in A_i = R_i \cap G_i$, there exists $(y_j)_{j \neq i} \in \prod_{j \neq i} R_j$ such that

$$s_i = -\sum_{j \neq i} y_j + \sum_{k \in I} \omega_k. \tag{12}$$

Then, $(\nu''/\nu')^2 \times$(11)+$(1 - (\nu''/\nu')^2) \times$(12) yields,

$$s_i = -\sum_{j \neq i} z_j + \sum_{k \in I} \omega_k - \frac{1}{(\nu')^2} \sum_{h \in I_{bd}} (s_h - \omega_h),$$

where

$$z_j = \left(\frac{\nu''}{\nu'}\right)^2 x_j + \left(1 - \left(\frac{\nu''}{\nu'}\right)^2\right) y_j \in R_j \quad \text{for each} \quad j \neq i.$$

Therefore,

$$s_i \in R_i \cap \left(G_i - \frac{1}{(\nu')^2} \sum_{h \in I_{bd}} \{s_h - \omega_h\} \right),$$

which contradicts (9). □

Proof of Claim 3. Note first that since $s_1 + \varepsilon_1^\nu \in R_1 \setminus A_1$ and $A_1 = R_1 \cap G_1$, we must have $s_1 + \varepsilon_1^\nu \notin G_1$.

Suppose that the assertion of the claim is not true. Then, there exists $(x_j)_{j \neq 1} \in \prod_{j \neq 1} R_j$ such that

$$s_1 = -\sum_{j \neq 1} x_j + \sum_{k \in I} \omega_k - \frac{1}{\nu^2} \sum_{h \in I_{bd}} (s_h - \omega_h) - 2\nu^2 \varepsilon_1^\nu.$$

Since $s_1 \in R_1 \cap (G_1 - \{\varepsilon_{out}^\nu\})$ (recall that $\nu \geq \overline{\nu} \geq \overline{\nu}_{in}$), there exists $(y_j)_{j \neq 1} \in \prod_{j \neq 1} R_j$ such that

$$s_1 = -\sum_{j \neq 1} y_j + \sum_{k \in I} \omega_k - \frac{1}{\nu^2} \sum_{h \in I_{bd}} (s_h - \omega_h).$$

Therefore, we have

$$s_1 = -\sum_{j\neq 1} z_j + \sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h) - 2\varepsilon_1^\nu, \tag{13}$$

where

$$z_j = \frac{1}{\nu^2}x_j + \left(1-\frac{1}{\nu^2}\right)y_j \in R_j \quad \text{for all} \quad j\neq 1.$$

Moreover, since $s_1 \in A_1 = R_1 \cap G_1$, there exists $(t_j)_{j\neq 1} \in \prod_{j\neq 1} R_j$ such that

$$s_1 = -\sum_{j\neq 1} t_j + \sum_{k\in I}\omega_k. \tag{14}$$

Multiplying (14) by $(1-(1/\nu)^2)$, we have

$$\left(1-\frac{1}{\nu^2}\right)s_1 = -\sum_{j\neq 1}\left(1-\frac{1}{\nu^2}\right)t_j + \left(1-\frac{1}{\nu^2}\right)\sum_{k\in I}\omega_k.$$

By adding this equation and $0 = -(1/\nu^2)\sum_{h\in I_{out}}(\omega_h-\omega_h)$ to (13),

$$\begin{aligned}
s_1+&\left(1-\frac{1}{\nu^2}\right)s_1\\
&= -\sum_{j\neq 1} z_j + \sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h) - 2\varepsilon_1^\nu\\
&\quad -\sum_{j\neq 1}\left(1-\frac{1}{\nu^2}\right)t_j + \left(1-\frac{1}{\nu^2}\right)\sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{out}}(\omega_h-\omega_h)\\
&= -\sum_{j\neq 1} z_j + \sum_{k\in I}\omega_k - \left(\frac{1}{\nu^2}s_1 + 2\varepsilon_1^\nu\right) - \sum_{h\in I_{bd}\setminus\{1\}}\frac{1}{\nu^2}s_h + \frac{1}{\nu^2}\sum_{h\in I_{bd}}\omega_h\\
&\quad -\sum_{h\in I_{out}}\left(1-\frac{1}{\nu^2}\right)t_h - \sum_{h\in I_{bd}\setminus\{1\}}\left(1-\frac{1}{\nu^2}\right)t_h + \left(1-\frac{1}{\nu^2}\right)\sum_{k\in I}\omega_k\\
&\quad -\sum_{h\in I_{out}}\frac{1}{\nu^2}\omega_h + \frac{1}{\nu^2}\sum_{h\in I_{out}}\omega_h\\
&= -\sum_{j\neq 1} z_j + \sum_{k\in I}\omega_k - \left(\frac{1}{\nu^2}s_1 + 2\varepsilon_1^\nu\right) - \sum_{h\in I_{out}}\left[\frac{1}{\nu^2}\omega_h + \left(1-\frac{1}{\nu^2}\right)t_h\right]\\
&\quad -\sum_{h\in I_{bd}\setminus\{1\}}\left[\frac{1}{\nu^2}s_j + \left(1-\frac{1}{\nu^2}\right)t_j\right] + \frac{1}{\nu^2}\sum_{k\in I}\omega_k + \left(1-\frac{1}{\nu^2}\right)\sum_{k\in I}\omega_k.
\end{aligned}$$

Rearranging the equation further, we obtain

$$2(s_1 + \varepsilon_1^\nu) = -\sum_{j\neq 1} a_j + 2\sum_{k\in I} \omega_k,$$

where for $j \in I_{out}$,

$$a_j = z_j + \left[\frac{1}{\nu^2}\omega_j + \left(1 - \frac{1}{\nu^2}\right)t_j\right] \in R_j + R_j,$$

and for $j \in I_{bd} \setminus \{1\}$,

$$a_j = z_j + \left[\frac{1}{\nu^2}s_j + \left(1 - \frac{1}{\nu^2}\right)t_j\right] \in R_j + R_j.$$

Therefore,

$$s_1 + \varepsilon_1^\nu = -\sum_{j\neq 1} \frac{1}{2}a_j + \sum_{k\in I} \omega_k.$$

Since $(1/2)a_j \in R_j$ for all $j \neq 1$, we have $s_1 + \varepsilon_1^\nu \in G_1$, which is a contradiction. $\square$

Proof of Claim 4. Note first that since $s_m + \varepsilon_m^\nu \in R_m \setminus A_m$ and $A_m = R_m \cap G_m$, we must have $s_m + \varepsilon_m^\nu \notin G_m$.

Suppose that there exists $(x_j)_{j\neq m} \in \prod_{j\neq m} R_j$ such that

$$s_m = -\sum_{j\neq m} x_j + \sum_{k\in I} \omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}} (s_h - \omega_h) - \sum_{q=1}^{m} 2^q \nu^{2q} \varepsilon_q^\nu.$$

Since $s_m \in R_m \cap (G_m - \{\varepsilon_{out}^\nu\})$, there exists $(y_j)_{j\neq m} \in \prod_{j\neq m} R_j$ such that

$$s_m = -\sum_{j\neq m} y_j + \sum_{k\in I} \omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}} (s_h - \omega_h).$$

Therefore, we have

$$s_m = -\sum_{j\neq m} z_j + \sum_{k\in I} \omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}} (s_h - \omega_h) - \sum_{q=1}^{m} 2^{q-(m-1)} \nu^{2q-2m} \varepsilon_q^\nu, \quad (15)$$

where

$$z_j = \frac{1}{2^{m-1}\nu^{2m}} x_j + \left(1 - \frac{1}{2^{m-1}\nu^{2m}}\right) y_j \in R_j \quad \text{for all} \quad j \neq m.$$

Moreover, since $s_m \in A_m = R_m \cap G_m$, there exists $(t_j)_{j \neq m} \in \prod_{j \neq m} R_j$ such that

$$s_m = -\sum_{j \neq m} t_j + \sum_{k \in I} \omega_k. \tag{16}$$

Multiplying (16) by $(1 - (1/\nu)^2)$, we have

$$\left(1 - \frac{1}{\nu^2}\right) s_m = -\sum_{j \neq m} \left(1 - \frac{1}{\nu^2}\right) t_j + \left(1 - \frac{1}{\nu^2}\right) \sum_{k \in I} \omega_k.$$

By adding this equation and $0 = -(1/\nu^2) \sum_{h \in I_{out}} (\omega_h - \omega_h)$ to (15),

$$\begin{aligned}
s_m + & \left(1 - \frac{1}{\nu^2}\right) s_m \\
&= -\sum_{j \neq m} z_j + \sum_{k \in I} \omega_k - \frac{1}{\nu^2} \sum_{h \in I_{bd}} (s_h - \omega_h) - \sum_{q=1}^{m} 2^{q-(m-1)} \nu^{2q-2m} \varepsilon_q^\nu \\
&\quad - \sum_{j \neq m} \left(1 - \frac{1}{\nu^2}\right) t_j + \left(1 - \frac{1}{\nu^2}\right) \sum_{k \in I} \omega_k - \frac{1}{\nu^2} \sum_{h \in I_{out}} (\omega_h - \omega_h) \\
&= -\sum_{j \neq m} z_j + \sum_{k \in I} \omega_k - \left(\frac{1}{\nu^2} s_m + 2\varepsilon_m^\nu\right) - \sum_{h \in I_{bd} \setminus \{m\}} \frac{1}{\nu^2} s_h + \frac{1}{\nu^2} \sum_{h \in I_{bd}} \omega_h \\
&\quad - \sum_{q=1}^{m-1} 2^{q-(m-1)} \nu^{2q-2m} \varepsilon_q^\nu - \sum_{h \in I_{out}} \left(1 - \frac{1}{\nu^2}\right) t_h - \sum_{h \in I_{bd} \setminus \{m\}} \left(1 - \frac{1}{\nu^2}\right) t_h \\
&\quad + \left(1 - \frac{1}{\nu^2}\right) \sum_{k \in I} \omega_k - \sum_{h \in I_{out}} \frac{1}{\nu^2} \omega_h + \frac{1}{\nu^2} \sum_{h \in I_{out}} \omega_h \\
&= -\sum_{j \neq m} z_j + \sum_{k \in I} \omega_k - \left(\frac{1}{\nu^2} s_m + 2\varepsilon_m^\nu\right) - \sum_{h \in I_{out}} \left[\frac{1}{\nu^2} \omega_h + \left(1 - \frac{1}{\nu^2}\right) t_h\right] \\
&\quad - \sum_{h \in I_{bd} \setminus \{m\}} \left[\frac{1}{\nu^2} s_h + \left(1 - \frac{1}{\nu^2}\right) t_h\right] - \sum_{q=1}^{m-1} 2^{q-(m-1)} \nu^{2q-2m} \varepsilon_q^\nu \\
&\quad + \frac{1}{\nu^2} \sum_{k \in I} \omega_k + \left(1 - \frac{1}{\nu^2}\right) \sum_{k \in I} \omega_k \\
&= -\sum_{j \neq m} z_j + 2 \sum_{k \in I} \omega_k - \left(\frac{1}{\nu^2} s_m + 2\varepsilon_m^\nu\right) \\
&\quad - \sum_{h \in I_{out}} \left[\frac{1}{\nu^2} \omega_h + \left(1 - \frac{1}{\nu^2}\right) t_h\right] - \sum_{h \in I_{bd} \setminus \{1, \cdots, m\}} \left[\frac{1}{\nu^2} s_h + \left(1 - \frac{1}{\nu^2}\right) t_h\right]
\end{aligned}$$

$$-\sum_{h\in\{1,\cdots,m-1\}}\left[\frac{1}{\nu^2}\left(s_h+2^{h-(m-1)}\nu^{h-m}\varepsilon_h^\nu\right)+\left(1-\frac{1}{\nu^2}\right)t_h\right].$$

Rearranging the equation further, we obtain

$$2(s_m+\varepsilon_m^\nu)=-\sum_{j\neq m}a_j+2\sum_{k\in I}\omega_k, \tag{17}$$

where for $j\in I_{out}$,

$$a_j=z_j+\left[\frac{1}{\nu^2}\omega_j+\left(1-\frac{1}{\nu^2}\right)t_j\right],$$

for $j\in I_{bd}\setminus\{1,\cdots,m-1\}$,

$$a_j=z_j+\left[\frac{1}{\nu^2}s_j+\left(1-\frac{1}{\nu^2}\right)t_j\right],$$

and for $j\in\{1,\cdots,m-1\}$,

$$a_j=z_j+\left[\frac{1}{\nu^2}\left(s_j+\frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}}\varepsilon_j^\nu\right)+\left(1-\frac{1}{\nu^2}\right)t_j\right].$$

Note that for $j\in\{1,\cdots,m-1\}$, since

$$0<\frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}}\leq 1,$$

we have

$$s_j+\frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}}\varepsilon_j^\nu\in R_j.$$

Then, by (17),

$$s_m+\varepsilon_m^\nu=-\sum_{j\neq m}\frac{1}{2}a_j+\sum_{k\in I}\omega_k.$$

Since $(1/2)a_j\in R_j$ for all $j\neq m$, we have $s_m+\varepsilon_m^\nu\in G_m$, which is a contradiction. □

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