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Axiomatizations of the Values for TU Games Using the Balanced Cycle Contributions Property

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Abstract

This paper presents axiomatizations of the Shapley and egalitarian values. The balanced cycle contributions property is the key axiom in this paper. It requires that, for any order of all the players, the sum of the claims from each player against his predecessor is balanced with the sum of the claims from each player against his successor. This property is satisfied not only by the Shapley value but also by some other values for TU games. Hence, it is a less restrictive requirement than the balanced contributions property introduced by Myerson (1980; International Journal of Game Theory 9, 169–182).

JEL classification: C71
Keywords: axiomatization; balanced cycle contributions property; Shapley value; egalitarian value

1 Introduction

An important criterion in allocation problems is fairness. The balanced contributions property introduced by Myerson (1980) is a widely-used fairness criterion in the cooperative game theory. This property requires that, for any pairs of players, the claim from one player against another is balanced with a counter claim from another against the player. Thus, if a solution satisfies this property, the outcome supported by the solution is fair in that no one has a higher number of claims against another. However, the property is rather restrictive because, along with efficiency, which is also an important criterion in allocation problems, the only solution satisfying the property is the Shapley value.

In this paper, we present a less restrictive property than the balanced contributions property. Considering a less restrictive and minimal requirement of one described by the balanced contributions property is relevant for the following two reasons. The first reason is that the balanced contributions property could be a very demanding property with regard to considering an application of the spirit of the condition to the real society. Since the population of the modern society is very large, balanced claims for *all* pairs of two individuals in the society are rarely expected. The other reason is related

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to a solution theory in cooperative game. Except for the Shapley value, there is no efficient solution satisfying the balanced contributions property. Thus, exploring a weaker condition than the balanced contributions property enriches the solution theory.

In our new less restrictive property, claims between two players cannot be balanced; however, claims among all players are balanced in a cyclical manner, i.e., for any order of players, the sum of the claims from each player against his predecessor is balanced with the sum of the claims from each player against his successor. This weaker balanced contributions property is satisfied by several values for TU games, such as the Shapley value, the egalitarian value, and the CIS (center of gravity of the imputation set) value (Driessen and Funaki 1991). Together with other basic axioms, the Shapley value and the egalitarian value are axiomatized, and these form our main results.

This paper is organized as follows. Section 2 states the notation and definitions are provided. The less restrictive fairness property is provided in Section 3. Sections 4 and 5 present axiomatizations of the Shapley and egalitarian values, respectively. In Section 6, our results are generalized to the situations where players are asymmetric. Section 7 concludes the paper. All the proofs of the propositions and theorems are relegated in the Appendix.

2 Preliminaries

Let $N \subseteq \mathbb{N}$ be a finite set of *players*, and let $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ be a *characteristic function*. A pair (N, v) is a *cooperative game with transferable utility*, or simply, a *game*. Let Γ be a set of all games and let $|N| = n$, where $|\cdot|$ represents the cardinality of the set. A non-empty subset $S \subseteq N$ is called a coalition, and $v(S)$ is the worth of the coalition. For simplicity, each singleton $\{i\} \subseteq N$ is represented as i when the possibility of confusion does not exist.

A game (N, v) is *zero-monotonic* if, for any $i \in N$ and for any $S \subseteq N \backslash i$, $v(S \cup i) \geq v(S) + v(i)$. A game (N, v) is *superadditive* if, for any $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. A game (N, v) is *convex* if, for any $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. The sets of all zero-monotonic games, all superadditive games, and all convex games are denoted by Γ^M, Γ^S, and Γ^C, respectively. In general, $\Gamma^C \subseteq \Gamma^S \subseteq \Gamma^M$.

For any coalition S, let $(N, u_S) \in \Gamma$ be the S-*unanimity game* defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

for any $T \subseteq N$. It is well known that any $(N, v) \in \Gamma$ is represented as a linear combination of unanimity games, i.e.

$$v = \sum_{S \subseteq N; S \neq \emptyset} \Delta_S^v u_S,$$

where $\Delta_S^v = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ is the *dividend* of S.

A value on Γ is a function that associates each game $(N, v) \in \Gamma$ with an n-dimensionalvector $x = (x_i)_{i \in N}$, which satisfies $\sum_{i \in N} x_i \leqq v(N)$. A value x on Γ is *efficient* if $\sum_{i \in N} x_i = v(N)$. One of the well-known efficient values is the *Shapley value* $\phi = (\phi_i)_{i \in N}$ (Shapley 1953); for any $(N, v) \in \Gamma$ and for any $i \in N$, it is defined as

$$\phi_i(N, v) = \sum_{S \subseteq N; S \ni i} \frac{\Delta_S^v}{|S|}.$$

Let φ be a value on Γ. The *balanced contributions property* (Myerson 1980) is the following. For any $(N, v) \in \Gamma$ and for any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v),$$

where $(N \setminus j, v)$ and $(N \setminus i, v)$ are restrictions of (N, v) on $N \setminus j$ and $N \setminus i$, respectively. Assume that a value φ is commonly accepted as a distribution rule in the society, and that the claim from i against j is measured by i's contribution to j, i.e., $c(i, j) = \varphi_j(N, v) - \varphi_j(N \setminus i, v)$. Then, the balanced contributions property is interpreted as a condition that claims between any two players are balanced with each other, i.e., $c(i, j) = c(j, i)$.

Myerson (1980) showed that the Shapley value is a unique efficient value on Γ, and it satisfies the balanced contributions property.

3 Balanced cycle contributions property

Let $e(i, j) = c(i, j) - c(j, i)$ be the *excess claim from i against j*. Then, the balanced contributions property is interpreted as a condition that, for any two individuals i and j, the excess claim from i against j is zero, i.e., $e(i, j) = 0$. A weaker and minimal requirement for balancedness of claims in a society is that the balancedness of the excess claims is attained as a whole of the members in a society. One of the possible expressions is that the sum of the excess claims among the society members is zero. That is, given a coalition S with $|S| = s$ and an order $(i_1, i_2, \ldots, i_s)$ on S, the following condition holds:

$$e(i_1, i_2) + e(i_2, i_3) + \cdots + e(i_{s-1}, i_s) + e(i_s, i_1) = \sum_{\ell=1}^{s} e(i_\ell, i_{\ell+1}) = 0, \qquad (1)$$

where $i_{s+1} = i_1$. In fact, this is a weaker requirement than the balancedness of claims for each pair of individuals because, if $e(i, j) = 0$ holds for each $\{i, j\} \subseteq N$, the above condition obviously holds. This can be interpreted as the condition that the sum of the excess claims of all the players is zero, or that the average of the excess claims among all the players is zero, irrespective of the order of players.

A similarity to the above condition is found in the general equilibrium theory of standard microeconomics. In this study, the excess demands of the individual economic agents are aggregated through the market, and the total excess demand, which is the sum of the individual demands of the economic agents, becomes zero at market equilibrium.

An order on S, $(i_1, i_2, \ldots, i_s)$, might be determined by some exogenous factors. Otherwise, both the group and the order might be endogenously determined. This can be explained by the following example. A player j_1 chooses one of the excess claims, say $e(j_1, j_2)$, which maximizes its excess claim. Then, the second player, j_2, also chooses one of its excess claims, say $e(j_2, j_3)$, and this process is continued until some player chooses the claim regarding one of the earlier players. As a result, we obtain coalition $S = \{i_1, i_2, \ldots, i_s\}$, whose members can be indexed by the above procedure.[1] Then, the above condition simply states that after the determination of

[1] This kind of process that constructs closed cycles among players can be seen as a variation of Gale's Top Trading Cycles algorithm that is used in several kinds of allocation problems such as housing markets (Shapley and Scarf 1974) and school choice problems (Abdulkadiroğlu and Sönmez 2003).

the coalition and the order, the sum of the excess claims among the members of the coalition according to the order should be zero.

Since $e(i_\ell, i_{\ell+1}) = c(i_\ell, i_{\ell+1}) - c(i_{\ell+1}, i_\ell)$, eq.(1) can be reduced to

$$\sum_{\ell=1}^{s} c(i_\ell, i_{\ell+1}) = \sum_{\ell=1}^{s} c(i_{\ell+1}, i_\ell).$$

Thus, given an order $(i_1, i_2, \ldots, i_s)$, the sum of the claims from each player against his predecessor is balanced with the sum of the claims by each player against his successor. The left-hand side of the above equation is called *cycle contributions with respect to an order* $(i_1, i_2, \ldots, i_s)$ and the right-hand side is called *cycle contributions with respect to the inverse order*. Hence, eq.(1) can be observed as the balancedness of cycle contributions with respect to the order and its inverse order.

Together with the assumption that a claim from i against j is measured by $\varphi_j(N, v) - \varphi_j(N \setminus i, v)$, we obtain the following axiom:

Group balanced cycle contributions property (GBCC): For any $(N, v) \in \Gamma$, any S with $s = |S| \geq 2$, and any order $(i_1, i_2, \ldots, i_s)$ on S,

$$\sum_{\ell=1}^{s} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{s} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_s$ and $i_{s+1} = i_1$.

The GBCC requires the balancedness of cycle contributions for any group. However, given the aim to attain only the balancedness of excess claims in a society as a whole, the balancedness of cycle contributions for only the grand coalition is justified. Thus, we get the following:

Balanced cycle contributions property (BCC): For any $(N, v) \in \Gamma$ and any order $(i_1, i_2, \ldots, i_n)$ on N,

$$\sum_{\ell=1}^{n} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{n} \left(\varphi_{i_\ell}(N, v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Since the term $\sum_{\ell=1}^{n} \varphi_{i_\ell}(N, v)$ is common to both sides, the condition described in the axiom is reduced to:

$$\sum_{\ell=1}^{n} \varphi_{i_\ell}(N \setminus i_{\ell-1}, v) = \sum_{\ell=1}^{n} \varphi_{i_\ell}(N \setminus i_{\ell+1}, v),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$. This is a more convenient representation of the BCC. Hence, we use this representation of BCC hereafter.

Note that, in a two person game $(\{i, j\}, v)$, the conditions required by the GBCC and BCC are automatically satisfied because both the left- and right-hand sides of the equations of the condition are $\varphi_i(i, v) + \varphi_j(j, v)$.

It is obvious that the BCC is weaker than the GBCC. The following shows the equivalence of the two axioms:

Proposition 1. *The GBCC and the BCC are equivalent.*

It is obvious that any value on Γ satisfying the balanced contributions property also satisfies the BCC. Moreover, it is clear that the *egalitarian value* $\psi = (\psi_i)_{i\in N}$, which is defined as, for any $(N,v) \in \Gamma$ and any $i \in N$, $\psi_i(N,v) = \frac{v(N)}{n}$, also satisfies the BCC because $\sum_{\ell=1}^{n} \psi_{i_\ell}(N \setminus i_{\ell-1}, v) = \sum_{\ell=1}^{n} \psi_{i_\ell}(N \setminus i_{\ell+1}, v) = \sum_{i\in N} \frac{v(N\setminus i)}{n-1}$. Since the egalitarian value is efficient, the BCC is definitely weaker than the balanced contributions property.

The BCC requires that cycle contributions among all players should be balanced between any order on the set of all players and its inverse order. Similarly, we can consider the property that cycle contributions among all groups of three (or more) players should be balanced between any order on the group and its inverse order as follows.

BCC for three players: For any $(N,v) \in \Gamma$ and for any three player coalition $\{i,j,k\} \subseteq N$

$$\varphi_i(N\setminus k,v)+\varphi_j(N\setminus i,v)+\varphi_k(N\setminus j,v) = \varphi_i(N\setminus j,v)+\varphi_j(N\setminus k,v)+\varphi_k(N\setminus i,v).$$

Note that in the above property, there is no need to specify the order of the players i,j,k because the above condition remains unchanged for any order and its inverse. It is straightforward that any value on Γ satisfying the balanced contributions property also satisfies the BCC for three players. Thus, the BCC for three players is a weaker property than the balanced contributions property as well as the BCC is. The relationships between the BCC and the BCC for three players is as follows.

Proposition 2. *The BCC and the BCC for three players are equivalent.*

Thus, this proposition implies the equivalence of the BCC and the BCC for three players.

In the above, we consider only groups of three players. In the following, we consider groups of r players, where $r \geq 4$.

BCC for *r* players: For any $(N,v) \in \Gamma$, any $S \subseteq N$ such that $|S| = r$ and any order $(i_1, i_2, \dots, i_r)$ on S,

$$\sum_{\ell=1}^{r} \left(\varphi_{i_\ell}(N,v) - \varphi_{i_\ell}(N \setminus i_{\ell-1}, v)\right) = \sum_{\ell=1}^{r} \left(\varphi_{i_\ell}(N,v) - \varphi_{i_\ell}(N \setminus i_{\ell+1}, v)\right),$$

where $i_0 = i_r$ and $i_{r+1} = i_1$.

The following proposition indicates that the BCC and the BCC for r players, where $r \geq 4$, are almost equivalent.

Proposition 3. *Let $r \geq 4$. (i) If φ satisfies the BCC, it also satisfies the BCC for r players. (ii) If φ satisfies the BCC for r players, it satisfies BCC when there are r or more players.*

So, the BCC for r players, where $r \geq 4$, is slightly weaker than the BCC. The difference between Propositions 2 and 3 comes from the fact that while the BCC for r players, where $r \geq 4$ is silent for a three person game (correctly speaking, a game with less than or equal to $r-1$ players), the BCC requires a non-trivial restriction to a game with three players.

4 Axiomatization of the Shapley value

Along with two other basic axioms, the BCC characterizes the Shapley value. The first axiom is a very fundamental one.

Efficiency (EF): For any $(N, v) \in \Gamma$,

$$\sum_{i \in N} \varphi_i(N, v) = v(N).$$

The second property, which was introduced by Derks and Haller (1999), is related to null players. A *null player* in (N, v) is a player $k \in N$, satisfying $v(S \cup k) = v(S)$ for any $S \subseteq N \setminus k$.

Null player out property (NPO): For any $(N, v) \in \Gamma$, if $k \in N$ is a null player in (N, v), then for any $i \in N \setminus k$,

$$\varphi_i(N, v) = \varphi_i(N \setminus k, v).$$

The NPO requires that a deletion of null players does not affect the payoffs of other players. Note that, in general, the NPO has no relationship with the general *null player property*.[2] However, along with EF, the NPO implies the null player property. It is easily verified that the Shapley value satisfies the NPO and EF. We obtain the following.

Theorem 1. *The Shapley value is the unique value on Γ that satisfies EF, the BCC, and NPO.*

For the independence of EF, the BCC, and NPO, see Table 1.

Table 1: Independence of the axioms in Theorem 1

Values / Properties	EF	BCC (for r)	NPO
The Banzhaf value (Banzhaf 1965)	–	+	+
The τ-value (Tijs 1987)[3]	+	–	+
The egalitarian value	+	+	–

+: satisfies, $-$: does not satisfy

Since the BCC and BCC for three players are equivalent, we obtain the fact that the Shapley value is a unique efficient value satisfying the BCC for three players and NPO as a corollary of Theorem 1. On the other hand, since the BCC for r players where $r \geq 4$ is weaker than the BCC, we cannot obtain the axiomatization of the Shapley value through the BCC for r players directly from Theorem 1 and Proposition 3. However, the following theorem shows that the Shapley value is axiomatized by EF, the BCC for r players and NPO.

Theorem 2. *Let $r \geq 3$. The Shapley value is the unique value on Γ that satisfies EF, the BCC for r players, and NPO.*

[2]The *null player property* requires that null players obtain nothing.
[3]Note that the τ-value is defined on the class of the quasi-balanced games.

A remark on Theorems 1 and 2 is that it is easy to check that the proofs of Theorems 1 and 2 (see Appendix) are applicable when we consider only restricted classes of games such as the zero-monotonic games, superadditive games, or convex games. Thus, Theorems 1 and 2 also hold, even though we replace Γ in the statement of the theorems by Γ^M, Γ^S, or Γ^C, respectively.

5 Axiomatization of the egalitarian value

Replacing NPO with the other property, the egalitarian value is characterized in a similar manner.

A *proportional player* in (N, v) is a player $k \in N$, satisfying $v(S \cup k) - v(S) = \frac{1}{|S|}v(S)$ for all non-empty $S \subseteq N \setminus k$. Thus, a proportional player is a player whose marginal contributions to any non-empty coalition is directly proportional to the worth of the coalition, and it is inversely proportional to the size of the coalition. Similar to NPO, the following is considered.

Proportional player out property (PPO): For any $(N, v) \in \Gamma$, if $k \in N$ is a proportional player in (N, v), then for any $i \in N \setminus k$,

$$\varphi_i(N, v) = \varphi_i(N \setminus k, v).$$

EF and PPO imply that any proportional player obtains an equal division of the worth of the grand coalition, since if $k \in N$ is a proportional player in (N, v),

$$\begin{aligned}\varphi_k(N, v) = v(N) - \sum_{i \in N \setminus k} \varphi_i(N, v) &= v(N) - \sum_{i \in N \setminus k} \varphi_i(N \setminus k, v) \\ &= v(N) - v(N \setminus k) = v(N) - \frac{n-1}{n} v(N) = \frac{v(N)}{n}.\end{aligned}$$

The egalitarian value ψ satisfies PPO, since if $k \in N$ is a proportional player in (N, v), then for any $i \in N \setminus k$,

$$\psi_i(N \setminus k, v) = \frac{v(N \setminus k)}{n-1} = \frac{(n-1)v(N)}{n} \cdot \frac{1}{n-1} = \frac{v(N)}{n}.$$

Following are the parallel results with Theorems 1 and 2.

Theorem 3. *The egalitarian value is the unique value on Γ that satisfies EF, the BCC, and PPO.*

For the independence of EF, the BCC, and PPO, see Table 2.

Theorem 4. *Let $r \geq 3$. The egalitarian value is the unique value on Γ that satisfies EF, the BCC for r players, and PPO.*

Unlike the case of the Shapley value, the proofs of Theorem 3 and 4 (see appendix) are not applicable when we consider only restricted classes of games, such as the zero-monotonic games, superadditive games, or convex games. These differences come from the differences between the properties of null players and proportional players. Given a zero-monotonic, superadditive, or convex game, when we add a player who is a null player in the null-extended game,[4] the extended game is zero-monotonic,

[4] For the definition of the null-extended game, see the proof of Theorem 1 in the Appendix.

Table 2: Independence of the axioms in Theorem 2

Values / Properties	EF	BCC (for r)	PPO
$\varphi = \begin{cases} \frac{\psi}{2} & \text{if } \lvert N\rvert = 1 \text{ and } v(N) > 0 \\ \psi & \text{otherwise} \end{cases}$	$-$	+	+
$\varphi_i(N, v) = \begin{cases} v(N) - v(N \setminus i) & \text{if } i \in P \\ \frac{(1-\lvert P\rvert)v(N)+\sum_{j\in P} v(N\setminus j)}{\lvert N\setminus P\rvert} & \text{otherwise} \end{cases}$	+	$-$	+
the Shapley value	+	+	$-$

P: a set of all proportional players, +: satisfies, $-$: does not satisfy

superadditive, or convex, respectively. However, when we add a player who is a proportional player in the proportional-extended game,[5] the extended game may not be zero-monotonic, superadditive, or convex. These are crucial in our proofs.

6 Weighted balanced cycle contributions property

In this section, we consider a non-symmetric generalization of the analysis in the previous sections. Let $w_i (> 0)$ denote a positive weight for a player i in the set of potential player $\mathbb{N}$. Let $w = (w_i)$, which is fixed throughout this section. The *w-weighted Shapley value* $\phi^w = (\phi_i^w)_{i\in N}$ for any $(N, v) \in \Gamma$ and any $i \in N$ is defined by

$$\phi_i^w(N, v) = \sum_{S\subseteq N; S\ni i} \frac{w_i}{\sum_{j\in S} w_j} \Delta_S^v.$$

When $w_i = w_j$ for all $i, j \in N$, the w-weighted Shapley value coincides with the Shapley value.

A non-symmetric generalization of the BCC is as follows.

Weighted balanced cycle contributions property (WBCC): For any $(N, v) \in \Gamma$ and any order $(i_1, i_2, \ldots, i_{|N|})$ on N,

$$\sum_{\ell=1}^{|N|} w_{i_\ell} \varphi_{i_\ell}(N \setminus i_{\ell-1}, v) = \sum_{\ell=1}^{|N|} w_{i_\ell} \varphi_{i_\ell}(N \setminus i_{\ell+1}, v),$$

where $i_0 = i_{|N|}$ and $i_{|N|+1} = i_1$.

Since the w-weighted Shapley value satisfies the weighed balanced contributions property that requires, $w_i(\phi_i^w(N, v) - \phi_i^w(N \setminus j, v)) = w_j(\phi_j^w(N, v) - \phi_j^w(N \setminus i, v))$, for each pair of players $i, j \in N$, it also satisfies the WBCC.

As the following theorem shows, the WBCC with EF and NPO axiomatizes the w-weighted Shapley value.

Theorem 5. *The w-weighted Shapley value is the unique value on Γ that satisfies EF, the WBCC, and NPO.*

[5] For the definition of the proportional-extended game, see the proof of Theorem 3 in the Appendix.

Similar to Theorem 2, the w-weighted Shapley value is also a unique value on Γ, satisfying EF, the WBCC for r players, and NPO. Moreover, these results also hold for restricted domains of games such as Γ^M, Γ^S and Γ^C.

The w-weighted egalitarian value ψ^w is defined by

$$\psi_i^w(N,v) = \frac{w_i}{\sum_{j\in N} w_j} v(N),$$

for each $i \in N$. When $w_i = w_j$ for all $i, j \in N$, the w-weighted egalitarian value coincides with the egalitarian value.

A *w-weighted proportional player* in (N, v) is a player $k \in N$ satisfying $v(S \cup k) - v(S) = \frac{w_k}{\sum_{j\in S} w_j} v(S)$ for all $S \subseteq N \setminus k$ with $S \neq \emptyset$. The following is a weighted version of PPO.

Weighted proportional player out (WPPO): For any $(N, v) \in \Gamma$, if $k \in N$ is a w-weighted proportional player in (N, v), then for any $i \in N \setminus k$,

$$\varphi_i(N, v) = \varphi_i(N \setminus k, v).$$

The following theorem holds.

Theorem 6. *The w-weighted egalitarian value is the unique value on Γ that satisfies EF, the WBCC, and WPPO.*

7 Concluding remarks

Except the Shapley and egalitarian values, there are several famous values that satisfy BCC such as the CIS value and the ENSC (egalitarian non-separable contribution) value (Driessen and Funaki 1991). The CIS value is characterized in a similar manner as we did in the note, while the ENSC value is not. For the case of the CIS value, we focus on the player $k \in N$ satisfying $v(S \cup k) - v(S) = \frac{1}{|S|}(v(S) - \sum_{i\in S} v(i))$ for all non-empty subset $S \subseteq N \setminus k$. EF, the BCC (or the BCC for r players) and the property that the elimination of the above-mentioned player does not affect the value of the other players, characterize the CIS value. For the case of the ENSC value, we can show that there exists no player whose elimination does not affect the value of the other players.

Between Theorems 1 and 3, the player on which we pay attention is different. Hence, we cannot generalize our results to α-egalitarian Shapley values (Joosten 1996), which are convex combinations of the egalitarian and Shapley values. If we pay attention to only null players and focus on the effect of the elimination of a null player in each value, all values we mention here (including the ENSC value) and all their convex combinations are characterized.

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References

Abdulkadiroğlu A, and Sönmez T (2003) School Choice: A Mechanism Design Approach, The American Economic Review 93:729–747

Banzhaf JF (1965) Weighted voting does not work: A mathematical analysis. Rutgers Law Review 19:317–343

Derks JJM, Haller HH (1999) Null players out? linear values for games with variable supports. International Game Theory Review 1:301–314

Driessen T, Funaki Y (1991) Coincidence of and collinearity between game theoretic solutions. OR Spektrum 13:15–30

Joosten R (1996) Dynamics, equilibria, and values. PhD dissertation, Faculty of Economics and Business Administration, Maastricht University. the Netherlands

Myerson RB (1980) Conference structures and fair allocation rules. International Journal of Game Theory 9:169–182

Shapley LS (1953) A value for n-person games. In: Roth AE (ed.) The Shapley value, Cambridge University Press pp. 41–48

Shapley LS, Scarf H (1974) On cores and indivisibility, Journal of Mathematical Economics 1:23–37

Tijs S (1987) An axiomatizations of the τ-value. Mathematical Social Sciences 13:177–181

Appendix: Proofs

Proof of Proposition 1. It is sufficient to show that the BCC implies the GBCC. Suppose that φ satisfies BCC. We will show that φ satisfies the balancedness of the cycle contributions for any coalition S with the fixed size s, inductively.

Consider the case for $s = 3$. When $n = 3$, the result is obvious. Hence, we consider the case where $n \geq 4$. For any $(N, v) \in \Gamma$ with $n \geq 4$, let $\{i, j, k\} \subseteq N$ and $a = (a_1, a_2, \ldots, a_m)$ be an order on the set $N \backslash \{i, j, k\}$. Hence, (i, j, k, a) is an order on N.

By BCC with respect to an order (i, j, k, a) on N,

$$\begin{aligned} &\varphi_i(N\backslash a_m, v)+\varphi_j(N\backslash i, v)+\varphi_k(N\backslash j, v)+\varphi_{a_1}(N\backslash k, v)+\cdots+\varphi_{a_m}(N\backslash a_{m-1}, v)\\ &= \varphi_i(N\backslash j, v)+\varphi_j(N\backslash k, v)+\varphi_k(N\backslash a_1, v)+\varphi_{a_1}(N\backslash a_2, v)+\cdots+\varphi_{a_m}(N\backslash i, v). \end{aligned} \tag{2}$$

By BCC with respect to an order (i, k, j, a),

$$\begin{aligned} &\varphi_i(N\backslash a_m, v)+\varphi_k(N\backslash i, v)+\varphi_j(N\backslash k, v)+\varphi_{a_1}(N\backslash j, v)+\cdots+\varphi_{a_m}(N\backslash a_{m-1}, v)\\ &= \varphi_i(N\backslash k, v)+\varphi_k(N\backslash j, v)+\varphi_j(N\backslash a_1, v)+\varphi_{a_1}(N\backslash a_2, v)+\cdots+\varphi_{a_m}(N\backslash i, v). \end{aligned} \tag{3}$$

eq.(2) $-$ eq.(3) equals

$$\begin{aligned} &\varphi_i(N \setminus k, v) + \varphi_j(N \setminus i, v) + 2\varphi_k(N \setminus j, v) + \varphi_{a_1}(N \setminus k, v) - \varphi_{a_1}(N \setminus j, v)\\ &= \varphi_i(N \setminus j, v) + 2\varphi_j(N \setminus k, v) + \varphi_k(N \setminus i, v) + \varphi_k(N \setminus a_1, v) - \varphi_j(N \setminus a_1, v). \end{aligned} \tag{4}$$

Similarly, by BCC with respect to two orders (j, k, i, a) and (j, i, k, a), we obtain

$$\begin{aligned}&\varphi_j(N \setminus i, v) + \varphi_k(N \setminus j, v) + 2\varphi_i(N \setminus k, v) + \varphi_{a_1}(N \setminus i, v) - \varphi_{a_1}(N \setminus k, v)\\ &= \varphi_j(N \setminus k, v) + 2\varphi_k(N \setminus i, v) + \varphi_i(N \setminus j, v) + \varphi_i(N \setminus a_1, v) - \varphi_k(N \setminus a_1, v),\end{aligned} \tag{5}$$

and by BCC with respect to two orders (k, i, j, a) and (k, j, i, a), we obtain

$$\begin{aligned}&\varphi_k(N \setminus j, v) + \varphi_i(N \setminus k, v) + 2\varphi_j(N \setminus i, v) + \varphi_{a_1}(N \setminus j, v) - \varphi_{a_1}(N \setminus i, v)\\ &= \varphi_k(N \setminus i, v) + 2\varphi_i(N \setminus j, v) + \varphi_j(N \setminus k, v) + \varphi_j(N \setminus a_1, v) - \varphi_i(N \setminus a_1, v).\end{aligned} \tag{6}$$

$\frac{1}{4}$(eq.(4) + eq.(5) + eq.(6)) equals

$$\varphi_j(N \setminus i, v) + \varphi_k(N \setminus j, v) + \varphi_i(N \setminus k, v) = \varphi_j(N \setminus k, v) + \varphi_k(N \setminus i, v) + \varphi_i(N \setminus j, v).$$

Thus, the balancedness of the cycle contributions holds for any coalition S with $s = 3$.

From the above results, we can present the case for $s = 4, \ldots, n-1$, inductively. Let $S = \{i_1, i_2, \ldots, i_s\}$. Adding the two equalities obtained from the GBCC for the two sets $\{i_1, i_2, \ldots, i_{s-1}\}$ and $\{i_1, i_{s-1}, i_s\}$, and rearranging it results in the equality of GBCC for S. □

Proof of Proposition 2: Let a value φ satisfy the BCC. Note that both conditions are trivially satisfied for any game with two players or less. Moreover, for any three players game, the BCC and the BCC for three players are equivalent. Thus, we consider the case $|N| \geq 4$.

First, we show that the BCC implies the BCC for three players. By Proposition 1, the BCC and the GBCC are equivalent and it is obvious that the GBCC implies the BCC for three players. Thus, the desired result is obtained.

Next, we show that the BCC for three players implies the BCC. Let a value φ satisfy the BCC for three players. For any $(N, v) \in \Gamma$ with $|N| \geq 4$, consider an order $(i_1, i_2, \ldots, i_{|N|})$ on N. By the BCC for three players with respect to (i_1, i_2, i_3),

$$\begin{aligned}&\varphi_{i_1}(N \setminus i_3, v) + \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v)\\ &\qquad = \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_1, v).\end{aligned} \tag{7}$$

By the BCC for three players with respect to (i_1, i_3, i_4),

$$\begin{aligned}&\varphi_{i_1}(N \setminus i_4, v) + \varphi_{i_3}(N \setminus i_1, v) + \varphi_{i_4}(N \setminus i_3, v)\\ &\qquad = \varphi_{i_1}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_1, v).\end{aligned} \tag{8}$$

eq.(7) + eq.(8) equals

$$\begin{aligned}&\varphi_{i_1}(N \setminus i_4, v) + \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v) + \varphi_{i_4}(N \setminus i_3, v)\\ &\qquad = \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_1, v).\end{aligned} \tag{9}$$

Next, by the BCC for three players with respect to (i_1, i_4, i_5),

$$\begin{aligned}&\varphi_{i_1}(N \setminus i_5, v) + \varphi_{i_4}(N \setminus i_1, v) + \varphi_{i_5}(N \setminus i_4, v)\\ &\qquad = \varphi_{i_1}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_5, v) + \varphi_{i_5}(N \setminus i_1, v).\end{aligned} \tag{10}$$

eq.(9) + eq.(10) equals

$$\varphi_{i_1}(N \setminus i_5, v) + \varphi_{i_2}(N \setminus i_1, v) + \varphi_{i_3}(N \setminus i_2, v) + \varphi_{i_4}(N \setminus i_3, v) + \varphi_{i_5}(N \setminus i_4, v)$$
$$= \varphi_{i_1}(N \setminus i_2, v) + \varphi_{i_2}(N \setminus i_3, v) + \varphi_{i_3}(N \setminus i_4, v) + \varphi_{i_4}(N \setminus i_5, v) + \varphi_{i_5}(N \setminus i_1, v).$$

Repeating a similar argument with respect to $(i_1, i_5, i_6), \ldots, (i_1, i_{|N|-1}, i_{|N|})$, we obtain

$$\sum_{r=1}^{n} \varphi_{i_r}(N \setminus i_{r-1}, v) = \sum_{r=1}^{n} \varphi_{i_r}(N \setminus i_{r+1}, v).$$

□

Proof of Proposition 3: First, we show (i). By (i) of Proposition 2, if φ satisfies the BCC, it also satisfies the BCC for three players. By the proof of (ii) of Proposition 2, it is clear that "if φ satisfies the BCC for three players, it satisfies the BCC for r players with respect to $r \geq 4$ when there are r or more players." If there are less than r players, it is trivial that φ satisfies the BCC for r players. Hence, (i) is obtained.

Next, we show (ii). Let $r \geq 4$. If φ satisfies the BCC for r players, and there are more than r players in a game, let $i, j, k \subseteq N$ and $a = (a_1, a_2, \ldots, a_{r-3})$ be an order on $S \subseteq N \backslash \{i, j, k\}$ satisfying $|S| = r - 3$. Applying BCC for r players with respect to orders (i, j, k, a), (i, k, j, a), (j, k, i, a), (j, i, k, a), (k, i, j, a), (k, j, i, a) on $S \cup \{i, j, k\}$, we have that φ satisfies the BCC for three players, i, j and k, as shown in the proof of (i) of Proposition 2. Thus, by (ii) of Proposition 2, it satisfies the BCC. □

Since Proposition 2 means the equivalence of the BCC and the BCC for three players, Theorem 1 is obtained as a corollary of Theorem 2. However, here we prove both theorems independently because it enables us to understand how the BCC and the BCC for r players prove the uniqueness of values.

Proof of Theorem 1: We have already shown that the Shapley value satisfies EF, the BCC and NPO. Hence, it is sufficient to show the uniqueness.

Let φ be a value on Γ satisfying the three properties. We show the uniqueness of the value by the induction with respect to the number of players.

When $|N| = 1$, by EF, $\varphi_i(N, v) = v(i)$ for $i \in N$.

Assume that, φ is uniquely determined, for any game with less than n players. We show that $\varphi(N, v)$ is uniquely determined when $N = \{1, 2, \ldots, n\}$.

Take any integer $k \in \mathbb{N} \setminus N$. Then, the null-extended game (N', w) of a game (N, v) with respect to k is defined as follows:

$$N' = N \cup k,$$

and for any $S \subseteq N'$,

$$w(S) = v(S \setminus k).$$

Clearly, k is a null player in (N', w) and $(N' \setminus j, w)$ for any $j \in N$. In addition, $(N' \setminus k, w) = (N, v)$ and $(N' \setminus \{j, k\}, w) = (N \setminus j, v)$ for any $j \in N$.

Consider an order $(1, k, 2, \ldots, n)$ on N'. By the BCC,

$$\varphi_1(N' \setminus n, w) + \varphi_k(N' \setminus 1, w) + \cdots + \varphi_n(N' \setminus (n-1), w)$$
$$= \varphi_1(N' \setminus k, w) + \varphi_k(N' \setminus 2, w) + \cdots + \varphi_n(N' \setminus 1, w). \quad (11)$$

By EF and NPO, $\varphi_k(N' \setminus 1, w) = \varphi_k(N' \setminus 2, w) = 0$. By NPO, $\varphi_i(N' \setminus j, w) = \varphi_i(N \setminus j, v)$ for any $\{i, j\} \subseteq N$. Therefore, (11) is equal to the following:

$$\begin{aligned}\varphi_1(N \setminus n, v) + \varphi_2(N, v) + \cdots + \varphi_n(N \setminus (n-1), v)& \\ = \varphi_1(N, v) + \varphi_2(N \setminus 3, v) + \cdots + \varphi_n(N \setminus 1, v),&\end{aligned}$$

or,

$$\begin{aligned}\varphi_1(N, v) - \varphi_2(N, v) = -\varphi_2(N \setminus 3, v) - \cdots - \varphi_n(N \setminus 1, v)& \\ + \varphi_1(N \setminus n, v) + \varphi_3(N \setminus 2, v) + \cdots + \varphi_n(N \setminus (n-1), v).&\end{aligned}$$

Let b_1 be the right-hand side of the above equation. By the induction hypothesis, b_1 is uniquely determined.

Applying the similar argument to the orders $(1, 2, k, 3, \ldots, , n)$, $(1, 2, 3, k, 4, \ldots, n)$, $\ldots$, and $(1, 2, \ldots, n-1, k, n)$, we obtain the following $(n-1)$ equations:

$$\begin{aligned}\varphi_1(N, v) - \varphi_2(N, v) &= b_1, \\ \varphi_2(N, v) - \varphi_3(N, v) &= b_2, \\ &\vdots \\ \varphi_{n-1}(N, v) - \varphi_n(N, v) &= b_{n-1},\end{aligned}$$

By EF,

$$\varphi_1(N, v) + \varphi_2(N, v) + \cdots + \varphi_n(N, v) = v(N).$$

Since these n equations are linear independent, $\varphi(N, v)$ is uniquely determined. □

Proof of Theorem 2: Let $(N, v) \in \Gamma$. Since we know that the Shapley value satisfies EF, NPO and the BCC for r players, we show that the value φ satisfying the three axioms is uniquely determined. In case $|N| = 1$, EF implies $v(i) = \psi_i(i, v)$. Consider $|N| \geq 2$. In what follows, we show that "if $|N| \geq 2$ and φ satisfies EF, NPO and BCC for r players, then it must satisfy the balanced contributions property introduced by Myerson (1980)."

Take any integer $k_1, k_2, \ldots, k_{r-2} \in \mathbb{N} \setminus N$, and let $K = \{k_1, k_2, \ldots, k_{r-2}\}$. Then, the null-extended game (N', w) of a game (N, v) with respect to K is defined as follows:

$$N' = N \cup K,$$

and for any $S \subseteq N'$,

$$w(S) = v(S \setminus K).$$

Clearly, $|N'| \geq r$ and each $k \in K$ is a null player in (N', w) and its any restriction (N'', w) where $N'' \subseteq N'$ such that $k \in N'$. In addition, $(N' \setminus K, w) = (N, v)$.

Take any $i, j \in N \subseteq N'$ and consider an order $(i, j, k_1, k_2, \ldots, k_{r-2})$. By the BCC for r players,

$$\begin{aligned}&\varphi_i(N' \setminus k_{r-2}, w) + \varphi_j(N' \setminus i, w) + \varphi_{k_1}(N' \setminus j, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus k_{r-3}, w) \\ &= \varphi_i(N' \setminus j, w) + \varphi_j(N' \setminus k_1, w) + \varphi_{k_1}(N' \setminus k_2, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus i, w).\end{aligned} \tag{12}$$

By EF ad NPO, eq.(12) equals

$$\varphi_i(N, v) + \varphi_j(N \setminus i, v) = \varphi_i(N \setminus j, v) + \varphi_j(N, v)$$

$$\iff \varphi_i(N,v) - \varphi_i(N \setminus j, v) = \varphi_j(N,v) - \varphi_j(N \setminus i, v).$$

$\square$

Proof of Theorem 3: The proof is similar to that of Theorem 1. The difference between the two proofs is that in the proof of Theorem 3, we consider the proportional-extended game (N', w) of a game (N, v) with respect to $k \in \mathbb{N} \setminus N$ defined as follows:

$$N' = N \cup k,$$

and for any $S \subseteq N'$,

$$w(S) = \begin{cases} 0 & \text{if } S = \{k\} \\ \frac{|S|}{|S| - |S \cap k|} v(S \setminus k) & \text{otherwise.} \end{cases}$$

Clearly, k is a proportional player in (N', w) and $(N' \setminus j, w)$ for any $j \in N$. In addition, $(N' \setminus k, w) = (N, v)$ and $(N' \setminus \{j, k\}, w) = (N \setminus j, v)$ for any $j \in N$.

Assume that φ is the egalitarian value, if there are less than or equal to $n-1$ players. Consider an order $(1, k, 2, \dots, n)$ on N'. By BCC, EF, PPO, the induction hypothesis and the definition of the proportional-extended game, we have

$$\varphi_1(N, v) - \varphi_2(N, v) = 0.$$

Applying a similar argument to the orders $(1, 2, k, 3, \dots, n), \dots, (1, 2, \dots, n-1, k, n)$, we have $\varphi_1(N, v) = \varphi_2(N, v) = \cdots = \varphi_n(N, v)$. By EF, we conclude $\varphi_i(N, v) = v(N)/n$ for all $i \in N$. $\square$

Proof of Theorem 4: We show the uniqueness of the value by the induction with respect to the number of players.

When $|N| = 1$, by EF, $\varphi_i(N, v) = v(i)$ for $i \in N$. Let $(N, v) \in \Gamma$, where $|N| = n$. Assume that, for any game with less than n players, φ is the egalitarian value.

Take any integer $k_1, k_2, \dots, k_{r-2} \in \mathbb{N} \setminus N$, and let $K = \{k_1, k_2, \dots, k_{r-2}\}$. Then, the proportional-extended game (N', w) of a game (N, v) with respect to K is defined as follows:

$$N' = N \cup K,$$

and for any $S \subseteq N'$,

$$w(S) = \begin{cases} 0 & \text{if } S \subseteq K, \\ \frac{|S|}{|S| - |S \cap K|} v(S \setminus K) & \text{otherwise.} \end{cases}$$

Then, each $k \in K$ is a proportional player in (N', w) since for any $k \in K$ and any $S \subseteq N' \setminus k$, if $S \subseteq K$, then $w(S \cup k) = w(S) = 0$, and otherwise,

$$\begin{aligned} w(S \cup k) - w(S) &= \frac{|S| + 1}{(|S| + 1) - |(S \cup k) \cap K|} v((S \cup k) \setminus K) - \frac{|S|}{|S| - |S \cap K|} v(S \setminus K) \\ &= \frac{1}{|S| - |S \cap K|} v(S \setminus K) = \frac{1}{|S|} w(S). \end{aligned}$$

Similarly, each $k \in K$ is a proportional player in any restricted game (N'', w), where $N'' \subseteq N'$ and $k \in N''$. In addition, $(N' \setminus K, w) = (N, v)$.

Take any $i, j \in N \subseteq N'$, and consider an order $(i, j, k_1, k_2, \ldots, k_{r-2})$. By the BCC for r players,

$$\varphi_i(N' \setminus k_{r-2}, w) + \varphi_j(N' \setminus i, w) + \varphi_{k_1}(N' \setminus j, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus k_{r-3}, w)$$
$$= \varphi_i(N' \setminus j, w) + \varphi_j(N' \setminus k_1, w) + \varphi_{k_1}(N' \setminus k_2, w) + \cdots + \varphi_{k_{r-2}}(N' \setminus i, w).$$

Repeatedly applying PPO, $\varphi_i(N' \setminus k_{r-2}, w) = \varphi_i(N, v)$, $\varphi_j(N' \setminus k_1, w) = \varphi_j(N, v)$, $\varphi_i(N' \setminus j, w) = \varphi_i(N \setminus j, v)$, and $\varphi_j(N' \setminus i, w) = \varphi_j(N \setminus i, v)$. By EF and PPO, $\varphi_{k_1}(N' \setminus j, w) = \frac{1}{|N'|-2} w(N' \setminus \{j, k_1\})$. Moreover, by repeatedly applying the definition of proportional players, $\frac{1}{|N'|-2} w(N' \setminus \{j, k_1\}) = \frac{v(N \setminus j)}{n-1}$. Similarly, $\varphi_{k_p}(N' \setminus k_{p-1}, w) = \frac{v(N)}{n}$ for each $p = 2, \ldots, r-2$, and $\varphi_{k_p}(N' \setminus k_{p+1}, w) = \frac{v(N)}{n}$ for each $p = 1, \ldots, r-3$.

Thus, the above equation can be reduced to

$$\varphi_i(N, v) + \varphi_j(N \setminus i, v) + \frac{v(N \setminus j)}{n-1} + (r-3)\frac{v(N)}{n}$$
$$= \varphi_i(N \setminus j, v) + \varphi_j(N, v) + \frac{v(N \setminus i)}{n-1} + (r-3)\frac{v(N)}{n}$$

Moreover, by the induction hypothesis, the above equation is

$$\varphi_i(N, v) = \varphi_j(N, v).$$

Together with EF, we have the desired result. □

Proofs of Theorems 5 and 6: The proofs of Theorems 5 and 6 are almost similar to the proofs of Theorems 1 and 3. □