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Maximin-optimal sustainable growth in a resource-based imperfect economy

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Maximin-optimal sustainable growth in a resource-based imperfect economy

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Abstract

I offer an approach linking a welfare criterion to the opportunities for sustainable development in an imperfect economy. The approach implies a dependence of the criterion on the economy's current state. The economy-linked criterion is constructed using an example with the maximin principle applied to a hybrid level-growth measure. This measure includes as special cases the conventional measures of consumption level and percent change as a measure of growth. The hybrid measure or geometrically weighted percent can be used for measuring sustainable growth as an alternative to percent. The problem is considered for the Dasgupta-Heal-Solow-Stiglitz model. Closed form solutions are obtained for the optimal paths including the paths dynamically consistent with the updates in reserve estimates.

Key words:
imperfect economy; dynamic preferences; essential nonrenewable resource; geometrically weighted percent; normative resource peak
JEL : O13; O47; Q32; Q38

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1. Introduction

Koopmans (1964) claimed that "a decision maker should wish to retain some flexibility with regard to his future preference ... to be able to make consistent responses to hypothetical choice situations." He argued that preferences should be adjusted to economic opportunities, "viewing physical assets as opportunities" (Koopmans 1964, p. 253), and that the economic specificity of normative problems "imposes mathematical limits on the autonomy of ethical thought" (Koopmans 1965, p. 254). Koopmans illustrated in a simple model with discounted criterion that the optimal path could not exist depending on the choice of the discount factor (preferences). His result implied that "ignoring realities in adopting 'principles' may lead one to search for a nonexistent optimum, or to adopt an 'optimum' that is open to unanticipated objections" (Koopmans 1965, p. 229). For example, it is known that sustainability of consumption over time depends on the initial value of capital for the maximin programs (Leininger 1985). A more recent example of unacceptable consequences of using a criterion that is "not adjusted to opportunities" can be found by applying the results of Stollery (1998) to an imperfect economy. Stollery examined the problem of a resource-extracting economy causing global warming and following the constant-utility path. One can easily check that this criterion is not compatible with the Cobb-Douglas technology in an imperfect economy when the initial state is not optimal. Plausible initial states (for example, constant extraction during an initial period) imply, in this framework, unsustainable extraction, rapid growth of temperature, and collapse of the economy. Arrow, Dasgupta and Mäler (2003) define imperfect economies as the "economies suffering from weak, or even bad, governance" (p. 648). Imperfection can also result from imperfect knowledge, say in justice theory or in information about the resource reserves, even when the decisions of a social planner are "perfect."

The phenomena of imperfection can arise, for example, when a social planner of a resource-based economy uses a recent theoretical result that implies the optimal initial rate of extraction, while the economy has been already extracting the resource for some time, and the current rate of extraction does not coincide with the optimal one. In this sense, a real economy cannot be perfect since it is unrealistic to expect a social planner to make a "perfect" long-run planning decision under the uncertainties in development of knowledge and/or in reserve estimates. Technically, the problem is connected with the derivation of the optimal path of extraction from a first-order differential equation that arises from the optimality conditions, implying that only one piece of information about the resource can be used in order to define uniquely the optimal path. The conventional approach uses the initial conditions of state variables (capital, resource reserve) and yields the initial rate of extraction as optimal or equilibrium. Then the path is inconsistent with almost any real pattern of extraction. On the other hand, defining the path by the initial rate of extraction will result in inconsistency with the reserve, i.e., in inefficiency or in unsustainable resource use. This contradiction can be resolved by using Koopmans's idea to express preferences in terms of existing opportunities.

In this paper, I introduce preferences in a general form that depends on a parameter, which is linked to the initial resource reserve, while the path of extraction is defined by the initial rate. This approach yields optimal sustainable (in a weak sense) paths that are efficient and, at the same time, consistent with the initial state of a specific imperfect economy. I assume here, following Koopmans, that "the *initial opportunity* is given by objective circumstances of technology and resources ... independently of the ordering" (Koopmans 1964, p. 251). This approach to formulation of a criterion is consistent with Bellman's Principle of Optimality, which claims that "an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman 1957 p. 83). Koopmans (1964) concentrated his attention only on uncertainty in the planner's future preferences, assuming complete certainty in physical assets of the economy. I consider here an example with a general form of preferences (criterion) that depends on reserve parametrically, while the reserve can be reevaluated over time. This approach implies flexibility of future preferences with respect to unpredictable changes in reserves.

The approach can be extended to defining an optimal path of saving consistent with the current pattern in an imperfect economy. For simplicity, I consider here only the problem of socially optimal extraction assuming that the pattern of saving is given since, as it is known, the path of resource extraction can be more important for sustainability than the pattern of saving (Bazhanov 2008b).

The economy-linked criterion is constructed here using an example of the maximin principle applied to a hybrid level-growth utility measure that I call "geometrically weighted percent." The use of the maximin in the problems of intergenerational justice implies that a social welfare measure should be maintained constant over time. Therefore, it is natural to use this convenient property of the maximin for formulating long-run programs of sustainable development.[1] The hybrid measure, to which the maximin is applied in this paper, includes as particular cases the level of consumption and the rate of growth. In the general case, geometrically weighted percent includes all intermediate forms for measuring the level and/or the rate of growth of consumption. This family of measures implies a corresponding family of patterns of optimal growth that can vary from stagnation to quasiarithmetic (including linear) and exponential growth. Using this approach, I answer the question: what is the "best" pattern of growth from this family that a specific imperfect economy with the given initial conditions can maintain forever? The approach differs from the conventional methodology in resource economics in that usually the optimal economy is being constructed under the given criterion.

The closed form solutions are obtained for the extended Dasgupta-Heal-

[1] Solow (1974) applied the (Rawls 1971) maximin to the level of consumption as a simple social welfare measure that implied the constant-per-capita-consumption criterion. On the other hand, there is a conventional practice to formulate some long-run development goals in terms of constant percent change of GDP (e.g., World 1987, p. 169, p. 173). I use here the words "growth" and "development" as synonyms because I consider an aggregate model of a growing economy. Growing aggregate capital includes, in the general case, new technologies implying that the economy is developing while per capita consumption is growing.

Solow-Stiglitz (DHSS) model (Dasgupta and Heal 1974; Solow 1974; Stiglitz 1974) with an essential nonrenewable resource under the standard Hartwick investment rule (Hartwick 1977). The extension is that the Hotelling Rule is modified by some phenomena whose total influence can be expressed in terms of an equivalent tax or subsidy.[2] It is shown that the feasible patterns of growth for this economy are between the constant consumption and quasiarithmetic growth with parameters depending on the technological properties of production function.

The paper is structured as follows. Sections 2 introduces the methodology of specification of a generalized criterion for given initial conditions; Section 3 describes the model; Section 4 provides the closed form solutions for the optimal paths in the DHSS model; Section 5 gives the condition defining the feasible patterns of sustainable growth; Section 6 examines the unacceptable consequences of applying the criterion beyond its feasible limits; Section 7 provides the numerical example; the optimal paths dynamically consistent with the updates in reserve estimates are constructed in Section 9. The conclusion is offered in Section 10.

2. The maximin variant of an economy-linked criterion

Criterion that is not adjusted to the economic opportunities can imply a nonexistent or unsustainable optimal path in a specific economy. For example,

[2] There is extensive literature on a discrepancy between the standard Hotelling Rule and the observed data. The Rule implies that the path of the resource extraction must be decreasing and the resource price must grow at the rate of interest. However, this is not the case in the real economy (e.g., Gaudet 2007). Gaudet (2007) considered different phenomena such as changes in the cost of extraction, durability, peculiarities of the market, and uncertainties. These phenomena can influence both the price dynamics and the paths of extraction, but they were not considered by Hotelling in his seminal paper (1931). Therefore, introduction of these effects into the model of Hotelling can reconcile his approach with empirical data for different kinds of resources, including oil.

the constant GDP percent change implies exponential growth that cannot be sustained infinitely under the assumptions of essential nonrenewable resource and a plausible pattern of technical change (Dasgupta and Heal 1979). Stollery (1998) considered another example combining consumption c in the constant-utility criterion $U(c,T) = c^{1-\gamma}T^{-1}/(1-\gamma) = const$ with the global temperature that rises exponentially $T(t) = T_0 \exp\left[\phi \int_0^t r(\xi) d\xi\right]$ with the resource extraction r. Assume that a social planner applies this criterion in an imperfect economy with constant extraction during a small initial period.[3] Then the criterion requires that T and c must grow exponentially over time, which is not possible, say, for the Cobb-Douglas production function with constant extraction. This combination implies unsustainable behavior of the economy unless the rate of extraction is very low and declines in the initial period. In Bazhanov (2008a), one more example demonstrated that an imperfect economy can enter an inferior path if it follows a criterion that is not linked to the "opportunities" of the economy expressed in the properties of production function and the initial state.

In order to avoid these unacceptable consequences, I construct here the economy-linked criterion using an example of the maximin principle applied to a generalized level-growth utility measure.[4] The use of the maximin in the problems of intergenerational justice implies that a social welfare measure must be constant over time. Therefore, it is natural to use this criterion for formulating the long-run programs of sustainable development.[5]

Solow (1974) showed that the maximin applied to the *level* of consumption implies constant consumption and no growth in output. The same approach can be applied to a more general measure that takes into account not only the *level*

[3] This pattern of r is close to the current path of the per capita world's oil extraction.

[4] This approach was also considered in Bazhanov (2007).

[5] One can claim that the overall wealth of an economy could be higher as a result of the alternate ups and downs; however, I will stick here to the evidence that "loss aversion favors social arrangements that provide a steady improvement of rewards or benefits over time, in preference to schedules in which the same total benefit is handed out in equal or diminishing quantities" (Kahneman and Varey 1991, p. 152).

of consumption but also the rate of its change.[6] As an example of a generalized measure of consumption, consider the one that includes as the specific cases conventional measures for the *level* or for the *growth* of consumption, depending on the values of parameters. Then the values of these parameters can be estimated for the "initial opportunities" in a specific imperfect economy. As an example of the imperfect economy, I use the DHSS economy with a nonrenewable resource, externality, and the tax, internalizing the externality in the optimal way. The closed form solutions for this economy are provided in Lemma 1, Proposition 1, and Corollary 1 (Section 4).

The expression $\dot{c}^{\gamma}c^{\mu}$ is considered here as an example of a hybrid level-growth measure. The maximin applied to this expression implies that already this expression, not consumption per se, must be kept constant over time. Assume for simplicity that $\mu = 1 - \gamma$. Then a variant of the constant-utility criterion or the criterion of just growth[7] of consumption has the form[8] of

$$\dot{c}^{\gamma}c^{1-\gamma} = \overline{U} = const, \tag{1}$$

[6]There are findings supporting the idea that, for estimating a consumer's perception of consumption and, consequently, the utility, it is not enough to calculate a vector of measurable static indicators. Lecomber (1979) noted that "people become accustomed to rising living standards and are dissatisfied with static ones" (p. 33). Scanlon (1991) further mentioned that "we can ask ... how well a person's life is going and whether that person is ... better off than he or she was a year ago" (p. 18). There is also evidence that has "documented the claim that people are relatively insensitive to steady states, but highly sensitive to changes" and that "the main carriers of value are gains and losses rather than overall wealth" (Kahneman and Varey 1991, p. 148). Here I take into account prehistory of consumption in the form of derivative $\dot{c}$.

[7]For $\gamma > 0$ this version of the criterion is applicable only to growth ($\dot{c} > 0$) because at the steady states ($\dot{c} = 0$) the expression $\dot{c}^{\gamma}c^{\mu}$ is always zero (not sensitive to the *level* of consumption).

[8]An additive form of a generalized measure $c + \gamma\dot{c}$ was considered by N.V. Long (2007, p. 303) in the problem of deriving an optimal pattern of saving. Long interpreted $\dot{c}$ as sympathy to future generation, while I treat it as person's (society's) consumption prehistory affecting the perception (utility) from the current consumption level c.

which yields the quasiarithmetic pattern of growth

$$c(t) = c_0(1 + \varphi t)^{\gamma}, \tag{2}$$

where $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$. Form (1) of no-envy principle can be rewritten as follows: $(\dot{c}/c)^{\gamma} c = \overline{U}$, which means that the decline in the *rate* of growth $\dot{c}/c$ is compensated by the growing *level* of consumption c.

In this approach, the question of justice between generations is measured in terms of parameters γ and $\overline{U}$. As Arrow, Dasgupta and Mäler (2003) put it: "Although intergenerational equity is nearly always discussed in terms of the rate at which future well-being is discounted ... equity would be more appropriately discussed in terms of the curvature of U" (p. 660).

Note that criterion (1) includes constant consumption as a specific case for $\gamma = 0$; more general expression $\dot{c}^{\gamma} c^{\mu}$ includes also the following:

(a) conventional function for measuring the utility of the *level* of growing with no limit consumption $c^{1-\eta}/(1-\eta)$ for $\gamma = 0$, $\mu = 1 - \eta$, and $\overline{U} = \widehat{U}(1-\eta)$;

(b) percent change as a conventional measure of the *growth* of consumption for $\gamma = 1$ and $\mu = -1$;

(c) a sample value function that relates value to an initial consumption c and to a change of consumption $\dot{c}$ (Kahneman and Varey 1991, p. 157): $V(\dot{c}, c) = b\dot{c}^{a}/c$ for $\dot{c} > 0$, where $a < 1$ and $b > 0; V(0, c) = 0; V(\dot{c}, c) = -Kb(-\dot{c})^{a}/c$ for $\dot{c} < 0$, where $K > 1$.

The important property of criterion (1) is that it allows for the growth of an economy and that the parameter γ can be specified for the economy's initial reserve while the path of extraction can be defined by the initial rate. This property means that the criterion can be used in numerical examples that resemble the behavior of real imperfect economies. The importance of the mechanism of adjusting the criterion to the context has been emphasized, for example, in Konow (2003): "The most significant challenge to ... any theory ... is to incorporate the impact of context on justice evaluation, and much work remains in this regard."

3. The model

The analysis is based on an example of a decentralized economy represented by the conventional, in this case, DHSS model with some externalities and government interventions expressed in a general form. I assume zero population growth, zero extraction cost, and the Cobb-Douglas technology[9]

$$q(t) = f(k(t), r(t)) = k^{\alpha}(t) r^{\beta}(t), \qquad (3)$$

where q - output, k - produced capital, r - current resource use, α, $\beta \in (0,1)$, $\alpha + \beta < 1$ are constants. The assumption that TFP $A(t)$ (Total Factor Productivity) exactly compensates for capital depreciation δk allows for considering the basic DHSS model with no capital depreciation and no TFP. At the same time, this assumption makes it possible to examine correctly various patterns of growth in the economy. The pattern of this specific TFP is provided in Section 8.

Without losing generality, assume that population equals unity. Then the lower-case variables are in per capita units: $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$). Prices of per capita capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$, where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$.

Solow (1974) – Hartwick (1977) approach showed that the resource rent investing rule $\dot{k} = r f_r = \beta q$ results in constant per capita consumption in this model under the standard Hotelling Rule $\dot{f}_r/f_r = f_k$. In this case, constant consumption implies linear path of capital: $k(t) = k_0 + \beta q_0 t$. Then the Hotelling Rule or the equation for $\dot{q}/q$ yields the equation for extraction: $\dot{r}/r = -\alpha q_0/k$ with the solution $r(t) = C_1(1 + At)^{-\alpha/\beta}$, where $A = \beta q_0/k_0$ and C_1 – constant

[9] There is mixed evidence about the elasticity of factor substitution between capital and resource including the results showing that this value is close to unity (Griffin and Gregory 1976; Pindyck 1979), which means that the use of the Cobb-Douglas technology is not implausible in this framework. However, plausibility is not the main reason for its use in this paper. As Asheim (2005) put it, "I do not claim that this model describes accurately ... production possibilities in the real world ... however, it is well-suited to illustrate how a small variation in the parameters ... may lead to very different consequences when combined with criteria for intergenerational justice" (p. 316).

of integration. It can be seen that $C_1 = r(0) = r_0$; however, the conventional approach defines C_1 via the initial reserve s_0 in order to make the extraction efficient: $s_0 = \int_0^\infty r(C_1, t)dt$. Integration gives $C_1 = s_0 q_0(\alpha - \beta)/k_0$. These two expressions for C_1 mean that the constant-consumption criterion requires that an economy must satisfy the "perfection condition"

$$s_0 = r_0 k_0 / [q_0(\alpha - \beta)] = r_0^{1-\beta} k_0^{1-\alpha} / (\alpha - \beta). \tag{4}$$

This condition can be interpreted in the two different ways:

1. If the owner of the resource (social planner) *has just discovered the oil field* or obtained it at an auction, then this condition gives the optimal initial extraction r_0 for the known resource reserve and initial capital. In this sense, the initial condition r_0 is considered as *"the future,"* and the economy is sustainable and efficient by construction.

2. If an economy *is already extracting the resource*, as it happens in most real economies, and the planner is going to apply the constant-consumption criterion at $t = 0$, then the initial value r_0 is treated as *"the past,"* and condition (4) shows how much reserve s_0 the economy needs in order to maintain constant consumption in the infinite horizon problem. Then the economy is either inefficient or unsustainable if actual reserve is larger or smaller than this value.

In the optimal control framework, when $r(t)$ is used as a control variable, the second situation means that the class of feasible control paths is restricted by the initial condition $r(0) = r_0$. The sources of economy imperfection, which lead to violation of condition (4) in real economies, can be conditionally divided in two groups: (a) objective and (b) subjective. Objective imperfections include uncontrollable natural processes (disasters) and imperfection of knowledge (e.g., knowledge in justice theory and knowledge about resource reserves). Subjective imperfections are connected with misgovernment.

Hence, a *problem of an imperfect extracting economy* can be formulated as follows: the optimal extraction can be either

(i) consistent with the reserve and inconsistent with the initial rate of extraction (*inapplicable* to the economy, violating Hadamard's (1902) principle requiring the continuity of a solution with respect to initial conditions for a correctly or well-posed mathematical problem) or

(ii) consistent with the initial rate of extraction and inconsistent with the reserve (applicable to the economy but either *inefficient or unsustainable*).

A solution to this problem is based here on the introduction of the parameter γ in criterion (1), which is to be linked to the reserve, defining a unique efficient and sustainable path. The implications of this approach are examined below for a specific imperfect extracting economy.

As an example of an imperfect economy with "the *initial opportunity* ... given by objective circumstances," consider the economy with the growing rates of extraction at the initial moment that is consistent with the world's oil extraction. This extraction is the result of the influence of various phenomena (including externalities and government policy)[10] that can be expressed in terms of tax $T(t)$ and that cause modification of the Hotelling Rule. This modification implies that if $p(t)$ is the "equilibrium Hotelling price" without imperfections and $f_r(t) \equiv f_r[p(t), T(t)] = p(t) + T(t)$ is the observable price with distortions, then the ratio $\dot{f}_r/f_r$ is not already equal to the rate of interest f_k.

I assume that

(1) The influence of imperfections can be compensated by institutional changes and environmental policies (Caillaud et al. 1988; Pezzey 2002), including tax in such a way that the resulting resource extraction will bring more social welfare to the economy.

(2) All the effects modifying the Hotelling Rule can be expressed in terms of equivalent amount of tax/subsidy.

For example, insecure property rights lead to shifting extraction from the future towards the present (Long 1975) or to "overexploitation" (in terms of consumption lost), which also happen in common property situations. I assume that the same effect can be obtained by subsidizing the oil-using production.

[10] For simplicity, I will call these phenomena here either "distortions," or "imperfections," or "externalities."

Thus, I will consider all the phenomena modifying the Hotelling Rule in the same terms of tax/subsidy including the subsidies themselves.[11]

The assumptions imply that, in the general case, the Hotelling Rule can be written as follows:

$$\frac{\dot{f}_r(t)}{f_r(t)} = f_k(t) + \tau(t), \tag{5}$$

where τ - distortion in terms of interest rate that aggregates the effects of the processes that cause imperfections of an economy.

A specific variant of the Hotelling Rule arrives usually from the first-order conditions in an optimal-control problem of a welfare maximization. The standard form of the Rule results, in various frameworks, from maximizing the present value of profit/utility of the owner of the resource stock by choosing the path of extraction (Dasgupta and Heal 1979, p. 157-158). A variant of modified Rule was obtained, for example, by Levhari and Liviatan (1977) in the problem of present value profit maximization when the cost of mining $C(r(t), s_0 - s(t))$ grows with cumulated extraction ($\partial C/\partial(s_0 - s) > 0$). In this case, the Rule takes the form of $[dM\pi(t)/dt]\,/M\pi(t) = f_k(t) - [\partial C/\partial(s_0 - s)]/M\pi(t)$, where $M\pi(t)$ - marginal profit. A more recent example is the case of irreversible global warming (Stollery 1998) affecting consumption and /or production. The Hotelling Rule, in this case, is $\dot{f}_r/f_r = f_k + (f_T + u_T/u_c)T_{s_0 - s(t)}/f_r$ (Hartwick 2009). Here, utility $u(c, T)$ is negatively affected ($u_T < 0$) by growing atmospheric temperature $T(s_0 - s)$, and temperature is rising due to oil use in the economy. In the case with temperature in the utility alone, the modifier is $\tau(t) = u_T T_{s_0 - s(t)}/(u_c f_r)$, which is negative when the resource is being extracted.

Assume also that the "initial opportunity" of a specific economy includes the pattern of saving, namely, that the economy follows the Hartwick saving rule. This assumption is consistent with the IMF data (world's saving, excluding the

[11] In fact, subsidies were being applied to stimulate oil use not only in the past but even today. Brown (2006) points out that "the world fossil fuel industry is still being subsidized by taxpayers at more than $210 billion per year."

U.S.A., has fluctuated between 0.24 and 0.26 of GDP since 1980). In this case, the standard Hotelling Rule ($\tau \equiv 0$) with the Hartwick rule $\dot{k} = rf_r = \beta q$ implies constant consumption over time (Hartwick 1977). In the general case, for $\tau \neq 0$, the saving rule with equation (5) implies

$$\frac{d\dot{k}}{dt} = \dot{r}f_r + r\dot{f}_r = \dot{r}f_r + r\left(f_k f_r + \tau f_r\right) \tag{6}$$

and $\dot{c} = f_k\dot{k} + f_r\dot{r} - \ddot{k}$. Substitution of (6) for $\ddot{k}$ yields the following: $\dot{c} = f_k\dot{k} + f_r\dot{r} - \dot{r}f_r - rf_kf_r - \tau rf_r = -\tau rf_r$, which goes to zero if expression $\tau/(rf_r) = \tau/(\beta q)$ goes to zero with $t \to \infty$. This means, first, that consumption is growing when $\tau < 0$; and second, that the extraction can approach a sustainable optimal path in a desirable way if there exists a corresponding tax that a social planner can apply in order to affect a path of τ, given the dependence between r and τ in the form of $\dot{r}/r = -\left[(1-w)f_k + \tau\right]/(1-\beta)$, where w is the saving rate (Bazhanov 2008b, p. 13).

Equation (6) and the saving rule imply also $\dot{f}_r/f_r = \beta\left[f_k + (\dot{r}/r)\left(1 - 1/\beta\right)\right] = f_k + \tau$ or $f_k(\beta-1) + (\dot{r}/r)(\beta-1) = \tau$ that yields

$$\alpha\frac{q}{k} + \frac{\dot{r}}{r} = \frac{\tau}{\beta - 1}. \tag{7}$$

Then

$$\frac{\dot{q}}{q} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{r}}{r} = \beta\left(\alpha\frac{q}{k} + \frac{\dot{r}}{r}\right) = \frac{\beta}{\beta - 1}\tau \tag{8}$$

implying that

1) growth of output is associated with negative $\tau(t)$ in the DHSS economy under the standard Hartwick rule;[12]

2) $\dot{q}/q \to 0$ with any $\tau(t) \to 0$.

According to the assumption, modifier $\tau(t)$ can be expressed in terms of tax/subsidy. This assumption implies that there exists a Pigovian tax $T(t)$

[12] The dependence between τ and the rate of growth in the DHSS economy is considered in Bazhanov (2008b) in more detail.

such that equation (5) takes the form[13]

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} = \frac{\dot{f}_r}{f_r} - \tau = f_k. \tag{9}$$

This equation can be rewritten as follows:

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} - \frac{\dot{f}_r - \tau f_r}{f_r} = 0$$

or, for $f_r(f_r+T) \neq 0$, it is equivalent to $\dot{f} f_r + \dot{T} f_r - \dot{f}_r f_r - T\dot{f}_r + \tau f_r(f_r+T) = 0$. The last expression (with $f_r \neq 0$) yields the dynamic condition for tax

$$\dot{T} - T f_k + \tau f_r = 0. \tag{10}$$

The general solution of (10) is

$$T(t) = e^{\int f_k(t)dt} \left[\widehat{T} - \int \tau f_r e^{-\int f_k(t)dt} dt \right]. \tag{11}$$

Solution (11) can be specified with an initial condition either for $T(0)$ or for $\dot{T}(0)$. If $T(t)$ is a new tax that compensates for imperfections and that (a) is continuous, and (b) was not applied before ($T(t) = 0$ for $t \leq 0$), then it should be assumed that $T(0) = T_0 = 0$, which yields $\dot{T}(0) = -\tau(0) f_r(0)$.

Hence, due to the link between τ and the rate of growth, a social planner can control the behavior of τ by applying such a tax, which implies optimality of the paths in the economy. The following section uses this link for obtaining the explicit solutions for the DHSS model.

4. Optimal paths in the DHSS economy

In the example below, a social planner runs an imperfect economy that has been extracting the resource for some time. At $t = 0$, the planner decides to keep the value of $\dot{c}^\gamma c^{1-\gamma}$ constant over time subject to the restricted extraction: $\int_0^\infty r(t)dt \leqslant s_0$. Production function is in the form of (3), the Hotelling Rule

[13] This dynamic efficiency condition was used, e.g., by Hamilton (1994) in the form $\dot{n}/n = f_k$ for the net rent per unit of resource $n = f_r - C - T$, where C is marginal cost of extraction.

is modified in the form of (5), and the saving rule is $\dot{k} = \beta q$. Capital, output, and consumption are nonnegative. The claim "the *initial opportunity* is given" implies that the initial values of all variables in the problem are known. In this framework, these values cannot be obtained as the optimal ones since then they could conflict with the values in a real imperfect economy. I assume that even the initial value of tax is zero (the tax is new) in order to obtain smooth continuations for all the paths in the economy, rendering them consistent with the initial state. Otherwise, discontinuous shift at $t = 0$ can change the *initial opportunity*, violating Koopmans's prerequisite, Bellman's Principle of Optimality, and Hadamard's principle of a well-posed problem. In this sense, I treat the initial state as "the past," implying that these values can be found from the last issues of some journals or from the last rows of some databases. The given initial state expresses "static" imperfection of an economy, in the sense that this state, in the general case, is not optimal with respect to a criterion not linked to the economy. Then this initial state implies the initial value of a "dynamic" imperfection expressed here in the path of the Hotelling Rule modifier. The optimal paths for the DHSS economy with the given initial conditions are provided in the following Lemma 1, Proposition 1, and Corollary 1.

Lemma 1. *For the economy $q = k^\alpha r^\beta$ with the saving rule $\dot{k} = \beta q$ and the Hotelling Rule in the form of $\dot{f}_r/f_r = f_k + \tau$, the unique path of the Hotelling Rule modifier*

$$\tau(t) = \frac{\beta - 1}{\beta} \frac{1}{\lambda_1 t + \lambda_0}$$

is socially optimal with respect to (1) with $\gamma = 1/\lambda_1$ and $\overline{U} = c_0/\lambda_0^{1/\lambda_1}$.

Proof. Condition (1) implies that $\dot{c}^\gamma c^{1-\gamma} = (1-\beta)^\gamma \dot{q}^\gamma (1-\beta)^{1-\gamma} q^{1-\gamma} = (1-\beta)\dot{q}^\gamma q^{1-\gamma} = \overline{U}$ or

$$\dot{q}^\gamma q^{1-\gamma} = \overline{U}/(1-\beta). \tag{12}$$

Equations (2) and (8) yield $q = c/(1-\beta) = c_0(1+\varphi t)^\gamma/(1-\beta)$ and $\dot{q} = \beta q \tau/(\beta - 1)$. Then equation (12) becomes

$$\left(\frac{\beta}{\beta - 1} q\tau\right)^\gamma q^{1-\gamma} = \left(\frac{\beta}{\beta - 1}\tau\right)^\gamma q = \frac{\overline{U}}{1-\beta}.$$

Substitution for q implies $[\tau\beta/(\beta-1)(1+\varphi t)]^{\gamma} = \overline{U}/c_0$ or

$$\tau = \left(\frac{\overline{U}}{c_0}\right)^{\frac{1}{\gamma}} \frac{\beta-1}{\beta}\frac{1}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{\varphi\gamma}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{1}{1/(\varphi\gamma)+t/\gamma},$$

where $\lambda_0 = 1/(\varphi\gamma)$ and $\lambda_1 = 1/\gamma$. Substitution for $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$ into the expression for λ_0 results in the formula for $\overline{U}$ via λ_0 and λ_1 ■

Proposition 1. *Let the economy $q = k^{\alpha}r^{\beta}$ follow the Hartwick rule $\dot{k} = \beta q$; the Hotelling Rule is $\dot{f}_r/f_r = f_k + \tau$, and the initial conditions are: $\dot{q}_0/q_0$ - the initial rate of growth; $q_0 = q(0) = k_0^{\alpha}r_0^{\beta}$ - the initial output, where $k_0 = k(0), r_0 = r(0)$, and $s_0 = s(0)$ are the initial values of capital, the resource extraction and the reserve estimate.*

Then the unique path of tax, introduced at $t = 0$ with $T(0) = 0$ in the following way:

$$T(t) = \beta\left\{[k(t)(1+\lambda_1)]^{\alpha} q_0^{\beta-1}\right\}^{1/\beta}\left[1-(t\lambda_1/\lambda_0+1)^{(\beta-1)/(\beta\lambda_1)}\right]$$

is socially optimal with respect to (1) with $\gamma = 1/\lambda_1$ and $\overline{U} = c_0/\lambda_0^{1/\lambda_1}$. The optimal tax implies the following paths of capital and the resource use:

$$\begin{aligned} k(t) &= k_0 + \frac{\beta q_0}{\lambda_0^{1/\lambda_1}(1+\lambda_1)}\left[(\lambda_1 t+\lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right], \\ r(t) &= \left(\frac{q_0}{\lambda_0^{1/\lambda_1}}\right)^{1/\beta}(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)} k^{-\alpha/\beta}, \end{aligned}$$

where $\lambda_0 = q_0/\dot{q}_0,\ \lambda_1 = \lambda_1(s_0)$.

Proof: Appendix 1.

The optimal paths, obtained in Proposition 1, are smooth continuations of the initial conditions. Indeed, the tax is zero at the initial moment since it is a new tax, "additional" to the already existing taxes or subsidies that are expressed in the Hotelling Rule modifier τ and in the corresponding distortion in price f_r. Another interesting property of the economy-linked solution is that the path of extraction r includes the growing multiplier $(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)}$ allowing for the growing extraction during a small period after the initial point.

Corollary 1. *In conditions of Proposition 1, the optimal path of consumption implied by (1) is*

$$c(t) = c_0 \left(1 + \frac{\dot{q}_0}{q_0} t\right)^{\gamma}$$

i.e. the optimal sustainable growth rate of consumption is defined by the initial GDP percent change $\dot{q}_0/q_0$ *and* $\gamma = 1/\lambda_1(s_0)$;

the expression for the Hotelling Rule is

$$\dot{f}_r(t)/f_r(t) = f_k(t) + \frac{\beta - 1}{\beta} \frac{1}{\lambda_1(s_0)t + q_0/\dot{q}_0},$$

where $\lambda_1(s_0)$ *is uniquely defined from the equation*

$$\begin{aligned} s_0 \quad = \quad & \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ & \times \left\{1 + (1 - \beta)\left(k_0 \beta \dot{q}_0 (\lambda_1 + 1) - \beta^2 q_0^2\right)\right. \qquad (13) \\ & \left. \times\ {}_2F_0 \left(\left[1, \frac{\beta(\lambda_1 + 2) - 1}{\beta(\lambda_1 + 1)}\right], [\,], \left(\beta q_0^2 - k_0 \dot{q}_0 (\lambda_1 + 1)\right)(\lambda_1 + 1)\beta^2\right)\right\}, \end{aligned}$$

where ${}_2F_0(\cdot)$ *is the hypergeometric function with 2 upper parameters and an empty list of lower parameters.*

Proof is the result of straightforward substitution of the expressions for $\overline{U}$, λ_0, and $\lambda_1(s_0)$ obtained in Lemma 1, Proposition 1, and Appendix 2 ■

Note that equation (13) defines a monotonically decreasing dependence between λ_1 and s_0 (Fig. 11). This result is intuitive since the larger the initial reserve s_0, the greater the optimal sustainable growth rate of consumption $\dot{c}/c = (1/\lambda_1)(\dot{q}_0/q_0)(1 + t\dot{q}_0/q_0)^{-1}$. Note also that the optimal tax results in the asymptotical satisfaction of the standard Hotelling Rule.

5. Compatibility of the criterion with initial conditions

Before considering numerical examples, I will examine possible limitations of criterion (1) that can cause unacceptable consequences if this criterion is used in an imperfect economy with a specific initial state. It is known that this criterion with $\gamma = 0$ (constant consumption) cannot be used in numerical examples with the data from a growing economy with the growing extraction $r(t)$ since, in this case, $\dot{r}(0)$ must be negative, and this value is strictly defined by the initial values of extraction $r(0)$, reserve $s(0)$, and parameters α and β. That is why an economy, pursuing this specific type of intergenerational justice, must adjust its extraction and saving during a transition period in order to switch to the optimal path in finite time (Bazhanov 2008a).

In the general case ($\gamma > 0$), an economy is already allowed to grow, and a specific type of growth corresponds to a specific set of initial data. This correspondence implies that the economy's initial conditions are already not strictly fixed by a criterion but they can belong to a range, or satisfy a restricting relationship. In Appendix 1, I have shown that, for the ratio $\dot{r}/r$ to be negative (declining extraction) in the long run, the value of λ_1 must be greater than $1/\alpha - 1$ implying $\gamma < 1/(1/\alpha - 1) = \alpha/(1-\alpha)$ ($\alpha = 0.3$ yields $\gamma < 0.43$). Now I will examine how the value of λ_1 is restricted by the requirement of convergence of the integral $\int_0^\infty r(t)dt$. Extraction $r(t)$ can be expressed as follows:

$$\begin{aligned} r(t) &= q^{1/\beta}k^{-\alpha/\beta} \\ &= \widehat{q}^{1/\beta}\left[\widehat{k}(\lambda_1 t+\lambda_0)^{-1/(\alpha\lambda_1)} + \frac{\beta\widehat{q}}{1+\lambda_1}(\lambda_1 t+\lambda_0)^{((\lambda_1+1)/\lambda_1 - 1/(\alpha\lambda_1))}\right]^{-\alpha/\beta}. \end{aligned}$$

Convergence of the integral is defined here by the behavior of the second term in bracket since $\lim_{t\to\infty}(\lambda_1 t+\lambda_0)^{-1/(\alpha\lambda_1)} = 0$. Then the convergence condition is $\left[\alpha^2(\lambda_1+1)-\alpha\right]/(\alpha\beta\lambda_1) > 1$ or

$$\lambda_1 > (1-\alpha)/(\alpha-\beta). \tag{14}$$

For example, this condition requires $\lambda_1 > 14$ ($\gamma < 0.0714$) for $\alpha = 0.3$ and $\beta = 0.25$, while the requirement of negative ratio $\dot{r}/r$ implies only $\lambda_1 > (1-\alpha)/\alpha = 2.333$. Note that the combination of condition (14) with the requirement of declining extraction ($\lambda_1 > (1-\alpha)/\alpha > 0$) implies $\alpha > \beta$ (Solow 1974).

Groth et al (2006) argued that the notion of regular growth should be more general than that of exponential growth. Inequality (14) shows that the value of γ must be less than $(\alpha-\beta)/(1-\alpha)$ in the DHSS economy regardless of the values of initial conditions. If $\alpha < 0.5$, this restriction prevents the economy from sustainable patterns of linear growth, let alone the exponential growth.

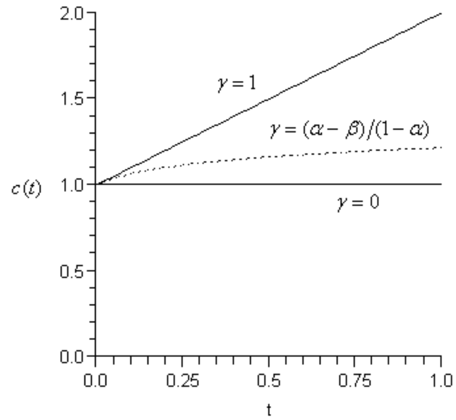

Figure 1: Patterns of feasible growth for the Cobb-Douglas economy with $\alpha = 0.3$ are between the constant ($\gamma = 0$) and the path with $\gamma = (\alpha - \beta)/(1 - \alpha)$.

The economy can realize only some variants of quasiarithmetic growth including stagnation ($\gamma = 0$). The set of these feasible sustainable paths is located in Figure 1 between the constant ($\gamma = 0$) and the path for $\gamma = (\alpha - \beta)/(1 - \alpha)$.

Condition (14) yields only the lower bound for λ_1. The exact value of λ_1 must be defined from the equation $\int_0^\infty r(t, \lambda_1)dt = s_0$. Therefore, the question of existence of the solution of this equation can be the source of incompatibility of criterion (1) with some sets of initial conditions. Hence, I will define the applicability of a criterion for formulating a long-run (sustainable) development program for an imperfect economy as follows.

Definition 1. A criterion is *applicable for a long-run development program*[14] in an economy $q = f(k, r)$ with the given initial state if there exists at least one optimal program $\langle q^*, k^*, r^* \rangle$ that satisfies the economy's initial conditions.

[14] A criterion can be *applicable* for comparing some paths from the set of feasible paths in an imperfect economy, but it can be *inapplicable* for a long-run development program because the optimal path that it implies can be *not realizable* in this economy in the long run.

The applicability of criterion (1) for a long-run program in the DHSS economy is considered in the following Proposition.

Proposition 2. *The criterion* $\dot{c}^{\gamma}c^{1-\gamma} = const$ *is applicable for a long-run development program in the economy* $q = k^{\alpha}r^{\beta}$ *with* $\dot{k} = \beta q$ *if the initial reserve* s_0 *satisfies the condition*

$$s_0 \geq \frac{k_0 r_0}{q_0(\alpha - \beta)} = \frac{k_0^{1-\alpha} r_0^{1-\beta}}{\alpha - \beta}, \tag{15}$$

where q_0, k_0, *and* r_0 *are the initial values of output, capital, and the rate of extraction.*

Proof. In Appendix 2, I have shown that the following formula can be used for defining λ_1 as a good approximation of the solution of the equation $\int_0^{\infty} r(t, \lambda_1)dt = s_0$ with respect to λ_1 :

$$\lambda_1 = \frac{(1-\alpha)s_0 q_0 + k_0 r_0}{(\alpha - \beta)s_0 q_0 - k_0 r_0}. \tag{16}$$

This formula captures the main peculiarities of behavior of the exact solution. In particular, it is seen that the denominator can be zero or that the value of λ_1 can go to infinity. So the formula implies that the denominator must be positive or $s_0 > k_0 r_0 / [q_0(\alpha - \beta)]$, which coincides with condition (15). Then the formula means that the value of $\lambda_1(s_0)$ is a function, decreasing from infinity at the minimal value of $s_0 = k_0 r_0 / [q_0(\alpha - \beta)]$ to the minimal value $\lambda_{1\,\min} = (1-\alpha)/(\alpha - \beta)$ for s_0 going to infinity (Fig. 2).

Indeed, the limiting case for the path of extraction with λ_1 going to infinity (corresponds to the smallest possible s_0) is given by

$$\begin{aligned} r_{\infty}(t) &\triangleq \lim_{\lambda_1 \to \infty} r(t, \lambda_1) = \lim_{\lambda_1 \to \infty} \left(\frac{q_0}{\lambda_0^{1/\lambda_1}} \right)^{1/\beta} (\lambda_1 t + \lambda_0)^{1/(\beta\lambda_1)} k^{-\alpha/\beta} \\ &= q_0^{1/\beta} \left[k_0 + \beta q_0 t \right]^{-\alpha/\beta}. \end{aligned}$$

The total amount of reserve, extracted along this path is

$$\int_0^{\infty} r_{\infty}(t)dt = \frac{q_0^{1/\beta}}{\beta q_0 \left(1 - \frac{\alpha}{\beta}\right)} \left[k_0 + \beta q_0 t \right]^{1-\alpha/\beta} |_0^{\infty} = -\frac{q_0^{1/\beta} k_0^{1-\alpha/\beta}}{q_0 (\beta - \alpha)} = \frac{k_0 r_0}{q_0 (\alpha - \beta)},$$

which is the greatest lower bound for feasible reserve s_0 ■

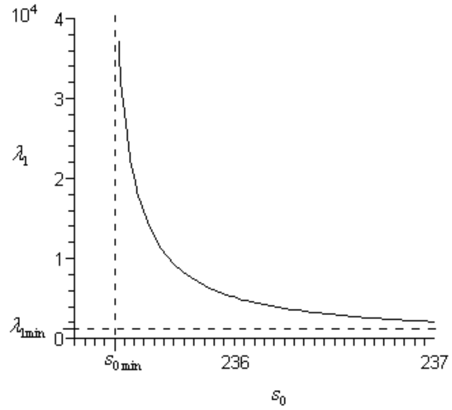

Figure 2: λ_1 as a function of the initial reserve s_0.

Solow (1974) - Hartwick (1977) is a particular case here for $\gamma = 0$ when condition (15) holds as equality, coinciding with the "perfection condition" (4). Hence, the introduction of the parameter γ in the generalized criterion resulted in the relaxation of the perfection condition. The modified criterion implies different patterns of *growth* depending on the available reserve and so the case of stagnation is included in (15) as a lower bound. However, it can happen that an imperfect economy does not satisfy this condition when the economy is overusing the resource. In this case, there is no sustainable growth path, constructed as a smooth continuation of the initial state. Then the economy needs a transition period with declining consumption for adjusting its patterns of extraction and saving in order to meet the minimum requirements expressed in (15). After the transition period, the economy can enter a sustainable growth path (Bazhanov 2008a). In a normative sense, this situation needs careful axiomatization in order to provide a criterion of just resource allocation in an overconsuming economy.

It would be interesting to analyze the applicability of the hybrid measure in the general form $\dot{c}^{\gamma}c^{\mu}$ if γ had been close to unity for plausible values of α. However, the analysis for the case with $\mu = 1 - \gamma$ and with the conventional value of $\alpha = 0.3$ (Nordhaus and Boyer 2000) shows that the DHSS economy, in this framework, can sustain only the patterns of quasiarithmetic growth that are closer to constant than to a linear function ($\gamma \ll 1$). Moreover, these patterns of sustainable growth, including constant consumption, are not affordable for any initial conditions. This result implies the impossibility of exponential growth for the basic DHSS model and, therefore, the inconvenience of percent as a measure for sustainable growth in this economy.

The impossibility of exponential growth implies an important practical application of the hybrid measure. This expression can be called *geometrically weighted percent,* and it can be used as a measure for sustainable growth of some economic indicators instead of regular percent. The rate of growth measured in regular percent declines if this growth is not exponential. The indefiniteness of the rate of decline makes regular percent an inconvenient and even a misleading measure for sustainable development. For example, this indicator was used as a necessary condition for sustainability even in such a seminal document for sustainable development as the Brundtland Report (World 1987), which claimed that "the key elements of sustainability are: a minimum of 3 percent per capita income growth in developing countries" (p. 169). Further, the Report suggested that "annual global per capita GDP growth rates of around 3 percent can be achieved. This growth is at least as great as that regarded in this report as a minimum for reasonable development" (p. 173). Besides contradictions with the environmental goals, which were noticed, for example, in Daly (1990) and Hueting (1990), measuring growth in GDP percent change can conflict with the theoretical possibility of realization of this program. In this sense, geometrically weighted percent in the form of (1) is more convenient for formulating the long-run economic goals because keeping this expression constant with the parameters linked to the economy implies that the optimal growth is always sustainable.

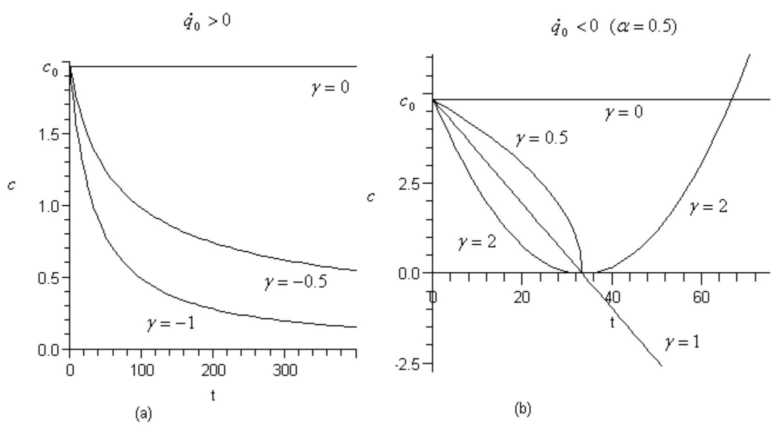

Figure 3: Unacceptable paths of consumption, resulted from criterion (1) in: (a) growing economy with $\gamma < 0$; (b) declining economy with $\gamma > 0$.

6. An economy with declining output and/or small reserve s_0

In order to complete the analysis of applicability of the economy-linked criterion, I will show that the criterion leads to unacceptable implications in the cases when an economy has declining output ($\dot{q}_0/q_0 < 0$) at $t = 0$ and/or $\gamma < 0$.[15] The optimal paths of consumption for these cases can be obtained by plotting the formula in Corollary 1.

For a growing economy ($\dot{q}_0/q_0 > 0$) with $\gamma < 0$, criterion (1) implies consumption paths asymptotically approaching zero (Fig. 3a). If the economy's output is declining at $t = 0$ and $\gamma > 0$, then the criterion yields the consumption decreasing to zero in finite time for all positive γ. However, for even integer values of $\gamma > 1$, the optimal path after hitting zero has unbounded polynomial growth (Fig. 3b). Note again that $\gamma > 1$ cannot be obtained in the DHSS model for the conventional values of α. In the last, presumably the most pessimistic case, when the economy declines with negative γ, criterion (1) requires the consumption to be growing to infinity in a finite period (Fig. 4). This scenario can

[15] It can be shown that negative γ is equivalent to violation of condition (15).

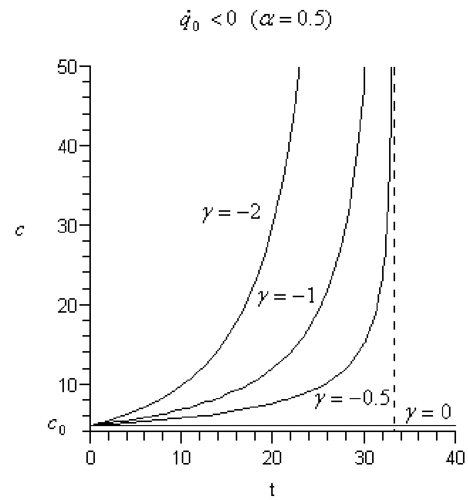

Figure 4: Unacceptable paths of consumption, resulted from criterion (1) in declining economy with $\gamma < 0$.

be realized only in the short run because growing consumption with decreasing output implies negative investment and subsequent collapse of the economy.

Hence, the only case when criterion (1) leads to ethically acceptable paths of consumption is growing output and satisfaction of condition (15). The paths of consumption, for this case, are depicted in Figure 1.

7. Numerical example

The primary initial values are $\alpha = 0.3$, $\beta = 0.25$,[16] $\dot{q}_0/q_0 = 0.03$ (GDP percent change), $r_0 = 3.624$ (the initial rate of extraction), $s_0 = 2 \cdot 180.472 = 360.944$ (the initial reserve).[17] The rate of extraction is growing with $\dot{r}_0 = 0.1$.

[16] This β implies a reasonable interest rate $f_k(0)$; at the same time, it is close to the world's pattern of saving given $\dot{k} = \beta q$.

[17] I use the world oil extraction on January 1, 2007 as r_0 and the doubled conventional world's reserves as s_0 (Radler, 2006): $r_0 = 72,486.5$ [1,000 bbl/day] $\times 365 = 26,457,572$ [1,000 bbl/year] or 3.624 bln t/year; $s_0 = 1,317,447,415$ [1,000 bbl] or 180.472 bln t. Ton of crude oil equals here 7.3 barrels. The report of Cambridge Energy Research Associates

Note that formula (8) gives a connection between the initial values, so k_0 can be expressed in terms of the readily available data:

$$k_0 = \left\{ \left[\frac{\dot{q}_0}{q_0} \frac{1}{\beta} - \frac{\dot{r}_0}{r_0} \right] / \left(\alpha r_0^{\beta} \right) \right\}^{\frac{1}{\alpha - 1}} = 8.5174.$$

Then $\lambda_0 = q_0/\dot{q}_0 = 33.333$. This yields $q_0 = k_0^{\alpha} r_0^{\beta} = 2.624$, $c_0 = (1-\beta)q_0 = 1.968$, $\dot{q}_0 = (\dot{q}_0/q_0)\, q_0 = 0.0787$, and $\tau(0) = (\dot{q}_0/q_0)\,(\beta - 1)/\beta = -0.09$. For these values, condition (15) is satisfied (in this case, $s_{0\,\min} = 235.3$);[18] the ratio q_0/k_0 equals 0.308, the rate of interest is $f_k(0) = \alpha q_0/k_0 = 0.092$. These values can be used in estimation of the optimal tax $T(t)$ and the paths of capital and extraction. The problem implies that there is no tax at the initial moment ($T_0 = 0$) that yields $\dot{T}_0 = 0.016$ (growing optimal tax). The value of λ_1, from the condition $\int_0^{\infty} r(t)dt = s_0$ is $\lambda_1 = 60.11$[19] (Appendix 2). This value implies the optimal path of capital that is very close to linear (solid line in Fig. 8): $k(t) = 8.16 + 0.0101 \cdot (60.11t + 33.33)^{1.0166}$, and the paths of extraction (solid line in Fig. 9) and tax (solid line in Fig. 7). Quasiarithmetic growth of consumption is depicted in the solid line in Fig. 10. Comparative analysis of these paths is provided in Section 9.

8. TFP compensating for capital depreciation

Uncertainty of technical change is reflected in a wide variety of models used in the literature. Optimistic approaches assume that this factor is exponentially growing in a form of TFP (Stiglitz 1974) while there are models with quasiarithmetic TFP (Asheim et al 2007) and with the TFP limited from above (Nordhaus

(CERA, 2006) claims that actual world's reserves (3.74 trn bbl) are about three times more than the conventional evaluation. I use here the "average" of the two estimates.

[18] If s_0 equals 180.472 bln t (conventional estimate) then condition (15) is violated, meaning impossibility of sustainable growth for this economy in the sense of criterion (1). Then the economy needs a transition period in order to adjust the initial state.

[19] Numerical calculation of the integral gives $\lambda_1 = 60.11$; the expression via the hypergeometric function (Appendix 2) implies $\lambda_1 = 72.33$, and the approximate formula (16) yields $\lambda_1 = 42.1$.

and Boyer 2000). The patterns of endogenous technical change are considered, for example, in Takayama (1980) or Grimaud and Rouge (2005). Since the main aim of this paper is to construct an example of an economy-linked criterion in an imperfect economy, I use here a simple assumption about a form of technical change that is somewhere between optimistic and pessimistic approaches. Some studies assume that technical change exactly compensates for the growing population (Dasgupta and Heal 1979; Stollery 1998). However, the assumption about constant population becomes more and more plausible with time.[20] This assumption implies that technical change (or a part of this change) can "compensate for" another negative factor of growth. I assume here that growth of TFP exactly compensates for capital depreciation since, unlike the growth of population, technical change and capital decay presumably will exist as long as human civilization and capital exist. The convenience of this assumption is linked with the correctness of the use of the basic DHSS model in the cases with unlimited growth in consumption.

The assumption about TFP $A(t)$ implies that $q(t) = A(t)k^{\alpha}r^{\beta} - \delta k = k^{\alpha}r^{\beta}$. Then $A(t)$ is given by $A(t) = 1+\delta k^{1-\alpha}r^{-\beta}$. Substitution for $r = \widehat{r}\left(\lambda_1 t + \lambda_0\right)^{1/(\beta\lambda_1)} k^{-\alpha/\beta}$, where $\widehat{r} = \widehat{q}^{1/\beta}$ and $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$ yields

$$A(t) = 1 + \frac{\delta}{\widehat{q}}\left[\frac{\widehat{k}}{\left(\lambda_1 t + \lambda_0\right)^{1/\lambda_1}} + \frac{\beta\widehat{q}}{1+\lambda_1}\left(\lambda_1 t + \lambda_0\right)\right],$$

which is asymptotically linear with the slope $\delta\beta/\left(1+\lambda_1\right)$. For the example above, given $\delta = 0.1$, the slope is $0.1 \cdot 0.25/(1+60.11) = 0.000409$ (Fig. 5).

The more optimistic the model of technical change, the less resources are left for the future generations along the optimal path of extraction. An illustration of this intuitive result is provided, for example, in Bazhanov (2008b, p. 29). The example considers an exponentially declining path of extraction that implies exponentially growing consumption in the model of Grimaud and Rouge

[20]The United Nations estimates that the world's population growth is going to flatten out at a level around 10 billion (UN 1999). Stabilization has already happened in developed countries, which are the main users of nonrenewable resources.

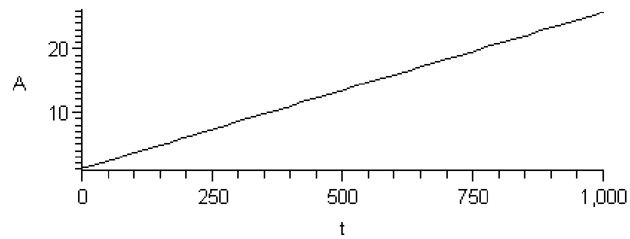

Figure 5: TFP compensating for capital depreciation.

(2005) with exponentially growing knowledge. It is shown that the same extraction yields consumption collapsing to zero in the case when technical change is represented by the TFP, compensating for capital decay. In this sense, underestimation of technical change causing inefficiency looks preferable to overestimation resulting in extinction. This preference implies the way to deal with uncertainty in the evolution of science when the long-run development programs are based on modest reliable patterns of technical change, and a growth criterion is adjusted with the updates in the stock of knowledge. The resulting paths are asymptotically efficient and prevent the extinction. This approach, applied to the uncertainty in the resource stock, is illustrated in the following section.

9. Variable reserves and dynamic preferences

The amount of reserve s_0 was considered so far as a constant, though in practice the value of the proven recoverable reserve is being updated annually. This value decreases because of the extraction and it can increase due to the discovery of new oil fields and due to the changes in oil prices and in extracting technologies. Nevertheless, in many theoretical problems, s_0 can be considered as all the amount of the reserve including proven, unproven, and as yet not discovered; therefore, it can be assumed correctly that s_0 is a constant in these problems. However, if the problem is to estimate numerically the path of tax that depends on s_0 and that controls the economy in the optimal way, then s_0

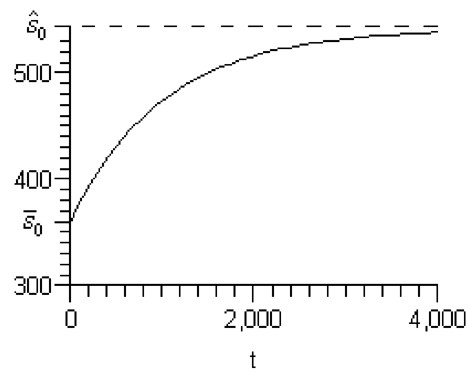

Figure 6: Updates in the estimate of s_0.

should be estimated as accurately as possible. Otherwise, the economy will be inefficient in the case of underestimation of s_0 or it will overconsume if s_0 is overestimated.

In this section, I will examine a procedure of dynamic policy correction that will depend on the information about the changes in the resource stock over time. The economy-linked criterion reflects these changes by recalculating the parameter $\gamma = 1/\lambda_1(s_0)$. When s_0 is reestimated, the new information implies the dynamic correction of the tax and of all the paths in the economy according to the changes in the criterion.

Assume that revaluation of s_0 is growing with time and asymptotically approaches a constant $\widehat{s}_0$, for example, in the following way (Fig. 6): [21]

$$s_0(t) = \widehat{s}_0 - e^{-wt}(\widehat{s}_0 - \overline{s}_0). \tag{17}$$

I will take for the numerical example $s_0(0) = \overline{s}_0 = 2 \cdot 180.47 = 360.94$ [bln t] and $\widehat{s}_0 = \lim_{t\to\infty} s_0(t) = 3 \cdot 180.47 = 541.41$[bln t] (CERA's reserve estimate). The parameter w equals 0.001. Substitution of (17) for s_0 in (16) and then substitution of the resulting expression into (1) yields the measure of the optimal sustainable growth dynamically responding to new information about

[21] $s_0(t)$ is treated here as piecewise constant with the periods of constancy going to zero.

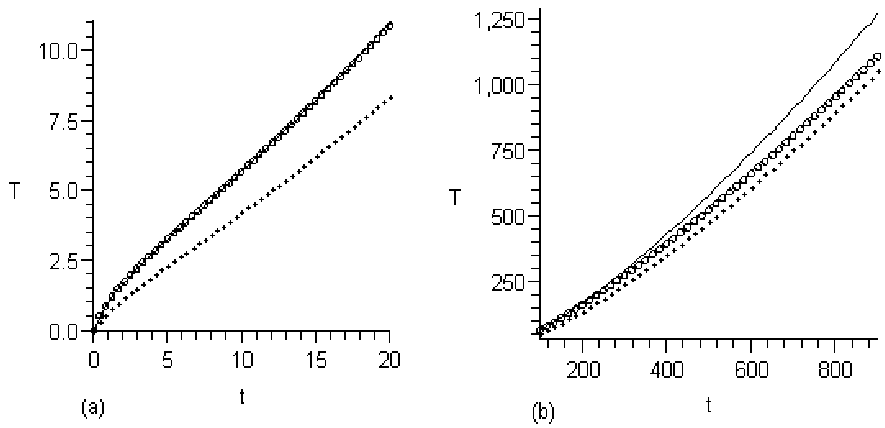

Figure 7: The optimal paths of tax: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0 = 1.5\overline{s}_0$ - in crosses; dynamically changing path - in circles.

the reserves. Then the dynamically changing $\lambda_1(s_0(t))$ implies corresponding updates in the paths of tax, capital, extraction, and consumption (Figs. 7 – 10, time in years). The paths, corresponding to the precommitment policy with the initial estimate $s_0(t) \equiv \overline{s}_0$, are depicted as a solid line, precommitment paths with $s_0(t) \equiv \widehat{s}_0$ (assuming full knowledge about reserves at the initial moment) are in crosses, and the dynamically updated paths are in circles.

The plots illustrate plausible reactions of the economy to the larger amount of the initial reserve ($s_0(t) \equiv \widehat{s}_0$, paths in crosses). The level of tax is lower, the levels of capital and the rates of extraction are higher and, as a result, the level of consumption is also higher. Note that the economy-linked criterion combined with the modified Hotelling Rule can imply hump-shaped optimal paths of extraction. This result implies the notion of *the normative resource peak.* This peak can be compared with the one being forecasted from the point of view of "physical possibility" of reaching the maximum level of extraction.[22]

[22] The theories of estimating the "physical" oil peak have been developing since the work of geologist M.K. Hubbert (1956). A methodology, different from Hubbert's oil-peak approach,

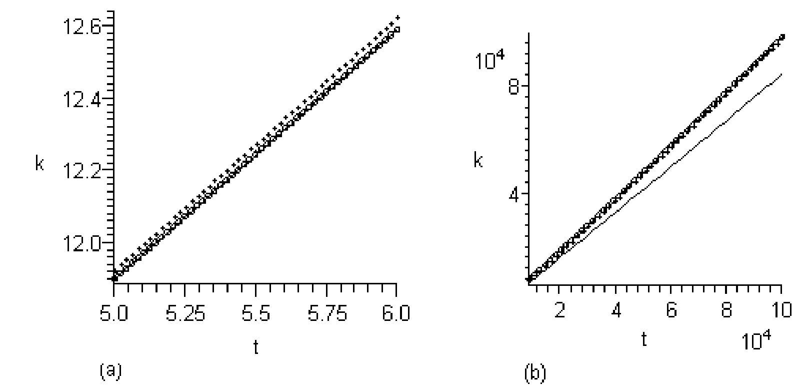

Figure 8: The optimal paths of capital: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles.

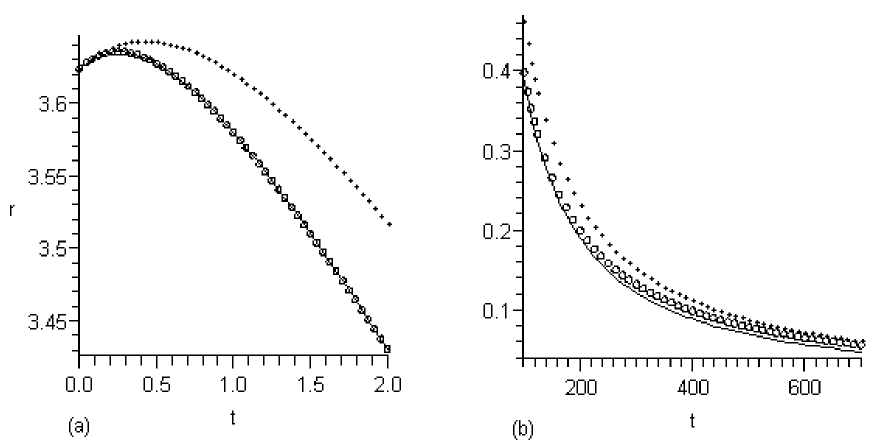

Figure 9: The optimal paths of extraction: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles.

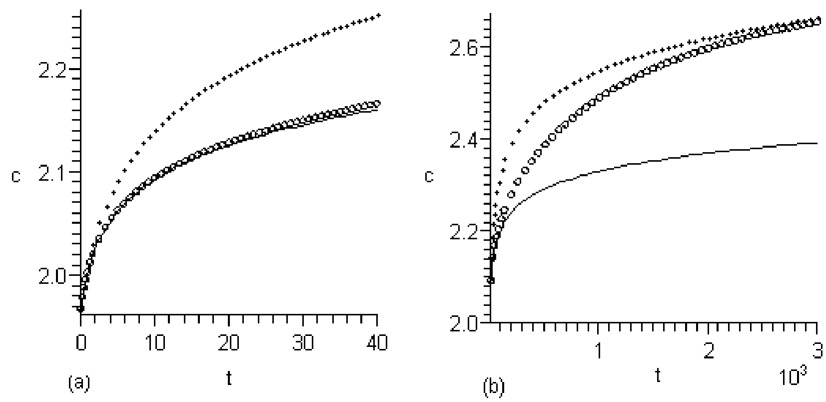

Figure 10: The optimal paths of consumption: (a) in the short run; (b) in the long run. For the initial reserve $\overline{s}_0$ - as a solid line; for the "full-knowledge" reserve $\widehat{s}_0$ - in crosses; dynamically updated path - in circles.

One could expect that, if an economy chooses an inferior path at the initial point due to lack of knowledge about the reserve, then the difference in consumption with respect to the optimal "full-knowledge" path (line in crosses, Fig. 10) will only increase with time under the given saving rule. However, the example shows that under the standard Hartwick rule the consumption in the economy with the dynamically defined preferences (line in circles) is asymptotically "catching-up" to the optimal level of consumption in the process of updating the information about the reserve. The maximum difference in consumption during this process is less than 5%.

Another implication of the dynamically updated preferences is that the level of $\overline{U}$ in criterion (1) becomes variable ($\overline{U}(t) = c_0/\lambda_0^{1/\lambda_1(s_0(t))}$). These changes

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was used in the CERA's report (CERA 2006), claiming that the world oil reserves are about three times larger than the conventional estimates, and that the "physical" oil peak is not expected before 2030. However, the optimal paths of extraction obtained in this paper imply that the normative oil peak must be much closer, namely, in 6 months, even for the CERA's reserve estimate.

in $\overline{U}$ could undermine the argument about the convenience of the geometrically weighted percent as a measure for sustainable growth. However, in the numerical example above with substantially changing information about the reserve, the change in $\overline{U}$ is nothing more than 5% (from $\overline{U}(0) = 1.81$ to $\overline{U}(\infty) = 1.71$), which looks negligible in comparison with the mismeasurements in the real economy.

10. Concluding remarks

Koopmans wrote that "the economist's traditional model of choice ... is based on an analytical separation of preference and opportunity" (Koopmans 1964, p. 243). This paper has offered an approach of linking a criterion (preference) to the opportunity of an imperfect economy. Koopmans assumed uncertain future preferences themselves with certain physical assets. In this paper, a general form of the criterion was defined by preferences, and this form was parametrically connected to the uncertain resource reserve and the technological properties of the economy. Using this economy-linked criterion, it has been shown that only the paths of quasiarithmetic growth can be sustainable in the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model, and that these paths are much closer to constant consumption than to linear function for the conventional value of capital share ($\alpha = 0.3$; Fig. 1). The DHSS model is extended here by assuming that the Hotelling Rule is modified by the phenomena whose total influence can be expressed in terms of an equivalent tax or subsidy (Section 3). I interpreted the absence of both technical change (TFP) and capital depreciation as presence of the specific TFP exactly compensating for capital decay (Section 8).

The example of an economy-linked criterion was constructed here for the maximin principle applied to a generalized level-growth measure (geometrically weighted percent). The parameter of this measure was linked to the economy's technological parameters and the initial conditions. The optimal paths were obtained in explicit form under the standard Hartwick rule (Section 4). The closed-form expression was derived for the dependance of the parameter, specifying the criterion, on the reserve estimate. This formula was used to examine the optimal paths dynamically responding to the updates in the reserve estimates (Section 9).

The assumption about the generalized form of the Hotelling Rule modifier made it possible to link the model to the world's oil extraction data (Sections 7 and 9). In particular, this modification allowed for nondecreasing extraction in the initial period. This property of the problem introduces the notion of the normative oil (resource) peak. It turned out that, in the framework of this paper, the optimal oil peak must be in 2-6 months depending on the amount of the reserve. In other words, the socially-optimal oil peak is much closer to the initial moment than the various forecasts of the "physical" oil peak that predict the maximum period of growth for the rates of extraction.

It would be interesting to apply in further studies

(1) the economy-linked criterion to the problems with specific externalities like Stollery's (1998) and Hamilton's (1994) global warming, where temperature affects not only the Hotelling Rule but also the utility and/or the production function;

(2) the methodology of linking a criterion to an imperfect economy for different hybrid measures and different criteria of justice;

(3) the methodology of linking a criterion to an imperfect economy with the specific patterns of endogenous technical change.

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12. Appendix 1 (Proof of Proposition 1)

Lemma 1 provides the optimal path of the Hotelling Rule modifier $\tau(t) = [(\beta - 1)/\beta]/(\lambda_1 t + \lambda_0)$. Indeed, equation (8) implies $\dot{q}/q = \tau\beta/(\beta - 1) = 1/(\lambda_1 t + \lambda_0)$, so that $\lambda_0 = q_0/\dot{q}_0$ (for $\dot{q}_0 \neq 0$), yielding $q(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$, where the constant of integration $\widehat{q}$ is defined from the initial condition $q(0) = q_0 : \widehat{q} = q_0/\lambda_0^{1/\lambda_1} = (\dot{q}_0/q_0)^{1/\lambda_1} q_0$. Then $\dot{q}(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1 - 1}$ and $q(t) = q_0\,(1 + t\lambda_1/\lambda_0)^{1/\lambda_1}$. The expression $\dot{q}^\gamma q^{1-\gamma}$ with $\gamma = 1/\lambda_1$ yields

$$\begin{aligned}
\dot{q}^\gamma q^{1-\gamma} &= \widehat{q}^{1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1/\lambda_1 - 1)/\lambda_1}\,\widehat{q}^{1-1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1-1/\lambda_1)/\lambda_1} \\
&= \widehat{q} = const = \overline{U}/(1-\beta).
\end{aligned}$$

The saving rule $\dot{k} = \beta\widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$ implies the path for capital $k(t) = \widehat{k} + [\beta\widehat{q}/(1 + \lambda_1)]\,(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)}$ with the constant of integration $\widehat{k} = k_0 - \beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1 + \lambda_1) = k_0 - \beta q_0\lambda_0/(1 + \lambda_1)$ defined by the initial condition $k(0) = k_0$. Then the capital path is

$$k(t) = k_0 + \frac{\beta\widehat{q}}{(1+\lambda_1)}\left[(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right].$$

The expressions for q and k result in the extraction path $r(t) = \widehat{r}(\lambda_1 t + \lambda_0)^{1/\beta\lambda_1} k^{-\alpha/\beta}$, which implies the following equation:[23]

$$\frac{\dot{r}}{r} = \frac{\widehat{k} + \beta\widehat{q}\left(\frac{1}{1+\lambda_1} - \alpha\right)(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)}}{\beta\widehat{k}\,(\lambda_1 t + \lambda_0) + \frac{\beta^2\widehat{q}}{1+\lambda_1}\,(\lambda_1 t + \lambda_0)^{(2+1/\lambda_1)}}. \tag{18}$$

The initial extraction r_0 defines the constant of integration $\widehat{r}$:

$\widehat{r} = r_0\lambda_0^{-1/\beta\lambda_1}\left[\widehat{k} + \beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1+\lambda_1)\right]^{\alpha/\beta}$. The more simple expression for $\widehat{r}$ can be obtained from the production function $q = k^\alpha r^\beta$, namely, $\widehat{r} = \widehat{q}^{1/\beta}$. Then, given the expression for $r(t)$, the parameter λ_1 can be estimated from the efficiency condition $s_0 = \int_0^\infty r(t)dt$ (Appendix 2).

Note that equation (18) implies that $\dot{r}/r \to 0$ with $t \to \infty$ and, in order to obtain feasible behavior of $r(t)$, it is necessary that the ratio $\dot{r}/r$ is negative

[23] The modified Hotelling Rule in form of (7) gives an equation for $\dot{r}/r$ that implies the same expression for r but in a more cumbersome way.

for large enough t. Assuming $\lambda_1 > 0$, one can see that, for large enough t, the denominator in (18) is positive, and the nominator is negative, if and only if $\alpha > 1/(1+\lambda_1)$ or $\lambda_1 > 1/\alpha - 1$, which justifies the assumption about the sign of λ_1 for $\alpha \in (0,1)$. This condition for $\lambda_1 = \lambda_1(s_0)$ can be interpreted as a condition of existence of the optimal, in the sense of criterion (1), paths in the economy with technological parameter α, and reserve s_0.

The explicit path of tax can be obtained from formula (11):

$$T(t) = \exp\left[\int f_k(t)dt\right]\left\{\widehat{T} - \int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt\right\}.$$

Consider the following integral, given the Hartwick rule: $\int f_k(t)dt = \alpha \int (q/k)dt = (\alpha/\beta)\int(\dot{k}/k)dt = (\alpha/\beta)\ln k + C_1$. This expression implies $\exp\left[\int f_k(t)dt\right] = C_2 k(t)^{\alpha/\beta}$ and

$$\int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt = \frac{1}{C_2}\left[\frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right] + C_3,$$

which yields

$$T(t) = k(t)^{\alpha/\beta}\left[\widehat{T} - \frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right], \qquad (19)$$

where $\widehat{T} = \widehat{T}(C_2, C_3)$. With $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$, $\widehat{r} = \widehat{q}^{1/\beta}$, and $T_0 = T(0)$, the expression for $\widehat{T}$ is: $\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta\widehat{q}^{1-1/\beta}(1+\lambda_1)^{\alpha/\beta}\lambda_0^{(\beta-1)/(\beta\lambda_1)}$ or $\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta q_0^{(\beta-1)/\beta}(1+\lambda_1)^{\alpha/\beta}$. Then formula (19) becomes

$$T(t) = k(t)^{\alpha/\beta}\left\{T_0 k_0^{-\alpha/\beta} + \beta(1+\lambda_1)^{\alpha/\beta} q_0^{(\beta-1)/\beta}\left[1 - (\frac{\lambda_1}{\lambda_0}t + 1)^{(\beta-1)/(\beta\lambda_1)}\right]\right\},$$

which, for $T_0 = 0$, yields the expression formulated in the proposition ■

13. Appendix 2 (Estimation of $\boldsymbol{\lambda_1(s_0)}$)

The condition $\int_0^\infty r(t,\lambda_1)dt = s_0$ gives the expression for $\lambda_1(s_0)$. Sequential integration by parts yields representation of s_0 as a series, which can be shown by expressing r in the following way: $r = q^{1/\beta}k^{-\alpha/\beta} = (1/\beta)^{1/\beta}\,\dot{k}^{1/\beta-1}\dot{k}k^{-\alpha/\beta}$. Denote $u = \dot{k}^{1/\beta-1}$ and $dv = k^{-\alpha/\beta}\dot{k}dt$. Then

$$\begin{aligned}\int_0^\infty rdt &= (1/\beta)^{1/\beta}\int_0^\infty udv = (1/\beta)^{1/\beta}\left[uv - \int_0^\infty vdu\right]\\ &= (1/\beta)^{1/\beta}\left[-\frac{\dot{k}_0^{1/\beta-1}k_0^{1-\alpha/\beta}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha}I_2\right],\end{aligned}$$

where $I_2 = \int_0^\infty k^{1-\alpha/\beta}\dot{k}^{1/\beta-2}\,\ddot{k}\,dt$. Substitution for $\ddot{k} = \beta\widehat{q}\,(\lambda_1 t+\lambda_0)^{1/\lambda_1-1} = (\beta\widehat{q})^{\lambda_1}\,\dot{k}^{1-\lambda_1}$ gives $I_2 = (\beta\widehat{q})^{\lambda_1}\,I_3$, where $I_3 = \int_0^\infty k^{1-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1}dt$. Since $k/\dot{k}^{(1+\lambda_1)} = \widehat{k}\dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)$, then $k^{1-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1} = k^{-\alpha/\beta}\dot{k}^{1/\beta}k/\dot{k}^{(1+\lambda_1)} = k^{-\alpha/\beta}\dot{k}^{1/\beta}\left[\widehat{k}\dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)\right]$. This expression implies that $I_3 = \widehat{k}\int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1}dt + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)\int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta}dt$. The second integral, expressed via the original one, equals $\beta^{1/\beta}\int_0^\infty rdt$. Then the original integral is

$$\begin{aligned}\int_0^\infty rdt &= (1/\beta)^{1/\beta}\left\{-\frac{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha}(\beta\widehat{q})^{\lambda_1}\right. \\ &\quad\left.\times\left[\frac{(\beta\widehat{q})^{-\lambda_1}}{(1+\lambda_1)}\beta^{1/\beta}\int_0^\infty rdt + \widehat{k}I_4\right]\right\},\end{aligned} \tag{20}$$

where $I_4 = \int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-(\lambda_1+1)}dt$. From (20), the reserve s_0 is given by

$$\begin{aligned}s_0 &= \int_0^\infty rdt = \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta+1)-1+1/\beta}(1/\beta)^{1/\beta} \\ &\quad\times\left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} - (1/\beta-1)(\beta\widehat{q})^{\lambda_1}\widehat{k}I_4\right\}.\end{aligned} \tag{21}$$

Integration of I_4 by parts, with $u = \dot{k}^{1/\beta-1-(\lambda_1+1)}$, $dv = k^{-\alpha/\beta}\dot{k}dt$, and the same substitutions, yields

$$\begin{aligned}I_4 &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta} \\ &\quad\times\left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - (1/\beta-1-(\lambda_1+1))(\beta\widehat{q})^{\lambda_1}\widehat{k}I_8\right],\end{aligned}$$

where $I_8 = \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta - 2(\lambda_1+1)} dt$. Then, using I_4 in (21), s_0 is given by

$$\begin{aligned} s_0 &= \int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1+1)(-\alpha/\beta+1) - 1 + 1/\beta} (1/\beta)^{1/\beta} \\ &\times \left\{ -k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1} - \frac{(\lambda_1+1)(1/\beta - 1)}{(\lambda_1+1)(-\alpha/\beta) - 1 + 1/\beta} (\beta \widehat{q})^{\lambda_1} \widehat{k} \right. \\ &\times \left. \left[-k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1 - (\lambda_1+1)} - (1/\beta - 1 - (\lambda_1+1)) (\beta \widehat{q})^{\lambda_1} \widehat{k} I_8 \right] \right\}. \end{aligned}$$

Integration of I_8 by parts, with $u = \dot{k}^{1/\beta - 1 - 2(\lambda_1+1)}$, and $dv = k^{-\alpha/\beta} \dot{k} dt$, yields

$$\begin{aligned} I_8 &= \frac{\lambda_1 + 1}{(\lambda_1+1)(-\alpha/\beta - 1) - 1 + 1/\beta} \\ &\times \left[-k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1 - 2(\lambda_1+1)} - (1/\beta - 1 - 2(\lambda_1+1)) (\beta \widehat{q})^{\lambda_1} \widehat{k} I_{12} \right]. \end{aligned}$$

Using the pattern of expressions for integrals $I_4, I_8, I_{12}, \ldots$ and multiplying fractions by $-\beta$, it can be shown that the original integral is

$$\int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1+1)(\alpha - \beta) - 1 + \beta} \cdot \beta^{(1-1/\beta)}$$

$$\times \left\{ k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1} + \frac{(\lambda_1+1)(1-\beta)}{(\lambda_1+1)\alpha - 1 + \beta} (\beta \widehat{q})^{\lambda_1} \widehat{k} \right.$$

$$\times \left[k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1 - (\lambda_1+1)} + \frac{(\lambda_1+1)(1 - \beta[1 + (\lambda_1+1)])}{(\lambda_1+1)(\alpha+\beta) - 1 + \beta} (\beta \widehat{q})^{\lambda_1} \widehat{k} \right.$$

$$\begin{aligned} &\times \left\{ k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1 - 2(\lambda_1+1)} + \frac{(\lambda_1+1)(1 - \beta[1 + 2(\lambda_1+1)])}{(\lambda_1+1)(\alpha+2\beta) - 1 + \beta} (\beta \widehat{q})^{\lambda_1} \widehat{k} \right. \\ &\times \left. \left. \left. \left[k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta - 1 - 3(\lambda_1+1)} + \ldots \right] \right\} \right] \right\}. \end{aligned}$$

Substitution for $\widehat{q}, \lambda_0$, and for $\dot{k}_0 = \beta k_0^\alpha r_0^\beta$ yields

$$\int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1+1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0}$$

$$\begin{aligned} &\times \left\{ 1 + \frac{(\lambda_1+1)(1-\beta)}{(\lambda_1+1)\alpha - 1 + \beta} \cdot \widehat{k} \cdot \left[\beta \dot{q}_0 \right. \right. \\ &+ \frac{(\lambda_1+1)(1 - \beta[1 + (\lambda_1+1)])}{(\lambda_1+1)(\alpha+\beta) - 1 + \beta} \cdot \widehat{k} \cdot \left\{ (\beta \dot{q}_0)^2 \right. \\ &+ \left. \left. \left. \frac{(\lambda_1+1)(1 - \beta[1 + 2(\lambda_1+1)])}{(\lambda_1+1)(\alpha+2\beta) - 1 + \beta} \cdot \widehat{k} \cdot \left[(\beta \dot{q}_0)^3 + \ldots \right] \right\} \right] \right\}. \end{aligned}$$

This expression is a closed form solution as a series:

$$\int_0^{\infty} r dt = \frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\frac{k_0 r_0}{q_0} \times\left\{1+\sum_{i=1}^{\infty}\left[\widehat{k}(\lambda_1)\beta\dot{q}_0(\lambda_1+1)\right]^i\cdot\prod_{j=0}^{i-1}\frac{1-\beta\left[1+j(\lambda_1+1)\right]}{(\lambda_1+1)(\alpha+j\beta)+\beta-1}\right\}.$$

The series can be expressed via special functions,[24] namely,

$$\prod_{j=0}^{i-1}\frac{1-\beta\left[1+j(\lambda_1+1)\right]}{(\lambda_1+1)(\alpha+j\beta)+\beta-1}=\left[-\beta\left(\lambda_1+1\right)\right]^i\Gamma\left(i-\frac{1-\beta}{\beta\left(\lambda_1+1\right)}\right)/\Gamma\left(-\frac{1-\beta}{\beta\left(\lambda_1+1\right)}\right)$$

and then

$$\begin{aligned}\int_0^{\infty} r dt &= \frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\frac{k_0 r_0}{q_0}\\ &\times\left\{1+(1-\beta)\beta\widehat{k}(\lambda_1)\dot{q}_0(\lambda_1+1)\right.\\ &\left.\times\, {}_2F_0\left(\left[1,\frac{\beta(\lambda_1+2)-1}{\beta(\lambda_1+1)}\right],[\,],-\widehat{k}(\lambda_1)\dot{q}_0\beta^2(\lambda_1+1)^2\right)\right\},\end{aligned} \tag{22}$$

where ${}_2F_0(\cdot)$ is the hypergeometric function with 2 upper parameters and an empty list of lower parameters. Substitution for $\widehat{k}=k_0-\beta q_0^2/\left[\dot{q}_0(1+\lambda_1)\right]$ (Appendix 1) results in equation (13) in Corollary 1. The second term in bracket $\{\cdot\}$ equals 0.247 for the numerical example (Section 7) and so, taking into account the existing uncertainty in the reserve estimate, the following formula can approximate the value of reserve

$$s_0=\int_0^{\infty} r dt=\frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\frac{k_0 r_0}{q_0},$$

which yields an explicit expression for $\lambda_1(s_0)$:

$$\lambda_1=\frac{(1-\alpha)s_0 q_0+k_0 r_0}{(\alpha-\beta)s_0 q_0-k_0 r_0}.$$

This formula captures the main peculiarities of behavior of the exact solution. Particularly, it has the same horizontal and vertical asymptotes as the closed form solution (22) (Fig. 11).

[24] The expression of the series via special functions can be obtained in Maple.

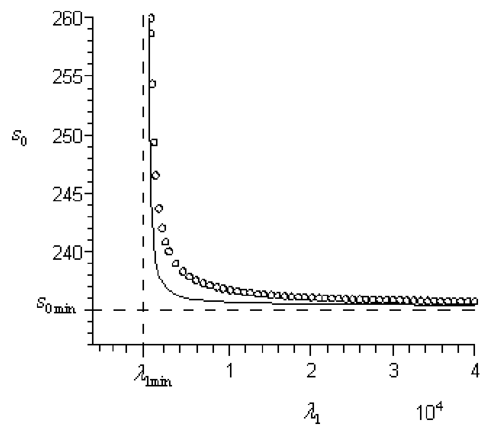

Figure 11: Dependence of reserve s_0 (the value of integral $\int_0^\infty r(t, \lambda_1)dt$) on λ_1: closed form solution (22) - in circles; approximate formula - solid line.