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Generalized Lattice Theoretical Comparative Statics

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Generalized Lattice Theoretical Comparative Statics

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Abstract

In this note, we consider the fundamental structure of the lattice theoretical comparative statics theorem, which is known as the necessary and sufficient relationship between monotonicity of the solutions of constrained optimization problems and quasisupermodularity of objective functions. Although the original theorem by Milgrom and Shannon (1994) [*Econometrica*, vol.62, pp157-180.] is constructed on a lattice, we show that this relationship can be extended to more general environments than a lattice. What is essentially required for the results in this paper is that two nonempty set-valued operations are defined on the domain of the objective function, whereas two single-valued operations with some special properties are defined on a lattice.

Keywords: Monotone comparative statics; Lattice; Quasisupermodularity

JEL Classification Numbers C02, C61

1 Introduction

The purpose of this note is to extend the theory of the lattice theoretical comparative statics to more general environment than ever known. The original theorem by Milgrom and Shannon (1994), which is widely applyed for economics and game theory[1], shows the necessary and sufficient condition for global monotonicity of comparative statics of the solution sets of constrained optimization problems. Generally, if one intend to perform comparative statics by applying Milgrom and Shannon's theorem, it is important to examine the following three points: The first is the strucuture of the domain of the objective function. The second is the behaviors of the feasible sets according to the changes of parameters. The third is the properties of the objective function. In the theory of the standard lattice theoretical comparative statics, it is required that the domain of objective

[1]See also Topkis (1998) for introduction to the lattice theoretical methods and its applications.

function is a *lattice* and that the changes of the feasible sets can be measured by *strong set order*. Under these conditions, *quasisupermodularity* of the objective function is the necessary and sufficient condition for the monotonicity of the solution set, which is nothig but the statement of Milgrom and Shannon's theorem.

Since, by its definition, a lattice is a partially ordered set with some special properties, one must define some appropriate binary relation with *reflexibility*, *antisymmetry*, and *transitivity* to apply Milgrom and Shannon's theorem. However, in some contexts in economics, defining a partial order is not an easy task. In the consumer theory, for instance, although Antoniadou (2007) defines the binary relation called "*the direct* (p, i)-*value order*" on the consumption set to analyze the normality of consumer demand by using the lattice theoretical monotone comparative statics theorem, if there exist more than two goods in the model, it is not a partial order but a preorder, that is, the direct (p, i)-value order cannot satisfy antisymmetry. To overcome this difficulty, Shirai (2008a) shows that the lack of antisymmetry does not cause any crucial problem: Milgrom and Shannon's theorem can be extended to the framework of preordered sets as well as the notions of a lattice, strong set order, and quasisupermodularity[2]. On the other hand, there exists another generalization of the theory of monotone comparative statics. Although, as stated in the beginning of this paragraph, most literature in this subject regard a lattice as a special case of a partially ordered set, one can also construct a lattice by defining two operations with some conditions on a set, that is, a lattice can be regarded as a special case of an algebra (See Grätzer (2000), for example.). Quah (2007) observes that Milgrom and Shannon's theorem remains true even if two operations on the domain of the objective function do not satisfy the condtions which make a set a lattice [Quah (2007), Theorem 1, pp.408.].

We consider unifying the above two streams of generalizations of the lattice theoretical comparative statics. That is, we extend Milgrom and Shannon's theorem to the environment which contains the ones considered in Shirai (2008a) and Quah (2007) as its special cases. The rest of this paper is organized as follows. First, we state the definition of the notions corresponding to a lattice, strong set order, and quasisupermodularity in the next section. In Section 3 and 4, we state the sufficient conditons and the necessary conditions for monotone comparative statics, respectively. It should be noted that, although Quah (2007) does not state explicitly, some kinds of order structure on the domain of the objective function plays a crucial role to derive the necessary condition.

[2] It should be noted that Mirman and Ruble (2003) shows the alternative way to deal with the models with more than two goods. They successfully define the partial order on the consumption set and make it possible to apply the original theorem by Milgrom and Shannon.

2 Definitions

2.1 $(\varDelta, \nabla)$-lattice structure

On the domains of functions, we define the fundamental platform on which our arguments explicated as follows.

Definition 1: A set X is said to be a $(\varDelta, \nabla)$-*lattice structure* if two nonempty sets $(x\varDelta y)$ and $(x\nabla y)$ are defined for every $x, y \in X$.

Note that this definition is general enough to contain the notions of the original lattice and some generalized versions of lattice which appears in recent studies. Indeed, if the both $\varDelta$ and ∇ are single-valued operations, a $(\varDelta, \nabla)$-lattice structure is equivalent with the notion which is considered in Quah (2007). On the other hand, if the set-valued operations $(\varDelta, \nabla)$ satisfy the "order structure" defined in follows, a $(\varDelta, \nabla)$-lattice structure can be constructed from some preorder and it coincides with a "*preordered lattice structure*" in Shirai (2008a), (2008b). Moreover, if $(\varDelta, \nabla)$ satisfy both single-valuedness and the order structure, then a $(\varDelta, \nabla)$-lattice strucutre can be constructed from some partial order and it coincides with a lattice. The formal definitions of the order structure, a preordered lattice structure, and a lattice are as follows.

Definition 2: The set-valued operations $(\varDelta, \nabla)$ on X satisfy the "*order structure* if the following conditions are satisfied.

1. *(Set-valued Idempotency)* $x \in I_x := (x\varDelta x) = (x\nabla x)$ for every $x \in X$.
2. *(Set-valied Commutativity)* $(x\varDelta y) = (y\varDelta x)$ and $(x\nabla y) = (y\nabla x)$ for every $x, y \in X$.
3. *(Set-valued Associativity)*

$$\forall t_{x,y} \in (x\varDelta y),\ t_{y,z} \in (y\varDelta z); (x\varDelta t_{y,z}) = (t_{x,y}\varDelta z)$$
$$\forall a_{x,y} \in (x\nabla y),\ a_{y,z} \in (y\nabla z); (x\nabla a_{y,z}) = (a_{x,y}\nabla z)$$

   for every $x, y, z \in X$.

4. *(Set-valued Absorption Identities)*

$$\forall a_{x,y} \in (x\nabla y); (x\varDelta a_{x,y}) = I_x$$
$$\forall t_{x,y} \in (x\varDelta y); (t_{x,y}\nabla y) = I_y.$$

   for every $x, y \in X$.

Definition 3: A set X is said to be a *preordered lattice structure* if two nonempty set-valued operations $(\varDelta, \nabla)$ with the order strucuture are defined.

Definition 4: A set X is said to be a *lattice* if two single-valued operations $(\varDelta, \nabla)$ with the order strucuture are defined.

Note that if (Δ, ∇) are single-valued, then the conditions stated in Definition 2 coincide with *Idempotency*, *Commutativity*, *Associativity*, and *Absorption Identities*, respectively. See Grätzer (2000) for the details of these algebraical notions. In the rest of this subsection, we state the alternative definitions of a preordered lattice structure and a lattice, which regard a preordered lattice structure and a lattice as a special case of preordered set and a partially ordered set, respectively.

Definition 5: Let X be a set endowed with a preorder $\preccurlyeq$. We say that an element u is an upper bound of x and y if $x \preccurlyeq u$ and $y \preccurlyeq u$. Let U be the set of upper bounds of x and y. We say that an element $a \in X$ is a supremum of x and y, if $a \in U$ and $a \preccurlyeq u$ for all $u \in U$. We write the set of supremums of x and y as $A_{x,y}$. In a similar fasion, we can define the set of infimums of x and y, and we write this as $T_{x,y}$. We say that X is a *preordered lattice structure* if both $T_{x,y}$ and $A_{x,y}$ are nonempty.

Definition 6: Let X be a set endowed with a partial order $\leq$. We say that X is a *lattice* if both the supremum and the infimum[3] of every two element $x, y \in X$ with respect to $\leq$ exist. The former is often represented by $x \vee y$ and the latter is represented by $x \wedge y$.

It is not difficult to check that the algebraical definitions and the order theoretical definitions are equivalent with each other. Indeed, with the order structure, one can define the preorder $\preccurlyeq$ as $x \preccurlyeq y \iff (x \nabla y) = I_y$, and construct a preordered lattice structure, with the set of the supremums of every $x, y \in X$ being equal to $(x \nabla y)$ and the set of the infimums being equal to $(x \Delta y)$. If (Δ, ∇) are single-valued, then a preorder $\preccurlyeq$ defined as above is, in fact, a partial order and X be a lattice under this partial order with $(x \Delta y) = x \wedge y$ and $(x \nabla y) = x \vee y$, respectively.

2.2 (Δ, ∇)-strong set order

Then, we define the notions on the changes of feasible sets and solution sets. By analogy of the notion of *strong set order*, which plays a crucial role in the traditional theory of lattice programming, we introduce the generalized strong set relation from the set-valued operations (Δ, ∇). In this case, we can consider the following four types of strong set orders.

Definition 7: Let X be a (Δ, ∇)-lattice structure and $S, S' \subset X$.

1. S' is said to be higher than S in *$ww(\Delta, \nabla)$-strong set order* if $(x \Delta y) \cap S \neq \emptyset$ and $(x \nabla y) \cap S' \neq \emptyset$ for every $x \in S$ and $y \in S'$. We write this as $S \leq_{ww(\Delta,\nabla)} S'$.
2. S' is said to be higher than S in *$ws(\Delta, \nabla)$-strong set order* if $(x \Delta y) \cap S \neq \emptyset$ and $(x \nabla y) \subset S'$ for every $x \in S$ and $y \in S'$. We write this as $S \leq_{ws(\Delta,\nabla)} S'$.
3. S' is said to be higher than S in sw-strong set order and write this as $S \leq_{sw(\Delta,\nabla)} S'$ if $(x \Delta y) \subset S$ and $(x \nabla y) \cap S' \neq \emptyset$ for every $x \in S$ and $y \in S'$

[3] The supremum of $x, y \in X$ is often called "join". Similarly, the infimum is called "meet".

4. S' is said to be higher than S in *ss*-strong set order and write this as $S \leq_{ss(\Delta,\nabla)} S'$ if $(x\Delta y) \subset S$ and $(x\nabla y) \subset S'$ for every $x \in S$ and $y \in S'$.

Note that *ss*-strong set relation implies both *sw*- and *ws*- strong set relations and *sw*- and *ws*-strong set relations imply than *ww*-strong set relation, respectively. It is also worth noting that these four concepts are all equivalent if both Δ and ∇ are single-valued operations. Following Quah (2007), in such cases, we call this relation (Δ, ∇)-strong set relation and write $S \leq_{(\Delta,\nabla)} S'$ when the set S' is higher than S. We give an intuitive example which is based on Antoniadou (2007), and Shirai (2008a).

Example 1: Consider the budget set in the consumer's utility maximization problem in which the consumption set of the consumer is $\mathbb{R}^n_+$. When the wealth level is $w > 0$, as long as the price vector p is fixed, the budget set of the consumer can be represented by $B(w) = \{x \mid px \leq w\}$. Antoniadou (2007) defines the preorder $\preccurlyeq_{(p,i)}$, which is named as *the direct* (p,i)*-value order* as follows: $x \preccurlyeq_{(p,i)} y \iff x_i \leq y_i,$ *and* $px \leq py$. In fact, with the (p,i)-value order endowed, the consumption set $\mathbb{R}^n_+$ is the preordered lattice structure. Hence, we can regard $(x\Delta y) = T_{x,y} = \{t \mid t_i = \min[x_i, y_i],\ pt = \min[px, py]\}$. Similarly, define $(x\nabla y) = A_{x,y} = \{a \mid a_i = \max[x_i, y_i],\ pa = \max[px, py]\}$. Then, the budget set $B(w)$ is monotone in the sense of *ss*-strong set order. That is, for all $w < w'$, we have $B(w) \leq_{ss(\Delta,\nabla)} B(w')$.

2.3 (Δ, ∇)-quasisupermodularity

Finally, we refer to the third point of what should be examined when one intend to perform lattice theoretical comparative statics: *quasisupermodularity* of functions. Similar to the case with strong set order, we have the following four types of quasisupermodularity.

Definition 8: Let X be a (Δ, ∇)-lattice structure. A function $f : X \to \mathbb{R}$ is said to be *ww-quasisupermodular* if it satisfies

$$\forall t \in (x\Delta y); f(x) \geq (>) f(t) \Rightarrow \exists a \in (x\nabla y); f(a) \geq (>) f(y)$$

for every $x, y \in X$. If f satisfies

$$\forall t \in (x\Delta y); f(x) \geq (>) f(t) \Rightarrow \forall a \in (x\nabla y); f(a) \geq (>) f(y),$$

then we say that f is *ws-quasisupermodular*. A function f is said to be *sw-quasisupermodular* if it satisfies

$$\exists t \in (x\Delta y); f(x) \geq (>) f(t) \Rightarrow \exists a \in (x\nabla y); f(a) \geq (>) f(y).$$

We say that f is *ss-quasisupermodular* if it satisfies

$$\exists t \in (x\Delta y); f(x) \geq (>) f(t) \Rightarrow \forall a \in (x\nabla y); f(a) \geq (>) f(y).$$

Again by analogy of strong set order, ss-quasisupermodularity implies both sw- and ws-quasisupermodularity, and sw-quasisupermodularity and ws-quasisupermodularity imply ww-quasisupermodularity, respectively. If both $\mathit{\Delta}$ and ∇ are single-valued operations, these four concepts are all equivalent and we call that $(\mathit{\Delta}, \nabla)$-quasisupermodularity. As is mentioned in Section 1, in the context of the standard lattice programming, quasisupermodularity of the objective function constitutes the necessary and sufficient conditon for the monotonicity of the solution set of optimization problems in the sense of the strong set order. On the other hand, in our generalized environments, the necessary conditions and the sufficient conditions have to be considered separately, since the former needs additional conditions on the structure of the set-valued operations $(\mathit{\Delta}, \nabla)$, while the latter does not.

3 Sufficient Conditions for Monotonicity

In this section, we consider the sufficient conditions for the monotonicity of the solution set of the constrained optimization problem. In the rest of this paper, we suppose that X is a $(\mathit{\Delta}, \nabla)$-lattice structure and that $f : X \to \mathbb{R}$. We deal with the optimization problem in the form of

$$\max_{x \in S \subset X} f(x)$$

and write the solution set of it as $M(S)$. We assume $M(S) \neq \emptyset$ for any $S \neq \emptyset$. We perform the comparative statics of $M(S)$ with respect to the change of the feasible set S and have the following four monotone comparative statics theorems depending on the degrees of the changes of the feasible set S, which are measured by the strong set orders defined in the previous section. Needless to say, the stronger the degree of the changes of the feasible sets, the milder condition on the objective function which assures the monotonicity of the solution sets.

Theorem 1: *Let X be a $(\mathit{\Delta}, \nabla)$-lattice structure, and $S, S' \subset X$. Suppose $S \leq_{ss(\mathit{\Delta},\nabla)} S'$. Then, we have the followings.*

1. *If f satisfies ww-quasisupermodularity, then $M(S) \leq_{ww(\mathit{\Delta},\nabla)} M(S')$.*
2. *If f satisfies ws-quasisupermodualarity, then $M(S) \leq_{ws(\mathit{\Delta},\nabla)} M(S')$.*
3. *If f satisfies sw-quasisupermodularity, then $M(S) \leq_{sw(\mathit{\Delta},\nabla)} M(S')$.*
4. *If f satisfies ss-quasisupermodualarity, then $M(S) \leq_{ss(\mathit{\Delta},\nabla)} M(S')$.*

Proof First, we show that ww-quasisupermodularity of objective function f implies $M(S) \leq_{ww(\mathit{\Delta},\nabla)} M(S')$. Suppose f is ww-quasisupermodular and $x \in M(S)$ and $y \in M(S')$. Then, it is obvious that $f(x) \geq f(t)$ for all $t \in (x\mathit{\Delta}y)$. By ww-quasisupermodularity, there exists at least one $a \in (x\nabla y)$ such that $f(a) \geq f(y)$, which implies $(x\nabla y) \cap M(S') \neq \emptyset$. Since $f(y) = f(a)$ for some $a \in (x\nabla y)$, by ww-quasisupermodularity, there exists at least one $t \in (x\Delta y)$ such that $f(x) = f(t)$, that is, $(x\Delta y) \cap M(S) \neq \emptyset$. This implies $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.

Then, suppose f is ws-quasisupermodular and $x \in M(S)$ and $y \in M(S')$. By ws-quasisupermodularity, we have $f(a) \geq f(y)$ for all $a \in (x\nabla y)$. This means $(x\nabla y) \subset M(S')$. Again by ws-quasisupermodularity, we have $f(x) = f(t)$ for at leaset one $t \in (x\Delta y)$, that is, $(x\Delta y) \cap M(S) \neq \emptyset$. This implies $M(S) \leq_{ws(\Delta,\nabla)} M(S')$.

Suppose that f is sw-quasisupermodular, $x \in M(S)$, and $y \in M(S')$. By the definition, $f(x) \geq f(t)$ for all $t \in (x\Delta y) \cap S \neq \emptyset$. This and the assumption of sw-quasisupermodularity implies that there exists some $a \in (x\nabla y) \subset S'$ such that $f(a) \geq f(y)$, which implies $(x\nabla y) \cap M(S') \neq \emptyset$. Again by sw-quasisupermodularity, for all $t \in (x\Delta y)$, we must have $f(x) = f(t)$, which completes the proof. The case with ss-quasisupermodularity, we use the fact that ss-quasisupermodularity implies both ws- and sw-quasisupermodularity of f. As is the same with the preceding paragraphs, suppose $x \in M(S)$, and $y \in M(S')$. It is obvious that $f(a) \geq f(y)$ for all $a \in (x\nabla y)$ by ws-quasisupermodularity. Hence, $(x\nabla y) \subset M(S')$. Since $f(a) = f(y)$ for all $a \in (x\nabla y)$, by sw-quasisupermodularity, it must follows that $f(x) = f(t)$ for all $t \in (x\Delta y)$, which means $(x\Delta y) \subset M(S)$. *[Q.E.D.]*

Remark 1: The last paragraph of the preceding proof implies that the joint assumption of ws-quasisupermodularity and sw-quasisupermodularity is, in fact, sufficient for $M(S) \leq_{ss(\Delta,\nabla)} M(S')$ under the condition of $S \leq_{ss(\Delta,\nabla)} S'$. Nevertheless, under the condition which enables us to prove the converse of Theorem 1 and other theorems in this section, ss-quasisupermodularity is equivalent with the joint assumption of ws- and sw-quasisupermodularity. We refer this point anew in the next section.

Theorem 2: *Let X be a (Δ,∇)-lattice structure, and $S, S' \subset X$. Suppose $S \leq_{sw(\Delta,\nabla)} S'$. Then, we have the followings.*

1. *If f has ws-quasisupermodularity, then $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.*
2. *If f has ss-quasisupermodular, then $M(S) \leq_{sw(\Delta,\nabla)} M(S')$.*

Proof Suppose that $x \in M(S)$ and $y \in M(S')$. Since $S \leq_{sw(\Delta,\nabla)} S'$, $(x\Delta y) \subset S$ and this implies that $f(x) \geq f(t)$ for all $t \in (x\Delta y)$. Since the fuction f is ws-quasisupermodular, we have $f(a) \geq f(y)$ for all $a \in (x\nabla y)$, which means $(x\nabla y) \cap M(S') \neq \emptyset$. Again by ws-quasisupermodularity, we must have $f(x) = f(t)$ for at least one $t \in (x\Delta y)$. This implies $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.

The case with ss-supermodularity, we already have $M(S) \leq_{ww(\Delta,\nabla)} M(S')$ by the previous paragraph of this proof. What we have to show is that $(x\Delta y) \subset M(S)$. Suppose $f(x) > f(t)$ for some $t \in (x\Delta y)$. By ss-quasisupermodularity, we have $f(a) > f(y)$ for all $a \in (x\nabla y)$ However, this contradicts the fact that $y \in M(S')$. *[Q.E.D.]*

Theorem 3: *Let X be a (Δ,∇)-lattice structure, and $S, S' \subset X$. Suppose $S \leq_{ws(\Delta,\nabla)} S'$. Then, we have the followings.*

1. *If f has sw-quasisupermodularity, then $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.*

2. If f has ss-quasisupermodularity, then $M(S) \leq_{ws(\Delta,\nabla)} M(S')$.

Proof Suppose $x \in M(S)$ and $y \in M(S')$. Since $(x\Delta y) \cap S \neq \emptyset$, there exists at least one $t \in (x\Delta y)$ such that $f(x) \geq f(t)$. By sw-quasisupermodularity of f implies that there exists at least one $a \in (x\nabla y)$ such that $f(a) \geq f(y)$. Since $(x\nabla y) \subset S'$, this means $a \in M(S')$. Again by sw-quasisupermodularity, we have $f(x) = f(t)$ for all $t \in (x\Delta y)$. This implies there exists at least one $t \in M(S)$, which means $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.

The case with ss-supermodularity, what we have to show is that $(x\nabla y) \subset M(S')$. Suppose $f(y) > f(a)$ for some $a \in (x\nabla y)$. By ss-quasisupermodularity, $f(t) > f(x)$ for all $t \in (x\Delta y)$, which contradicts the assumption of $x \in M(S)$. *[Q.E.D]*

Theorem 4: *Let X be a (Δ, ∇)-lattice structure, and $S, S' \subset X$. Suppose $S \leq_{ww(\Delta,\nabla)} S'$. If f is ss-quasisupermodular, then we have $M(S) \leq_{ww(\Delta,\nabla)} M(S')$.*

Proof Suppose $x \in M(S)$ and $y \in M(S')$. Since $(x\Delta y) \cap S \neq \emptyset$, there exists at least one $t \in (x\Delta y)$ such that $f(x) \geq f(t)$. By ss-quasisupermodularity of f, we have $f(a) \geq f(y)$ for all $a \in (x\nabla y)$. Since $(x\nabla y) \cap S' \neq \emptyset$, this means $(x\nabla y) \cap M(S') \neq \emptyset$. Suppose $f(x) > f(t)$ for some $t \in (x\Delta y)$, then $f(a) > f(t)$ for all $a \in (x\nabla y)$, which contradicts the fact that $y \in M(S')$, hence $f(x) = f(t)$ for all $t \in (x\Delta y)$. Since $(x\Delta y) \cap S \neq \emptyset$, this means $(x\Delta y) \cap M(S) \neq \emptyset$. Now, we have $M(S) \leq_{ww(\Delta,\nabla)} M(S')$. *[Q.E.D.]*

We can summarize the above results as follows. For example, one can read off that ss-quasisupermodularity causes ss-strong set monotonicity of $M(S)$ under $S \leq_{ss(\Delta,\nabla)} S'$, and does sw-strong set monotonicity of $M(S)$ under $S \leq_{sw(\Delta,\nabla)} S'$ from the first and second components of the first row in Table 1 below.

Table 1	Set Relation			
Quasisupermodularity	ss	sw	ws	ww
ss-quasisupermodular	ss	sw	ws	ww
sw-quasisupermodular	sw	×	ww	×
ws-quasisupermodular	ws	ww	×	×
ww-quasisupermodular	ww	×	×	×

4 Necessary Conditions for Monotonicity

Although the arguments on the sufficient conditions for the monotonicity do not require any additional conditions on the set-valued operations (Δ, ∇), to derive the necessary conditions for the monotonicity of the solution set, we need weakend version of the order structure, which we call "*the pseudo order structure* on these operations. The formal definition of the pseudo order structure is as follows.

Definition 9: Let X be a (Δ, ∇)-lattice structure. We say that (Δ, ∇) satisfies *the pseudo order structure* if the conditions

1. $(x' \varDelta y) = (x \varDelta y)$ and $(x' \nabla y) = (x \nabla y)$ for every $x' \in I_x$ and $y \in X$.
2. $\forall t \in (x \varDelta y); (t \varDelta y) = (x \varDelta y)$ and $\forall a \in (x \nabla y); (x \nabla a) = (x \nabla y)$ for every $x, y \in X$.
3. $\forall t \in (x \varDelta y), a \in (x \nabla y)$; $(t \varDelta a) = (x \varDelta y)$ and $(t \nabla a) = (x \nabla y)$ for every $x, y \in X$.

are satisfied in addition to the set valued reflexibility and the set-valued absorption identities.

The naming of "the pseudo order structure" originates from the fact that $(\varDelta, \nabla)$ defined by some preorder satisfies this property. However, the satisfaction of orderedlike structure does not imply that $(\varDelta, \nabla)$ is constructed from some order. See the following example.

Example 2: Let $X = \mathbb{R}^2$. Define the operations $(\nabla, \varDelta)$ as follows.

$$
\begin{aligned}
x \nabla y &= y && if\ x_1 \leq y_1 \\
&= \lambda x + (1-\lambda)(x \vee_E y) && if\ x_1 > y_1 \\
x \varDelta y &= x && if\ x_1 \leq y_1 \\
&= \lambda y + (1-\lambda)(x \wedge_E y) && if\ x_1 > y_1
\end{aligned}
$$

where $\lambda \in [0,1]$, and $\vee_E$ and $\wedge_E$ designate the supremum and the infimum, respectively with respect to Euclidean order. Note that the above definition of $(\varDelta, \nabla)$ is the same as that of $(\varDelta_1^\lambda, \nabla_1^\lambda)$ in Quah (2007). These $(\nabla, \varDelta)$ satisfy the pseudo order structure. However, both $\varDelta$ and ∇ do not satisfy commutativity. Indeed, for $x = (1,0)$, $y = (0,1)$, and $\lambda = \frac{1}{2}$, we have

$$
\begin{aligned}
(x \nabla y) &= \frac{1}{2}((1,0) + (1,1)) = (1, \frac{1}{2}) \\
(y \nabla x) &= (1,0) \\
(x \varDelta y) &= \frac{1}{2}((0,1) + (0,0)) = (0, \frac{1}{2}) \\
(y \varDelta x) &= (0,1).
\end{aligned}
$$

Then, we state the counterparts of Theorem 1-4, that is, the necessary conditions for monotone comparative statics.

Theorem 5: *Let X be a $(\varDelta, \nabla)$-lattice structure, and $(\varDelta, \nabla)$ satisfy the pseudo order structure. Suppose $S, S' \subset X$ and $S \leq_{ss(\varDelta,\nabla)} S'$. Then, $M(S) \leq_{ww(\varDelta,\nabla)} M(S')$ implies ww-quasisupermodularity of f. If $M(S) \leq_{ws(\varDelta,\nabla)} M(S')$, then ws-quasisupermodularity is implied. Similarly, we have that $M(S) \leq_{sw(\varDelta,\nabla)} M(S')$ and $M(S) \leq_{ss(\varDelta,\nabla)} M(S')$ implies sw-quasisupermodularity and ss-supermodularity, respectively.*

Proof Throughout this proof, we set $S = I_x \cup (x \varDelta y)$ and $S' = I_y \cup (x \nabla y)$ for some $x, y \in X$. Note that, by the pseudo order structure, $S \leq_{ss(\varDelta,\nabla)} S'$. First, we show the case with $M(S) \leq_{ww(\Delta,\nabla)} M(S')$. Let $x \in M(S)$, that is, $f(x) \geq f(t)$ for all $t \in (x\Delta y)$. Since $M(S) \leq_{ww(\Delta,\nabla)} M(S')$, we have $(x\nabla z') \cap M(S') \neq \emptyset$ for every $z' \in M(S')$. By the pseudo order structure, $(x\nabla z') = (x\nabla y)$, which implies that

$$\forall t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \exists a \in (x\nabla y); f(a) \geq f(y).$$

Then, suppose $f(x) > f(t)$ for all $t \in (x\Delta y)$ and $f(y) = f(a)$ for all $a \in (x\nabla y)$. Since $M(S) \leq_{ww(\Delta,\nabla)} M(S')$, $(x\Delta y) \cap M(S) \neq \emptyset$, which is contradiction. Thus, we have

$$\forall t \in (x\Delta y); f(x) > f(t) \Rightarrow \exists a \in (x\nabla y); f(a) > f(y).$$

Second, we show the case with $M(S) \leq_{ws(\Delta,\nabla)} M(S')$. Suppose $x \in M(S)$. Then, by the assumption of $M(S) \leq_{ws(\Delta,\nabla)} M(S')$, we have $(x\nabla z) \subset M(S')$ for every $z' \in M(S')$. By pseudo order structure, we have $(x\nabla z') = (x\nabla y)$, which implies that

$$\forall t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \forall a \in (x\nabla y); f(a) \geq f(y).$$

Then, suppose $f(x) > f(t)$ for all $t \in (x\Delta y)$ and $f(y) = f(a)$ for some $a \in (x\nabla y)$. By $M(S) \leq_{ws(\Delta,\nabla)} M(S')$, we must have $(x\Delta y) \cap M(S) \neq \emptyset$. This contradicts the assumption of $f(x) > f(t)$. Thus, we have

$$\forall t \in (x\Delta y); f(x) > f(t) \Rightarrow \forall a \in (x\nabla y); f(a) > f(y).$$

Third, we deal with the case with $M(S) \leq_{sw(\Delta,\nabla)} M(S')$. Suppose $f(a) < f(y)$ for all $a \in (x\nabla y)$, that is, $M(S') \subset I_y$. By the pseudo order structure, we have $(z\nabla z') = (z\nabla y)$ for every $z \in M(S)$ and $z' \in M(S')$. Note that $M(S) \cap I_x = \emptyset$. Suppose not. Then, $(z\nabla z') = (x\nabla y) \cap M(S') \neq \emptyset$, which contradicts our assumption. Again by the pseudo order structure, we have $(z\Delta z') = (x\Delta y) \subset M(S)$. This implies that $M(S) = (x\Delta y)$, that is,

$$\forall a \in (x\nabla y); f(a) < f(y) \Rightarrow \forall t \in (x\Delta y); f(x) < f(t).$$

Taking the contraposition, we have

$$\exists t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \exists a \in (x\nabla y); f(a) \geq f(y).$$

The case with strict inequalities can be proven as follows. Suppose $f(x) > f(t)$ for some $t \in (x\Delta y)$. Then, $M(S') \cap I_y$ must be empty. Indeed, if there exists $z' \in I_y \cap M(S')$, by the pseudo order structure, we must have $(z\Delta z') = (x\Delta y) \subset M(S)$ for every $z \in M(S)$, contradiction. This implies that

$$\exists t \in (x\Delta y); f(x) > f(t) \Rightarrow \exists a \in (x\nabla y); f(a) > f(y).$$

Finally, we show that $M(S) \leq_{ss(\Delta,\nabla)} M(S')$ leads ss-quasisupermodularity of f. Suppose $f(a) < f(y)$ for some $a \in (x\nabla y)$. Note that $M(S) \cap I_x$ must be empty. Indeed, if there exists some $z \in M(S) \cap I_x$, by the assumption of $M(S) \leq_{ss(\Delta,\nabla)} M(S')$ and the pseudo order structure, $(z\nabla z') = (x\nabla y) \subset M(S')$ for every $z' \in M(S')$, which is contradiction. Moreover, we have $(z\Delta z') = (x\Delta y) \subset M(S)$, hence $(x\Delta y) = M(S)$. Thus, we have

$$\exists a \in (x\nabla y); f(a) < f(y) \Rightarrow \forall t \in (x\Delta y); f(x) < f(t).$$

Taking the contraposition,

$$\exists t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \forall a \in (x\nabla y); f(a) \geq f(y).$$

The case with strong inequalities can be proven as follows. Suppose $f(x) > f(t)$ for some $t \in (x\Delta y)$. As is the same with the previous paragraph, we can show that $M(S') \cap I_y = \emptyset$. Thus, $M(S') \subset (x\nabla y)$. Moreover, by the pseudo order structure and the assumption of $M(S) \leq_{ss(\Delta,\nabla)} M(S')$, we have $(z\nabla z') = (x\nabla y) \subset M(S')$ for any pair of $z \in M(S)$ and $z' \in M(S')$. This implies that

$$\exists t \in (x\Delta y); f(x) > f(t) \Rightarrow \forall a \in (x\nabla y); f(a) > f(y).$$

This completes the proof. *[Q.E.D.]*

Corollary 1: *Let X be a (Δ,∇)-lattice structure and $f : X \to \mathbb{R}$. Suppose that (Δ,∇) satisfy the pseudo order structure. Then, the objective function f satisfies ss-quasisupermodularity if and only if it satisfies both ws-quasisupermodularity and sw-quasisupermodularity.*

Proof By the definitions, only if part is obvious. It is already refered, at Remark 1, that the joint assumption of *ws*-quasisupermodularity and *sw*-quasisupermodularity implies $M(S) \leq_{ss(\Delta,\nabla)} M(S')$. The preceding theorem proposes that $M(S) \leq_{ss(\Delta,\nabla)} M(S')$ under the condition of $S \leq_{ss(\Delta,\nabla)} S'$ implies *ss*-quasisupermodularity, whence our claim follows. *[Q.E.D.]*

Theorem 6: *Let X be a (Δ,∇)-lattice structure, and (Δ,∇) satisfy the pseudo order structure. Suppose $S, S' \subset X$ and $S \leq_{sw(\Delta,\nabla)} S'$. Then, $M(S) \leq_{ww(\Delta,\nabla)} M(S')$ implies ws-quasisupermodularity of f. If $M(S) \leq_{sw(\Delta,\nabla)} M(S')$, then ss-quasisupermodularity is implied.*

Proof Let $S = I_x \cup (x\Delta y)$ and $S'_{\bar{a}} = I_y \cup \{\bar{a}\}$ for some $\bar{a} \in (x\nabla y)$. By the pseudo order structure, $S \leq_{sw(\Delta,\nabla)} S'_{\bar{a}}$. Since *ss*-strong set relation implies *sw*-strong set relation, and by the previous theorem, it is already known that $M(S) \leq_{ww(\Delta,\nabla)} M(S'_{\bar{a}})$ leads *ww*-quasisupermodularity. Suppose $x \in M(S)$. By the assumption of $M(S) \leq_{ww(\Delta,\nabla)} M(S'_{\bar{a}})$ and the pseudo order structure, $\bar{a}$ must be contained in $M(S'_{\bar{a}})$, that is, $f(\bar{a}) \geq f(y)$. Since the choice of $\bar{a}$ is arbitrary, we have $f(a) \geq f(y)$ for all $a \in (x\nabla y)$. Thus, we have

$$\forall t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \forall a \in (x\nabla y); f(a) \geq f(y).$$

The case with strong inequalities can be shown as follows. Suppose $f(x) > f(t)$ for all $t \in (x\Delta y)$ and $f(\bar{a}) = f(y)$. Then, by the assumption of $M(S) \leq_{ww(\Delta,\nabla)} M(S'_{\bar{a}})$, we must have $(x\Delta y) \cap M(S) \neq \emptyset$, contradiction.

On the case with $M(S) \leq_{sw(\Delta,\nabla)} M(S')$, by the previous theorem and the preceding paragraph of this proof, it is already known that f must satisfy both sw-quasisupermodularity and ws-quasisupermodularity. Hence our claim follows from Corollary 1. *[Q.E.D.]*

Theorem 7: *Let X be a (Δ,∇)-lattice structure, and (Δ,∇) satisfy the pseudo order structure. Suppose $S, S' \subset X$ and $S \leq_{ws(\Delta,\nabla)} S'$. Then, $M(S) \leq_{ww(\Delta,\nabla)} M(S')$ implies sw-quasisupermodularity of f. If $M(S) \leq_{ws(\Delta,\nabla)} M(S')$, then ss-quasisupermodularity is implied.*

Proof Let $S_{\bar{t}} = I_x \cup \{\bar{t}\}$ for some $\bar{t} \in (x\Delta y)$ and $S' = I_y \cup (x\nabla y)$. By the pseudo order strucutre, $S_{\bar{t}} \leq_{ws(\Delta,\nabla)} S'$. By Theorem 5, the necessity of ww-quasisupermodularity is already known. Suppose $f(a) < f(y)$ for all $a \in (x\nabla y)$ and $I_x \cap M(S_{\bar{t}}) \neq \emptyset$. Then, by the pseudo order structure, we must have $(x\nabla y) \cap M(S') \neq \emptyset$, which is contradiction. Thus, $f(x) < f(\bar{t})$. Since the choice of $\bar{t} \in (x\Delta y)$ is arbitrary, $f(x) < f(t)$ for all $t \in (x\Delta y)$. Hence, taking the contraposition,

$$\exists t \in (x\Delta y); f(x) \geq f(t) \Rightarrow \exists a \in (x\nabla y); f(a) \geq f(y).$$

Suppose $f(a) \leq f(y)$ for all $a \in (x\nabla y)$. Then, by the pseudo order structure, we have $(x\Delta y) \cap M(S) \neq \emptyset$, that is, $f(x) \leq f(\bar{t})$. Since the choice of $\bar{t}$ is arbitrary, by taking the contraposition,

$$\exists t \in (x\Delta y); f(x) > f(t) \Rightarrow \exists a \in (x\nabla y); f(a) > f(y).$$

The rest of the proof, to show that $M(S_{\bar{t}}) \leq_{ws(\Delta,\nabla)} M(S')$ leads ss-quasisupermodularity, can be constructed by the same way as the previous theorem. *[Q.E.D.]*

Theorem 8: *Let X be a (Δ,∇)-lattice structure, and (Δ,∇) satisfy the pseudo order structure. Suppose $S, S' \subset X$ and $S \leq_{ww(\Delta,\nabla)} S'$. Then, $M(S) \leq_{ww(\Delta,\nabla)} M(S')$ implies ss-quasisupermodularity of f.*

Proof By the preceding theorems, f must satisfies both ws-supermodularity and sw-quasisupermodularity. Hence, by Corollary 1, our claim follows. *[Q.E.D.]*

The assumption of the pseudo order structure on (Δ,∇) plays a crucial role in the proofs of Theorem 5-8. Indeed, without this assumption, we can present a counter example [4]. See the following example.

[4] Note that the existence of this counter example does not cause any difficulties for the main result in Quah (2007).

Example 3: Let $X = \{-1, 0, 1\}$ and define $\varDelta$ and ∇ as

$$
\begin{aligned}
x \varDelta y &= xy \\
x \nabla y &= 1 & if\ x = y = 1 \\
&= -1 & if\ x = y = -1 \\
&= x + y & otherwise.
\end{aligned}
$$

Define a function

$$
\begin{aligned}
f(x) &= 1 & if\ x = -1 \\
&= 2 & if\ x = 0 \\
&= 3 & if\ x = 1.
\end{aligned}
$$

This function does *not* satisfy $(\nabla, \varDelta)$-quasisupermodularity. Indeed, when $x = -1$ and $y = 1$, we have both $f(x) = 1 = f(x\varDelta y)$ and $f(x\nabla y) = 2 < 3 = f(y)$. Although f does not satisfy $(\nabla, \varDelta)$-quasisupermodularity, we can show that $M(S) \leq_{(\nabla,\varDelta)} M(S')$ for every $S \leq_{(\nabla,\varDelta)} S'$ as follows.

If $\{0\} = M(S)$, it is easy to check that $M(S) \leq_{(\nabla,\varDelta)} M(S')$. Indeed, for an arbitrary element $y \in M(S')$, $0\varDelta y = 0 \in M(S)$ and $0\nabla y = y \in M(S')$. Suppose $\{1\} = M(S)$. In this case, under the condition of $S \leq_{(\nabla,\varDelta)} S'$, if $0 \in S' \Rightarrow 1 \in S'$ since $1 \in S$. Similarly, $-1 \in S' \Rightarrow 0 \in S'$. As a result, we have $1 \in S'$, hence $\{1\} = M(S')$, which implies that $M(S) \leq_{(\nabla,\varDelta)} M(S')$. Finally, we show that, under the condition of $S \leq_{(\nabla,\varDelta)} S'$, -1 cannot be the element of $M(S)$. It is obvious that if $-1 \in M(S)$, S must be equal to $\{-1\}$. Suppose there exists some $S' \subset X$ such that $\{-1\} \leq_{(\nabla,\varDelta)} S'$. Let y be an arbitrary element of S'. Then, since $-1\varDelta y = -y$ must be equal to -1, we have $S' = \{1\}$. However, $-1\nabla 1 = 0 \notin S'$, contradiction. Thus, we show that $S \leq_{(\nabla,\varDelta)} S' \Rightarrow M(S) \leq_{(\nabla,\varDelta)} M(S')$ despite the lack of $(\nabla, \varDelta)$-quasisupermodularity of f.

Finally, as the corollaries of Theorem 1-8, we state the necessary and sufficient relationship between the monotonicity of solution set of constrained optimization problems and quasisupermodularity under the pseudo order structure of set-valued operations on the domain of objective function.

Corollary 2: *Let X be a $(\varDelta, \nabla)$-lattice structure, $S, S' \subset X$, and $S \leq_{ss(\varDelta,\nabla)} S'$. Suppose $(\varDelta, \nabla)$ satisfies the pseudo order structure. Then, we have the following necessary and sufficient relations.*

1. *$M(S) \leq_{ww(\varDelta,\nabla)} M(S')$ if and only if f satisfies ww-quasisupermodularity.*
2. *$M(S) \leq_{ws(\varDelta,\nabla)} M(S')$ if and only if f satisfies ws-quasisupermodularity.*
3. *$M(S) \leq_{sw(\varDelta,\nabla)} M(S')$ if and only if f satisfies sw-quasisupermodularity.*
4. *$M(S) \leq_{ss(\varDelta,\nabla)} M(S')$ if and only if f satisfies ss-quasisupermodularity.*

Corollary 3: *Let X be a (Δ, ∇)-lattice structure, $S, S' \subset X$, and $S \leq_{sw(\Delta,\nabla)} S'$. Suppose (Δ, ∇) satisfies the pseudo order structure. Then, we have the following necessary and sufficient relations.*

1. *$M(S) \leq_{ww(\Delta,\nabla)} M(S')$ if and only if f satisfies ws-quasisupermodularity.*
2. *$M(S) \leq_{sw(\Delta,\nabla)} M(S')$ if and only if f satisfies ss-quasisupermodularity.*

Corollary 4: *Let X be a (Δ, ∇)-lattice structure, $S, S' \subset X$, and $S \leq_{ws(\Delta,\nabla)} S'$. Suppose (Δ, ∇) satisfies the pseudo order structure. Then, we have the following necessary and sufficient relations.*

1. *$M(S) \leq_{ww(\Delta,\nabla)} M(S')$ if and only if f satisfies sw-quasisupermodularity.*
2. *$M(S) \leq_{ws(\Delta,\nabla)} M(S')$ if and only if f satisfies ss-quasisupermodularity.*

Corollary 5: *Let X be a (Δ, ∇)-lattice structure, $S, S' \subset X$, and $S \leq_{ww(\Delta,\nabla)} S'$. Suppose (Δ, ∇) satisfies the pseudo order structure. Then, $M(S) \leq_{ww(\Delta,\nabla)} M(S')$ if and only if f satisfies ss-quasisupermodularity.*

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