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On the leximin and utilitarian overtaking criteria with extended anonymity

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On the leximin and utilitarian overtaking criteria with extended anonymity*

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Abstract

This paper studies extensions of the leximin and utilitarian overtaking criteria on the set of infinite utility streams. We propose new leximin and utilitarian overtaking criteria, called the $\mathcal{S}$-W-leximin and the $\mathcal{S}$-overtaking social welfare relations (SWRs) respectively, each of which satisfies the extended anonymity called $\mathcal{S}$-Anonymity (or Fixed-Step Anonymity). The axiomatic characterizations (in terms of subrelation) of these SWRs are established. We also show that the $\mathcal{S}$-W-leximin SWR is equivalent to the fixed-step extension à la Lauwers [Aust J Philos **75**, 222-233 (1997)] and Fleurbaey and Michel [J Math Econ **39**, 777-802 (2003)] of the leximin overtaking criterion. On the other hand, the $\mathcal{S}$-overtaking SWR is a subrelation of the fixed-step extension of the utilitarian overtaking criterion and they are not equivalent. To explain this contrast, we also characterize the $\mathcal{S}$-W-leximin SWR and the fixed-step extension of the utilitarian overtaking criterion by using an extended consistency axiom.

Keywords Extended Anonymity; Overtaking criterion; Leximin principle; Utilitarianism

JEL Classification Numbers D63; D71

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1 Introduction

One useful way to evaluate very long-term economic policies affecting future generations (e.g. greenhouse gas abatement programs) is, like in optimal growth theory, modeled by the comparison of infinite utility streams which represent attainable welfare levels of an infinite number of generations. In the study of evaluation of infinite utility streams, Strong Pareto and Finite Anonymity have been usually employed as the basic axioms requiring efficiency and impartiality of evaluation. From the view point of selectivity to be realized in evaluation criteria, the most desirable goal we would like to pursue would be to construct a social welfare ordering (SWO) or, more ambitiously, a social welfare function (SWF) satisfying the two axioms since these criteria enable us to compare *all* utility streams. However, as shown by Basu and Mitra (2003), there is no SWF satisfying the two basic axioms, and as for a SWO, Lauwers (2006) and Zame (2007) recently confirm the conjecture by Fleurbaey and Michel (2003) that any SWO satisfying the axioms must involve the use of non-constructive mathematics, i.e., it cannot have an explicit description and is of no use for a practical purpose.[1] In view of these impossibility results, it becomes more important than ever before to work on social welfare relations (SWRs), transitive but not necessarily complete binary relations, to construct an evaluation criterion satisfying the two basic axioms.[2]

In the case of a SWR, we indeed obtain several constructible criteria, e.g. the *Suppes-Sen* SWR (Svensson 1980).[3] Hence, the task we should address is to extend those constructible but incomplete SWRs to more selective ones by adding some plausible axioms. In the literature, there have been many attempts to construct SWRs satisfying some additional properties as well as the two basic axioms. Analogous to the finite population case, most of such attempts have been done by reformulating utilitarianism and the leximin principle in the context of ranking infinite utility streams.[4] Among the existing infinite-horizon reformulations of these principles, the *utilitarian* SWR proposed by Basu and Mitra (2007a) and its leximin counterpart, called the *leximin* SWR, in Bossert, Sprumont and Suzumura (2007) can be seen as the most basic ones since their axiomatic characterizations are given (in terms of subrelation) by the infinite-horizon variants of the axioms which characterize the finite-horizon leximin and utilitarian orderings: the informational invariance axiom called Partial Translation-Scale Invariance and the equity axiom called Hammond Equity respectively are added to the

[1] In Basu and Mitra (2007b), their impossibility result is strengthened with Weak Pareto on a certain rich domain.

[2] Instead of laying down completeness, Fleurbaey and Michel (2003) and Sakai (2008) examine another route by weakening transitivity to quasi-transitivity.

[3] The Suppes-Sen SWR is the infinite-horizon variant of the grading principle due to Suppes (1966) and Sen (1970). It is characterized (in terms of subrelation) by the two basic axioms (Asheim, Buchholz and Tungodden 2001).

[4] Among the existing reformulations other than those we will mention below, see, for example, the recent work by Asheim, d'Aspremont and Banerjee (2008).

two basic axioms (Basu and Mitra 2007a; Bossert, Sprumont and Suzumura 2007).[5] In view of their axiomatic foundations, the use of the utilitarian and leximin SWRs seems to be quite plausible. However, selectivity inherent in these SWRs remains in an unsatisfactory level because both of these SWRs compare only those utility streams whose tail parts are Pareto comparable beyond a certain finite time. Consequently, once an infinite number of generations come into conflict over increase and decrease in their utilities across two streams, those utility streams are declared to be non-comparable by them.

The purpose of this paper is to extend the utilitarian and leximin SWRs to deal with the conflicts involving an infinite number of generations. For this purpose, we employ two additional axioms. One is the axiom of (weak) consistency employed by Basu and Mitra (2007a), and the other is the extended anonymity which we call $\mathcal{S}$-Anonymity. The (weak) consistency axiom prescribes a consistent way to transform infinite-horizon evaluation to an infinite-number of comparisons of finite-horizon truncated streams. On the other hand, $\mathcal{S}$-Anonymity, which is first introduced by Lauwers (1997b) under the name Fixed-Step Anonymity, formalizes stronger impartiality than Finite Anonymity by allowing for the use of particular infinite permutations called fixed-step permutations. These two axioms have been independently employed in the literature. In Basu and Mitra (2007a), they characterize the *overtaking* SWR due to von Weizsäcker (1965) by adding consistency to the axioms of the utilitarian SWR. The leximin counterpart of it is also formulated by Asheim and Tungodden (2004) with the name *W-leximin* SWR and characterized with an axiom quite similar to consistency.[6] As to $\mathcal{S}$-Anonymity, Banerjee (2006) characterizes the extended utilitarian SWR which we call the $\mathcal{S}$*-utilitarian* SWR by replacing Finite Anonymity with $\mathcal{S}$-Anonymity in the axioms of the utilitarian SWR, and its leximin counterpart called the $\mathcal{S}$*-lexmin* SWR is characterized by Kamaga and Kojima (2008) in a similar manner. For these two types of extended leximin and utilitarian principles, we have, not surprisingly, the trade-off due to differences between the additional axioms that some utility streams are comparable by the overtaking and the W-leximin SWRs but not by the $\mathcal{S}$-utilitarian and the $\mathcal{S}$-leximin SWRs and vice versa. In this paper, we impose both two additional axioms on SWRs and clarify the classes of the extended leximin and utilitarian SWRs satisfying both additional axioms. Clearly, such SWRs resolve the aforementioned trade-off on possible comparisons.

In this paper, we also discuss the strong version of consistency which Basu and Mitra (2007a) use to characterize the *catching-up* SWR due to Atsumi (1965) and

[5] Partial Translation-Scale Invariance is called Partial Unit Comparability in Basu and Mitra (2007a).

[6] Asheim and Tungodden (2004) characterize the W-leximin and the overtaking SWRs with the axiom called Weak Preference Continuity which is similar to Basu and Mitra's consistency axiom. This axiom will also be discussed in Sect. 2.2.

von Weizsäcker (1965).[7] It is known that the catching-up SWR violates $\mathcal{S}$-Anonymity (Banerjee 2006), and this means that it is impossible to obtain the extended leximin and utilitarian SWRs satisfying both strong consistency and $\mathcal{S}$-Anonymity. Our first result shows that this impossibility result is ascribed to the incompatibility of only the three axioms in a SWR: Strong Pareto, strong consistency and $\mathcal{S}$-Anonymity. Thus, the impossibility of the $\mathcal{S}$-anonymous extension of the catching-up SWR can be understood as a continuation of a series of studies on, what we call, Pareto-anonymity dilemma. This point will be discussed in more detail in Sect. 3.1.

In contrast, weak consistency and $\mathcal{S}$-Anonymity are compatible even in the presence of the axioms of the leximin and utilitarian SWRs. We formulate the extended leximin and utilitarian overtaking criteria, called the *$\mathcal{S}$-W-leximin* SWR and the *$\mathcal{S}$-overtaking* SWR, and show that the $\mathcal{S}$-W-leximin and the $\mathcal{S}$-overtaking SWRs are characterized (in terms of subrelation) by weak consistency, $\mathcal{S}$-Anonymity and the axioms of the leximin and utilitarian SWRs respectively.

The $\mathcal{S}$-W-leximin and the $\mathcal{S}$-overtaking SWRs are defined by extending the W-leximin and the overtaking SWRs with an application of fixed-step permutations as in the constructions of the $\mathcal{S}$-utilitarian and the $\mathcal{S}$-leximin SWRs by Banerjee (2006) and Kamaga and Kojima (2008). In the literature, Lauwers (1997b) proposes another extension method to formulate $\mathcal{S}$-anonymous SWRs, and following his extension method, Fleurbaey and Michel (2003) define a variant of the catching-up SWR, called *type 2* SWR, which compares infinite utility streams by successive comparisons of utility sums of certain fixed-periodic truncated streams. Their extension method, which we will refer to as fixed-step extension, provides an alternative route to extend a finitely anonymous SWR to the one satisfying $\mathcal{S}$-Anonymity. In this paper, we clarify the relation between our $\mathcal{S}$-anonymous extensions of the W-leximin and the overtaking SWRs and the fixed-step extensions of them. The results we obtain for the leximin and utilitarian cases are quite contrasting. The $\mathcal{S}$-W-leximin SWR turns out to be equivalent to the fixed-step extension of the W-leximin SWR while the $\mathcal{S}$-overtaking SWR is a subrelation of the fixed-step extension of the overtaking SWR which we will call the *fixed-step overtaking* SWR, and they are *not* equivalent. To explain and highlight the contrast between the leximin and utilitarian cases, we also provide the characterizations of the $\mathcal{S}$-W-leximin and the fixed-step overtaking SWRs by using extended consistency called Weak Fixed-Step Consistency.

The rest of the paper is organized as follows. Section 2 introduces notation and definitions. The axioms we impose on SWRs are also presented there. Section 3 provides the impossibility results regarding strong consistency and also establishes the charac-

[7] The catching-up SWR and its leximin counterpart are also characterized by Asheim and Tungodden (2004) with the strong version of preference continuity. The strong preference continuity axiom is also discussed in Sect. 2.2.

terizations of the $\mathcal{S}$-W-leximin and the $\mathcal{S}$-overtaking SWRs. In Section 4, we prove the equivalence and non-equivalence results for our new $\mathcal{S}$-anonymous SWRs and the fixed-step extensions of the W-leximin and the overtaking SWRs. The characterizations of the $\mathcal{S}$-W-leximin and the fixed-step overtaking SWRs are also established. Section 5 concludes with some remarks.

2 Preliminary

2.1 Notation and definitions

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ the set of all positive integers. Throughout this paper excepting Sect. 3.1, we let $X = \mathbb{R}^{\mathbb{N}}$ be the set of all utility streams $\boldsymbol{x} = (x_1, x_2, \dots)$. For all $i \in \mathbb{N}$, x_i is interpreted as the utility level of the ith generation. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, we write $\boldsymbol{x}^{-n} = (x_1, \dots, x_n)$ and $\boldsymbol{x}^{+n} = (x_{n+1}, x_{n+2}, \dots)$. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, $\left(x_{(1)}^{-n}, \dots, x_{(n)}^{-n}\right)$ denotes a rank-ordered permutation of $\boldsymbol{x}^{-n}$ such that $x_{(1)}^{-n} \leq \cdots \leq x_{(n)}^{-n}$, ties being broken arbitrarily.

Negation of a statement is indicated by the symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} \geqslant \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, and $\boldsymbol{x} > \boldsymbol{y}$ if $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given two sets A and B, we write $A \subseteq B$ to mean A is a subset of B and $A \subsetneq B$ to mean $A \subseteq B$ and $A \neq B$.

A binary relation $\succsim$ on X is a subset of $X \times X$. For convenience, the fact that $(\boldsymbol{x}, \boldsymbol{y}) \in \succsim$ will be symbolized by $\boldsymbol{x} \succsim \boldsymbol{y}$. An asymmetric part of $\succsim$ is denoted by $\succ$ and a symmetric part by $\sim$, i.e. $\boldsymbol{x} \succ \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ holds but $\boldsymbol{y} \succsim \boldsymbol{x}$ does not, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{x}$. A SWR is a reflexive and transitive binary relation on X, i.e. a quasi-ordering.[8] A SWR $\succsim_A$ is said to be a subrelation of a SWR $\succsim_B$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \sim_A \boldsymbol{y}$ implies $\boldsymbol{x} \sim_B \boldsymbol{y}$ and (ii) $\boldsymbol{x} \succ_A \boldsymbol{y}$ implies $\boldsymbol{x} \succ_B \boldsymbol{y}$.

We represent any permutation on the set $\mathbb{N}$ by a permutation matrix. A permutation matrix is an infinite matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ satisfying the following properties:

(i) for each $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$;

(ii) for each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Let $\mathcal{P}$ be the set of all permutation matrices. Note that, for all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{P}$, the product $\boldsymbol{P}\boldsymbol{x} = (Px_1, Px_2, \dots)$ belongs to X, where $Px_i = \sum_{k \in \mathbb{N}} p_{ik} x_k$ for all

[8] A binary relation $\succsim$ on X is (i) reflexive if, for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \succsim \boldsymbol{x}$, and (ii) transitive if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succsim \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$.

$i \in \mathbb{N}$. For any $\boldsymbol{P} \in \mathcal{P}$, let $\boldsymbol{P}'$ be the inverse of $\boldsymbol{P}$ satisfying $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$, where $\boldsymbol{I}$ is the infinite identity matrix.[9] For all $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ and all $n \in \mathbb{N}$, let $\boldsymbol{P}(n)$ denote the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\ldots,n\}}$. A matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ is a finite permutation matrix if there exists $n \in \mathbb{N}$ such that $p_{ii} = 1$ for all $i > n$. Let $\mathcal{F}$ be the set of all finite permutation matrices. We denote by $\mathcal{S}$ the following set of permutation matrices:

$$\mathcal{S} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{l} \text{there exists } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \\ \boldsymbol{P}(nk) \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

The class $\mathcal{S}$ is exactly the set of all fixed-step permutations which was first introduced by Lauwers (1997b).[10]

2.2 Axioms

We introduce the basic axioms which characterize the leximin and utilitarian SWRs proposed by Bossert, Sprumont and Suzumura (2007) and Basu and Mitra (2007a).[11]

Strong Pareto (SP) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} > \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

$\mathcal{F}$-Anonymity (FA) *For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{F}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.*

Hammond Equity (HE) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all $i, j \in \mathbb{N}$, if $y_i < x_i < x_j < y_j$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $\boldsymbol{x} \succsim \boldsymbol{y}$.*

Partial Translation-Scale Invariance (PTSI) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, all $\boldsymbol{a} \in \mathbb{R}^{\mathbb{N}}$ and all $n \in \mathbb{N}$, if $\boldsymbol{x}^{+n} = \boldsymbol{y}^{+n}$ and $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{a} \succsim \boldsymbol{y} + \boldsymbol{a}$.*

FA is also called Finite Anonymity. **HE** is an infinite-horizon variant of the equity axiom due to Hammond (1976), which asserts that an order-preserving change which diminishes inequality of utilities between conflicting two generations is socially preferable.[12] **PTSI** postulates the invariance property corresponding to the assumption that utility differences of generations are comparable but utility levels are not.[13]

We now move to additional axioms to be imposed on a SWR.

$\mathcal{S}$-Anonymity (SA) *For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{S}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.*

Weak Consistency (WC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$,* (i) *if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \ldots) \succ (\boldsymbol{y}^{-n}, 0, 0, \ldots)$, then $\boldsymbol{x} \succ \boldsymbol{y}$;* (ii) *if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \ldots) \sim (\boldsymbol{y}^{-n}, 0, 0, \ldots)$, then $\boldsymbol{x} \sim \boldsymbol{y}$.*

[9] For any $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{P}$, the product $\boldsymbol{P}\boldsymbol{Q}$ is defined by $(r_{ij})_{i,j \in \mathbb{N}}$ with $r_{ij} = \sum_{k \in \mathbb{N}} p_{ik} q_{kj}$.

[10] On the class $\mathcal{S}$, see also the discussion in Sect. 5.

[11] See Footnote 18 for the definitions of these SWRs.

[12] The weaker version of **HE** called Hammond Equity for the Future is proposed by Asheim and Tungodden (2005).

[13] For the detailed explanation of informational invariance axioms, we refer the reader to d'Aspremont and Gevers (2002) and Bossert and Weymark (2004).

Strong Consistency (SC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$,* (i) *if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succsim (\boldsymbol{y}^{-n}, 0, 0, \dots)$, then $\boldsymbol{x} \succsim \boldsymbol{y}$;* (ii) *if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succsim (\boldsymbol{y}^{-n}, 0, 0, \dots)$ and for all $n \in \mathbb{N}$, there exists $n' \geq n$ such that $(\boldsymbol{x}^{-n'}, 0, 0, \dots) \succ (\boldsymbol{y}^{-n'}, 0, 0, \dots)$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

Since $\mathcal{F} \subsetneq \mathcal{S}$, **SA** (also called Fixed-Step Anonymity) is stronger than **FA**.[14] The two versions of consistency axioms are employed by Basu and Mitra (2007a). The interpretation of these axioms may depend on what the zero utility level means in a domain being considered.[15] However, in the presence of the basic axioms: **SP**, **FA** and **HE** or **PTSI**, the independence property implied by these axioms makes irrelevant the utility value taken in the constant tail parts, and we can interpret these axioms as saying that our evaluation of infinite-horizon utility streams should be consistent with an infinite number of evaluation of their truncated streams.[16] It should be noted that Asheim and Tungodden (2004) consider the following similar axioms:

Weak Preference Continuity: For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succ \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

Strong Preference Continuity: For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succsim \boldsymbol{y}$, and for all $n \in \mathbb{N}$, there exists $n' \geq n$ such that $(\boldsymbol{x}^{-n'}, \boldsymbol{y}^{+n'}) \succ \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

Since Weak (resp. Strong) Preference Continuity and **WC** (resp. **SC**) become equivalent under the basic axioms of the leximin and utilitarian SWRs, we will employ only **WC** and **SC** in the sequel. All our results stated with **WC** and **SC** can also be established with these preference continuity axioms.

3 $\mathcal{S}$-anonymous overtaking criteria

3.1 Impossibility results

We now examine the extended leximin and utilitarian SWRs satisfying both the consistency and the extended anonymity. As we noted in Sect. 1, there is no extended utilitarian SWR satisfying both **SA** and **SC**. Before proceeding to the analysis of the case of **WC**, we provide further examination of this impossibility result.

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[14]For the researches employing **SA**, see Lauwers (1997b; 2006), Fleurbaey and Michel (2003), Banerjee (2006), Mitra and Basu (2007), Kamaga and Kojima (2008) and Sakai (2008).

[15]Basu and Mitra (2007a) define these axioms in spirit to Axiom 3 in Brock (1970). In their analysis, the domain of a SWR is taken by $X = [0, 1]^{\mathbb{N}}$. Fleurbaey and Michel (2003) propose a similar axiom under the name Limit Ranking.

[16]The independence property implied by the basic axioms is given by: for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in X$ and all $n \in \mathbb{N}$, $(\boldsymbol{x}^{-n}, \boldsymbol{z}^{+n}) \succsim (\boldsymbol{y}^{-n}, \boldsymbol{z}^{+n})$ iff $(\boldsymbol{x}^{-n}, \boldsymbol{w}^{+n}) \succsim (\boldsymbol{y}^{-n}, \boldsymbol{w}^{+n})$.

As shown in the following proposition, the impossibility in the case of **SC** is ascribed to incompatibility between **SA** and **SC** in a strongly Paretian SWR.

Proposition 1. *Let $X \supseteq \{a, b\}^{\mathbb{N}}$ with a and b being distinct real numbers. Then, there is no SWR $\succsim$ on X satisfying **SP**, **SA** and **SC**.*

Proof. Without loss of generality, we assume that $a = 0$ and $b = 1$. The proof is done by contradiction. Suppose that there exists a SWR $\succsim$ satisfying **SP**, **SA** and **SC**. Let $\boldsymbol{x}, \boldsymbol{y} \in X$ be such that $\boldsymbol{x} = (1, 0, 1, 0, \dots)$ and $\boldsymbol{y} = (0, 1, 0, 1, \dots)$. By **SA**, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \sim (\boldsymbol{y}^{-n}, 0, 0, \dots)$ for all even $n \in \mathbb{N}$. By **SA** and **SP**, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \sim (\boldsymbol{y}^{-(n+2)}, 0, 0, \dots)$ for all odd $n \in \mathbb{N}$ and $(\boldsymbol{x}^{-(n+2)}, 0, 0, \dots) \succ (\boldsymbol{x}^{-n}, 0, 0, \dots)$ for all $n \in \mathbb{N}$, thus by transitivity, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succ (\boldsymbol{y}^{-n}, 0, 0, \dots)$ for all odd $n \geq 3$. By **SC**, $\boldsymbol{x} \succ \boldsymbol{y}$ follows, while $\boldsymbol{x} \sim \boldsymbol{y}$ is obtained by **SA**, a contradiction. □

Remark 1. In view of the proof of Proposition 1, even if **SP** is weakened to the following Paretian axiom, we still obtain the same impossibility result:

Weak Dominance: For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

The trade-offs between some forms of Paretian axioms and certain anonymity requirements have been extensively analyzed in the literature.[17] It is known that **SA** itself is compatible with **SP** in a SWR (Lauwers 1997b), while Strong Anonymity defined by $\mathcal{P}$ (which we may call $\mathcal{P}$-Anonymity) comes in conflict with **SP** even without any rationality condition of a binary relation $\succsim$ (van Liedekerke 1995; Lauwers 1997a). On the other hand, the weaker anonymity **FA** is compatible even with both **SP** and **SC** (and Strong Preference Continuity): e.g. the catching-up SWR (Asheim and Tungodden 2004; Basu and Mitra 2007a). However, as shown by Proposition 1, it is impossible to strengthen **FA** back to **SA** in all such SWRs even in the quite restricted domain, so-called binary domain. Consequently, seeing **SP** and **FA** as indispensable properties of SWRs, our choice of the additional requirement from **SA** and **SC** becomes a branching point in exploring more selective SWRs.

3.2 Possibility results: Characterizations

We now return to our main concern and examine the extended leximin and utilitarian SWRs satisfying both **SA** and **WC**. We begin by introducing the W-leximin and the overtaking SWRs formulated by Asheim and Tungodden (2004) and von Weizsäcker (1965) respectively and the $\mathcal{S}$-leximin and the $\mathcal{S}$-utilitarian SWRs due to Kamaga and

[17] Among numerous researches on this topic other than those mentioned here, we refer the reader to Diamond (1965), Svensson (1980), Campbell (1985), Shinotsuka (1998), Fleurbaey and Michel (2003), Sakai (2003; 2006), Banerjee and Mitra (2008) for the cases where a certain topological continuity axiom is added.

Kojima (2008) and Banerjee (2006), which will play a role of a steppingstone to the new extended criteria we are seeking now. Let $\succsim_L^n$ denote the finite-horizon leximin ordering defined on $\mathbb{R}^n$ for each $n \in \mathbb{N}$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, $\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n}$ if and only if $(x_{(1)}^{-n}, \ldots, x_{(n)}^{-n}) = (y_{(1)}^{-n}, \ldots, y_{(n)}^{-n})$ or there exists an integer $m < n$ such that $(x_{(1)}^{-n}, \ldots, x_{(m)}^{-n}) = (y_{(1)}^{-n}, \ldots, y_{(m)}^{-n})$ and $x_{(m+1)}^{-n} > y_{(m+1)}^{-n}$.

The W-leximin and the overtaking SWRs, denoted by $\succsim_{Lw}$ and $\succsim_O$ respectively, are defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{Lw} \boldsymbol{y} \text{ iff (i) there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succ_L^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}$$
$$\text{or (ii) there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \sim_L^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}; \tag{1}$$
$$\boldsymbol{x} \succsim_O \boldsymbol{y} \text{ iff (i) there exists } \bar{n} \in \mathbb{N} \text{ such that } \textstyle\sum_{i=1}^n x_i > \sum_{i=1}^n y_i \text{ for all } n \geq \bar{n}$$
$$\text{or (ii) there exists } \bar{n} \in \mathbb{N} \text{ such that } \textstyle\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \text{ for all } n \geq \bar{n}. \tag{2}$$

Next, we introduce the $\mathcal{S}$-leximin and the $\mathcal{S}$-utilitarian SWRs, denoted by $\succsim_{SL}$ and $\succsim_{SU}$ respectively: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{SL} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P} \in \mathcal{S} \text{ and } n \in \mathbb{N} \text{ such that}$$
$$\boldsymbol{P}\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{P}\boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}; \tag{3}$$
$$\boldsymbol{x} \succsim_{SU} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P} \in \mathcal{S} \text{ and } n \in \mathbb{N} \text{ such that}$$
$$\textstyle\sum_{i=1}^n (Px)_i \geq \sum_{i=1}^n y_i \text{ and } \boldsymbol{P}\boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}. \tag{4}$$

The SWRs $\succsim_{SL}$ and $\succsim_{SU}$ extend the leximin SWR proposed by Bossert, Sprumont and Suzumura (2007) and the utilitarian SWR by Basu and Mitra (2007) with an application of fixed-step permutations.[18]

We now formulate the $\mathcal{S}$-anonymous extensions of the W-leximin and the overtaking SWRs by applying the extension method used in $\succsim_{SL}$ and $\succsim_{SU}$ to $\succsim_{Lw}$ and $\succsim_O$. We define $\succsim_{SLw}$ and $\succsim_{SO}$ as the following binary relations on X: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{SLw} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S} \text{ such that } \boldsymbol{P}\boldsymbol{x} \succsim_{Lw} \boldsymbol{Q}\boldsymbol{y}. \tag{5}$$
$$\boldsymbol{x} \succsim_{SO} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S} \text{ such that } \boldsymbol{P}\boldsymbol{x} \succsim_O \boldsymbol{Q}\boldsymbol{y}. \tag{6}$$

We will call $\succsim_{SLw}$ and $\succsim_{SO}$ the $\mathcal{S}$-*W-leximin* SWR and the $\mathcal{S}$-*overtaking* SWR respectively. Indeed, in the following lemma, we verify that $\succsim_{SLw}$ and $\succsim_{SO}$ are well-defined as a SWR on X. Furthermore, the simple characterizations of their asymmetric and symmetric parts are established.

[18] The leximin and utilitarian SWRs, denoted $\succsim_L$ and $\succsim_U$, are defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_L \boldsymbol{y} \text{ iff there exists } n \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n};$$
$$\boldsymbol{x} \succsim_U \boldsymbol{y} \text{ iff there exists } n \in \mathbb{N} \text{ such that } \textstyle\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}.$$

Lemma 1. *$\succsim_{SLw}$ and $\succsim_{SO}$ are reflexive and transitive. Furthermore, for all $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\begin{cases} \boldsymbol{x} \succ_{SLw} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S} \text{ such that } \boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}; & \text{(7a)} \\ \boldsymbol{x} \sim_{SLw} \boldsymbol{y} \text{ iff there exists } \boldsymbol{P} \in \mathcal{S} \text{ such that } \boldsymbol{Px} \sim_{Lw} \boldsymbol{y}, & \text{(7b)} \end{cases}$$

and

$$\begin{cases} \boldsymbol{x} \succ_{SO} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S} \text{ such that } \boldsymbol{Px} \succ_{O} \boldsymbol{Qy}; & \text{(8a)} \\ \boldsymbol{x} \sim_{SO} \boldsymbol{y} \text{ iff there exists } \boldsymbol{P} \in \mathcal{S} \text{ such that } \boldsymbol{Px} \sim_{O} \boldsymbol{y}. & \text{(8b)} \end{cases}$$

Proof. See Appendix. □

Using (7a,b) (resp. (8a,b)), one can verify that $\succsim_{SLw}$ (resp. $\succsim_{SO}$) includes each of $\succsim_{Lw}$ and $\succsim_{SL}$ (resp. $\succsim_{O}$ and $\succsim_{SU}$) as a subrelation. It should be noted that in contrast to (3) and (4), we need an application of two fixed-step permutations in (5) and (6) to make $\succsim_{SLw}$ and $\succsim_{SO}$ transitive relations. This is due to that the asymmetric parts of $\succsim_{Lw}$ and $\succsim_{O}$ violate the invariance property which is verified for their symmetric parts in Lemma 2 in Appendix.[19]

We are ready to state our main results. The following theorems show that if we add both **SA** and **WC** to the axioms of the leximin and utilitarian SWRs respectively, then the classes of logically permissible SWRs are characterized as all the SWRs including $\succsim_{SLw}$ and $\succsim_{SO}$ respectively as a subrelation.

Theorem 1. *A SWR $\succsim$ satisfies **SP**, **SA**, **HE** and **WC** if and only if $\succsim_{SLw}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

Theorem 2. *A SWR $\succsim$ satisfies **SP**, **SA**, **PTSI** and **WC** if and only if $\succsim_{SO}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

Theorem 1 (resp. 2) is interpreted as saying that $\succsim_{SLw}$ (resp. $\succsim_{SO}$) is the least element (with respect to set inclusion) in the class of SWRs satisfying the axioms.[20]

To demonstrate higher selectivity inherent in $\succsim_{SLw}$ (resp. $\succsim_{SO}$) than $\succsim_{Lw}$ and $\succsim_{SL}$ (resp. $\succsim_{O}$ and $\succsim_{SU}$), we provide the following example.

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[19] On this, see also Example 1 we will present later. Asheim, d'Aspremont and Banerjee (2008) discuss several versions of invariance property with respect to an application of permutations.

[20] For the formal explanation of this interpretation, see Banerjee (2006) and Basu and Mitra (2007a). As usually discussed in the literature, we can conclude from Arrow's (1963) variant of Szpilrajn's (1930) Theorem that there exists a complete SWR in these classes of SWRs. However, as we mentioned in Sect. 1, those SWOs can never be explicitly described (Lauwers 2006; Zame 2007).

Example 1. Consider the following utility streams $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x} = (1, 1, \tfrac{1}{3}, \tfrac{1}{3}, \ \tfrac{1}{3^2}, \tfrac{1}{3^2}, \dots)$$
$$\boldsymbol{y} = (1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3^2}, \tfrac{2}{3^2}, \tfrac{2}{3^3}, \dots).$$

These streams are constructed as: $(x_1, y_1) = (1, 1)$; $(x_n, y_n) = \left(\frac{3}{\sqrt{3}^n}, \frac{2}{\sqrt{3}^n}\right)$ if n is even; and $(x_n, y_n) = \left(\frac{\sqrt{3}}{\sqrt{3}^n}, \frac{2\sqrt{3}}{\sqrt{3}^n}\right)$ if n ($\neq 1$) is odd. Note that we have

$$\begin{cases} x_{(1)}^{-n} > y_{(1)}^{-n} \text{ for all even } n, \text{ and } x_{(1)}^{-n} < y_{(1)}^{-n} \text{ for all odd } n > 1; \\ \sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i \text{ for all even } n, \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \text{ for all odd } n. \end{cases}$$

Thus, $\succsim_{Lw}$ and $\succsim_{O}$ cannot compare $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, $\succsim_{SL}$ and $\succsim_{SU}$ cannot, either, since there is no $\boldsymbol{P}$ in $\mathcal{S}$ with which we have the Pareto dominance between tails of $\boldsymbol{Px}$ and $\boldsymbol{y}$. On the other hand, using $\bar{\boldsymbol{P}} \in \mathcal{S}$ corresponding to the permutation π on $\mathbb{N}$ defined as: $\pi(n) = n + 1$ for all odd n and $\pi(n) = n - 1$ for all even n, we obtain $\boldsymbol{x} \succ_{Lw} \bar{\boldsymbol{P}}\boldsymbol{y}$ and $\boldsymbol{x} \succ_{O} \bar{\boldsymbol{P}}\boldsymbol{y}$. Thus, $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$ and $\boldsymbol{x} \succ_{SO} \boldsymbol{y}$. □

Notice that the streams $\boldsymbol{x}$ and $\bar{\boldsymbol{P}}\boldsymbol{y}$ in the example are comparable by $\succsim_{Lw}$ and $\succsim_{O}$, whereas they are not by $\succsim_{SL}$ and $\succsim_{SU}$. On the other hand, the streams $(1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, \dots)$ are comparable by $\succsim_{SL}$ and $\succsim_{SU}$, but not by $\succsim_{Lw}$ and $\succsim_{O}$. Such a trade-off on possible comparisons is now resolved by our new criteria, $\succsim_{SLw}$ and $\succsim_{SO}$.

4 Fixed-Step overtaking criteria

4.1 Equivalence and non-equivalence results

In this section, we introduce the alternative $\mathcal{S}$-anonymous extension method proposed by Lauwers (1997b) and Fleurbaey and Michel (2003) and clarify the relation between our extension method and theirs.

For the overtaking SWR, the $\mathcal{S}$-anonymous extension à la Lauwers, Fleurbaey and Michel, denoted by $\succsim_{FO}$, can be formulated as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{FO} \boldsymbol{y} \text{ iff } \begin{cases} \text{there exists } k \in \mathbb{N} \text{ such that } \sum_{i=1}^{nk} x_i > \sum_{i=1}^{nk} y_i \text{ for all } n \in \mathbb{N} \\ \text{or there exists } k \in \mathbb{N} \text{ such that } \sum_{i=1}^{nk} x_i = \sum_{i=1}^{nk} y_i \text{ for all } n \in \mathbb{N}. \end{cases} \quad (9)$$

We call $\succsim_{FO}$ the *fixed-step overtaking* SWR and refer to this type of extension as fixed-step extension.[21] One can easily verify that $\succsim_{FO}$ is reflexive and transitive and

[21] Lauwers (1997b) and Fleurbaey and Michel (2003) formulate the variants of catching-up SWR by using

the asymmetric and symmetric parts of it are characterized as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\begin{cases} \boldsymbol{x} \succ_{FO} \boldsymbol{y} \text{ iff there exists } k \in \mathbb{N} \text{ such that } \sum_{i=1}^{nk} x_i > \sum_{i=1}^{nk} y_i \text{ for all } n \in \mathbb{N}; & \text{(10a)} \\ \boldsymbol{x} \sim_{FO} \boldsymbol{y} \text{ iff there exists } k \in \mathbb{N} \text{ such that } \sum_{i=1}^{nk} x_i = \sum_{i=1}^{nk} y_i \text{ for all } n \in \mathbb{N}. & \text{(10b)} \end{cases}$$

As we will show later, $\succsim_{FO}$ includes $\succsim_{SO}$ as a subrelation, and they are definitely different criteria.

For the W-leximin SWR, we can also define the fixed-step extension of it in a similar manner to (9). The reader may think that in analogy with $\succsim_{FO}$, the fixed-step extension of $\succsim_{Lw}$ extends $\succsim_{SLw}$ and achieves higher selectivity. However, surprisingly, they turn out to be equivalent. We provide the following proposition.

Proposition 2. *For all $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\begin{cases} \boldsymbol{x} \succ_{SLw} \boldsymbol{y} \textit{ iff there exists } k \in \mathbb{N} \textit{ such that } \boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk} \textit{ for all } n \in \mathbb{N}; & \text{(11a)} \\ \boldsymbol{x} \sim_{SLw} \boldsymbol{y} \textit{ iff there exists } k \in \mathbb{N} \textit{ such that } \boldsymbol{x}^{-nk} \sim_L^{nk} \boldsymbol{y}^{-nk} \textit{ for all } n \in \mathbb{N}. & \text{(11b)} \end{cases}$$

Proof. See Appendix. □

In view of this proposition, the equivalent reformulation of $\succsim_{SLw}$ is given as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{SLw} \boldsymbol{y} \text{ iff } \begin{cases} \text{there exists } k \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk} \text{ for all } n \in \mathbb{N} \\ \text{or there exists } k \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-nk} \sim_L^{nk} \boldsymbol{y}^{-nk} \text{ for all } n \in \mathbb{N}. \end{cases}$$

By the formulae (11a,b), the procedure for ranking infinite utility streams by $\succsim_{SLw}$ now becomes much simpler to use than the one described in (5) or (7a,b).

Next, we clarify the relation between $\succsim_{SO}$ and $\succsim_{FO}$. The following proposition shows that the asymmetric parts of $\succsim_{SO}$ and $\succsim_{FO}$ coincide with each other, whereas the symmetric parts of them do not.

Proposition 3. *For all $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\boldsymbol{x} \succ_{SO} \boldsymbol{y} \textit{ iff there exists } k \in \mathbb{N} \textit{ such that } \sum_{i=1}^{nk} x_i > \sum_{i=1}^{nk} y_i \textit{ for all } n \in \mathbb{N}. \quad (12)$$

Furthermore, $\sim_{SO} \subsetneq \sim_{FO}$.

Proof. See Appendix. □

From this proposition, one can see that $\succsim_{FO}$ includes $\succsim_{SO}$ as a subrelation. Now, the characterizations of the asymmetric and symmetric parts of $\succsim_{SO}$ are also given by (8b) and (12). The enhanced selectivity of $\succsim_{FO}$ is illustrated in the following example.

this extension method. One can verify that $\succsim_{FO}$ is a subrelation of the variant by Fleurbaey and Michel (2003), which they call type 2 SWR, and they are definitely different criteria.

Example 2. Consider the streams $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2}, \ldots)$ and $\boldsymbol{y} = (0, 1, 0, 1, \ldots)$. By (10b), $\boldsymbol{x} \sim_{FO} \boldsymbol{y}$. However, from (8b) and (12), $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-comparable by $\succsim_{SO}$. □

We can explain the contrast between the leximin and utilitarian cases in terms of the difference in the symmetric parts of the finite-horizon leximin and utilitarian orderings. From the argument regarding (14) in Appendix, we can verify that $\boldsymbol{x} \sim_{SO} \boldsymbol{y}$ iff there exists $k \in \mathbb{N}$ such that $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$ and $(x_{nk+1}, \ldots, x_{(n+1)k})$ is a permutation of $(y_{nk+1}, \ldots, y_{(n+1)k})$ for all $n \in \mathbb{N}$. Clearly, it is impossible to derive the formula like (10b) for $\succsim_{SO}$ from this equivalence assertion. On the other hand, by the parallel argument, we also obtain that $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$ iff there exists $k \in \mathbb{N}$ such that $\boldsymbol{x}^{-k} \sim_L^k \boldsymbol{y}^{-k}$ and $(x_{nk+1}, \ldots, x_{(n+1)k})$ is a permutation of $(y_{nk+1}, \ldots, y_{(n+1)k})$ for all $n \in \mathbb{N}$. In contrast, we can now derive (11b) by the definition of indifference relation of $\succsim_L^{nk}$.

4.2 Characterizations

To give an axiomatic explanation of the contrast between the leximin and utilitarian fixed-step extensions, we now characterize $\succsim_{SLw}$ and $\succsim_{FO}$ by using an extended consistency axiom. We begin by introducing the following extended consistency.

Weak Fixed-Step Consistency (WFC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$,* (i) *if there exists $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $(\boldsymbol{x}^{-nk}, 0, 0, \ldots) \succ (\boldsymbol{y}^{-nk}, 0, 0, \ldots)$, then $\boldsymbol{x} \succ \boldsymbol{y}$;* (ii) *if there exists $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $(\boldsymbol{x}^{-nk}, 0, 0, \ldots) \sim (\boldsymbol{y}^{-nk}, 0, 0, \ldots)$, then $\boldsymbol{x} \sim \boldsymbol{y}$.*

WFC requires that evaluation of infinite-horizon utility streams should be consistent with a certain k-periodic successive comparison of their truncated streams.[22] Clearly, **WFC** implies **WC**. Furthermore, **FA** together with **WFC** imply **SA**. Thus, from Proposition 1, **WFC** and **SC** are incompatible in any SWR satisfying **SP** and **FA**.

Using **WFC**, we obtain the following characterizations of $\succsim_{SLw}$ and $\succsim_{FO}$.

Theorem 3. *A SWR $\succsim$ on X satisfies **SP**, **FA**, **HE** and **WFC** if and only if $\succsim_{SLw}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

Theorem 4. *A SWR $\succsim$ satisfies **SP**, **FA**, **PTSI** and **WFC** if and only if $\succsim_{FO}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

From Theorems 3 and 4, the contrast between Propositions 2 and 3 can be ascribed to the difference in the consistency properties of $\succsim_{SLw}$ and $\succsim_{SO}$: $\succsim_{SLw}$ satisfies **WFC**

[22] Fleurbaey and Michel (2003) introduce the similar axiom called Limit Ranking in Fixed Step.

as well as **WC** while $\succsim_{SO}$ does only **WC**. Consequently, one of the differences between $\succsim_{SLw}$ and $\succsim_{SO}$ is now explained in terms of **WFC**, apart from **HE** and **PTSI**.

Recall that **WC** and Weak Preference Continuity by Asheim and Tungodden (2004) are equivalent in the presence of the axioms of the leximin and the utilitarian SWRs and they are interchangeable in Theorems 1 and 2. Now, the natural question to ask is whether **WFC** and the following extended preference continuity axiom are interchangeable in Theorems 3 and 4.

Weak Fixed-Step Preference Continuity: For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $(\boldsymbol{x}^{-nk}, \boldsymbol{y}^{+nk}) \succ \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

The answer to the question is negative. Replacing **WFC** with this extended preference continuity, **FA** no longer implies **SA** even in the presence of the other axioms in Theorems 3 and 4. As the purpose of this paper is concerned, we will not provide further discussion about the permissible SWRs in the case where Weak Fixed-Step Preference Continuity is employed in place of **WFC**.[23]

In Table 1, we summarize Theorems 1 to 4 and compare them with the existing results by Asheim and Tungodden (2004) (abbreviated as A&T in Table 1), Banerjee (2006) (Bn), Basu and Mitra (2007a) (B&M) and Kamaga and Kojima (2008) (K&K).[24] For each row in Table 1, the class of SWRs that includes the SWR stated in the first column as a subrelation is characterized by the axioms indicated by $\oplus$, and furthermore, each SWR of the class satisfies (resp. violates) the axioms indicated by + (resp. –). Compared to the existing characterization results, one can see that Theorems 1 to 4 provide the refinements of permissible SWRs by using the additional axioms, **SA**, **WC** and **WFC** (4th, 5th and 7th columns) respectively. Proposition 1 gives the symbol – in the 6th column. The 1st (or 2nd) and 6th rows highlight a difference between $\succsim_{SLw}$ and $\succsim_{SO}$ in terms of **WFC**.

5 Concluding remarks

In this paper, we employed the two additional axioms, **SA** and **WC**, and established the extensions of the basic infinite-horizon reformulations of the leximin and utilitarian principles proposed by Bossert, Sprumont and Suzumura (2007) and Basu and Mitra (2007a). Our new extended criteria, $\succsim_{SLw}$ and $\succsim_{SO}$, realize higher selectivity than the existing ones each corresponding to the case where a single additional requirement is chosen from the above axioms. We also clarified the relation between our extended

[23] Further results regarding the case where we use Weak Fixed-Step Preference Continuity are available upon request.

[24] In Asheim and Tungodden (2004), the characterizations of $\succsim_{Lw}$ and $\succsim_{O}$ are established by using Weak Preference Continuity instead of **WC** and the invariance axiom called 2-Genaration Unit Comparability in place of **PTSI**. All our results can also be established by using these axioms.

Table 1: Characterizations of the extended leximin and utilitarian SWRs

SWR	**SP**	**FA**	**SA**	**WC**	**SC**	**WFC**	**HE**	**PTSI**	*characterization*
$\succsim_{SLw}$	$\oplus$	$\oplus$	$+$	$+$	$-$	$\oplus$	$\oplus$	$-$	Theorem 3
	$\oplus$	$+$	$\oplus$	$\oplus$	$-$	$+$	$\oplus$	$-$	Theorem 1
$\succsim_{Lw}$	$\oplus$	$\oplus$		$\oplus$			$\oplus$	$-$	A&T
$\succsim_{SL}$	$\oplus$	$+$	$\oplus$		$-$		$\oplus$	$-$	K&K
$\succsim_{FO}$	$\oplus$	$\oplus$	$+$	$+$	$-$	$\oplus$	$-$	$\oplus$	Theorem 4
$\succsim_{SO}$	$\oplus$	$+$	$\oplus$	$\oplus$	$-$		$-$	$\oplus$	Theorem 2
$\succsim_{O}$	$\oplus$	$\oplus$		$\oplus$			$-$	$\oplus$	A&T/B&M
$\succsim_{SU}$	$\oplus$	$+$	$\oplus$		$-$		$-$	$\oplus$	Bn

criteria and the fixed-step extensions of the leximin and utilitarian overtaking criteria in terms of **WFC**. In the leximin case, we found that these two types of extensions are equivalent.

The technique employed to prove Lemma 1 (and Lemmata 2 and 3 which are used to prove the lemma) is quite general and is applicable to the extensions of all other SWRs defined by a sequence of finite-horizon orderings satisfying certain moderate properties (See Appendix). Such a general approach is initiated by d'Aspremont (2007) to "exploit the bulk of social choice theory as developed for the finite case (d'Aspremont 2007, p.114)" and is followed by Asheim, d'Aspremont and Banerjee (2008), Kamaga and Kojima (2008) and Sakai (2008).

In this paper, we employ $\mathcal{S}$-Anonymity (Fixed-Step Anonymity) as an extended anonymity axiom. As shown by Mitra and Basu (2007), any (and only) group(s) of cyclic permutations can define the anonymity axiom consistent with a strongly Paretian SWR.[25] The axiom **SA** is a special case of $\mathcal{Q}$-Anonymity defined by a group $\mathcal{Q}$ of cyclic permutations, corresponding to the case $\mathcal{Q} = \mathcal{S}$. Recent contribution by Lauwers (2006) shows that the class $\mathcal{S}$ is not a maximal group of cyclic permutations. An issue to be addressed in future work is to examine the possibility of extending our results to an arbitrary group of cyclic permutations. We leave this issue for future research.

Appendix

Proof of Lemma 1. We prove the lemma for the case of $\succsim_{SLw}$ by only using properties common to the finite-horizon leximin and utilitarian orderings (**P1** to **P3** below). Thus, the same argument can be directly applied to the case of $\succsim_{SO}$, and we omit it.

Let $\succsim_U^n$ denote the finite-horizon utilitarian ordering defined on $\mathbb{R}^n$ for each $n \in \mathbb{N}$: for all $x^{-n}, y^{-n} \in \mathbb{R}^n$, $x^{-n} \succsim_U^n y^{-n}$ if and only if $\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$. Each

[25] We refer the reader to Mitra and Basu (2007) for detailed explanation of a group of cyclic permutations.

of the sequences of the finite-horizon leximin and utilitarian orderings, $\{\succsim_L^n\}_{n\in\mathbb{N}}$ and $\{\succsim_U^n\}_{n\in\mathbb{N}}$, satisfies the following three properties:[26]

P1: For all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, if $\boldsymbol{x}^{-n} > \boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \succ^n \boldsymbol{y}^{-n}$;

P2: For all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, if $\boldsymbol{x}^{-n}$ is a permutation of $\boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \sim^n \boldsymbol{y}^{-n}$;

P3: For all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$ and all $r \in \mathbb{R}$, $(\boldsymbol{x}^{-n}, r) \succsim^{n+1} (\boldsymbol{y}^{-n}, r)$ if and only if $\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n}$,

where $\succsim^n$ denotes an ordering on $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

First, we prove the formulae (7a) and (7b). We use the following lemma.

Lemma 2. *For all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all $\boldsymbol{P} \in \mathcal{S}$, $\boldsymbol{x} \sim_{Lw} \boldsymbol{y}$ if and only if $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Py}$.*

Proof of Lemma 2. First, we prove the only-if-part. Let $\boldsymbol{P} \in \mathcal{S}$ and suppose that $\boldsymbol{x} \sim_{Lw} \boldsymbol{y}$. Since $\boldsymbol{P} \in \mathcal{S}$, there exists $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{P}(nk)$ is a finite-dimensional permutation matrix. By (1), we can find $\bar{m} \in \mathbb{N}$ such that

$$\begin{cases} \bar{m} = nk \text{ for some } n \in \mathbb{N}; & (13a) \\ \boldsymbol{x}^{-m} \sim_L^m \boldsymbol{y}^{-m} \text{ for all } m \geq \bar{m}. & (13b) \end{cases}$$

We show, by contradiction, that

$$x_m = y_m \text{ for all } m > \bar{m}. \tag{14}$$

Suppose that (14) does not hold. Let m' be the smallest integer such that $m' > \bar{m}$ and $x_{m'} \neq y_{m'}$. Without loss of generality, we assume $x_{m'} > y_{m'}$. By **P3**, $(\boldsymbol{x}^{-(m'-1)}, y_{m'}) \sim_L^{m'} \boldsymbol{y}^{-m'}$. By **P1**, $\boldsymbol{x}^{-m'} \succ_L^{m'} (\boldsymbol{x}^{-(m'-1)}, y_{m'})$. By transitivity, $\boldsymbol{x}^{-m'} \succ_L^{m'} \boldsymbol{y}^{-m'}$. a contradiction to (13b). Thus, (14) holds.

By (13a), $\boldsymbol{P}(\bar{m})$ is a finite-dimensional permutation matrix. Thus, by **P2**, we have $\boldsymbol{x}^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Px})^{-\bar{m}}$ and $\boldsymbol{y}^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Py})^{-\bar{m}}$. By transitivity and (13b), $(\boldsymbol{Px})^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Py})^{-\bar{m}}$. Note that $(\boldsymbol{Px})^{+\bar{m}} = (\boldsymbol{Py})^{+\bar{m}}$ by (14). Thus, by **P3**, $(\boldsymbol{Px})^{-m} \sim_L^m (\boldsymbol{Py})^{-m}$ holds for all $m \geq \bar{m}$. By (1), $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Py}$.

The if-part is proved by using the inverse $\boldsymbol{P}' \in \mathcal{S}$ and the only-if-part. □

We now prove the equivalence assertions in (7a) and (7b). The only-if-part of (7a) and the if-part of (7b) are straightforward from (5), and we omit them.

[26] **P1** is the finite-horizon version of **SP**. **P2** is the standard anonymity axiom in a finite-horizon framework. **P3** is a kind of independence requirement similar to *Extended Independence of the Utilities of Unconcerned Individuals* introduced by Blackorby, Bossert and Donaldson (2002) in the variable population social choice, which requires social ranking to be independent of the existence of a utility-unconcerned generation.

[If-part of (7a)]: Suppose that there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$. Then, by (5), $\boldsymbol{x} \succsim_{SLw} \boldsymbol{y}$. We show, by contradiction, that $\neg(\boldsymbol{y} \succsim_{SLw} \boldsymbol{x})$. Suppose $\boldsymbol{y} \succsim_{SLw} \boldsymbol{x}$. By (5), there exist $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{S}$ such that $\boldsymbol{Ry} \succsim_{Lw} \boldsymbol{Sx}$. Since $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} \in \mathcal{S}$, there exist $p, q, r, s \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{P}(np), \boldsymbol{Q}(nq), \boldsymbol{R}(nr)$ and $\boldsymbol{S}(ns)$ are finite dimensional permutation matrices. By (1), there exist $\bar{n}, \bar{n}' \in \mathbb{N}$ such that

$$\boldsymbol{Px}^{-n} \succ_L^n \boldsymbol{Qy}^{-n} \text{ for all } n \geq \bar{n}, \tag{15}$$

and

$$\boldsymbol{Ry}^{-n} \succ_L^n \boldsymbol{Sx}^{-n} \text{ for all } n \geq \bar{n}' \text{ or } \boldsymbol{Ry}^{-n} \sim_L^n \boldsymbol{Sx}^{-n} \text{ for all } n \geq \bar{n}'. \tag{16}$$

Let $k = p \cdot q \cdot r \cdot s$, and choose $\hat{n} \in \mathbb{N}$ such that $\hat{n}k \geq \max\{\bar{n}, \bar{n}'\}$. By **P2**, $\boldsymbol{Qy}^{-\hat{n}k} \sim_L^{\hat{n}k} \boldsymbol{Ry}^{-\hat{n}k}$. By (15), (16) and transitivity, we obtain $\boldsymbol{Px}^{-\hat{n}k} \succ_L^{\hat{n}k} \boldsymbol{Sx}^{-\hat{n}k}$, a contradiction to **P2**.

[Only-if-part of (7b)]: Suppose that $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$. By definition, there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$ such that $\boldsymbol{Px} \succsim_{Lw} \boldsymbol{Qy}$. If we have $\neg(\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Px})$, then, by (7a), we have $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$, a contradiction to $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$. Thus, $\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Px}$ holds, and $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Qy}$ follows. By Lemma 2, $\boldsymbol{Q}'\boldsymbol{Px} \sim_{Lw} \boldsymbol{Q}'\boldsymbol{Qy} (= \boldsymbol{y})$. Since $\boldsymbol{Q}'\boldsymbol{P} \in \mathcal{S}$, the proof is completed.

Next, we prove that $\succsim_{SLw}$ is reflexive and transitive. We use the following lemma.

Lemma 3. *For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, if $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{SLw} \boldsymbol{z}$, then $\boldsymbol{x} \succ_{SLw} \boldsymbol{z}$, i.e., $\succsim_{SLw}$ is quasi-transitive.*

Proof of Lemma 3. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ be such that $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{SLw} \boldsymbol{z}$. By (7a), there exist $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} \in \mathcal{S}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$ and $\boldsymbol{Ry} \succ_{Lw} \boldsymbol{Sz}$. Since $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} \in \mathcal{S}$, there exist $p, q, r, s \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{P}(np), \boldsymbol{Q}(nq), \boldsymbol{R}(nr)$, and $\boldsymbol{S}(ns)$ are finite dimensional permutation matrices. Let $k = p \cdot q \cdot r \cdot s$. By (1), we can find $K \in \mathbb{N}$ such that $K = nk$ for some $n \in \mathbb{N}$ and

$$(\boldsymbol{Px})^{-m} \succ_L^m (\boldsymbol{Qy})^{-m} \text{ and } (\boldsymbol{Ry})^{-m} \succ_L^m (\boldsymbol{Sz})^{-m} \text{ for all } m \geq K. \tag{17}$$

By **P2**, $(\boldsymbol{Qy})^{-nK} \sim_L^{nK} (\boldsymbol{Ry})^{-nK}$ for all $n \in \mathbb{N}$. By (17) and transitivity,

$$(\boldsymbol{Px})^{-nK} \succ_L^{nK} (\boldsymbol{Sz})^{-nK} \text{ for all } n \in \mathbb{N}. \tag{18}$$

We now prove the following claim.

Claim 1. For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $k \in \mathbb{N}$ such that $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$, then there exists $\boldsymbol{P} \in \mathcal{S}$ such that $\boldsymbol{Px}^{-m} \succ_L^m \boldsymbol{Py}^{-m}$ for all $m \in \mathbb{N}$.

If we have $\boldsymbol{x}^{-m} \succ_L^m \boldsymbol{y}^{-m}$ for all $m \in \mathbb{N}$, $\boldsymbol{P} = \boldsymbol{I}$ trivially proves the claim. We now consider the other cases. Fix $n \in \mathbb{N}$ arbitrarily, and let $i \in \{(n-1)k+1, \dots, nk-1\}$ be the smallest integer for which $\neg(\boldsymbol{x}^{-i} \succ_L^i \boldsymbol{y}^{-i})$ follows. By completeness of $\succsim_L^i$,

$$\boldsymbol{y}^{-i} \succsim_L^i \boldsymbol{x}^{-i}. \tag{19}$$

Since $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$, there exists $j \in \{i+1, \dots, nk\}$ such that

$$x_j > y_j. \tag{20}$$

Otherwise, by (19), **P1** and **P3**, we have $\boldsymbol{y}^{-nk} \succsim_L^{nk} \boldsymbol{x}^{-nk}$, a contradiction. In what follows, we construct a permutation $\boldsymbol{P}$ that entails the property stated in the claim. Let $\boldsymbol{T}_1 \in \mathcal{F}$ be a permutation exchanging only i and j, i.e., for all $\boldsymbol{z} \in X$,

$$(T_1 z)_i = z_j,\ (T_1 z)_j = z_i,\ \text{and } (T_1 z)_h = z_h \text{ for all } h \in \mathbb{N} \setminus \{i, j\}. \tag{21}$$

By (20), **P1** and **P3**, we have

$$(\boldsymbol{T}_1\boldsymbol{x})^{-m} \succ_L^m (\boldsymbol{T}_1\boldsymbol{y})^{-m} \text{ for all } m \in \{(n-1)k+1, \dots, i\}.$$

Moreover, by **P2**,

$$(\boldsymbol{T}_1\boldsymbol{x})^{-nk} \sim_L^{nk} \boldsymbol{x}^{-nk} \text{ and } (\boldsymbol{T}_1\boldsymbol{y})^{-nk} \sim_L^{nk} \boldsymbol{y}^{-nk}.$$

Thus, by the assumption of the claim and transitivity,

$$(\boldsymbol{T}_1\boldsymbol{x})^{-nk} \succ_L^{nk} (\boldsymbol{T}_1\boldsymbol{y})^{-nk}.$$

Using the same argument repeatedly at most finite t times (with i being redefined for the permuted streams, say $\boldsymbol{T}_1\boldsymbol{x}$ and $\boldsymbol{T}_1\boldsymbol{y}$, each time), we obtain t permutations $\boldsymbol{T}_1, \cdots, \boldsymbol{T}_t$ such that, for all $m \in \{(n-1)k+1, \dots, nk\}$,

$$(\boldsymbol{T}_t \cdots \boldsymbol{T}_1\boldsymbol{x})^{-m} \succ_L^m (\boldsymbol{T}_t \cdots \boldsymbol{T}_1\boldsymbol{y})^{-m}. \tag{22}$$

Since n is arbitrarily chosen in the above argument, we obtain the same conclusion for all $n \in \mathbb{N}$ with $t(n)$ permutations $\boldsymbol{T}_{1(n)}, \cdots, \boldsymbol{T}_{t(n)}$ which are now conditioned by n. Using the sequence of those permutations, $\{\boldsymbol{T}_{1(1)}, \boldsymbol{T}_{2(1)}, \dots, \boldsymbol{T}_{t(1)}, \boldsymbol{T}_{1(2)}, \boldsymbol{T}_{2(2)}, \dots\}$, we can define an infinite dimensional matrix $\boldsymbol{P}$ as follows: for all $n \in \mathbb{N}$,

$$\boldsymbol{P}(nk) = \boldsymbol{T}_n(nk),\ \text{where } \boldsymbol{T}_n \text{ is the composition } \boldsymbol{T}_{t(n)} \cdots \boldsymbol{T}_{2(2)}\boldsymbol{T}_{1(2)}\boldsymbol{T}_{t(1)} \cdots \boldsymbol{T}_{1(1)}.$$

By (21), $\boldsymbol{P}(nk)$ is well-defined as a finite dimensional permutation matrix for all $n \in \mathbb{N}$. Thus, $\boldsymbol{P} \in \mathcal{S}$. Since (22) holds for any case of $n \in \mathbb{N}$, we obtain $\boldsymbol{Px}^{-m} \succ_L^m \boldsymbol{Py}^{-m}$ for all $m \in \mathbb{N}$. Thus, we complete the proof of the claim.

From Claim 1 and (18), there exists $\tilde{\boldsymbol{P}} \in \mathcal{S}$ such that $(\tilde{\boldsymbol{P}}\boldsymbol{Px})^{-m} \succ_L^m (\tilde{\boldsymbol{P}}\boldsymbol{Sz})^{-m}$ for all $m \in \mathbb{N}$. By (1), $\tilde{\boldsymbol{P}}\boldsymbol{Px} \succ_{Lw} \tilde{\boldsymbol{P}}\boldsymbol{Sz}$. Since $\tilde{\boldsymbol{P}}\boldsymbol{P}, \tilde{\boldsymbol{P}}\boldsymbol{S} \in \mathcal{S}$, $\boldsymbol{x} \succ_{SLw} \boldsymbol{z}$ by (7a). □

We now prove that $\succsim_{SLw}$ is reflexive and transitive. Reflexivity is obvious. We only prove transitivity. By Lemma 3, $\succsim_{SLw}$ is quasi-transitive. We show that $\boldsymbol{x} \succsim_{SLw} \boldsymbol{z}$ follows for each of the three cases: (i) $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$ and $\boldsymbol{y} \sim_{SLw} \boldsymbol{z}$; (ii) $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{SLw} \boldsymbol{z}$; and (iii) $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$ and $\boldsymbol{y} \sim_{SLw} \boldsymbol{z}$. Consider the case (i). By (7a) and (7b), there exist $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathcal{S}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$ and $\boldsymbol{Ry} \sim_{Lw} \boldsymbol{z}$. Since $\boldsymbol{QR}' \in \mathcal{S}$, $(\boldsymbol{Qy} =)\boldsymbol{QR}'\boldsymbol{Ry} \sim_{Lw} \boldsymbol{QR}'\boldsymbol{z}$ follows from Lemma 2. By transitivity, $\boldsymbol{Px} \succ_{Lw} \boldsymbol{QR}'\boldsymbol{z}$. By (7a), $\boldsymbol{x} \succ_{SLw} \boldsymbol{z}$. By the similar argument, we can prove the other cases, and we omit them. □

Proofs of Theorems 1 and 2. We prove only Theorem 1. Using the characterization of $\succsim_O$ by Basu and Mitra (2007a), Theorem 2 is proved by the same argument.

The if-part is straightforward, and we omit it. We prove the only-if-part. Let $\succsim$ be a SWR satisfying **SP**, **SA**, **HE** and **WC**. From the characterization of $\succsim_{Lw}$ by Ahseim and Tungodden (2004), $\succsim_{Lw}$ is a subrelation of $\succsim$. To verify $\succsim_{SLw}$ is a subrelation of $\succsim$, we suppose that $x \succ_{SLw} y$. By (7a), there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$. Since $\succsim_{Lw}$ is a subrelation of $\succsim$, we have $\boldsymbol{Px} \succ \boldsymbol{Qy}$. By **SA**, $\boldsymbol{x} = \boldsymbol{P}'\boldsymbol{Px} \sim \boldsymbol{Px}$ and $\boldsymbol{y} = \boldsymbol{Q}'\boldsymbol{Qy} \sim \boldsymbol{Qy}$. By transitivity, $\boldsymbol{x} \succ \boldsymbol{y}$. By the same argument, we can prove that if $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$ then $\boldsymbol{x} \sim \boldsymbol{y}$, and we omit it. □

Proof of Proposition 2. **[If-part of (11a)]** Suppose that there exists $k \in \mathbb{N}$ such that $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. From Claim 1 in the proof of Lemma 3, there exists $\boldsymbol{P} \in \mathcal{S}$ such that $\boldsymbol{Px}^{-m} \succ_L^m \boldsymbol{Py}^{-m}$ for all $m \in \mathbb{N}$. Thus, $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Py}$, and $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$ by (7a).

[Only-if-part of (11a)] Suppose that $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$. By (7a), there exists $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$ such that $\boldsymbol{Px}^{-n} \succ_L^n \boldsymbol{Qy}^{-n}$ for all $n \geq \bar{n}$ for some $\bar{n} \in \mathbb{N}$. Since $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$, there exist $p, q \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{P}(np)$ and $\boldsymbol{Q}(nq)$ are finite dimensional permutation matrices. Let $k \in \mathbb{N}$ be such that $k = p \cdot q \cdot n$ for some $n \in \mathbb{N}$ and $k \geq \bar{n}$. By **P2**, $\boldsymbol{x}^{-nk} \sim_L^{nk} \boldsymbol{Px}^{-nk}$ and $\boldsymbol{y}^{-nk} \sim_L^{nk} \boldsymbol{Qy}^{-nk}$ for all $n \in \mathbb{N}$. By transitivity, $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$.

[If-part of (11b)] Suppose that there exists $k \in \mathbb{N}$ such that $\boldsymbol{x}^{-nk} \sim_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. By the definition of $\succsim_L^n$, $\boldsymbol{x}^{-nk}$ is a permutation of $\boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. Thus, there exists $\boldsymbol{P} \in \mathcal{S}$ such that $\boldsymbol{P}(nk)$ is a finite dimensional permutation matrix for all

$n \in \mathbb{N}$ and $\boldsymbol{Px} = \boldsymbol{y}$. By reflexivity, $\boldsymbol{Px} \sim_{Lw} \boldsymbol{y}$. By (7b), $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$.

[Only-if-part of (11b)] The proof is easy and the outline is provided in Sect. 4.1. □

Proof of Proposition 3. Recall that Claim 1 in the proof of Lemma 3 is directly applicable to the case of the finite-horizon utilitarian ordering. Thus, the formula (12) is proved by the same argument as in the proof of (11a). The set inclusion $\sim_{SO} \subseteq \sim_{FO}$ is verified by the argument in the final part of Sect. 4.1. Example 2 verifies $\sim_{SO} \neq \sim_{FO}$. □

Proofs of Theorems 3 and 4. We prove only Theorem 3. By using the characterization of $\succsim_O$ by Basu and Mitra (2007a), Theorem 4 is proved by the same argument.

[Only-if-part] From the characterization of $\succsim_{Lw}$ by Asheim and Tungodden (2004), $\succsim_{Lw}$ is a subrelation of $\succsim$. Suppose that $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$. By (11a), there exists $k \in \mathbb{N}$ such that $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. By (1), $(\boldsymbol{x}^{-nk}, 0, 0, \dots) \succ_{Lw} (\boldsymbol{y}^{-nk}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. Since $\succsim_{Lw}$ is a subrelation of $\succsim$, $(\boldsymbol{x}^{-nk}, 0, 0, \dots) \succ (\boldsymbol{y}^{-nk}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. By **WFC**, $\boldsymbol{x} \succ \boldsymbol{y}$. By the same argument, we can prove that if $\boldsymbol{x} \sim_{SLw} \boldsymbol{y}$ then $\boldsymbol{x} \sim \boldsymbol{y}$.

[If-part] From the characterization of $\succsim_{Lw}$ by Asheim and Tungodden (2004), $\succsim$ satisfies **SP**, **FA** and **HE**. We show that $\succsim$ satisfies **WFC**. Let $\boldsymbol{x}, \boldsymbol{y} \in X$, and suppose that there exists $k \in \mathbb{N}$ such that $(\boldsymbol{x}^{-nk}, 0, 0, \dots) \succ (\boldsymbol{y}^{-nk}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. First, we show, by contradiction, that $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. Suppose that $\neg(\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk})$ for some $n \in \mathbb{N}$. By completeness of $\succsim_L^{nk}$, $\boldsymbol{y}^{-nk} \succsim_L^{nk} \boldsymbol{x}^{-nk}$. Then, we have $(\boldsymbol{y}^{-nk}, 0, 0, \dots) \succsim_{SLw} (\boldsymbol{x}^{-nk}, 0, 0, \dots)$. Since $\succsim_{SLw}$ is a subrelation of $\succsim$, $(\boldsymbol{y}^{-nk}, 0, 0, \dots) \succsim (\boldsymbol{x}^{-nk}, 0, 0, \dots)$ holds, a contradiction. Thus, $\boldsymbol{x}^{-nk} \succ_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$. By (7a), $\boldsymbol{x} \succ_{SLw} \boldsymbol{y}$. Since $\succsim_{SLw}$ is a subrelation of $\succsim$, $\boldsymbol{x} \succ \boldsymbol{y}$.

Next, suppose that there exists $k \in \mathbb{N}$ such that $(\boldsymbol{x}^{-nk}, 0, 0, \dots) \sim (\boldsymbol{y}^{-nk}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. By the same argument, we can show that $\boldsymbol{x}^{-nk} \sim_L^{nk} \boldsymbol{y}^{-nk}$ for all $n \in \mathbb{N}$, and $\boldsymbol{x} \sim \boldsymbol{y}$ is obtained. □

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