21COE-GLOPE

21COE-GLOPE Working Paper Series

On the extensions of the infinite-horizon leximin and the overtaking criteria

Kohei Kamaga and Takashi Kojima

Revised version of No.36:

The earlier version of this manuscript was entitled "Q-Anonymity and preference continuity." Main results of the earlier draft are restated in a different form.

Working Paper No. 48

On the extensions of the infinite-horizon leximin and the overtaking criteria*

Kohei Kamaga† Takashi Kojima‡

First version, December 18, 2007;
Current version, March 31, 2008

Abstract

The purpose of this paper is to formulate and characterize the infinte-horizon variants of the leximin principle and utilitarianism that satisfy both Preference-continuity or Consistency and $\mathcal{Q}$-Anonymity. We formulate new extended leximin and utilitarian social welfare relations (SWRs), called $\mathcal{Q}$-W-leximin SWR and $\mathcal{Q}$-overtaking criterion respectively, and show that Weak Preference-continuity (or Weak Consistency) and $\mathcal{Q}$-Anonymity together with Strong Pareto and Hammond Equity (resp. Partial Unit Comparability) characterize all SWRs that include the $\mathcal{Q}$-W-leximin SWR (resp. the $\mathcal{Q}$-overtaking criterion) as a subrelation. We also show that there exists no SWR satisfying Strong Pareto, Strong Preference-continuity (or Strong Consistency) and $\mathcal{Q}$-Anonymity.

Keywords $\mathcal{Q}$-Anonymity; Preference-continuity; Consistency; Leximin principle; Overtaking criterion

JEL Classification Numbers D63; D70

*This paper was previously entitled "$\mathcal{Q}$-Anonymity and preference continuity." The authors are grateful for helpful comments from Yoshio Kamijo, Tetsuro Okazaki, Hiroo Sasaki, and Koichi Suga. Remaining errors are our own. Financial support from the Ministry of Education, Culture, Sports, Science and Technology of Japan under Waseda University 21st COE-GLOPE project is gratefully acknowledged.

†Graduate School of Economics, Waseda University, 1-6-1, Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan (E-mail: k-kmg@ruri.waseda.jp)

‡Graduate School of Economics, Waseda University, 1-6-1, Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan (E-mail: tkojima@ruri.waseda.jp)

1 Introduction

In evaluating infinite-horizon utility streams, Strong Pareto and Finite Anonymity are the most common principles employed in the literature. The former is the requirement of efficiency (or sensitivity) and the latter of impartiality among generations. These basic principles lead us to the infinite-horizon variant of the Suppes-Sen grading principle (Svensson 1980; Asheim et al. 2001).[1] The infinite-horizon Suppes-Sen grading principle evaluates the relative goodness of two utility streams only by the Pareto dominance after a transformation by a suitable finite permutation. Hence, what the Suppes-Sen grading principle by itself asserts on our evaluation is quite weak and many utility streams will be declared to be non-comparable.

Further comparability beyond the Suppes-Sen grading principle have been pursued along two rival principles of justice, Rawlsian lexicographic maximin principle and utilitarianism. Basu and Mitra (2007) formulate and characterize the infinite-horizon variant of utilitarianism, henceforth *utilitarian social welfare relation* (SWR), which applies the well-established finite-horizon utilitarian ordering to the first n generations' utilities and the Pareto principle to the utilities of infinitely many future generations.[2] In a similar manner, the infinite-horizon variant of leximin principle, called *leximin SWR*, is formalized and characterized by Bossert et al. (2007) with the finite-horizon leximin ordering and the Pareto principle. These SWRs are characterized by the infinite-horizon variants of the axioms characterizing the finite-horizon utilitarian and leximin orderings respectively, Partial Unit Comparability (in the case of the utilitarian SWR) and Hammond Equity (in the case of the leximin SWR) as well as Strong Pareto and Finite Anonymity. Although both two exhibit higher level of comparability than the Suppes-Sen grading principle, utility streams involving a conflict among *infinitely* many generations are still non-comparable since the Pareto principle, applied to future generations' utilities, is an incomplete quasi-ordering.[3]

To give a resolution to conflicts involving infinitely many generations, two different kinds of extensions of the leximin and utilitarian SWRs have been proposed in the literature.[4] The first one is the extensions considered by Asheim and Tungodden (2004) and Basu and Mitra (2007). They respectively employ an additional axiom called *Preference-continuity* or *Consistency*. Preference-continuity and Consistency are quite similar and both basically require that our comparisons of infinite-horizon

[1]The Suppes-Sen grading principle is originally formulated in a finite population setting. See Suppes (1966) and Sen (1970).

[2]This type of SWR is generically referred to as *simplified criterion* in d'Aspremont (2007).

[3]It should be noted that Basu and Mitra (2007) show that in a certain class of intertemporal economic models, the utilitarian SWR will suffice for deriving a unique greatest path.

[4]The extensions we introduce here do not exhaust all the existing ones. Focusing on the notions of time-invariance and stationarity, Asheim and Banerjee (2008) recently propose the *generalized time-invariant overtaking criterion*. The leximin and utilitarian versions of their extended criterion exhibit higher level of comparability than the leximin and utilitarian SWRs respectively.

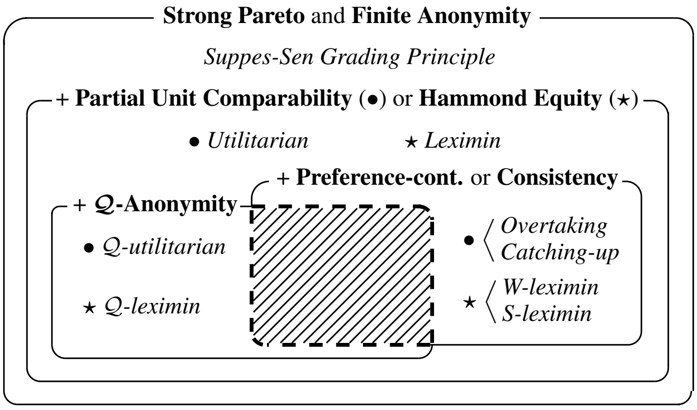

Figure 1: Characterizations of admissible class of SWRs and minimal elements

utility streams should be consistent with an *infinite* number of comparisons of *finite-horizon* truncated paths. Adding Weak (resp. Strong) Preference-continuity, Asheim and Tungodden (2004) characterize the extended leximin SWR called *W-leximin* (resp. *S-leximin*) *SWR* and the well-known extended utilitarian SWR called *overtaking* (resp. *catching-up*) *criterion*. Basu and Mitra (2007) also characterize the overtaking and catching-up criteria with two versions of consistency.

The other type of extension is proposed by Banerjee (2006) and also analyzed in Kamaga and Kojima (2008).[5] They strengthen the notion of impartiality from Finite Anonymity to $\mathcal{Q}$-Anonymity. $\mathcal{Q}$-Anonymity is first introduced by Lauwers (1997b) under the name Fixed Step Anonymity and is defined by a specific type of infinite permutations as well as finite permutations.[6] The most common example that illustrates the difference between Finite Anonymity and $\mathcal{Q}$-Anonymity is the streams $\boldsymbol{x} = (1, 0, 1, 0, \dots)$ and $\boldsymbol{y} = (0, 1, 0, 1, \dots)$. While Finite Anonymity cannot provide a definite ranking of $\boldsymbol{x}$ and $\boldsymbol{y}$, $\mathcal{Q}$-Anonymity declares them to be indifferent. Banerjee (2006) characterizes the *$\mathcal{Q}$-utilitarian SWR* with $\mathcal{Q}$-Anonymity, and its leximin counterpart, called *$\mathcal{Q}$-leximin SWR*, is characterized in Kamaga and Kojima (2008). The existing characterizations we mentioned here are summarized in Figure 1.

Both the extension employing Preference-continuity or Consistency and that using $\mathcal{Q}$-Anonymity have merits and demerits respectively. Since the W-leximin SWR and the overtaking criterion (and also the S-leximin SWR and the catching-up criterion) are defined as an infinite number of application of the finite-horizon leximin and utilitarian orderings respectively, these SWRs make further comparisons beyond the limits of the leximin and utilitarian SWRs and will provide more selected maximal paths.

[5] See also Mitra and Basu (2007).

[6] See also Fleurbaey and Michel (2003) and Sakai (2008), where other related anonymity axioms are also introduced in a comprehensive manner.

However, they do not ensure social indifference between the streams $\boldsymbol{x}$ and $\boldsymbol{y}$ we noted above. Indeed, the S-leximin SWR and the catching-up criterion conclude $\boldsymbol{x}$ is strictly preferable to $\boldsymbol{y}$, and the W-leximin SWR and the overtaking criterion declare them non-comparable.[7] On the other hand, according to the $\mathcal{Q}$-utilitarian and $\mathcal{Q}$-leximin SWRs, $\boldsymbol{x}$ and $\boldsymbol{y}$ is declared to be socially indifferent. However, since these $\mathcal{Q}$-anonymous extensions still apply the Pareto principle to future generations' utilities, there will be room for improvement in their incompleteness, which could be dealt with by invoking Preference-continuity or Consistency.

The purpose of this paper is to formulate and characterize new extended leximin and utilitarian SWRs satisfying both Preference-continuity or Consistency and $\mathcal{Q}$-Anonymity, i.e. those incorporating both merits of the extensions by Asheim and Tungodden (2004) and Basu and Mitra (2007) and by Banerjee (2006) and Kamaga and Kojima (2008). In Figure 1, the shaded area corresponds to the class we are interested in. As we have noted earlier, it is impossible to additionally impose $\mathcal{Q}$-Anonymity on the S-leximin SWR and the catching-up criterion. Consequently, the shaded area in Figure 1 is empty in the case of Strong Preference-continuity or Strong Consistency. We show that this impossibility can be ascribed to the incompatibility of $\mathcal{Q}$-Anonymity and Strong Preference-continuity (or Strong Consistency) in a strongly Paretian SWR. This impossibility result tells that our choice of $\mathcal{Q}$-Anonymity or Strong Preference-continuity (or Strong Consistency) is a branching point in exploring the SWRs that make further comparisons beyond the Suppes-Sen grading principle.

In contrast to the cases of Strong Preference-continuity and Strong Consistency, it is possible to define the extended leximin and utilitarian SWRs satisfying both Weak Preference-continuity or Weak Consistency and $\mathcal{Q}$-Anonymity. We formulate $\mathcal{Q}$-anonymous extensions of the W-leximin SWR and the overtaking criterion, called *$\mathcal{Q}$-W-leximin SWR* and *$\mathcal{Q}$-overtaking criterion* respectively. We show that if Weak Preference-continuity (or Weak Consistency) and $\mathcal{Q}$-Anonymity are added to the basic axioms, Strong Pareto, Finite Anonymity and Hammond Equity or Partial Unit Comparability, then all SWRs that include the $\mathcal{Q}$-W-leximin SWR or the $\mathcal{Q}$-overtaking criterion respectively as a sub-relation will be characterized. In other words, under these axioms, the $\mathcal{Q}$-W-leximin SWR and the $\mathcal{Q}$-overtaking criterion respectively are the least restrictive SWRs and we must respect the comparisons obtained by these SWRs respectively.

The rest of the paper is organized as follows. Section 2 presents notation and definitions. The axioms we impose on SWRs are also introduced. Section 3 provides the results obtained in this paper. In Section 4, we compare our new SWRs with some well-established ones. Section 5 concludes with some remarks.

[7] Banerjee (2006) is the first who observes the catching-up criterion violates $\mathcal{Q}$-Anonymity. On this, see Example 1 in Banerjee (2006).

2 Preliminary

2.1 Notation and definitions

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ be the set of all positive integers $\{1, 2, \dots\}$. We let $X = \mathbb{R}^{\mathbb{N}}$ be the domain of infinite utility streams. An infinite-dimensional vector $\boldsymbol{x} = (x_1, x_2, \dots)$ is a typical element of X and, for each $i \in \mathbb{N}$, x_i is interpreted as utility of the ith generation. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, we denote $(x_1, \dots, x_n)$ by $\boldsymbol{x}^{-n}$ and $(x_{n+1}, x_{n+2}, \dots)$ by $\boldsymbol{x}^{+n}$. Thus, given any $\boldsymbol{x} \in X$ and any $n \in \mathbb{N}$, we can write $\boldsymbol{x} = (\boldsymbol{x}^{-n}, \boldsymbol{x}^{+n})$. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, $\left(x^{-n}_{(1)}, \dots, x^{-n}_{(n)}\right)$ denotes a rank-ordered permutation of $\boldsymbol{x}^{-n}$ such that $x^{-n}_{(1)} \leq \cdots \leq x^{-n}_{(n)}$, ties being broken arbitrarily.

A SWR, denoted by $\succsim$, is a reflexive and transitive binary relation on X, i.e. a quasi-ordering.[8] An asymmetric component of $\succsim$ is denoted by $\succ$ and a symmetric component by $\sim$, i.e. $\boldsymbol{x} \succ \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ holds but $\boldsymbol{y} \succsim \boldsymbol{x}$ does not, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{x}$. A SWR $\succsim_A$ is said to be a subrelation of a SWR $\succsim_B$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \sim_A \boldsymbol{y}$ implies $\boldsymbol{x} \sim_B \boldsymbol{y}$ and (ii) $\boldsymbol{x} \succ_A \boldsymbol{y}$ implies $\boldsymbol{x} \succ_B \boldsymbol{y}$.

Following Mitra and Basu (2007) and Banerjee (2006), we represent any permutation on the set $\mathbb{N}$ by a permutation matrix. A permutation matrix is an infinite matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ satisfying the following properties:

(i) for each $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$;

(ii) for each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given any permutation matrix $\boldsymbol{P}$, we denote by $\boldsymbol{P}'$ its unique inverse which satisfies $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the infinite identity matrix. Let $\mathcal{P}$ be the set of all permutation matrices. Given any $\boldsymbol{P} \in \mathcal{P}$ and any $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\dots,n\}}$ by $\boldsymbol{P}(n)$. A finite permutation matrix is a permutation matrix $\boldsymbol{P}$ such that $p_{ii} = 1$ for all $i > n$ for some $n \in \mathbb{N}$. The set of all finite permutation matrices is denoted by $\mathcal{F}$.

As in Mitra and Basu (2007) and Banerjee (2006), we focus on a certain class of *cyclic* permutations which defines a *group* under the usual matrix multiplication.[9] A

[8]A binary relation $\succsim$ on X is (i) reflexive if, for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \succsim \boldsymbol{x}$, and (ii) transitive if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succsim \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$.

[9]Let $\mathcal{G}$ be a set of permutation matrices. $\mathcal{G}$ is said to define a group under the usual matrix multiplication if it satisfies the following four properties: (i) for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}$, $\boldsymbol{P}\boldsymbol{Q} \in \mathcal{G}$, (ii) there exists $\boldsymbol{I} \in \mathcal{G}$ such that for all $\boldsymbol{P} \in \mathcal{G}$, $\boldsymbol{I}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{I} = \boldsymbol{P}$, (iii) for all $\boldsymbol{P} \in \mathcal{G}$, there exists $\boldsymbol{P}' \in \mathcal{G}$ such that $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$, and (iv) for all $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathcal{G}$, $(\boldsymbol{P}\boldsymbol{Q})\boldsymbol{R} = \boldsymbol{P}(\boldsymbol{Q}\boldsymbol{R})$. Mitra and Basu (2007) show that a class of permutations by which an anonymity axiom compatible with Strong Pareto can be defined if and only if the class consists solely of cyclic permutations and defines a group with respect to the matrix multiplication, where we use the

permutation matrix $\boldsymbol{P} \in \mathcal{P}$ is said to be cyclic if, for any unit vector $\boldsymbol{e} = (0, \ldots, 0, 1, 0, \ldots) \in X$, there exists $k \in \mathbb{N}$ such that k-times repeated application of $\boldsymbol{P}$ to $\boldsymbol{e}$ generates $\boldsymbol{e}$ again, i.e. $\overbrace{\boldsymbol{P} \ldots \boldsymbol{P}}^{k} \boldsymbol{e} = \boldsymbol{e}$. Throughout the paper, we let $\mathcal{Q}$ be the following subclass of $\mathcal{P}$:

$$\mathcal{Q} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{c} \text{there exists } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \\ \boldsymbol{P}(nk) \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

The class $\mathcal{Q}$ is exactly the set of all fixed step permutations which is first introduced by Lauwers (1997b). It is easily checked that $\mathcal{Q}$ is the class of cyclic permutations and defines a group with respect to the matrix multiplication, and also that $\mathcal{F} \subset \mathcal{Q}$.

Negation of a statement is indicated by the logic symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} \geqslant \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, and $\boldsymbol{x} > \boldsymbol{y}$ if $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$.

2.2 Axioms

2.2.1 Basic axioms

We introduce some basic axioms that provide axiomatic foundations of the infinite-horizon variants of the leximin and utilitarian orderings.

We begin with two guiding principles of sensitivity and impartiality.

Strong Pareto (SP) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} > \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

$\mathcal{F}$-Anonymity (FA) *For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{F}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.*

FA is also called *Finite* (or *Weak*) *Anonymity*. **SP** and **FA** characterize the infinite-horizon Suppes-Sen grading principle defined by the permutations in $\mathcal{F}$ (Svensson 1980; Asheim et al. 2001).

The next one is an infinite-horizon variant of the well-known consequentialist equity axiom introduced by Hammond (1976).

Hammond equity (HE) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all $i, j \in \mathbb{N}$, if $y_i < x_i < x_j < y_j$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $\boldsymbol{x} \succsim \boldsymbol{y}$.*

HE asserts that an order-preserving change which diminishes inequality of utilities between conflicting two generations is socially preferable. The leximin SWR is characterized by **SP**, **FA** and **HE** (Bossert et al. 2007).[10] The definition of the leximin SWR is available in Sect. 4.

term anonymity axiom to refer to the condition which asserts that a SWR must conclude $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$ for all $\boldsymbol{x} \in X$ and all $\boldsymbol{P}$ in an adopted class of permutations.

[10] On this, see also the argument in the proof of Proposition 1 in Asheim and Tungodden (2004).

We move to the following two informational invariance axioms.

Partial unit comparability (PUC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, all $\boldsymbol{a} \in \mathbb{R}^{\mathbb{N}}$ and all $n \in \mathbb{N}$, if $\boldsymbol{x}^{+n} = \boldsymbol{y}^{+n}$ and $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{a} \succsim \boldsymbol{y} + \boldsymbol{a}$.*

2-Generation unit comparability (2UC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, all $i, j \in \mathbb{N}$, and all $\boldsymbol{a} \in \mathbb{R}^{\mathbb{N}}$ if, for all $k \neq i, j$, $a_k = 0$ and $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{a} \succsim \boldsymbol{y} + \boldsymbol{a}$.*

PUC is employed in Basu and Mitra (2007) and **2UC** in Asheim and Tungodden (2004). Although the definitions of them are slightly different, both two basically assert that utility differences of generations are comparable but utility levels are not.[11] **PUC** (or **2UC**) together with **SP** and **FA** characterizes the utilitarian SWR (Basu and Mitra 2007).[12] For the formal definition of the utilitarian SWR, see Sect. 4.

2.2.2 Additional axioms

We now introduce additional axioms that are used to characterize the extended leximin and utilitarian SWRs.

We begin with the axioms employed by Asheim and Tungodden (2004) and Basu and Mitra (2007). Asheim and Tungodden (2004) consider two versions of preference-continuity axioms.

Weak preference-continuity (WPC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succ \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

Strong preference-continuity (SPC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succsim \boldsymbol{y}$, and for all $\bar{n} \in \mathbb{N}$, there exists $n \geq \bar{n}$ such that $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succ \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

Basu and Mitra (2007) employ the following consistency axioms.

Weak consistency (WC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, (a) if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succ (\boldsymbol{y}^{-n}, 0, 0, \dots)$, then $\boldsymbol{x} \succ \boldsymbol{y}$; (b) if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \sim (\boldsymbol{y}^{-n}, 0, 0, \dots)$, then $\boldsymbol{x} \sim \boldsymbol{y}$.*

Strong consistency (SC) *For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succsim (\boldsymbol{y}^{-n}, 0, 0, \dots)$, then $\boldsymbol{x} \succsim \boldsymbol{y}$, and if there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succsim (\boldsymbol{y}^{-n}, 0, 0, \dots)$ and for all $\bar{n} \in \mathbb{N}$, there exists $n \geq \bar{n}$ such that $(\boldsymbol{x}^{-n}, 0, 0, \dots) \succ (\boldsymbol{y}^{-n}, 0, 0, \dots)$, then $\boldsymbol{x} \succ \boldsymbol{y}$.*

[11] Since **HE** assumes at least ordinally measurable and level comparable utilities, it is incompatible with **2UC** and **PUC**. For the detailed explanation of informational invariance axioms, we refer the reader to Bossert and Weymark (2004) and d'Aspremont and Gevers (2002).

[12] For the case of **2UC**, see the argument in the proof of Proposition 4 in Asheim and Tungodden (2004).

Both **WPC** and **WC** (and also **SPC** and **SC**) are defined similarly in spirit to Axiom 3 in Brock (1970) and basically assert that our comparison of infinite-horizon utility streams should be consistent with the comparisons of their finite-horizon truncated paths if the length of truncations are large enough. Indeed, these axioms are equivalent in the class of SWRs that include the leximin or utilitarian SWR as a subrelation in both cases of strong and weak versions of them.[13]

Next, we introduce the axiom employed by Banerjee (2006) and Kamaga and Kojima (2008). Instead of **FA**, they impose the following stronger anonymity axiom.

$\mathcal{Q}$-Anonymity (QA) *For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{Q}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.*

QA is also called *Fixed Step Anonymity*. It formalizes a stronger notion of impartiality than **FA** by employing the class $\mathcal{Q}$ composed of all fixed step permutation matrices as well as all finite permutation matrices.[14]

For each of the additional axioms, the characterizations of the extended leximin and utilitarian SWRs are already established: *W-leximin SWR* and *overtaking criterion* with **WPC** or **WC** and *S-leximin SWR* and *catching-up criterion* with **SPC** or **SC** (Asheim and Tungodden 2004; Basu and Mitra 2007) and *$\mathcal{Q}$-utilitarian* and *$\mathcal{Q}$-leximin SWRs* with **QA** (Banerjee 2006; Kamaga and Kojima 2008). See Sections 3 and 4 (and the footnote 19) for the definitions of these SWRs.

3 Further extensions and characterizations

The principal task of this paper is to establish characterizations of the extended leximin and utilitarian SWRs that satisfy both of the two different kinds of additional axioms, one of the four axioms of preference-continuity or consistency and **QA**, i.e. the characterizations of those extended criteria which incorporate the merits of the extensions by Asheim and Tungodden (2004) and Basu and Mitra (2007) and by Banerjee (2006) and Kamaga and Kojima (2008). Since, as we noted earlier, it is impossible to formulate the extensions of the leximin and utilitarian SWRs satisfying **QA** and **SPC** or **SC**. our interest lies particularly on the possibility of the extended leximin and utilitarian SWRs that satisfy both **QA** and **WPC** or **WC**.

Before proceeding to the main issue, we show that the impossibility for cases of the stronger versions of preference-continuity and consistency is ascribed to the incompatibility between **QA** and **SPC** or **SC** in a strongly Paretian SWR.

[13]It should be noted that **SP** and the following independence implied by any of the utilitarian and leximin SWRs suffice for this equivalence: for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z} \in X$, if there exists $n \in \mathbb{N}$ such that $\boldsymbol{x}^{-n} = \boldsymbol{w}^{-n}$ and $\boldsymbol{y}^{-n} = \boldsymbol{z}^{-n}$, and $\boldsymbol{x}^{+n} = \boldsymbol{y}^{+n}$ and $\boldsymbol{w}^{+n} = \boldsymbol{z}^{+n}$, then $\boldsymbol{x} \succsim \boldsymbol{y}$ iff $\boldsymbol{w} \succsim \boldsymbol{z}$.

[14]**SP** and **QA** characterize the extension of the Suppes-Sen grading principle defined by $\mathcal{Q}$ (Banerjee 2006; Mitra and Basu 2007).

Proposition 1. (i) *There exists no SWR $\succsim$ satisfying **SP**, **QA**, and **SPC**.* (ii) *There exists no SWR $\succsim$ satisfying **SP**, **QA**, and **SC**.*

Proof. See Appendix. □

The trade-off between efficiency formalized as Paretian axioms and impartiality done by anonymity axioms has been intensively analyzed in the literature. As we noted in the preceding section, **QA** itself is compatible with **SP**, whereas the anonymity defined by all possible permutations on $\mathbb{N}$ comes in conflict with **SP** (van Liedekerke 1995; Lauwers 1997a). Furthermore, weakening **QA** to **FA**, it is possible to add **SPC** or **SC** as well. However, as shown in Proposition 1, if we strengthen the notion of impartiality **FA** to **QA** in such SWRs, we must go back to impossibility again.[15] Therefore, under two basic principles, **SP** and **FA**, our choice of additional axioms **QA** or **SPC** (or **SC**) becomes a branching point in exploring admissible SWRs exhibiting higher level of comparability than the Suppes-Sen grading principle.

We will now return to our main concern. To establish the characterizations of the extended leximin and the extended utilitarian SWRs both two additional axioms **QA** and **WPC** (or **WC**), we will formulate $\mathcal{Q}$-anonymous extensions of the W-leximin relation and the overtaking criterion respectively. For this purpose, we begin with the definitions of the W-leximin and the overtaking criteria. Let $\succsim_L^n$ denote the finite-horizon leximin ordering defined on $\mathbb{R}^n$ for each $n \in \mathbb{N}$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, $\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n}$ if and only if $(x_{(1)}^{-n}, \ldots, x_{(n)}^{-n}) = (y_{(1)}^{-n}, \ldots, y_{(n)}^{-n})$ or there exists an integer $m < n$ such that $(x_{(1)}^{-n}, \ldots, x_{(m)}^{-n}) = (y_{(1)}^{-n}, \ldots, y_{(m)}^{-n})$ and $x_{(m+1)}^{-n} > y_{(m+1)}^{-n}$.

The W-leximin relation $\succsim_{Lw}$ is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\begin{cases} \boldsymbol{x} \succ_{Lw} \boldsymbol{y} \text{ iff there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succ_L^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}; \\ \boldsymbol{x} \sim_{Lw} \boldsymbol{y} \text{ iff there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \sim_L^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}. \end{cases}$$

Similarly, the overtaking criterion $\succsim_O$ is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\begin{cases} \boldsymbol{x} \succ_O \boldsymbol{y} \text{ iff there exists } \bar{n} \in \mathbb{N} \text{ such that } \sum_{i=1}^n x_i > \sum_{i=1}^n y_i \text{ for all } n \geq \bar{n}; \\ \boldsymbol{x} \sim_O \boldsymbol{y} \text{ iff there exists } \bar{n} \in \mathbb{N} \text{ such that } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \text{ for all } n \geq \bar{n}. \end{cases}$$

We formally state the characterizations of $\succsim_{Lw}$ and $\succsim_O$ established by Asheim and Tungodden (2004) and Basu and Mitra (2007), which will be used to prove our main results later.[16]

[15]Fleurbaey and Michel (2003) provide comprehensive analysis of the trade-offs between **SP** and some well-established anonymity axioms. They also obtain a similar impossibility to Proposition 1 with Limit Ranking. Limit Ranking is similar to our preference-continuity or consistency axioms but there is no logical relationship between them.

[16]In their original characterizations, Asheim and Tungodden (2004) use **WPC** and **2UC** and Basu and

Proposition 2 (Asheim and Tungodden 2004, Proposition 2)**.** *A SWR $\succsim$ satisfies **SP**, **FA**, any of $\{$**WPC**, **WC**$\}$, and **HE** if and only if $\succsim_{Lw}$ is a subrelation of $\succsim$.*

Proposition 3 (Asheim and Tungodden 2004, Proposition 5; Basu and Mitra 2007, Theorem 3)**.** *A SWR $\succsim$ satisfies **SP**, **FA**, any of $\{$**WPC**, **WC**$\}$, and any of $\{$**2UC**, **PUC**$\}$ if and only if $\succsim_O$ is a subrelation of $\succsim$.*

We now introduce the $\mathcal{Q}$-anonymous extensions of $\succsim_{Lw}$ and $\succsim_O$, which we will call *$\mathcal{Q}$-W-leximin SWR* and *$\mathcal{Q}$-overtaking Criterion* respectively. The $\mathcal{Q}$-W-leximin relation, denoted by $\succsim_{QLw}$, is defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{QLw} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim_{Lw} \boldsymbol{Qy}. \tag{1}$$

Similarly, the $\mathcal{Q}$-overtaking criterion, $\succsim_{QO}$, is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{QO} \boldsymbol{y} \text{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim_{O} \boldsymbol{Qy}. \tag{2}$$

The following proposition tells that each of $\succsim_{QLw}$ and $\succsim_{QO}$ is well-defined as a SWR on X and the strict relation and the indifference relation corresponding to them are more simply characterized.

Proposition 4. *Each of $\succsim_{QLw}$ and $\succsim_{QO}$ is well-defined as a SWR on X, i.e. reflexive and transitive, and satisfies the following: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\begin{cases} \boldsymbol{x} \succ_{QLw} \boldsymbol{y} \textit{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q} \textit{ such that } \boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}; & \text{(3a)} \\ \boldsymbol{x} \sim_{QLw} \boldsymbol{y} \textit{ iff there exists } \boldsymbol{P} \in \mathcal{Q} \textit{ such that } \boldsymbol{Px} \sim_{Lw} \boldsymbol{y}, & \text{(3b)} \end{cases}$$

and

$$\begin{cases} \boldsymbol{x} \succ_{QO} \boldsymbol{y} \textit{ iff there exist } \boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q} \textit{ such that } \boldsymbol{Px} \succ_{O} \boldsymbol{Qy}; & \text{(4a)} \\ \boldsymbol{x} \sim_{QO} \boldsymbol{y} \textit{ iff there exists } \boldsymbol{P} \in \mathcal{Q} \textit{ such that } \boldsymbol{Px} \sim_{O} \boldsymbol{y}. & \text{(4b)} \end{cases}$$

Proof. See Appendix. □

By (3b) and (4b), $\succsim_{QLw}$ and $\succsim_{QO}$ satisfy **QA**.[17] Furthermore, from (3a), (3b) and the fact that $\boldsymbol{I} \in \mathcal{Q}$, it follows that $\succsim_{Lw}$ is a subrelation of $\succsim_{QLw}$, and the same is true for $\succsim_O$ and $\succsim_{QO}$ by (4a) and (4b). Thus, from Propositions 2 and 3, $\succsim_{QLw}$ and $\succsim_{QO}$ also satisfy all the axioms characterizing $\succsim_{Lw}$ and $\succsim_O$ respectively. Therefore,

Mitra (2007) employ **WC** and **PUC**. It is easily checked that **WPC** and **WC** are interchangeable and so are the two invariance axioms **2UC** and **PUC**. In the statements of Propositions 2 and 3 and Theorems 1 and 2, we follow d'Aspremont and Gevers (2002) and use the expression "any of {...}" to mean the axioms in {...} are interchangeable.

[17] Notice that $\boldsymbol{Px} \sim_{QLw} \boldsymbol{x}$ follows from the fact that $(\boldsymbol{x} =)\boldsymbol{P}'\boldsymbol{Px} \sim_{Lw} \boldsymbol{x}$.

Table 1: Characterizations of $\mathcal{F}$-anonymous SWRs and $\mathcal{Q}$-extensions

SWR	Axioms							
(least restrictive)	**SP**	**FA**	**QA**	**SPC/SC**	**WPC/WC**	**HE**	**2UC/PUC**	*characterization*
$\mathcal{Q}$-W-leximin	$\oplus$	$+$	$\oplus$	$-$	$\oplus$	$\oplus$	$-$	Theorem 1
W-leximin	$\oplus$	$\oplus$			$\oplus$	$\oplus$	$-$	AT (2004)
S-leximin	$\oplus$	$\oplus$	$-$	$\oplus$	$+$	$\oplus$	$-$	AT (2004)
$\mathcal{Q}$-overtaking	$\oplus$	$+$	$\oplus$	$-$	$\oplus$	$-$	$\oplus$	Theorem 2
overtaking	$\oplus$	$\oplus$			$\oplus$	$-$	$\oplus$	AT (2004) and BM (2007)
catching-up	$\oplus$	$\oplus$	$-$	$\oplus$	$+$	$-$	$\oplus$	AT (2004) and BM (2007)

$\succsim_{QLw}$ and $\succsim_{QO}$ certainly belong to $\mathcal{Q}$-anonymous subclasses of those characterized in Propositions 2 and 3.

Our main results show that the classes of all SWRs satisfying both **QA** and **WPC** (and **WC**) as well as the basic axioms (i.e. the shaded area in Figure 1) coincide with all SWRs that include $\succsim_{QLw}$ and $\succsim_{QO}$ respectively as a subrelation.

Theorem 1. *A SWR $\succsim$ satisfies **SP**, **QA**, any of $\{$**WPC**, **WC**$\}$, and **HE** if and only if $\succsim_{QLw}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

Theorem 2. *A SWR $\succsim$ satisfies **SP**, **QA**, any of $\{$**WPC**, **WC**$\}$, and any of $\{$**2UC**, **PUC**$\}$ if and only if $\succsim_{QO}$ is a subrelation of $\succsim$.*

Proof. See Appendix. □

Theorem 1 (resp. 2) is interpreted as saying that $\succsim_{QLw}$ (resp. $\succsim_{QO}$) is the *least restrictive* SWR among all the SWRs satisfying **SP**, **QA**, **WPC** (or **WC**), and **HE** (resp. **2UC** (or **PUC**)). Formally, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, we have

$$\begin{cases} \boldsymbol{x} \succsim_{QLw} \boldsymbol{y} \text{ if and only if } \boldsymbol{x} \succsim \boldsymbol{y} \text{ for all } \succsim \in \Xi_{QLw}; \\ \boldsymbol{x} \succsim_{QO} \boldsymbol{y} \text{ if and only if } \boldsymbol{x} \succsim \boldsymbol{y} \text{ for all } \succsim \in \Xi_{QO}, \end{cases}$$

where Ξ_{QLw} (resp. Ξ_{QO}) is the set of all SWRs that satisfy **SP**, **QA**, any of $\{$**WPC**, **WC**$\}$, and **HE** (resp. any of $\{$**2UC**, **PUC**$\}$).[18] From Arrow's (1963) variant of Szpilrajn's (1930) lemma, each of Ξ_{QLw} and Ξ_{QO} contains at least one complete SWR, i.e. social welfare ordering.

[18]For each of the two equivalence assertions, the only if part follows from the only if statement of the corresponding theorem, and the if part is also straightforward from the fact that $\succsim_{QLw} \in \Xi_{QLw}$ (resp. $\succsim_{QO} \in \Xi_{QO}$).

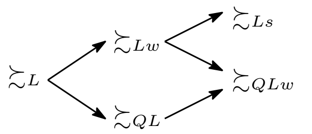

Figure 2: Extended leximin SWRs

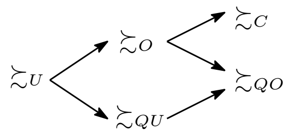

Figure 3: Extended utilitarian SWRs

Table 1 summarizes the characterizations in Theorems 1 and 2 and compares them with those established by Asheim and Tungodden (2004) and Basu and Mitra (2007) (in Table 1, AT (2004) and BM (2007) respectively). For each row in Table 1, the class of SWRs that includes the SWR stated in the first column as a subrelation is characterized by the axioms indicated by ⊕, and furthermore, each SWR out of the class satisfies (resp. violates) the axioms indicated by + (resp. –). Compared to the characterizations in Asheim and Tungodden (2004) and Basu and Mitra (2007), our results are regarded as the refinements of admissible SWRs by using the stronger notion of impartiality, **QA**, than **FA**. The impossibilities in Proposition 1 give "–" in the 4th and 5th column in Table 1. Consequently, it can be said that it is possible to reflect the stronger notion of impartiality **QA**, but it comes at a cost of the stronger versions of preference-continuity and consistency properties, **SPC** and **SC**.

4 Comparison with some well-established SWRs

In this section, we compare our new SWRs $\succsim_{QLw}$ and $\succsim_{QO}$ with some relevant ones in the literature. We begin with the formal definitions of the leximin SWR (Bossert et al. 2007) and the utilitarian SWR (Basu and Mitra 2007) and also of their $\mathcal{Q}$-anonymous extensions, $\mathcal{Q}$-leximin SWR (Kamaga and Kojima 2008) and $\mathcal{Q}$-utilitarian SWR (Banerjee 2006).

- The leximin SWR $\succsim_L$ and the utilitarian SWR $\succsim_U$:

  $\boldsymbol{x} \succsim_L \boldsymbol{y}$ iff there exists $n \in \mathbb{N}$ such that $\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n}$ and $\boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}$.

  $\boldsymbol{x} \succsim_U \boldsymbol{y}$ iff there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$ and $\boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}$.

- The $\mathcal{Q}$-leximin SWR $\succsim_{QL}$ and the $\mathcal{Q}$-utilitarian SWR $\succsim_{QU}$:

  $\boldsymbol{x} \succsim_{QL} \boldsymbol{y}$ iff there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim_L \boldsymbol{y}$.

  $\boldsymbol{x} \succsim_{QU} \boldsymbol{y}$ iff there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim_U \boldsymbol{y}$.

Figures 2 and 3 summarize the relationships among the SWRs we discussed so far, where $\succsim_{Ls}$ and $\succsim_C$ denote the S-leximin SWR and the catching-up criterion respectively and we write $\succsim \to \succsim'$ to mean $\succsim$ is a subrelation of $\succsim'$.[19]

The following example shows that our new SWRs $\succsim_{QLw}$ and $\succsim_{QO}$ respectively can make further comparisons of streams beyond the limits of their subrelations $\succsim_L$, $\succsim_U$, $\succsim_{Lw}$, $\succsim_{QL}$, $\succsim_O$, and $\succsim_{QU}$.

Example 1. Consider the following utility streams $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x} = (1, 1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3^2}, \tfrac{1}{3^2}, \tfrac{1}{3^3}, \dots)$$
$$\boldsymbol{y} = (1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3^2}, \tfrac{2}{3^2}, \tfrac{2}{3^3}, \tfrac{2}{3^3}, \dots).$$

One can generate the streams $\boldsymbol{x}$ and $\boldsymbol{y}$ in the following way: $x_1 = 1$ and $y_1 = 1$ and, for all $n \geq 2$

$$x_n = \begin{cases} \frac{3}{\sqrt{3}^n} & \text{if } n \text{ is even} \\ \frac{\sqrt{3}}{\sqrt{3}^n} & \text{otherwise,} \end{cases} \quad \text{and} \quad y_n = \begin{cases} \frac{2}{\sqrt{3}^n} & \text{if } n \text{ is even} \\ \frac{2\sqrt{3}}{\sqrt{3}^n} & \text{otherwise.} \end{cases}$$

Clearly, x and y are non-comparable according to $\succsim_{Lw}$, since

$$\begin{cases} \min\{x_1, \dots, x_n\} < \min\{y_1, \dots, y_n\} & \text{for all even } n, \\ \min\{x_1, \dots, x_n\} > \min\{y_1, \dots, y_n\} & \text{for all odd } n. \end{cases}$$

Moreover, $\succsim_O$ also declares them non-comparable, since

$$\begin{cases} \sum_{i=1}^n x_i > \sum_{i=1}^n y_i & \text{for all even } n, \\ \sum_{i=1}^n x_i = \sum_{i=1}^n y_i & \text{for all odd } n. \end{cases}$$

As to $\succsim_{QL}$ and $\succsim_{QU}$, they still declare $\boldsymbol{x}$ and $\boldsymbol{y}$ non-comparable, since any of the permutations $\boldsymbol{P}$ in $\mathcal{Q}$ cannot give the Pareto dominance between $\boldsymbol{Px}$ and $\boldsymbol{y}$. Thus, $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-comparable according to any of $\succsim_{Lw}$, $\succsim_{QL}$, $\succsim_O$, and $\succsim_{QU}$ (thus, $\succsim_L$ and $\succsim_U$ either).

However, using the 2-period cyclic permutation $\bar{\boldsymbol{P}} \in \mathcal{Q}$ corresponding to the permutation π defined as: $\pi(n) = n + 1$ if n is odd, and $\pi(n) = n - 1$ if n is even, we have $\boldsymbol{x} \succ_{Lw} \bar{\boldsymbol{P}}\boldsymbol{y}$ and $\boldsymbol{x} \succ_O \bar{\boldsymbol{P}}\boldsymbol{y}$. Thus, according to $\succsim_{QLw}$ or $\succsim_{QO}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are comparable and $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$ and $\boldsymbol{x} \succ_{QO} \boldsymbol{y}$. □

[19] The S-leximin SWR $\succsim_{Ls}$ and the catching-up criterion $\succsim_C$ are defined as: $\boldsymbol{x} \succsim_{Ls} \boldsymbol{y}$ iff there exists $\bar{n} \in \mathbb{N}$ such that $\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n}$ for all $n \geq \bar{n}$; $\boldsymbol{x} \succsim_C \boldsymbol{y}$ iff there exists $\bar{n} \in \mathbb{N}$ such that $\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$ for all $n \geq \bar{n}$.

The streams $\boldsymbol{x}$ and $\bar{\boldsymbol{P}}\boldsymbol{y}$ can be an example of the case where $\succsim_{Lw}$ and $\succsim_O$ can compare those streams, but $\succsim_{QL}$ and $\succsim_{QU}$ cannot. As noted in the introduction, the streams $(1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, \dots)$ give us an example of the converse case. Since our new criteria $\succsim_{QLw}$ and $\succsim_{QO}$ incorporate both two merits of $\succsim_{QL}$ or $\succsim_{QU}$ and $\succsim_{Lw}$ or $\succsim_O$ respectively, they can resolve the trade-off in the choice of the $\mathcal{Q}$-anonymous extensions $\succsim_{QL}$ and $\succsim_{QU}$ or the preference-continuous or consistent relations $\succsim_{Lw}$ and $\succsim_O$.

5 Concluding remarks

We have characterized the classes of all SWRs satisfying not only the basic axioms which give axiomatic foundations of the infinite-horizon variants of leximin principle and utilitarianism, $\succsim_L$ and $\succsim_U$, respectively (**SP**, **FA** and **HE** or **2UC** (or **PUC**)) but also the two additional requirements, the weak version of preference-continuity or consistency (**WPC** or **WC**) and the stronger notion of impartiality than Finite Anonymity (**QA**). In these classes of SWRs, our new extended SWRs $\succsim_{QLw}$ and $\succsim_{QO}$ respectively are the least restrictive ones. Therefore, our two characterization theorems tell that under the axioms stated above, our evaluation of intergenerational welfare distribution must be based on the comparisons according to $\succsim_{QLw}$ and $\succsim_{QO}$ respectively. As we have observed in Sect. 4, $\succsim_{QLw}$ and $\succsim_{QO}$ can lead us to further comparisons of streams beyond the limits of the well-established extended SWRs $\succsim_{Lw}$, $\succsim_O$, $\succsim_{QL}$ and $\succsim_{QU}$.

Both $\succsim_{QLw}$ and $\succsim_{QO}$ are formulated as the extensions of $\succsim_{Lw}$ and $\succsim_O$ by using permutations of the class $\mathcal{Q}$ and are characterized by strengthening **FA** to **QA** in the lists of the axioms characterizing $\succsim_{Lw}$ and $\succsim_O$ respectively. As will be shown in Appendix A.2, these results are generalizable to any SWR defined by using a sequence of finite-horizon orderings satisfying certain moderate properties in the same way as in $\succsim_{Lw}$ and $\succsim_O$. Such an general approach to the analysis of infinite-horizon criteria is initiated by d'Aspremont (2007) and also taken by Asheim and Banerjee (2008) and Kamaga and Kojima (2008).

Finally, we should discuss the issue, raised by Banerjee (2006), on the rankings of summable streams derived by extended utilitarian SWRs. As he discussed in his Example 3, $\succsim_{QU}$ declares the following two summable sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ to be non-comparable: $\boldsymbol{u} = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^3}, \dots)$ and $\boldsymbol{v} = (1, 1, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^4}, \dots)$. Since we have $\sum_{i=1}^{\infty} u_i = 7/3 < 8/3 = \sum_{i=1}^{\infty} v_i$, if we follow the spirit of utilitarianism, then we should conclude $\boldsymbol{v}$ is strictly better than $\boldsymbol{u}$. Our extended utilitarian relation $\succsim_{QO}$, which satisfies **WPC** (and **WC**), can compare any two summable sequences in terms of their sums of utilities if their utility sums are *different*. [20] In this respect, $\succsim_{QO}$ is a quite appealing infinite-horizon formulation of utilitarianism. However, it still fails to rank summable sequences according to their sums of utilities if the total sums are *equal*. Notice that $\boldsymbol{x}$ and $\boldsymbol{y}$ considered in Sect. 4 are summable and $\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} y_i = 3$. For these streams, $\succsim_{QO}$ concludes that $\boldsymbol{x}$ is strictly better than $\boldsymbol{y}$. To formulate and characterize an extended utilitarian relation that completely reflects the utilitarian doctrine for all summable sequences, we must lay down **WPC** and **WC**. We leave this issue for future research.

Appendix

A.1. Proof of Proposition 1

Proof of Proposition 1. First, we prove (i) by contradiction. Suppose that $\succsim$ satisfies **SP**, **QA**, and **SPC**. Let $\boldsymbol{x} = (1, 0, 1, 0, \dots)$ and $\boldsymbol{y} = (0, 1, 0, 1, \dots)$. By **QA**,

$$(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \sim \boldsymbol{y} \text{ for all even } n \in \mathbb{N}, \tag{5}$$

and also $(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \sim (x_1, \boldsymbol{y}^{+1})$ for all odd $n \in \mathbb{N}$. By **SP**, $(x_1, \boldsymbol{y}^{+1}) \succ \boldsymbol{y}$. By transitivity,

$$(\boldsymbol{x}^{-n}, \boldsymbol{y}^{+n}) \succ \boldsymbol{y} \text{ for all odd } n \in \mathbb{N}. \tag{6}$$

From (5) and (6), **SPC** gives $\boldsymbol{x} \succ \boldsymbol{y}$, while $\boldsymbol{x} \sim \boldsymbol{y}$ is obtained by **QA**.

Next, we prove (ii). The proof is similar to that of (i). Suppose $\succsim$ satisfies **SP**, **QA**, **SC**. Let $\boldsymbol{x} = (1, 0, 1, 0, \dots)$ and $\boldsymbol{y} = (0, 1, 0, 1, \dots)$. By **QA**,

$$(\boldsymbol{x}^{-n}, 0, 0, \dots) \sim (\boldsymbol{y}^{-n}, 0, 0, \dots) \text{ for all even } n \in \mathbb{N}, \tag{7}$$

and $(\boldsymbol{x}^{-n}, 0, 0, \dots) \sim (\boldsymbol{y}^{-(n+2)}, 0, 0, \dots)$ for all odd $n \in \mathbb{N}$. By **SP**, $(\boldsymbol{x}^{-(n+2)}, 0, 0, \dots) \succ (\boldsymbol{x}^{-n}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. Since $\succsim$ is transitive,

$$(\boldsymbol{x}^{-(2n+1)}, 0, 0, \dots) \succ (\boldsymbol{y}^{-(2n+1)}, 0, 0, \dots) \text{ for all odd } n \in \mathbb{N}. \tag{8}$$

From (7) and (8), **SC** gives $\boldsymbol{x} \succ \boldsymbol{y}$, while, by **QA**, $\boldsymbol{x} \sim \boldsymbol{y}$. □

[20] This result is generalizable to any two sequences such that the cumulative sums of difference between the streams converge in $\mathbb{R}_{++}$.

A.2. Proof of Proposition 4

First, we introduce the finite-horizon utilitarian relation $\succsim_U^n$ defined on $\mathbb{R}^n$ for each $n \in \mathbb{N}$: for all $x^{-n}, y^{-n} \in \mathbb{R}^n$, $x^{-n} \succsim_U^n y^{-n}$ if and only if $\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$. Note that both of the finite-horizon leximin and utilitarian relations $\succsim_L^n$ and $\succsim_U^n$ are orderings on $\mathbb{R}^n$ for all $n \in \mathbb{N}$, and moreover, each of the sequences of them, $\{\succsim_L^n\}_{n\in\mathbb{N}}$ and $\{\succsim_U^n\}_{n\in\mathbb{N}}$, satisfies the following three properties:[21] for all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

(α) If $\boldsymbol{x}^{-n} > \boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \succ^n \boldsymbol{y}^{-n}$;

(β) If $\boldsymbol{x}^{-n}$ is a permutation of $\boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \sim^n \boldsymbol{y}^{-n}$;

(γ) For all $r \in \mathbb{R}$, $(\boldsymbol{x}^{-n}, r) \succsim^{n+1} (\boldsymbol{y}^{-n}, r)$ if and only if $\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n}$,

where $\succsim^n$ denotes an ordering on $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

We provide the proof of Proposition 4 for the case of $\succsim_{QLw}$ by only using the properties (α), (β), and (γ). Thus, the same argument can be directly applied to the case of $\succsim_{QO}$, and we omit it.

First, we prove the equivalence assertions in (3a) and (3b). To prove them, we use the following Lemma.

Lemma 1. *For all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all $\boldsymbol{P} \in \mathcal{Q}$,*

$$\boldsymbol{x} \sim_{Lw} \boldsymbol{y} \text{ if and only if } \boldsymbol{P}\boldsymbol{x} \sim_{Lw} \boldsymbol{P}\boldsymbol{y}. \tag{9}$$

Proof. (only if part) Assume $\boldsymbol{x} \sim_{Lw} \boldsymbol{y}$, and consider any $\boldsymbol{P} \in \mathcal{Q}$. Since $\boldsymbol{P} \in \mathcal{Q}$, there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\boldsymbol{P}(nk)$ is a finite-dimensional permutation matrix. By the definition of $\succsim_{Lw}$, we can find $\bar{m} \in \mathbb{N}$ such that

$$\boldsymbol{x}^{-m} \sim_L^m \boldsymbol{y}^{-m} \text{ for all } m \geq \bar{m} \text{ with } \bar{m} = nk \text{ for some } n \in \mathbb{N}. \tag{10}$$

We show, by contradiction, that

$$x_m = y_m \text{ for all } m > \bar{m}. \tag{11}$$

Suppose that (11) does not hold. Let m' be the smallest integer such that $m' > \bar{m}$ and $x_{m'} \neq y_{m'}$. Without loss of generality, we assume $x_{m'} > y_{m'}$. By (γ),

[21] Property (α) is the finite-horizon version of **SP**. Property (β) is a well-known anonymity axiom in a finite-horizon framework. Property (γ) is a kind of separability requirement similar to *Extended Independence of the Utilities of Unconcerned Individuals* introduced by Blackorby et al. (2002) in the framework of variable population social choice, which requires our evaluation to be independent of the existence of an unconcerned generation.

$(\boldsymbol{x}^{-(m'-1)}, y_{m'}) \sim_L^{m'} \boldsymbol{y}^{-m'}$. By ($\alpha$), $\boldsymbol{x}^{-m'} \succ_L^{m'} (\boldsymbol{x}^{-(m'-1)}, y_{m'})$. The transitivity of $\succsim_L^{m'}$ gives $\boldsymbol{x}^{-m'} \succ_L^{m'} \boldsymbol{y}^{-m'}$, which contradicts (10). Thus, (11) holds. Since $\boldsymbol{P}(\bar{m})$ is a finite-dimensional permutation matrix, by (β), $\boldsymbol{x}^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Px})^{-\bar{m}}$ and $\boldsymbol{y}^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Py})^{-\bar{m}}$. Then, by the transitivity of $\succsim^{\bar{m}}$ and (10), $(\boldsymbol{Px})^{-\bar{m}} \sim_L^{\bar{m}} (\boldsymbol{Py})^{-\bar{m}}$. Note that, by (11), $(\boldsymbol{Px})^{+\bar{m}} = (\boldsymbol{Py})^{+\bar{m}}$. Thus, by ($\gamma$), $(\boldsymbol{Px})^{-m} \sim_L^{m} (\boldsymbol{Py})^{-m}$ holds for all $m \geq \bar{m}$. By the definition of $\succsim_{Lw}$, $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Py}$.

(if part) Take any $\boldsymbol{x}, \boldsymbol{y} \in X$ and any $\boldsymbol{P} \in \mathcal{Q}$, and assume $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Py}$. Since $\boldsymbol{P}' \in \mathcal{Q}$, the only if part of the lemma gives $(\boldsymbol{x} =)\boldsymbol{P}'\boldsymbol{Px} \sim_{Lw} \boldsymbol{P}'\boldsymbol{Py}(= \boldsymbol{y})$. □

We are ready to prove the equivalence assertions in (3a) and (3b).

(only if (3a)): Suppose $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$. Then, by definition, there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim_{Lw} \boldsymbol{Qy}$. Moreover, for all $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{Q}$, $\neg(\boldsymbol{Ry} \succsim_{Lw} \boldsymbol{Sx})$. Otherwise, we have a contradiction to $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$. Thus, $\neg(\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Px})$.

(if (3a)): Suppose that there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$. Then, by definition, $\boldsymbol{x} \succsim_{QLw} \boldsymbol{y}$. We show, by contradiction, that $\neg(\boldsymbol{y} \succsim_{QLw} \boldsymbol{x})$. Assume $\boldsymbol{y} \succsim_{QLw} \boldsymbol{x}$. Then, there exist $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{Q}$ such that $\boldsymbol{Ry} \succsim_{Lw} \boldsymbol{Sx}$. Let $p, q, r, s \in \mathbb{N}$ be period of cycle in $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$, and $\boldsymbol{S}$ respectively. By the definition of $\succsim_{Lw}$, there exist $\bar{n}, \bar{n}' \in \mathbb{N}$ such that

$$\boldsymbol{Px}^{-n} \succ_L^n \boldsymbol{Qy}^{-n} \text{ for all } n \geq \bar{n}, \tag{12}$$

and

$$\begin{cases} \boldsymbol{Ry}^{-n} \succ_L^n \boldsymbol{Sx}^{-n} \text{ for all } n \geq \bar{n}', \\ \text{or} \\ \boldsymbol{Ry}^{-n} \sim_L^n \boldsymbol{Sx}^{-n} \text{ for all } n \geq \bar{n}'. \end{cases} \tag{13}$$

Let $k = p \times q \times r \times s$, and choose $\hat{n} \in \mathbb{N}$ such that $\hat{n}k \geq \max\{\bar{n}, \bar{n}'\}$. Note that $\boldsymbol{P}(\hat{n}k)$, $\boldsymbol{Q}(\hat{n}k)$, $\boldsymbol{R}(\hat{n}k)$, and $\boldsymbol{S}(\hat{n}k)$ are finite-dimensional permutation matrices. By (β), $\boldsymbol{Qy}^{-\hat{n}k} \sim_L^{\hat{n}k} \boldsymbol{Ry}^{-\hat{n}k}$. Then, from (12) and (13), the transitivity of $\succsim^{\hat{n}k}$ gives $\boldsymbol{Px}^{-\hat{n}k} \succ_L^{\hat{n}k} \boldsymbol{Sx}^{-\hat{n}k}$, which contradicts ($\beta$).

(only if (3b)): Suppose $\boldsymbol{x} \sim_{QLw} \boldsymbol{y}$. By definition, there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim_{Lw} \boldsymbol{Qy}$. If we have $\neg(\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Px})$, then, by (3a), $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$, and a contradiction is obtained. Thus, $\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Px}$ must hold, and $\boldsymbol{Px} \sim_{Lw} \boldsymbol{Qy}$ follows. By Lemma 1, $\boldsymbol{Q}'\boldsymbol{Px} \sim_{Lw} \boldsymbol{Q}'\boldsymbol{Qy}(= \boldsymbol{y})$. Since $\boldsymbol{Q}'\boldsymbol{P} \in \mathcal{Q}$, the proof is completed.

(if (3b)): Suppose that there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \sim_{Lw} \boldsymbol{y}$. Since $\boldsymbol{P}, \boldsymbol{I} \in \mathcal{Q}$, $\boldsymbol{x} \succsim_{QLw} \boldsymbol{y}$. If we have $\neg(\boldsymbol{y} \succsim_{QLw} \boldsymbol{x})$, then, by definition, for all $\boldsymbol{Q}, \boldsymbol{R} \in \mathcal{Q}$, $\neg(\boldsymbol{Qy} \succsim_{Lw} \boldsymbol{Rx})$, which contradicts $\boldsymbol{Px} \sim_{Lw} \boldsymbol{y}$. Thus, $\boldsymbol{y} \succsim_{QLw} \boldsymbol{x}$.

Next, we prove that $\succsim_{QLw}$ is a SWR on X. To prove this, we begin with the following lemma.

Lemma 2. *$\succsim_{QLw}$ is quasi-transitive, i.e. for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, if $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{QLw} \boldsymbol{z}$, then $\boldsymbol{x} \succ_{QLw} \boldsymbol{z}$.*

Proof. Assume that $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{QLw} \boldsymbol{z}$. By (3a), there exist $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$ and $\boldsymbol{Ry} \succ_{Lw} \boldsymbol{Sz}$. Let $p, q, r, s \in \mathbb{N}$ be period of cycle in $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$, and $\boldsymbol{S}$ respectively and also $k = p \times q \times r \times s$. Then, for all $n \in \mathbb{N}$, each of $\boldsymbol{P}(nk)$, $\boldsymbol{Q}(nk)$, $\boldsymbol{R}(nk)$, and $\boldsymbol{S}(nk)$ is a finite-dimensional permutation matrix. By the definition of $\succsim_{Lw}$, we can find $\bar{m} \in \mathbb{N}$ such that $\bar{m} = nk$ for some $n \in \mathbb{N}$ and, for all $m \geq \bar{m}$,

$$(\boldsymbol{Px})^{-m} \succ_L^m (\boldsymbol{Qy})^{-m} \text{ and } (\boldsymbol{Ry})^{-m} \succ_L^m (\boldsymbol{Sz})^{-m}. \tag{14}$$

By (β), $(\boldsymbol{Qy})^{-n\bar{m}} \sim_L^{n\bar{m}} (\boldsymbol{Ry})^{-n\bar{m}}$ for all $n \in \mathbb{N}$. Then, by (14), the transitivity of $\succsim_L^{n\bar{m}}$ gives

$$(\boldsymbol{Px})^{-n\bar{m}} \succ_L^{n\bar{m}} (\boldsymbol{Sz})^{-n\bar{m}} \text{ for all } n \in \mathbb{N}. \tag{15}$$

We show that there exist $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{S}} \in \mathcal{Q}$ such that $(\tilde{\boldsymbol{P}}\boldsymbol{x})^{-m} \succ_L^m (\tilde{\boldsymbol{S}}\boldsymbol{z})^{-m}$ for all $m \geq \bar{m}$. Then, by the definition of $\succsim_{Lw}$ and (3a), $x \succ_{QLw} z$ is obtained as desired. If $(\boldsymbol{Px})^{-m} \succ_L^m (\boldsymbol{Sz})^{-m}$ for all $m \geq \bar{m}$, $\boldsymbol{P} = \tilde{\boldsymbol{P}}$ and $\boldsymbol{S} = \tilde{\boldsymbol{S}}$ trivially follow. We now consider the other cases. For any $n \in \mathbb{N}$, let $i(n) \in \{n\bar{m}+1, \ldots, (n+1)\bar{m}-1\}$ be the smallest integer for which $\neg((\boldsymbol{Px})^{-i(n)} \succ_L^{i(n)} (\boldsymbol{Sz})^{-i(n)})$ holds. Since $\succsim_L^{i(n)}$ is complete, this is equivalent to

$$(\boldsymbol{Sz})^{-i(n)} \succsim_L^{i(n)} (\boldsymbol{Px})^{-i(n)}. \tag{16}$$

By (15), there must be $j(n) \in \{i(n)+1, \ldots, (n+1)\bar{m}\}$ such that

$$(Px)_{j(n)} > (Sz)_{j(n)}. \tag{17}$$

Otherwise, by (16), (α) and (γ), we have $(\boldsymbol{Sz})^{-(n+1)\bar{m}} \succsim_L^{(n+1)\bar{m}} (\boldsymbol{Px})^{-(n+1)\bar{m}}$, which contradicts (15). We construct $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{Q}}$ as follows. Let $\boldsymbol{T}_{1(n)} \in \mathcal{F}$ be a transposition of $i(n)$ and $j(n)$, i.e.

$$\begin{cases} (T_{1(n)}(T_{1(n)}x))_{i(n)} = (T_{1(n)}x)_{j(n)} = x_{i(n)}, \text{ and} \\ (T_{1(n)}x)_k = x_k \text{ for all } k \in \mathbb{N} \setminus \{i(n), j(n)\}. \end{cases} \tag{18}$$

By (17), (α) and (γ), we have $(\boldsymbol{T}_{1(n)}\boldsymbol{Px})^{-N} \succ_L^N (\boldsymbol{T}_{1(n)}\boldsymbol{Sz})^{-N}$ for all $N \in \{n\bar{m} +$

$1,\dots,i(n)\}$. Moreover, by (β), $(\boldsymbol{T}_{1(n)}\boldsymbol{P}\boldsymbol{x})^{-(n+1)\bar{m}} \sim_L^{(n+1)\bar{m}} (\boldsymbol{P}\boldsymbol{x})^{-(n+1)\bar{m}}$, and $(\boldsymbol{T}_{1(n)}\boldsymbol{S}\boldsymbol{z})^{-(n+1)\bar{m}} \sim_L^{(n+1)\bar{m}} (\boldsymbol{S}\boldsymbol{z})^{-(n+1)\bar{m}}$. Thus, by (15) and the transitivity of $\succsim_L^{(n+1)\bar{m}}$,

$$(\boldsymbol{T}_{1(n)}\boldsymbol{P}\boldsymbol{x})^{-(n+1)\bar{m}} \succ_L^{(n+1)\bar{m}} (\boldsymbol{T}_{1(n)}\boldsymbol{S}\boldsymbol{z})^{-(n+1)\bar{m}}.$$

Using the same argument repeatedly at most $k(n) := (n+1)\bar{m} - i(n)$ times (redefine $i(n)$ for the streams obtained after the transpositions for each case), we have, for all $N \in \{n\bar{m}+1,\dots,(n+1)\bar{m}\}$,

$$(\boldsymbol{T}_{k(n)}\dots\boldsymbol{T}_{2(n)}\boldsymbol{T}_{1(n)}\boldsymbol{P}\boldsymbol{x})^{-N} \succ_L^N (\boldsymbol{T}_{k(n)}\dots\boldsymbol{T}_{2(n)}\boldsymbol{T}_{1(n)}\boldsymbol{S}\boldsymbol{z})^{-N}. \tag{19}$$

Using the sequence of transpositions $\{\boldsymbol{T}_{1(1)}, \boldsymbol{T}_{2(1)}, \dots, \boldsymbol{T}_{k(n)}, \dots\}$, we define infinite-dimensional matrices $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{S}}$ as: for all $n \in \mathbb{N}$,

$$\begin{cases} \tilde{\boldsymbol{P}}(n\bar{m}) = [\boldsymbol{T}_{k(n)}\dots\boldsymbol{T}_{2(1)}\boldsymbol{T}_{1(1)}\boldsymbol{P}](n\bar{m}); \\ \tilde{\boldsymbol{S}}(n\bar{m}) = [\boldsymbol{T}_{k(n)}\dots\boldsymbol{T}_{2(1)}\boldsymbol{T}_{1(1)}\boldsymbol{S}](n\bar{m}). \end{cases}$$

By (18), $\tilde{\boldsymbol{P}}(n\bar{m})$ and $\tilde{\boldsymbol{S}}(n\bar{m})$ are well-defined as finite-dimensional permutation matrices for all $n \in \mathbb{N}$. Thus, $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{S}} \in \mathcal{Q}$. By (19) where $n \in \mathbb{N}$ is arbitrarily chosen, we obtain $(\tilde{\boldsymbol{P}}\boldsymbol{x})^{-m} \succ_L^m (\tilde{\boldsymbol{S}}\boldsymbol{z})^{-m}$ for all $m \geq \bar{m}$. □

We now show that $\succsim_{QLw}$ is a SWR on X.

(Reflexivity): Since $\boldsymbol{I} \in \mathcal{Q}$ and $\succsim_{Lw}$ is reflexive, $\succsim_{QLw}$ is also reflexive.

(Transitivity): In view of Lemma 2, we are enough to show that for each of the three cases: (a) $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$ and $\boldsymbol{y} \sim_{QLw} \boldsymbol{z}$; (b) $\boldsymbol{x} \sim_{QLw} \boldsymbol{y}$ and $\boldsymbol{y} \succ_{QLw} \boldsymbol{z}$; and (c) $\boldsymbol{x} \sim_{QLw} \boldsymbol{y}$ and $\boldsymbol{y} \sim_{QLw} \boldsymbol{z}$, we have $\boldsymbol{x} \succsim_{QLw} \boldsymbol{z}$. In case (a), by (3a) and (3b), there exist $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \succ_{Lw} \boldsymbol{Q}\boldsymbol{y}$ and $\boldsymbol{R}\boldsymbol{y} \sim_{Lw} \boldsymbol{z}$. Since $\boldsymbol{Q}\boldsymbol{R}' \in \mathcal{Q}$, we have $(\boldsymbol{Q}\boldsymbol{y} =)\boldsymbol{Q}\boldsymbol{R}'\boldsymbol{R}\boldsymbol{y} \sim_{Lw} \boldsymbol{Q}\boldsymbol{R}'\boldsymbol{z}$ by Lemma 1. The transitivity of $\succsim_{Lw}$ gives $\boldsymbol{P}\boldsymbol{x} \succ_{Lw} \boldsymbol{Q}\boldsymbol{R}'\boldsymbol{z}$, and $\boldsymbol{x} \succ_{QLw} \boldsymbol{z}$ follows from (3a). In the case of (b), by (3a) and (3b), there exist $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \sim_{Lw} \boldsymbol{y}$ and $\boldsymbol{Q}\boldsymbol{y} \succ_{Lw} \boldsymbol{R}\boldsymbol{z}$. Since $\boldsymbol{Q}\boldsymbol{P} \in \mathcal{Q}$, we can prove $\boldsymbol{x} \succ_{QLw} \boldsymbol{z}$ by the similar argument to case (a), and we omit it. Finally, we consider case (c). By (3b), there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \sim_{Lw} \boldsymbol{y}$ and $\boldsymbol{Q}\boldsymbol{y} \sim_{Lw} \boldsymbol{z}$. Since $\boldsymbol{Q}\boldsymbol{P} \in \mathcal{Q}$, we obtain $\boldsymbol{x} \sim_{QLw} \boldsymbol{z}$ by the similar argument to the case (a), and we omit it.

A.3. Proofs of Theorems 1 and 2

Proof of Theorem 1. The if part is obvious from the argument in Sect. 3. We provide the proof of the only if part. Assume that a SWR $\succsim$ satisfies **SP**, **QA**, **WPC**, and **HE**.

We show that (a) if $\boldsymbol{x} \succ_{QLw} \boldsymbol{y}$ then $\boldsymbol{x} \succ \boldsymbol{y}$ and (b) if $\boldsymbol{x} \sim_{QLw} \boldsymbol{y}$ then $\boldsymbol{x} \sim \boldsymbol{y}$. From Proposition 2, $\succsim_{Lw}$ is now a subrelation of $\succsim$.

(a) Assume $x \succ_{QLw} y$. By (3a), there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succ_{Lw} \boldsymbol{Qy}$. Since $\succsim_{Lw}$ is a subrelation of $\succsim$, $\boldsymbol{Px} \succ \boldsymbol{Qy}$. By **QA**, $\boldsymbol{x} = \boldsymbol{P}'\boldsymbol{Px} \sim \boldsymbol{Px}$ and $\boldsymbol{y} = \boldsymbol{Q}'\boldsymbol{Qy} \sim \boldsymbol{Qy}$. By the transitivity of $\succsim$, $\boldsymbol{x} \succ \boldsymbol{y}$.

(b) Assume $\boldsymbol{x} \sim_{QLw} \boldsymbol{y}$. By (3b), there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \sim_{Lw} \boldsymbol{y}$. Since $\succsim_{Lw}$ is a subrelation of $\succsim$, $\boldsymbol{Px} \sim \boldsymbol{y}$. By **QA**, $\boldsymbol{x} = \boldsymbol{P}'\boldsymbol{Px} \sim \boldsymbol{Px}$. Since $\succsim$ is transitive, $\boldsymbol{x} \sim \boldsymbol{y}$. □

Proof of Theorem 2. Using Proposition 3, the same argument as in the proof of Theorem 1 can be applied to prove Theorem 2, and we omit it. □

References

Arrow, K.J.: Social Choice and Individual Values. Wiley, New York (1963)

Asheim, G.B., Banerjee, K.: Generalized time-invariant overtaking. mimeo, University of Oslo (2008)

Asheim, G.B., Buchholz, W., Tungodden, B.: Justifying sustainability. J Environ Econ Manage **41**, 252-268 (2001)

Asheim, G.B., Tungodden, B.: Resolving distributional conflicts between generations. Econ Theory **24**, 221-230 (2004)

Banerjee, K.: On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams. Soc Choice Welf **27**, 327-339 (2006)

Basu, K., Mitra, T.: Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization. J Econ Theory **133**, 350-373 (2007)

Blackorby, C., Bossert, W., Donaldson, D.: Utilitarianism and the theory of justice. In: Arrow, K.J., Sen, A.K., Suzumura, K. (eds.) Handbook of Social Choice and Welfare I, pp. 543-596. North-Holland, Amsterdam (2002)

Bossert, W., Sprumont, Y., Suzumura, K.: Ordering infinite utility streams. J Econ Theory **135**, 579-589 (2007)

Bossert, W., Weymark, J.A.: Utility in social choice. In: Barbara, S., Hammond, P.J., Seidl, C. (eds.) Handbook of Utility Theory, Extensions, II, pp. 1099-1177. Kluwer, Boston (2004)

Brock, W.A.: An axiomatic basis for the Ramsey-Weizsäcker overtaking criterion. Econometrica **38**, 927-929 (1970) d'Aspremont, C.: Formal welfarism and intergenerational equity. In: Roemer, J., Suzumura, K. (eds.) Intergenerational Equity and Sustainability, pp. 113-130. Palgrave, London (2007)

d'Aspremont, C., Gevers, L.: Social welfare functionals and interpersonal comparability. In: Arrow, K.J., Sen, A.K., Suzumura, K. (eds.) Handbook of Social Choice and Welfare I, pp. 459-541. North-Holland, Amsterdam (2002)

Fleurbaey, M., Michel, P.: Intertemporal equity and the extension of the Ramsey criterion. J Math Econ **39**, 777-802 (2003)

Hammond, P.J.: Equity, Arrow's conditions, and Rawls' difference principle. Econometrica **44**, 793-804 (1976)

Kamaga, K., Kojima, T.: $\mathcal{Q}$-anonymous social welfare relations on infinite utility streams. 21COE-GLOPE Working Paper Series **47**, Waseda University (2008)

Lauwers, L.: Rawlsian equity and generalised utilitarianism with and infinite population. Econ Theory **9**, 143-150 (1997a)

Lauwers, L.: Infinite utility: insisting on strong monotonicity. Aust J Philos **75**, 222-233 (1997b)

Mitra, T., Basu, K.: On the existence of Paretian social welfare quasi-orderings for infinite utility streams with extended anonymity. In: Roemer, J., Suzumura, K. (eds.) Intergenerational Equity and Sustainability, pp. 85-99. Palgrave, London (2007)

Sakai, T.: Intergenerational equity and an explicit construction of welfare criteria. mimeo, Yokohama National University (2008)

Sen, A.K.: Collective Choice and Social Welfare. Holden-Day, Amsterdam (1970)

Suppes, P.: Some formal models of grading principles. Synthese **6**, 284-306 (1966)

Svensson, L.G.: Equity among generations. Econometrica **48**, 1251-1256 (1980)

Szpilrajn, E.: Sur l'extension de l'ordre partiel. Fundam Math **16**, 386-389 (1930)

van Liedekerke, L.: Should utilitarians be cautious about an infinite future. Aust J Philos **73**, 405-407 (1995)