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Q-anonymous social welfare relations on infinite utility streams

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$\mathcal{Q}$-anonymous social welfare relations on infinite utility streams*

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Abstract

We examine whether $\mathcal{Q}$-Anonymity, the axiom of impartiality stronger than Finite Anonymity, and Pigou-Dalton Equity or Hammond Equity, the well-known consequentialist equity axioms, are compatible in a strongly Paretian social welfare relation (SWR) on infinite utility streams. For each of the two equity axioms, we provide the characterization of the equitable subclass of $\mathcal{Q}$-anonymous and strongly Paretian SWRs: in the case of Pigou-Dalton Equity, all SWRs to which the extended generalized Lorenz SWR is a subrelation, and in the case of Hammond Equity, all SWRs to which the extended leximin SWR is a subrelation. To make clear the difference between these SWRs and the extended utilitarian SWR in terms of their equity properties, we also provide a new characterization of the extended utilitarian SWR using an equity axiom stated in the similar situations to those considered in Pigou-Dalton Equity and Hammond Equity.

Keywords: $\mathcal{Q}$-Anonymity, Intergenerational equity, Generalized Lorenz criterion, Leximin principle, Utilitarianism, Simplified criterion

JEL Classification Numbers: D63, D70

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1 Introduction

Consider a situation where we need to choose one among several alternative policies which will affect infinitely many future generations as well as the present generation (e.g. greenhouse gas abatement programs). If we are concerned only with each generation's welfare measured in terms of utility, we can deal with such an intergenerational decision problem by using a social welfare relation (SWR), a reflexive and transitive binary relation, defined on infinite utility streams.[1] In exploring the SWRs which will be acceptable not only for the present generation but also for future generations, two basic principles are usually considered. One is Strong Pareto, the axiom of efficiency, and the other is Finite Anonymity, the axiom of impartiality, which asserts that two utility streams related by permuting *finitely* many generations are socially indifferent. It is known that these two basic axioms together are equivalent to the infinite-horizon variant of the well-known justice principle called Suppes-Sen grading principle (Svensson 1980; Asheim et al. 2001).[2] Although this justice principle "does squeeze out as much as possible out of the use of dominance (or vector inequality)" (Sen 1970, p.151), there still be problems to be resolved. Asheim and Tungodden (2004) point out the following two problems. One is that what the Suppes-Sen grading principle by itself asserts on the intergenerational decision-making may be insufficient to resolve distributional conflicts among generations in many cases, and the other is that Finite Anonymity is too weak to realize impartial treatment of *infinitely* many generations in a satisfactory manner.

To resolve the former issue, i.e. the problem of distributional conflicts among generations, Asheim and Tungodden (2004) and Bossert et al. (2007) examine the possibility of additionally imposing a consequentialist equity requirement. Asheim and Tungodden (2004) examine the case of adding the infinite-horizon variant of Hammond Equity and show that the two versions of the leximin principle are characterized in terms of Strong Pareto, Finite Anonymity, Hammond Equity, and one of two alternative preference-continuity axioms. Bossert et al. (2007) consider not only the case of adding Hammond Equity but also the case of imposing the weaker equity axiom called Pigou-Dalton Equity.[3] They characterize the infinite-horizon generalized Lorenz criterion in terms of Strong Pareto, Finite Anonymity, and Pigou-Dalton Equity and also show that the infinite-horizon leximin principle is characterized if Pigou-Dalton Equity

[1]Some readers may think that a complete SWR, i.e. social welfare ordering (SWO), is more desirable since it can compare any two utility streams. Nevertheless, a non-complete SWR can play a sufficient role in selecting optimal or maximal paths in some economic models (Mitra 2005; Basu and Mitra 2007). Moreover, using Arrow's (1963) variant of Szpilrajn's (1930) theorem, our results stated in terms of a SWR ensure the existence of the SWO that respects the comparisons obtained by the SWR in question in a non-constructive manner.

[2]The Suppes-Sen grading principle is originally defined in a finite population setting. For details, we refer the reader to Suppes (1966) and Sen (1970).

[3]Pigou-Dalton Equity is also employed by Sakai (2003a; 2003b; 2006) and Hara et al. (2007).

is strengthened to Hammond Equity.[4] These existing characterizations surely provide the possibility of equitable resolutions to distributional conflicts among generations.

The purpose of this paper is to explore a further resolution beyond those established in Asheim and Tungodden (2004) and Bossert et al. (2007) by not only adding the equity axiom, Pigou-Dalton Equity or Hammond Equity, but also strengthening the impartiality requirement in the strongly Paretian and finitely anonymous SWRs. In other words, we explore a resolution not only to the distributional conflict but also to the weakness of Finite Anonymity, i.e. the latter issue mentioned above. Since the axiom of Strong Anonymity defined by all logically possible permutations of infinitely many generations inevitably comes in conflict with Strong Pareto (van Liedekerke 1995; Lauwers 1997a), we must consider the intermediate between Finite Anonymity and Strong Anonymity. In this paper, we focus on the anonymity axiom called $\mathcal{Q}$-Anonymity. $\mathcal{Q}$-Anonymity was first introduced in Lauwers (1997b) by the name Fixed Step Anonymity.[5] $\mathcal{Q}$-Anonymity is defined in terms not only of finite permutations but also of infinite but cyclic permutations, and thus it is logically stronger than Finite Anonymity but weaker than Strong Anonymity. Although many existing works show the possibilities of $\mathcal{Q}$-anonymous and strongly Paretian SWRs (Lauwers 1997b; Fleurbaey and Michel 2003; Banerjee 2006; Mitra and Basu 2007), the compatibility of $\mathcal{Q}$-anonymity and the equity requirements in the strongly Paretian SWRs is still uncovered in the literature. In this paper, we examine the possibility of additionally imposing $\mathcal{Q}$-Anonymity on the SWR that satisfies Strong Pareto and Pigou-Dalton Equity or Hammond Equity.

The results obtained in this paper are positive. We define the extensions of the generalized Lorenz and the leximin SWRs, called $\mathcal{Q}$-generalized Lorenz criterion and $\mathcal{Q}$-leximin principle respectively, in the same way as Banerjee (2006a) does for the Suppes-Sen and the utilitarian SWRs. Then, we show that each of the $\mathcal{Q}$-generalized Lorenz criterion and the $\mathcal{Q}$-leximin principle is well-defined as a SWR on infinite utility streams and that the former is characterized by Strong Pareto, $\mathcal{Q}$-Anonymity and Pigou-Dalton Equity and the latter in terms of Strong Pareto, $\mathcal{Q}$-Anonymity and Hammond Equity. Compared to the characterizations established in Bossert et al. (2007) with Finite Anonymity, our results can be regarded as the refinement of the impartial subclasses of the strongly Paretian and equitable SWRs by using the stronger impartiality axiom, $\mathcal{Q}$-Anonymity.

As we briefly noted above, the $\mathcal{Q}$-anonymous extension of a finitely anonymous SWR is already proposed by Banerjee (2006a) for the Suppes-Sen and the utilitar-

[4] The logical relationship among the two versions of the leximin principle characterized by Asheim and Tungodden (2004) and the leximin principle in Bossert et al. (2007) is the same as the one among the chatching-up criterion, the overtaking criterion, and the utilitarian SWR in Basu and Mitra (2007).

[5] See also Fleurbaey and Michel (2003).

ian SWRs. He characterizes the extended Suppes-Sen SWR by Strong Pareto and $\mathcal{Q}$-Anonymity and also does the extended utilitarian SWR, called $\mathcal{Q}$-utilitarian SWR, in terms of Strong Pareto, $\mathcal{Q}$-Anonymity, and an informational invariance axiom called Partial Translation Scale Invariance. Compared to Banerjee's (2006a) results, our results can also be interpreted as the characterizations of the equitable subclasses of the strongly Paretian and $\mathcal{Q}$-anonymous SWRs by using Pigou-Dalton Equity and Hammond Equity. To make clear the difference between our newly established SWRs, the $\mathcal{Q}$-generalized Lorenz and the $\mathcal{Q}$-leximin SWRs, and the $\mathcal{Q}$-utilitarian SWR in terms of their equity properties, we also provide an alternative characterization of the $\mathcal{Q}$-utilitarian SWR focusing on its equity property. We show that the $\mathcal{Q}$-utilitarian SWR is also characterized by Strong Pareto, $\mathcal{Q}$-Anonymity, and Incremental Equity. Incremental Equity is the axiom that prescribes a resolution to the conflicting situations similar to those considered in Pigou-Dalton Equity and Hammond Equity, and it requires that, for any utility transfer between two generations, a pre-transfer utility stream and a post-transfer stream must be socially indifferent. Our characterizations of the three $\mathcal{Q}$-anonymous SWRs, the $\mathcal{Q}$-generalized Lorenz, the $\mathcal{Q}$-leximin, and the $\mathcal{Q}$-utilitarian SWRs, together tell us how the logically admissible SWRs will change in accordance with which of the equity axioms, Pigou-Dalton Equity, Hammond Equity, and Incremental Equity, is additionally imposed on the strongly Paretian and $\mathcal{Q}$ -anonymous SWRs.

The rest of the paper is organized as follows. Section 2 introduces notation and definitions. In Section 3, we present the axioms we impose on a SWR and also provide the formal definitions of the $\mathcal{Q}$-generalized Lorenz criterion and the $\mathcal{Q}$-leximin principle. Then, the characterizations of them are established. Section 4 provides the characterization of the $\mathcal{Q}$-utilitarian SWR using Incremental Equity. Section 5 concludes. All proofs are available in Appendix.

2 Notation and definitions

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ be the set of all positive integers $\{1, 2, \dots\}$. We let $X = \mathbb{R}^{\mathbb{N}}$ be the domain of infinite utility streams. An infinite-dimensional vector $\boldsymbol{x} = (x_1, x_2, \dots)$ is a typical element of X and, for each $i \in \mathbb{N}$, x_i is interpreted as utility of the ith generation. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, we denote $(x_1, \dots, x_n)$ by $\boldsymbol{x}^{-n}$ and $(x_{n+1}, x_{n+2}, \dots)$ by $\boldsymbol{x}^{+n}$. Thus, given any $\boldsymbol{x} \in X$ and any $n \in \mathbb{N}$, we can write $\boldsymbol{x} = (\boldsymbol{x}^{-n}, \boldsymbol{x}^{+n})$. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, $\left(x^{-n}_{(1)}, \dots, x^{-n}_{(n)}\right)$ denotes a rank-ordered permutation of $\boldsymbol{x}^{-n}$ such that $x^{-n}_{(1)} \leq \cdots \leq x^{-n}_{(n)}$, ties being broken arbitrarily.

A SWR, denoted by $\succsim$, is a reflexive and transitive binary relation on X, i.e. a quasi-ordering.[6] An asymmetric component of $\succsim$ is denoted by $\succ$ and a symmetric component by $\sim$ respectively, i.e. $\boldsymbol{x} \succ \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ holds but $\boldsymbol{y} \succsim \boldsymbol{x}$ does not, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{x}$. A SWR $\succsim_A$ is said to be a subrelation to a SWR $\succsim_B$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \sim_A \boldsymbol{y}$ implies $\boldsymbol{x} \sim_B \boldsymbol{y}$ and (ii) $\boldsymbol{x} \succ_A \boldsymbol{y}$ implies $\boldsymbol{x} \succ_B \boldsymbol{y}$.

Following Mitra and Basu (2007) and Banerjee (2006a), we represent any permutation on the set of generations $\mathbb{N}$ by a permutation matrix. A permutation matrix is an infinite matrix $\boldsymbol{P} = (p_{ij})_{i,j\in\mathbb{N}}$ satisfying the following properties:

(i) for each $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$;

(ii) for each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given any permutation matrix $\boldsymbol{P}$, we denote by $\boldsymbol{P}'$ its unique inverse which satisfies $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the infinite identity matrix. Let $\mathcal{P}$ be the set of all permutation matrices. Given any $\boldsymbol{P} \in \mathcal{P}$ and any $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j\in\{1,\ldots,n\}}$ by $\boldsymbol{P}(n)$. A finite permutation matrix is a permutation matrix $\boldsymbol{P}$ such that $p_{ii} = 1$ for all $i > n$ for some $n \in \mathbb{N}$. The set of all finite permutation matrices is denoted by $\mathcal{F}$.

As in Mitra and Basu (2007) and Banerjee (2006a), we focus on the class of *cyclic* permutations which defines a *group* under the usual matrix multiplication.[7] A permutation matrix $\boldsymbol{P} \in \mathcal{P}$ is said to be cyclic if, for any unit vector $\boldsymbol{e} = (0, \ldots, 0, 1, 0, \ldots) \in X$, there exists $k \in \mathbb{N}$ such that k-times repeated application of $\boldsymbol{P}$ to $\boldsymbol{e}$ generates $\boldsymbol{e}$ again, i.e. $\overbrace{\boldsymbol{P} \cdots \boldsymbol{P}}^{k} \boldsymbol{e} = \boldsymbol{e}$. Throughout the paper, we let $\mathcal{Q}$ be the following subclass of $\mathcal{P}$:

$$\mathcal{Q} = \left\{ \boldsymbol{P} \in \mathcal{P} : \begin{array}{c} \text{there exists } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \\ \boldsymbol{P}(nk) \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

It is easily checked that $\mathcal{Q}$ is the class of cyclic permutations and defines a group with respect to matrix multiplication, and also that $\mathcal{F} \subset \mathcal{Q}$.

Negation of a statement is indicated by the logic symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} \geqslant \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, and $\boldsymbol{x} > \boldsymbol{y}$ if $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$.

[6] A binary relation $\succsim$ on X is (i) reflexive if, for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \succsim \boldsymbol{x}$, and (ii) transitive if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succsim \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$.

[7] Let $\mathcal{G}$ be a set of permutation matrices. $\mathcal{G}$ is said to define a group under the usual matrix multiplication if it satisfies the following four properties: (i) for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{G}$, $\boldsymbol{PQ} \in \mathcal{G}$, (ii) there exists $\boldsymbol{I} \in \mathcal{G}$ such that for all $\boldsymbol{P} \in \mathcal{G}$, $\boldsymbol{IP} = \boldsymbol{PI} = \boldsymbol{P}$, (iii) for all $\boldsymbol{P} \in \mathcal{G}$, there exists $\boldsymbol{P}' \in \mathcal{G}$ such that $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{PP}' = \boldsymbol{I}$, and (iv) for all $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathcal{G}$, $(\boldsymbol{PQ})\boldsymbol{R} = \boldsymbol{P}(\boldsymbol{QR})$.

3 Impartiality and consequentialist equity

In this section, we examine the possibility of a strongly Paretian and equitable SWR that satisfies an impartiality axiom stronger than Finite Anonymity. We begin with the formal definition of Strong Pareto.

Strong Pareto (SP): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} > \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

The requirement of impartial treatment of generations is usually formalized by using permutations of generations. In accordance with an adopted class of permutations, a different notion of impartiality will be formalized. We consider the following two alternative impartiality axioms.

$\mathcal{F}$-Anonymity (FA): For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{F}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.

$\mathcal{Q}$-Anonymity (QA): For all $\boldsymbol{x} \in X$ and all $\boldsymbol{P} \in \mathcal{Q}$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$.

FA and **QA** are also called *Finite* (or *Weak*) *Anonymity* and *Fixed Step Anonymity* respectively. Since $\mathcal{F} \subset \mathcal{Q}$, **QA** implies **FA**. While the anonymity axiom defined in terms of all permutations in $\mathcal{P}$ is not compatible with **SP** (van Liedekerke 1995; Lauwers 1997a), both **FA** and **QA** are compatible with **SP**. **SP** and **FA** characterize the infinite-horizon Suppes-Sen grading principle defined in terms of $\mathcal{F}$ (Svensson 1980; Asheim et al. 2001), and replacing **FA** with **QA**, its variant defined by $\mathcal{Q}$ is characterized (Banerjee 2006a).[8]

Although the basic principles of efficiency and impartiality formalized by **SP** and **FA** or **QA** are fairly appealing and intuitive in dealing with intergenerational welfare issues, what they assert on our intergenerational decision-making is quite weak and may be insufficient to resolve distributional conflicts among generations in many cases. To resolve distributional conflicts, Asheim and Tungodden (2004) and Bossert et al. (2007) examine the possibility of adding a consequentialist equity axiom in a strongly Paretian and finitely anonymous SWR. Asheim and Tungodden (2004) consider the infinite-horizon extension of Hammond Equity.[9] Bossert et al. (2007) examine the case of the weaker equity requirement called Pigou-Dalton Equity as well as Hammond Equity.

Pigou-Dalton Equity (PDE): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \leq x_j < y_j$ and $x_i - y_i = y_j - x_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

[8] Mitra and Basu (2007) shows that a class of permutations by which an anonymity axiom compatible with **SP** can be defined if and only if the class consists solely of cyclic permutations and defines *group* with respect to matrix multiplication, where we use the term anonymity axiom refer to the condition which asserts that a SWR must conclude $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$ for all $\boldsymbol{x} \in X$ and all $\boldsymbol{P}$ in an adopted class of permutations.

[9] Hammond Equity is originally formulated by Hammond (1976) in a finite population setting.

Hammond Equity (HE): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \leq x_j < y_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

Both two axioms deal with distributional conflicts between two generations.[10] **PDE** asserts that an order-preserving transfer from a relatively better-off generation to a relatively worse-off generation is strictly socially preferable, i.e. it formalizes the well-known transfer principle à la Pigou (1912) and Dalton (1920). On the other hand, **HE** asserts that an order-preserving change which diminishes inequality of utilities between conflicting two generations is strictly socially improving. **PDE** agrees to such a value judgment by **HE** as long as the absolute value of utility change are equal for the conflicting two generations. Hence, **HE** is stronger than **PDE**.

Bossert et al. (2007) show that **SP**, **FA**, and **PDE** together characterize the infinite-horizon generalized Lorenz criterion and also that the infinite-horizon leximin principle is characterized if **PDE** is strengthened to **HE**. We now present the formal definitions of these two infinite-horizon SWRs. For all $n \in \mathbb{N}$, let $\succsim_G^n$ denote the finite-horizon generalized Lorenz SWR defined on $\mathbb{R}^n$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

$$\boldsymbol{x}^{-n} \succsim_G^n \boldsymbol{y}^{-n} \Leftrightarrow \textstyle\sum_{i=1}^k x_{(i)}^{-n} \geq \sum_{i=1}^k y_{(i)}^{-n} \text{ for all } k \in \{1, \ldots, n\}.$$

The generalized Lorenz SWR, denoted $\succsim_G$, is defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_G \boldsymbol{y} \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_G^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}. \quad (1)$$

Next, we introduce the leximin SWR. For all $n \in \mathbb{N}$, let $\succsim_L^n$ denote the finite-horizon leximin SWR defined on $\mathbb{R}^n$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

$$\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \Leftrightarrow \begin{cases} \boldsymbol{x}^{-n} \text{ is a permutation of } \boldsymbol{y}^{-n}, \text{ or} \\ \text{there exists } m \in \{1, \ldots, n\} \text{ such that} \\ \quad x_{(i)}^{-n} = y_{(i)}^{-n} \text{ for all } i < m \text{ and } x_{(m)}^{-n} > y_{(m)}^{-n}. \end{cases}$$

The leximin SWR, denoted $\succsim_L$, is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_L \boldsymbol{y} \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}. \quad (2)$$

We now formally state the characterizations by Bossert et al. (2007), which will be used to establish our main results later.

Proposition 1 (Bossert et al. 2007, Theorem 1)**.** *A SWR $\succsim$ on X satisfies **SP**, **FA**, and **PDE** if and only if $\succsim_G$ is a subrelation to $\succsim$.*

[10]These conditions are generically referred to as two-person equity axioms in the finite population framework. On this, see d'Aspremont and Gevers (2002) and Bossert and Weymark (2004).

Proposition 2 (Bossert et al. 2007, Theorem 2)**.** *A SWR $\succsim$ on X satisfies **SP**, **FA**, and **HE** if and only if $\succsim_L$ is a subrelation to $\succsim$.*

Now, the natural question to ask is whether it is possible to realize the stronger notion of impartiality formalized as **QA** in a strongly Paretian and equitable (in the sense of **PDE** or **HE**) SWR. This paper provides a positive answer to this question. We define $\mathcal{Q}$-anonymous extensions of the generalized Lorenz and the leximin SWRs in the same way as Banerjee (2006a) does for the Suppes-Sen SWR and the utilitarian SWR.[11] The $\mathcal{Q}$-generalized Lorenz criterion, denoted $\succsim_{QG}$, is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{QG} \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim_G \boldsymbol{y}. \tag{3}$$

The $\mathcal{Q}$-leximin principle $\succsim_{QL}$ is defined as: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{QL} \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim_L \boldsymbol{y}. \tag{4}$$

In what follows, we refer to Banerjee's (2006a) way of extension as *$\mathcal{Q}$-extension*. As will be shown later, each of the $\mathcal{Q}$-generalized Lorenz criterion and the $\mathcal{Q}$-leximin principle is also well-defined as a SWR on X, and the former satisfies all of **SP**, **QA**, and **PDE** and the latter satisfies **HE** as well as these three axioms. Thus, these two $\mathcal{Q}$-extensions not only provide the resolution to the distributional conflicts among generations but also realize the stronger notion of impartiality than **FA** in the intergenerational welfare evaluation, i.e. can be the resolution to both two problems we mentioned earlier.

Once the possibility of a strongly Paretian, $\mathcal{Q}$-anonymous and equitable SWR is established, we should proceed to specify the class of those SWRs in the next step. Our main theorems provide the characterizations of the equitable subclasses of the $\mathcal{Q}$-anonymous and strongly Paretian SWRs for each of **PDE** and **HE**.

Theorem 1. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **PDE** if and only if $\succsim_{QG}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

Theorem 2. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **HE** if and only if $\succsim_{QL}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

As discussed by Basu and Mitra (2007) and Banerjee (2006a), Theorems 1 and 2 are interpreted as saying that $\succsim_{QG}$ (resp. $\succsim_{QL}$) is *the least restrictive* SWR among all

[11] Sect. 4 provides the formal definitions of the utilitarian SWR and the $\mathcal{Q}$-anonymous extension of it. It should be noted that $\mathcal{Q}$-anonymous SWRs are also proposed by Lauwers (1997b) and Fleurbaey and Michel (2003).

the SWRs that satisfy **SP**, **QA**, and **PDE** (resp. **HE**). Formally, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, we have

$$\begin{cases} \boldsymbol{x} \succsim_{QG} \boldsymbol{y} \text{ if and only if } \boldsymbol{x} \succsim \boldsymbol{y} \text{ for all } \succsim \in \Xi_{QG}; & \text{(5a)} \\ \boldsymbol{x} \succsim_{QL} \boldsymbol{y} \text{ if and only if } \boldsymbol{x} \succsim \boldsymbol{y} \text{ for all } \succsim \in \Xi_{QL}, & \text{(5b)} \end{cases}$$

where Ξ_{QG} (resp. Ξ_{QL}) is the set of all SWRs that satisfy **SP**, **QA**, and **PDE** (resp. **HE**). The only if part of (5a) (resp. (5b)) is obvious from the only if statement of Theorem 1 (resp. Theorem 2). The if part of (5a) (resp. (5b)) is also straightforward from the fact that $\succsim_{QG} \in \Xi_{QG}$ (resp. $\succsim_{QL} \in \Xi_{QL}$).

4 $\mathcal{Q}$-utilitarian SWR and 2-generation conflicts

In Theorems 1 and 2, we characterize the equitable subclasses of $\mathcal{Q}$-anonymous and strongly Paretian SWRs with the equity axioms **PDE** and **HE**. Both of **PDE** and **HE** prescribe a resolution to the 2-generation conflicts where two generations i and j are in conflict ($x_i > y_i$ and $x_j < y_j$) and the other generations are indifferent between two utility distributions ($x_k = y_k$ for all $k \neq i, j$). Thus, Theorems 1 and 2 tell us how the admissible class of SWRs will change in accordance with which of the resolution to the 2-generation conflicts, **PDE** or **HE**, is required to the $\mathcal{Q}$-anonymous and strongly Paretian SWRs.

Another plausible SWR that satisfies both of **SP** and **QA** is the $\mathcal{Q}$-extension of the utilitarian SWR. The $\mathcal{Q}$-extension of the utilitarian SWR is proposed by Banerjee (2006a) under the name $\mathcal{Q}$-utilitarian SWR. He characterizes it (precisely, all SWRs including it as a subrelation) in terms of **SP**, **QA**, and the informational invariance axiom called *Partial Translation Scale Invariance*.

Partial Translation Scale Invariance (PTSI): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, all $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$, and all $n \in \mathbb{N}$, if $\boldsymbol{x}^{+n} = \boldsymbol{y}^{+n}$ and $\boldsymbol{x} \succsim \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{\alpha} \succsim \boldsymbol{y} + \boldsymbol{\alpha}$.

PTSI is a different kind of axiom from those which prescribe a resolution to the 2-generation conflicts such as **PDE** and **HE**. [12]

The purpose of this section is to make clear the difference among the three $\mathcal{Q}$-anonymous SWRs, the $\mathcal{Q}$-generalized Lorenz, the $\mathcal{Q}$-leximin, and the $\mathcal{Q}$-utilitarian SWRs, in terms of their resolutions to the 2-generation conflicts. For this purpose, we provide an alternative characterization of the $\mathcal{Q}$-utilitarian SWR using an equity axiom stated in the 2-generation conflicts.

[12] **PTSI** is also called *Partial Unit Comparability*. This axiom is interpreted as saying that utility differences of finitely many generations are comparable but utility levels are not. For the detailed discussion about the invariance axioms, we refer the reader to Bossert and Weymark (2004) and d'Aspremont and Gevers (2002).

We begin with the formal definitions of the utilitarian SWR and its $\mathcal{Q}$-extension. Let $\succsim_U^n$ denote the finite-horizon utilitarian SWR defined on $\mathbb{R}^n$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

$$\boldsymbol{x}^{-n} \succsim_U^n \boldsymbol{y}^{-n} \Leftrightarrow \textstyle\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i.$$

The utilitarian SWR $\succsim_U$ is defined by the finite-horizon utilitarian SWR and the Pareto criterion: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_U \boldsymbol{y} \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_U^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}. \qquad (6)$$

The $\mathcal{Q}$-utilitarian SWR $\succsim_{QU}$ is defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{QU} \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{P}\boldsymbol{x} \succsim_U \boldsymbol{y}. \qquad (7)$$

We now introduce an axiom stated in the 2-generation conflicts. Both of the utilitarian and the $\mathcal{Q}$-utilitarian SWRs prescribe the following resolution to the 2-generation conflicts.[13]

Incremental Equity (IE): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $x_i - y_i = y_j - x_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $\boldsymbol{x} \sim \boldsymbol{y}$.

In contrast to **PDE**, this axiom asserts that, for any transfer between two generations, the pre-transfer utility stream and the post-transfer stream are considered to be equally good without any reference to the relative utility levels of conflicting two generations. Thus, **IE** is interpreted as saying that the intergenerational decision making must be neutral with respect to any utility transfer between two generations. Since a transposition of two generations' utilities is a special case of utility transfer between two generations, **IE** implies **FA**.[14] Furthermore, the following proposition tells that **IE** clearly distinguishes the SWRs including the utilitarian SWR as a subrelation from the other strongly Paretian SWRs.[15]

Proposition 3. *A SWR $\succsim$ on X satisfies* ***SP*** *and* ***IE*** *if and only if $\succsim_U$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

In view of the proposition, and from the fact that **IE** implies **FA**, it may seem that **IE** is quite strong and such a strong axiom may not be suitable for describing the characteristics of a SWR in general. However, the purpose of this section is to make

[13]**IE** was first proposed by Blackorby et al. (2002) in a finite population framework. See also d'Aspremont and Gevers (2002) and Blackorby et al. (2005).

[14]Note that any finite permutation is equivalently represented as a finite composition of transpositions.

[15]In Basu and Mitra (2007), $\succsim_U$ is also characterized in terms of **SP**, **FA**, and **PTSI**.

Table 1: Characterizations of $\mathcal{F}$-anonymous SWRs and $\mathcal{Q}$-extensions

SWR	*efficiency*	*impartiality*		*equity*			*invariance*	
(least restrictive)	**SP**	**FA**	**QA**	**PDE**	**HE**	**IE**	**PTSI**	*characterized in ...*
$\mathcal{Q}$-gener'd Lorenz	$\oplus$	$+$	$\oplus$	$\oplus$		$-$	$-$	Theorem 1
Gener'd Lorenz	$\oplus$	$\oplus$		$\oplus$		$-$	$-$	Bossert et al. (2007)
$\mathcal{Q}$-leximin	$\oplus$	$+$	$\oplus$	$+$	$\oplus$	$-$	$-$	Theorem 2
Leximin	$\oplus$	$\oplus$		$+$	$\oplus$	$-$	$-$	Bossert et al. (2007)
$\mathcal{Q}$-utilitarian	$\oplus$	$+$	$\oplus$	$-$	$-$	$\oplus$	$+$	Theorem 3
	$\oplus$	$+$	$\oplus$	$-$	$-$	$+$	$\oplus$	Banerjee (2006a)
Utilitarian	$\oplus$	$+$		$-$	$-$	$\oplus$	$+$	Proposition 3
	$\oplus$	$\oplus$		$-$	$-$	$+$	$\oplus$	Basu and Mitra (2007)

clear the 2-generation equity property of the $\mathcal{Q}$-utilitarian SWR and compare it with those of the $\mathcal{Q}$-generalized Lorenz and the $\mathcal{Q}$-leximin SWRs. In this respect, **IE** is a useful axiom.

We now state an alternative characterization of the $\mathcal{Q}$-utilitarian SWR. As shown in the following theorem, the $\mathcal{Q}$-utilitarian SWR is also characterized in terms of **SP**, **QA**, and **IE**.

Theorem 3. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **IE** if and only if $\succsim_{QU}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

From Theorems 1, 2, and 3, we can categorize the admissible subclasses of $\mathcal{Q}$-anonymous and strongly Paretian SWRs in terms of the three alternative resolutions to the 2-generation conflicts, **PDE**, **HE**, and **IE**. Table 1 summarizes the three characterizations and compares them with those established by Banerjee (2006a), Basu and Mitra (2007), and Bossert et al. (2007). For each row in Table 1, the class of SWRs that includes the SWR in the first column as a subrelation is characterized by the axioms indicated by $\oplus$, and furthermore, each SWR out of the class satisfies (resp. violates) the axioms indicated by + (resp. –). Compared to the characterizations in Bossert et al. (2007), our results are regarded as establishing (i) the possibility of reflecting the stronger notion of impartiality than **FA** in the strongly Paretian and equitable SWRs, and compared to Banerjee's (2006a) work, (ii) the possibility of realizing the consequentialist equities in the strongly Paretian and $\mathcal{Q}$-anonymous SWRs.

5 Conclusion

In this paper, we examined the possibility of adding the strong impartiality axiom formalized as **QA** in the strongly Paretian and equitable SWRs, and we obtained positive results. In particular, we show that the $\mathcal{Q}$-extension of the generalized Lorenz SWR is characterized with **SP**, **QA**, and **PDE** and the $\mathcal{Q}$-extension of the leximin SWR in terms of **SP**, **QA**, and **HE**. We also provide the characterization of the $\mathcal{Q}$-utilitarian SWR using the equity axiom **IE**. Our characterizations together enable us to categorize the admissible subclasses of the $\mathcal{Q}$-anonymous and strongly Paretian SWRs in terms of the equity axioms, **PDE**, **HE**, and **IE**. From Arrow's (1963) variant of Szpilrajn's (1930) theorem, we can conclude that there exists an ordering on X which satisfies **SP**, the axiom of efficiency, **QA**, the axiom of impartiality, and **PDE**, **HE**, or **IE**, the axioms of consequentialist equity. These results provide plausible resolutions to the existing impossibilities of an impartial or equitable social welfare ordering (e.g. Diamond 1965; Sakai 2003a, 2003b, 2006; Banerjee 2006b; Hara et al. 2007) and also strengthen the resolution established by Bossert et al. (2007) with the weaker impartiality requirement **FA**.[16]

All the three extended criteria $\succsim_{QG}$, $\succsim_{QL}$ and $\succsim_{QU}$ are characterized by strengthening **FA** to **QA** in the lists of the axioms characterizing $\succsim_G$, $\succsim_L$ and $\succsim_U$ respectively. As will be shown in Appendix, this results are generalizable to any SWR defined by using a sequence of finite-horizon SWRs satisfying certain moderate properties in the same way as in $\succsim_G$, $\succsim_L$ and $\succsim_U$. Such an general approach to the analysis of infinite-horizon criteria is initiated by d'Aspremont (2007) and also taken by Asheim and Banerjee (2008) and Kamaga and Kojima (2008).

We should also note that, compared to the admissible class of SWRs obtained with **FA**, incompleteness of the least restrictive SWR is improved in the class characterized with **QA**. Further level of comparability beyond $\succsim_{QL}$ and $\succsim_{QU}$ could be achieved by additionally imposing the axiom of preference-continuity considered in Asheim and Tungodden (2004) or that of consistency used in Basu and Mitra (2007). In fact, as shown by Kamaga and Kojima (2008), this approach is promising. Further exploration of this route and a similar approach with other plausible axioms will be interesting issues.

[16]In Banerjee (2006b), a social welfare ordering (SWO) is required to be representable as the Bergson-Samuelson social welfare function, and in Diamond (1965), Sakai (2003a; 2003b; 2006) and Hara et al. (2007) a certain continuity axiom is employed.

Appendix

We begin with two important lemmata that shed light on properties of the $\mathcal{Q}$-extension of a $\mathcal{F}$-anonymous SWR. Each of the lemmata is stated for the class of SWRs that satisfy three basic properties common to $\succsim_G$, $\succsim_L$, and $\succsim_U$. Note that each of $\succsim_G$, $\succsim_L$, and $\succsim_U$ is defined in terms of the Pareto criterion and the sequence of finite-horizon SWRs, $(\succsim_G^n)_{n\in\mathbb{N}}$, $(\succsim_L^n)_{n\in\mathbb{N}}$, and $(\succsim_U^n)_{n\in\mathbb{N}}$ respectively. Each of the sequences satisfies the following properties:[17]

Property 1 (P1): For all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$ if $x_i^{-n} \geq y_i^{-n}$ for all $i \in \{1, \ldots, n\}$ and $\boldsymbol{x}^{-n} \neq \boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \succ^n \boldsymbol{y}^{-n}$;

Property 2 (P2): For all $n \in \mathbb{N}$, all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, and all $r \in \mathbb{R}$, if $\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n}$ then $(\boldsymbol{x}^{-n}, r) \succsim^{n+1} (\boldsymbol{y}^{-n}, r)$;

Property 3 (P3): For all $n \in \mathbb{N}$ and all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$, if $\boldsymbol{x}^{-n}$ is a permutation of $\boldsymbol{y}^{-n}$, then $\boldsymbol{x}^{-n} \sim^n \boldsymbol{y}^{-n}$.

The lemmata are stated for the class of infinite-horizon SWRs, denoted Ξ, each of which is defined in terms of a sequence of finite-horizon SWRs satisfying **P1**, **P2**, and **P3** and of the Pareto crietrion. We now formally define the class Ξ. Let $(\succsim^n)_{n\in\mathbb{N}}$ be a sequence of finite-horizon SWRs which consists of one finite-horizon SWR $\succsim^n$ for each possible length of time horizon $n \in \mathbb{N}$. Let $\mathcal{S}$ be the set of all possible sequences of finite-horizon SWRs $(\succsim^n)_{n\in\mathbb{N}}$ that satisfies the properties **P1** to **P3**. For each $(\succsim^n)_{n\in\mathbb{N}} \in \mathcal{S}$, we can associate a binary relation $\succsim$ on X as follows: for any $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim \boldsymbol{y} \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}.^{18} \qquad (8)$$

Let φ denote the mapping that associates to any $(\succsim^n)_{n\in\mathbb{N}} \in \mathcal{S}$ a binary relation on X defined in (8). We define Ξ as

$$\Xi = \varphi(\mathcal{S}),$$

i.e. the set of binary relations on X each of which is associated with a sequence of finite-horizon relations in $\mathcal{S}$ in a way of (8). As will be shown in Claim 1, every binary relation in Ξ is well-defined as a SWR on X. Moreover, by **P3** and (8), each binary

[17] **P1** is the finite-horizon version of **SP**. **P2** is a kind of separability requirement similar to *Extended Independence of the Utilities of Unconcerned Individuals* (or *Existence Independence*) introduced by Blackorby et al. (2002; 2005) in the framework of variable population social choice, which requires our evaluation to be independent of the addition of an unconcerned generation. **P3** is a well-known anonymity axiom in a finite-horizon framework.

[18] d'Aspremont (2007) refers to this type of binary relation as *simplified criterion*.

relation in Ξ is $\mathcal{F}$-anonymous. For each $\succsim \in \Xi$, its $\mathcal{Q}$-extension, denoted $\succsim_Q$, is defined as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_Q \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim \boldsymbol{y}. \tag{9}$$

We are ready to state the following lemmata. We owe a lot to Banerjee's (2006a) work in establishing the lemmata. Our results are regarded as the generalizations of his results established for the Suppes-Sen and the utilitarian SWRs.

Lemma 1. *For any $\succsim \in \Xi$, its $\mathcal{Q}$-extension $\succsim_Q$ is also a SWR on X.*

Lemma 2. *For any $\succsim \in \Xi$, its $\mathcal{Q}$-extension $\succsim_Q$ has the following property: for any $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\begin{cases} \boldsymbol{x} \succ_Q \boldsymbol{y} \text{ if and only if there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succ \boldsymbol{y}; & \text{(10a)} \\ \boldsymbol{x} \sim_Q \boldsymbol{y} \text{ if and only if there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \sim \boldsymbol{y}. & \text{(10b)} \end{cases}$$

Proof of Lemma 1. The proof proceeds through two claims.

Claim 1. *Every $\succsim \in \Xi$ is reflexive and transitive, i.e. well-defined as a SWR on X.*

Proof of Claim 1. Reflexivity is straightforward. To check the transitivity of $\succsim$, consider any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ such that $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$. We will show $\boldsymbol{x} \succsim \boldsymbol{z}$ holds. By (8), there exist $n, n' \in \mathbb{N}$ such that (i) $\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n}$ and $\boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}$ and (ii) $\boldsymbol{y}^{-n'} \succsim^{n'} \boldsymbol{z}^{-n'}$ and $\boldsymbol{y}^{+n'} \geqslant \boldsymbol{z}^{+n'}$. Let $\bar{n} = \max\{n, n'\}$. We demonstrate the proof for the case of $\bar{n} = n'$. The same argument can be applied to the case of $\bar{n} = n$. Notice that $\boldsymbol{x}^{+\bar{n}} \geqslant \boldsymbol{z}^{+\bar{n}}$. Thus, we are enough to show that $\boldsymbol{x}^{-\bar{n}} \succsim^{\bar{n}} \boldsymbol{z}^{-\bar{n}}$. By **P2**, $\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n}$ implies $(\boldsymbol{x}^{-n}, y_{n+1}, \ldots, y_{\bar{n}}) \succsim^{\bar{n}} \boldsymbol{y}^{-\bar{n}}$. By **P1** and the reflexivity of $\succsim^{\bar{n}}$, $\boldsymbol{x}^{-\bar{n}} \succsim^{\bar{n}} (\boldsymbol{x}^{-n}, y_{n+1}, \ldots, y_{\bar{n}})$. Since $\succsim^{\bar{n}}$ is transitive, $\boldsymbol{x}^{-\bar{n}} \succsim^{\bar{n}} \boldsymbol{y}^{-\bar{n}}$ holds, which together with $\boldsymbol{y}^{-\bar{n}} \succsim^{\bar{n}} \boldsymbol{z}^{-\bar{n}}$ and the transitivity of $\succsim^{\bar{n}}$ gives $\boldsymbol{x}^{-\bar{n}} \succsim^{\bar{n}} \boldsymbol{z}^{-\bar{n}}$.

Claim 2. *For any $\succsim \in \Xi$, we have the following: for any $\boldsymbol{P} \in \mathcal{Q}$ and any $\boldsymbol{x}, \boldsymbol{y} \in X$,*

$$\begin{cases} \boldsymbol{x} \succ \boldsymbol{y} \text{ if and only if } \boldsymbol{Px} \succ \boldsymbol{Py}; & \text{(11a)} \\ \boldsymbol{x} \sim \boldsymbol{y} \text{ if and only if } \boldsymbol{Px} \sim \boldsymbol{Py}. & \text{(11b)} \end{cases}$$

Proof of Claim 2. We will show that $\boldsymbol{x} \succsim \boldsymbol{y}$ if and only if $\boldsymbol{Px} \succsim \boldsymbol{Py}$, from which the equivalence assertions in (11a) and (11b) immediately follow.

(*only if* part) Assume $\boldsymbol{x} \succsim \boldsymbol{y}$. By (8), there exists $n \in \mathbb{N}$ such that

$$\boldsymbol{x}^{-n} \succsim^n \boldsymbol{y}^{-n} \text{ and } \boldsymbol{x}^{+n} \geqslant \boldsymbol{y}^{+n}. \tag{12}$$

Take $\boldsymbol{P} \in \mathcal{Q}$ arbitrarily. We want to show that $\boldsymbol{Px} \succsim \boldsymbol{Py}$. Without loss of generality, let $\boldsymbol{P}$ be a k-period cyclic permutation matrix. Then, we can find $\hat{n} \in \mathbb{N}$ such that

$k\hat{n} \geq n$. Let $\bar{n} = k\hat{n}$. Note that $\boldsymbol{P}(\bar{n})$ is a finite-dimensional permutation matrix. Now, consider the following profiles $\boldsymbol{w}, \boldsymbol{z} \in X$:

$$\boldsymbol{w} = (\boldsymbol{x}^{-\bar{n}}, (\boldsymbol{Px})^{+\bar{n}}) \text{ and } \boldsymbol{z} = (\boldsymbol{y}^{-\bar{n}}, (\boldsymbol{Py})^{+\bar{n}}).$$

From the definitions of $\boldsymbol{w}$ and $\boldsymbol{z}$ and (12), it follows that

$$\begin{cases} \boldsymbol{w}^{-n} \succsim^n \boldsymbol{z}^{-n}, \text{ and} \\ \boldsymbol{w}^{+n} = (x_{n+1}, \ldots, x_{\bar{n}}, (\boldsymbol{Px})^{+\bar{n}}) \geqslant (y_{n+1}, \ldots, y_{\bar{n}}, (\boldsymbol{Py})^{+\bar{n}}) = \boldsymbol{z}^{+n}. \end{cases}$$

Thus, by (8), $\boldsymbol{w} \succsim \boldsymbol{z}$. Note that, by (8) and **P3**, $\succsim$ satisfies **FA**. By **FA**, $\boldsymbol{w} \sim \boldsymbol{Px}$ and $\boldsymbol{z} \sim \boldsymbol{Py}$. Since, by Claim 1, $\succsim$ is transitive, $\boldsymbol{Px} \succsim \boldsymbol{Py}$ is obtained as desired.

(*if* part) Assume $\boldsymbol{Px} \succsim \boldsymbol{Py}$. Since $\boldsymbol{P}' \in \mathcal{Q}$, we obtain $\boldsymbol{x} = \boldsymbol{P}'(\boldsymbol{Px}) \succsim \boldsymbol{P}'(\boldsymbol{Py}) = \boldsymbol{y}$ by the "only if" part of the lemma.

Proof of Lemma 1. Take $\succsim \in \Xi$ arbitrarily and let $\succsim_Q$ be its $\mathcal{Q}$-extension. Reflexivity is straightforward from the fact that $\boldsymbol{I} \in \mathcal{Q}$ and $\succsim$ is reflexive. We show that $\succsim_Q$ is transitive. Assume that $\boldsymbol{x} \succsim_Q \boldsymbol{y}$ and $\boldsymbol{y} \succsim_Q \boldsymbol{z}$. Then, by (9), there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim \boldsymbol{y}$ and $\boldsymbol{Qy} \succsim \boldsymbol{z}$. By Claim 2, $\boldsymbol{Px} \succsim \boldsymbol{y}$ if and only if $\boldsymbol{Q}(\boldsymbol{Px}) \succsim \boldsymbol{Qy}$. By Claim 1, $\succsim$ is transitive, and thus $\boldsymbol{Q}(\boldsymbol{Px}) \succsim \boldsymbol{z}$ holds. Since $\boldsymbol{QP} \in \mathcal{Q}$, $\boldsymbol{x} \succsim_Q \boldsymbol{z}$ follows from (9). ■

Proof of Lemma 2. First, we prove the equivalence assertion in (10a). Note that, by the definiton of $\succsim_Q$, $\boldsymbol{x} \succ_Q \boldsymbol{y}$ is equivalent to

$$\begin{cases} \text{there exists } \boldsymbol{P} \in \mathcal{Q} \text{ such that } \boldsymbol{Px} \succsim \boldsymbol{y}, \text{ and} & \text{(13a)} \\ \text{for all } \boldsymbol{Q} \in \mathcal{Q}, \neg(\boldsymbol{Qy} \succsim \boldsymbol{x}). & \text{(13b)} \end{cases}$$

(*only if* part of (10a)) Assume $\boldsymbol{x} \succ_Q \boldsymbol{y}$. The proof is done by contradiction. Suppose that there is no $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succ \boldsymbol{y}$, or equivalently, such that $\boldsymbol{Px} \succsim \boldsymbol{y}$ and $\neg(\boldsymbol{y} \succsim \boldsymbol{Px})$. From (13a), there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succsim \boldsymbol{y}$. Thus, for this $\boldsymbol{P}$, we must have $\boldsymbol{Px} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{Px}$, i.e. $\boldsymbol{Px} \sim \boldsymbol{y}$. Then, by (11b), we have $\boldsymbol{x} = \boldsymbol{P}'(\boldsymbol{Px}) \sim \boldsymbol{P}'\boldsymbol{y}$, which contradicts (13b).

(*if* part of (10a)) Assume that there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{Px} \succ \boldsymbol{y}$. By (9), $\boldsymbol{x} \succsim_Q \boldsymbol{y}$. We want to show $\neg(\boldsymbol{y} \succsim_Q \boldsymbol{x})$. We prove this by contradiction. Suppose that $\boldsymbol{y} \succsim_Q \boldsymbol{x}$ holds. Then, by (9), there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{Qy} \succsim \boldsymbol{x}$. By Claim 2, $\boldsymbol{P}(\boldsymbol{Qy}) \succsim \boldsymbol{Px}$. Let $\boldsymbol{R}$ denote the product $\boldsymbol{PQ}$. Since $\succsim$ is transitive, $\boldsymbol{Px} \succ \boldsymbol{y}$ and $\boldsymbol{Ry} \succsim \boldsymbol{Px}$ give $\boldsymbol{Ry} \succ \boldsymbol{y}$. Then, by (8), there exists $n \in \mathbb{N}$ such that

$$(\boldsymbol{Ry})^{-n} \succsim^n \boldsymbol{y}^{-n} \text{ and } (\boldsymbol{Ry})^{+n} \geqslant \boldsymbol{y}^{+n}.$$

Note that $\boldsymbol{R}$ belongs to $\mathcal{Q}$. Without loss of generality, let $\boldsymbol{R}$ be a k-period cyclic permutation matrix, i.e., for any $\hat{n} \in \mathbb{N}$, $\boldsymbol{R}(k\hat{n})$ is a finite-dimensional permutation matrix. Consider $\bar{n} \in \mathbb{N}$ such that $\bar{n} \geq n$ and $\bar{n} = k\hat{n}$ for some $\hat{n} \in \mathbb{N}$. By **P3** and the fact that $\bar{n} \geq n$, we have

$$(\boldsymbol{R}\boldsymbol{y})^{-\bar{n}} \sim^{\bar{n}} \boldsymbol{y}^{-\bar{n}} \text{ and } (\boldsymbol{R}\boldsymbol{y})^{+\bar{n}} \geqslant \boldsymbol{y}^{+\bar{n}}.$$

If we have $(\boldsymbol{R}\boldsymbol{y})^{+\bar{n}} = \boldsymbol{y}^{+\bar{n}}$, $\boldsymbol{R}\boldsymbol{y} \sim \boldsymbol{y}$ follows, and this contradicts $\boldsymbol{R}\boldsymbol{y} \succ \boldsymbol{y}$. But, in the case of $(\boldsymbol{R}\boldsymbol{y})^{+\bar{n}} > \boldsymbol{y}^{+\bar{n}}$, we can find $m \in \mathbb{N}$ such that $(Ry_{\bar{n}+1}, \ldots, Ry_{k(\hat{n}+m)}) > (y_{\bar{n}+1}, \ldots, y_{k(\hat{n}+m)})$, which contradicts the fact that $\boldsymbol{R}$ is a k-period cyclic permutation matrix. Thus, in either case, we obtain a contradiction.

Next, we prove the equivalence assertion in (10b).

(*only if* part of (10b)) Assume $\boldsymbol{x} \sim_Q \boldsymbol{y}$. Then, by (9), we can find $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \succsim \boldsymbol{y}$. If we had $\boldsymbol{P}\boldsymbol{x} \succ \boldsymbol{y}$, it would follow from (10a) that $\boldsymbol{x} \succ_Q \boldsymbol{y}$ holds, which contradicts $\boldsymbol{x} \sim_Q \boldsymbol{y}$. Thus, we must have $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{y}$.

(*if* part of (10b)) Assume that there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{y}$. Then, by (9), we have $\boldsymbol{x} \succsim_Q \boldsymbol{y}$. Moreover, by (11b), $\boldsymbol{x} = \boldsymbol{P}'(\boldsymbol{P}\boldsymbol{x}) \sim \boldsymbol{P}'\boldsymbol{y}$ holds. Then, by (9) again, we also obtain $\boldsymbol{y} \succsim_Q \boldsymbol{x}$. Thus, $\boldsymbol{x} \sim_Q \boldsymbol{y}$ follows. ■

We now provide the proofs of Theorems 1 and 2.

Proof of Theorem 1. (*only if* part) Assume that a SWR $\succsim$ on X satisfies **SP**, **QA**, and **PDE**. Since **QA** implies **FA**, it follows from Proposition 1 that $\succsim_G$ is a subrelation to $\succsim$. First, we show that if $\boldsymbol{x} \succ_{QG} \boldsymbol{y}$ then $\boldsymbol{x} \succ \boldsymbol{y}$. Assume $\boldsymbol{x} \succ_{QG} \boldsymbol{y}$. By (10a), there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \succ_G \boldsymbol{y}$. Since $\succsim_G$ is a subrelation to $\succsim$, $\boldsymbol{P}\boldsymbol{x} \succ \boldsymbol{y}$ holds. By **QA**, $\boldsymbol{x} \sim \boldsymbol{P}\boldsymbol{x}$. By the transitivity of $\succsim$, $\boldsymbol{x} \succ \boldsymbol{y}$ follows. Next, assume $\boldsymbol{x} \sim_{QG} \boldsymbol{y}$. We want to show $\boldsymbol{x} \sim \boldsymbol{y}$. By (10b), there exists $\boldsymbol{P} \in \mathcal{Q}$ such that $\boldsymbol{P}\boldsymbol{x} \sim_G \boldsymbol{y}$. Since $\succsim_G$ is a subrelation to $\succsim$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{y}$. By **QA**, $\boldsymbol{x} \sim \boldsymbol{P}\boldsymbol{x}$. The transitivity of $\succsim$ gives $\boldsymbol{x} \sim \boldsymbol{y}$.

(*if* part) Assume that $\succsim_{QG}$ is a subrelation to $\succsim$. From Lemma 2 and the fact that $\boldsymbol{I} \in \mathcal{Q}$, $\succsim_G$ is a subrelation to $\succsim_{QG}$, which means that $\succsim_G$ is a subrelation to $\succsim$. Thus, from Proposition 1, $\succsim$ satisfies **SP** and **PDE**. As for **QA**, it is checked as follows. Since $\succsim_G$ is reflexive, $\boldsymbol{x} = \boldsymbol{P}'(\boldsymbol{P}\boldsymbol{x}) \sim_G \boldsymbol{x}$ holds for any $\boldsymbol{x} \in X$ and any $\boldsymbol{P} \in \mathcal{Q}$. By (10b), $\boldsymbol{P}\boldsymbol{x} \sim_{QG} \boldsymbol{x}$. Since $\succsim_{QG}$ is a subrelation to $\succsim$, $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{x}$. ■

Proof of Theorem 2. Using Proposition 2, the only if part is proved in the same way as in the proof of Theorem 1. The if part is also proved by the same argument as in the proof of Theorem 1. ■

Next, we prove Proposition 3 and Theorem 3.

Proof of Proposition 3. The if part is straightforward and we omit it. We prove the only if part in two steps.

Step 1.[19] We show that $\boldsymbol{x} \sim \boldsymbol{y}$ holds whenever there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\boldsymbol{x}^{+n} = \boldsymbol{y}^{+n}$. The case of $n = 1$ is trivial. If $n \geq 2$, consider the following operation: fix $\bar{n} \in \{1, \dots, n-1\}$ arbitrarily and construct $\bar{\boldsymbol{x}} \in X$ as follows: $\bar{x}_{\bar{n}} = y_{\bar{n}}$ for $\bar{n}$; $\bar{x}_n = x_n + x_{\bar{n}} - y_{\bar{n}}$ for n; $\bar{x}_k = x_k$ for all $k \in \mathbb{N} \setminus \{\bar{n}, n\}$. By **IE**, $\boldsymbol{x} \sim \bar{\boldsymbol{x}}$. Applying the above operation once for each $\bar{n} \in \{1, \dots, n-1\}$ repeatedly, we can construct the profile $\boldsymbol{y}$ from $\boldsymbol{x}$. Then, the transitivity of $\succsim$ gives $\boldsymbol{x} \sim \boldsymbol{y}$.

Step 2. From Step 1 and (6), it is obvious that if $\boldsymbol{x} \sim_U \boldsymbol{y}$ then $\boldsymbol{x} \sim \boldsymbol{y}$. We show that if $\boldsymbol{x} \succ_U \boldsymbol{y}$ then $\boldsymbol{x} \succ \boldsymbol{y}$. Assume $\boldsymbol{x} \succ_U \boldsymbol{y}$. By (6), there exists $n \in \mathbb{N}$ such that $(\sum_{i=1}^{n} x_i, \boldsymbol{x}^{+n}) > (\sum_{i=1}^{n} y_i, \boldsymbol{y}^{+n})$. Then, we can find $\bar{n} \geq n$ such that

$$\sum_{i=1}^{\bar{n}} x_i > \sum_{i=1}^{\bar{n}} y_i \text{ and } \boldsymbol{x}^{+\bar{n}} \geqslant \boldsymbol{y}^{+\bar{n}}.$$

Take $\boldsymbol{z} \in X$ such that $z_1 = y_1 + \sum_{i=1}^{\bar{n}}(x_i - y_i)$, $z_i = y_i$ for all $i \in \{2, \dots, \bar{n}\}$, and $z_j = x_j$ for all $j \in \{\bar{n}+1, \bar{n}+2, \dots\}$. From Step 1, $\boldsymbol{x} \sim \boldsymbol{z}$ follows. By **SP**, $\boldsymbol{z} \succ \boldsymbol{y}$. The transitivity of $\succsim$ gives $\boldsymbol{x} \succ \boldsymbol{y}$. ■

Proof of Theorem 3. Using Proposition 3, the proof is done in the same way as in the proof of Theorem 1. ■

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[19]The result demonstrated in this step is well-known fact. See, for example, Blackorby et al. (2002, 2005) for the finite population case, and also Asheim and Tungodden (2004) for the infinite-horizon case.

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