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Q-anonymous social welfare relations on infinite utility streams

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$\mathcal{Q}$-anonymous social welfare relations on infinite utility streams

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Abstract

We examine whether $\mathcal{Q}$-Anonymity, the axiom of impartiality stronger than Finite Anonymity, and Pigou-Dalton Equity or Hammon Equity, the well-known consequentialist equity axioms, are compatible in a strongly Paretian social welfare relation (SWR) on infinite utility streams. For each case of these two equity axioms, we provide the characterization of the equitable subclass of $\mathcal{Q}$-anonymous and strongly Paretian SWRs: in the case of Pigou-Dalton Equity, all SWRs to which the extended generalized Lorenz SWR is a subrelation, and in the case of Hammond Equity, all SWRs to which the extended leximin SWR is a subrelation. We also provide a new axiomatization of the extended utilitarian SWR using an equity axiom that requires utility transfer among two generations to be socially indifferent.

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1 Introduction

Consider a situation where we need to choose optimal one(s) among several alternative policies which will affect infinitely many future generations as well as the present generation (e.g. environmental policies). If we are concerned only with each generation's welfare measured in terms of utility, we can deal with such an intergenerational decision problem by using a social welfare relation (SWR), a reflexive and transitive binary relation, defined on infinite utility streams.[1] In exploring the SWRs which will be acceptable not only for the present generation but also for future generations, two basic principles are usually considered. One is Strong Pareto, the axiom of efficiency, and the other is Finite Anonymity, the axiom of impartiality, which asserts that two utility streams related by permuting *finitely* many generations are socially indifferent. It is known that these two basic axioms together are equivalent to the infinite-horizon variant of the well-known justice principle called Suppes-Sen grading principle.[2] Although this justice principle "does squeeze out as much as possible out of the use of dominance (or vector inequality)" (Sen (1970), p.151), as argued by Asheim and Tungodden (2004), there still be at least two problems to be resolved. One is that what the Suppes-Sen grading principle by itself asserts on the intergenerational decision-making may be insufficient to resolve distributional conflicts among generations in many cases, and the other is that Finite Anonymity is too weak to realize impartial treatment of *infinitely* many generations in a satisfactory manner.

To resolve the former issue, i.e. the problem of distributional conflicts among generations, Asheim and Tungodden (2004) and Bossert et al. (2007) examine the possibility of additionally imposing a consequentialist equity requirement. Asheim and Tungodden (2004) examine the case of adding the infinite-horizon variant of Hammond Equity and show that the two versions of the leximin principle are characterized in terms of Strong Pareto, Finite Anonymity, Hammond Equity, and one of two alternative preference-continuity axioms. Bossert et al. (2007) consider not only the case of adding Hammond Equity but also the case of imposing the weaker equity axiom called Pigou-Dalton Equity. They charac-

[1] Some readers may think that a complete SWR, i.e. social welfare ordering (SWO), is more desirable since it can compare any two utility streams. Nevertheless, a non-complete SWR is sufficient even to determine a unique optimal policy in some economic models (see, for example, Mitra (2005) and Basu and Mitra (2007)). Moreover, as will be shown later, using Arrow's (1963) variant of Szpilrajn's (1930) theorem, our results stated in terms of a SWR ensure the existence of the SWO that respects the comparisons by the SWR in question in an inconstructive way.

[2] On this, see Svennson (1980) and Asheim et al. (2001). The Suppes-Sen grading principle is originally defined in the finite population setting. For details, we refer the reader to Suppes (1966) and Sen (1970).

terize the infinite-horizon generalized Lorenz criterion in terms of Strong Pareto, Finite Anonymity, and Pigou-Dalton Equity and also show that the infinite-horizon leximin principle is characterized if Pigou-Dalton Equity is strengthened to Hammond Equity.[3] These existing characterizations surely provide the possibility of equitable resolutions to distributional conflicts among generations.

The purpose of this paper is to explore a further resolution beyond those established in Asheim and Tungodden (2004) and Bossert et al. (2007) by not only adding the equity axiom, Pigou-Dalton Equity or Hammond Equity, but also strengthening the impartiality requirement in the strongly Paretian and finitely anonymous SWRs to resolve the latter issue mentioned above, i.e. the weakness of Finite Anonymity, as well. Since, as shown in van Liedekerke (1995) and Lauwers (1997a), the axiom of Strong Anonymity defined by all logically possible permutations of infinitely many generations inevitably comes in conflict with Strong Pareto, we must consider the intermediate between Finite Anonymity and Strong Anonymity. In this paper, we focus on the anonymity axiom called $\mathcal{Q}$-Anonymity. $\mathcal{Q}$-Anonymity was first introduced in Lauwers (1997b) by the name Fixed Step Anonymity.[4] $\mathcal{Q}$-Anonymity is defined in terms not only of finite permutations but also of infinite but cyclic permutations, and thus it is logically stronger than Finite Anonymity but weaker than Strong Anonymity. In their recent paper, Mitra and Basu (2007) show that $\mathcal{Q}$-Anonymity is compatible with Strong Pareto and, moreover, that the class of permutations considered in $\mathcal{Q}$-Anonymity is the largest class of permutations in terms of which the anonymity axiom that is compatible with Strong Pareto is well-defined.[5] In the subsequent sections, we examine the possibility of imposing $\mathcal{Q}$-Anonymity on the strongly Paretian and equitable SWRs.

The results obtained in this paper are positive. We define the extensions of the generalized Lorenz and the leximin SWRs, called $\mathcal{Q}$-generalized Lorenz criterion and $\mathcal{Q}$-leximin principle respectively, in the same way as Banerjee (2006a) does for the Suppes-Sen and the utilitarian SWRs. Then, we show that each of the $\mathcal{Q}$-generalized Lorenz criterion and the $\mathcal{Q}$-leximin principle is well-defined as a SWR on infinite utility streams and that the former is characterized by Strong Pareto, $\mathcal{Q}$-Anonymity and Pigou-Dalton Equity and the latter in terms of Strong Pareto, $\mathcal{Q}$-Anonymity and Hammond Equity.

[3]The logical relationship among the two versions of the leximin principles characterized by Asheim and Tungodden (2004) and the leximin principle in Bossert et al. (2007) is the same as the one among the chatching-up criterion, the overtaking criterion, and the utilitarian SWR. For the latter three SWRs, we refer the reader to Basu and Mitra (2007).

[4]See also Fleurbaey and Michel (2003).

[5]In this paper, the term "anonymity axiom" is used to refer to the condition which asserts that two utility streams related by a permutation of a certain class must be socially indifferent.

As we briefly noted above, the $\mathcal{Q}$-anonymous extension of a finitely anonymous SWR is already proposed by Banerjee (2006a) for the Suppes-Sen and the utilitarian SWRs. He characterizes the extended Suppes-Sen SWR by Strong Pareto and $\mathcal{Q}$-anonymity and does the extended utilitarian SWR, called $\mathcal{Q}$-utilitarian SWR, in terms of Strong Pareto, $\mathcal{Q}$-Anonymity, and an informational invariance axiom called Partial Translation Scale Invariance. In view of Banerjee's (2006a) results, our characterizations of the $\mathcal{Q}$-generalized Lorenz and the $\mathcal{Q}$-leximin SWRs are also interpreted as showing the possibility of adding the equity requirement, Pigou-Dalton Equity or Hammond Equity, in the strongly Paretian and $\mathcal{Q}$-anonymous SWRs. To make clear the linkage between our analysis focusing on a consequentialist equity and Banerjee's (2006a) work, we also provide a new characterization of the $\mathcal{Q}$-utilitarian SWR focusing on its equity property. We show that the $\mathcal{Q}$-utilitarian SWR is also characterized by Strong Pareto, $\mathcal{Q}$-Anonymity, and Incremental Equity which is stated for the conflicting situations similar to those considered in Pigou-Dalton Equity and Hammond Equity and requires that any utility transfer among two generations is socially indifferent. Our characterizations of the $\mathcal{Q}$-generalized Lorenz, the $\mathcal{Q}$-leximin, and the $\mathcal{Q}$-utilitarian SWRs together tell us how the logically admissible SWRs will change in accordance with which of the equity axioms, Pigou-Dalton Equity, Hammond Equity, and Incremental Equity, is additionally imposed on the strongly Paretian and $\mathcal{Q}$-anonymous SWRs.

The rest of the paper is organized as follows. Section 2 introduces notation and definitions. In Section 3, we provide the axioms we impose on a SWR and the formal definitions of the $\mathcal{Q}$-generalized Lorenz criterion and the $\mathcal{Q}$-leximin principle. Then, we state the characterizations of them. Section 4 establishes the characterization of the $\mathcal{Q}$-utilitarian SWR using Incremental Equity. Section 5 concludes. All proofs are available in Appendix.

2 Notation and definitions

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ the set of all positive integers $\{1, 2, \dots\}$. We let $X \equiv \mathbb{R}^{\mathbb{N}}$ be the domain of infinite utility streams. An infinite-dimensional vector $x = (x_1, x_2, \dots)$ is a typical element of X and x_i is interpreted as utility of ith generation for each $i \in \mathbb{N}$. For all $x \in X$ and all $n \in \mathbb{N}$, we denote $(x_1, x_2, \dots, x_n)$ by x^{-n} and $(x_{n+1}, x_{n+2}, \dots)$ by x^{+n}. Thus, given any $x \in X$ and any $n \in \mathbb{N}$, we can write $x = (x^{-n}, x^{+n})$. For all $n \in \mathbb{N}$ and all $x \in X$, $\left(x^{-n}_{(1)}, x^{-n}_{(2)}, \dots, x^{-n}_{(n)}\right)$ denote a rank-ordered permutation of x^{-n} such that $x_{(1)}^{-n} \leq x_{(2)}^{-n} \leq \cdots \leq x_{(n)}^{-n}$, ties being broken arbitrarily.

A SWR, $\succsim$, is a reflexive and transitive binary relation on X, i.e. a quasi-ordering.[6] An asymmetric component of $\succsim$ is denoted by $\succ$ and a symmetric component by $\sim$ respectively, i.e. $x \succ y$ if and only if $x \succsim y$ holds but $y \succsim x$ does not, and $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. A SWR $\succsim_A$ is said to be a subrelation to a SWR $\succsim_B$ if, for all $x, y \in X$, (i) $x \sim_A y$ implies $x \sim_B y$ and (ii) $x \succ_A y$ implies $x \succ_B y$.

Following Mitra and Basu (2007) and Banerjee (2006a), we represent any permutation on the set of generations $\mathbb{N}$ by a permutation matrix. A permutation matrix is an infinite matrix $P = (p_{ij})_{i,j\in\mathbb{N}}$ satisfying the following properties:

(i) for each $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$;

(ii) for each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given any permutation matrix P, we denote by P' its unique inverse which satisfies $P'P = PP' = I$, where I denotes the infinite identity matrix. $\mathcal{P}$ is the set of all permutation matrices. Given a permutation matrix $P \in \mathcal{P}$ and $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j\in\{1,2,\ldots,n\}}$ by $P(n)$. A finite permutation matrix is a permutation matrix P such that there exists $n \in \mathbb{N}$ such that $(Px)^{+n} = x^{+n}$. The set of all finite permutation matrices is denoted by $\mathcal{F}$.

As in Mitra and Basu (2007) and Banerjee (2006a), we focus on the class of *cyclic* permutations which defines a *group* under the usual matrix multiplication.[7] A permutation matrix $P \in \mathcal{P}$ is said to be cyclic if, for any unit vector $e = (0, \ldots, 0, 1, 0, \ldots) \in X$, there exists $k \in \mathbb{N}$ such that k-times repeated application of P to e generates e again, i.e. $\overbrace{P \cdots P}^{k} e = e$. Throughout the paper, we let $\mathcal{Q}$ be the following subclass of $\mathcal{P}$:

$$\mathcal{Q} = \left\{ P \in \mathcal{P} : \begin{array}{c} \text{there exists } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \\ P(nk) \text{ is a finite-dimensional permutation matrix} \end{array} \right\}.$$

[6] A binary relation $\succsim$ on X is (i) reflexive if, for all $x \in X$, $x \succsim x$, and (ii) transitive if, for all $x, y, z \in X$, $x \succsim z$ holds whenever $x \succsim y$ and $y \succsim z$.

[7] Let $\mathcal{G}$ be a set of permutation matrices. $\mathcal{G}$ is said to define a group under the usual matrix multiplication if it satisfies the following four properties: (i) for all $P, Q \in \mathcal{G}$, $PQ \in \mathcal{G}$, (ii) there exists $I \in \mathcal{G}$ such that for all $P \in \mathcal{G}$, $IP = PI = P$, (iii) for all $P \in \mathcal{G}$, there exists $P' \in \mathcal{G}$ such that $P'P = PP' = I$, and (iv) for all $P, Q, R \in \mathcal{G}$, $(PQ)R = P(QR)$.

It is easily checked that $\mathcal{Q}$ is the class of cyclic permutations and defines a group (with respect to matrix multiplication), and also that $\mathcal{F} \subseteq \mathcal{Q}$.

Negation of a statement is indicated by the logic symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $x, y \in X$, $x \geqslant y$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, and $x > y$ if $x \geqslant y$ and $x \neq y$.

3 Impartiality and consequentialist equity

In this section, we examine the possibility of a strongly Paretian and equitable SWR that satisfies an impartiality axiom stronger than Finite Anonymity. We begin with the formal definition of Strong Pareto.

Strong Pareto (SP): For all $x, y \in X$ with $x > y$, $x \succ y$.

The requirement of impartial treatment of generations is usually formalized by using permutations of generations. In accordance with an adopted class of permutations, different notion of impartiality will be formalized. We consider the following two alternative impartiality axioms.

$\mathcal{F}$-Anonymity (FA): For all $x \in X$ and all $P \in \mathcal{F}$, $Px \sim x$.

$\mathcal{Q}$-Anonymity (QA): For all $x \in X$ and all $P \in \mathcal{Q}$, $Px \sim x$.

FA and **QA** are also called *Finite (or Weak) Anonymity* and *Fixed Step Anonymity* respectively. Since $\mathcal{F} \subseteq \mathcal{Q}$, **QA** implies **FA**. It is known that while the anonymity axiom defined in terms of all permutations in $\mathcal{P}$ is not compatible with **SP**, both **FA** and **QA** are compatible with **SP**. **SP** and **FA** (resp. **QA**) characterize the infinite-horizon Suppes-Sen grading principle defined in terms of $\mathcal{F}$ (resp. $\mathcal{Q}$).[8] Furthermore, in Mitra and Basu (2007), they show that $\mathcal{Q}$ is the largest class of permutations by which **SP**-compatible anonymity axiom can be defined, where we use the term anonymity axiom to refer to the condition which asserts that $Px \sim x$ for all $x \in X$ and all P in an adopted class of permutations.

Although the basic principles of efficiency and impartiality formalized by **SP** and **FA** or **QA** are fairly appealing and intuitive in dealing with intergenerational welfare issues, what they assert on our intergenerational decision-making is quite weak and may be insufficient to resolve distributional conflicts among generations in many cases. To resolve distributional conflicts, Asheim and Tungodden

[8] For the characterizations of the two versions of the infinite-horizon Suppes-Sen grading principle, see Svensson (1980), Asheim et al. (2001), and Banerjee (2006a).

(2004) and Bossert et al. (2007) examine the possibility of adding a consequentialist equity axiom in a strongly Paretian and finitely anonymous SWR. Asheim and Tungodden (2004) consider the infinite-horizon extension of Hammond Equity.[9] Bossert et al. (2007) also examine a weaker equity axiom called Pigou-Dalton Equity as well as Hammond Equity.

Pigou-Dalton Equity (PDE): For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \leq x_j < y_j$ and $x_i - y_i = y_j - x_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Hammond Equity (HE): For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \leq x_j < y_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Both two axioms deal with a distributional conflict between two generations.[10] **PDE** asserts that an order-preserving transfer from a relatively better-off generation to a relatively worse-off generation, i.e. the well-known Pigou-Dalton transfer, is strictly socially preferable. On the other hand, **HE** asserts that an order-preserving change which diminishes inequality of utilities between conflicting two generations is strictly socially improving. **PDE** agrees to such a value judgment by **HE** as long as utility differences of conflicting two generations are equal. Hence, **HE** is stronger than **PDE**.

Bossert et al. (2007) show that **SP**, **FA**, and **PDE** together characterize the infinite-horizon generalized Lorenz criterion and also that the infinite-horizon leximin principle is characterized if **PDE** is strengthened to **HE**. We now present the formal definitions of these two infinite-horizon SWRs. For all $n \in \mathbb{N}$, let $\succsim_G^n$ denote the finite-horizon generalized Lorenz SWR defined on $\mathbb{R}^n$: for all $x, y \in \mathbb{R}^n$,

$$x \succsim_G^n y \Leftrightarrow \textstyle\sum_{t=1}^k x_{(t)} \geq \sum_{t=1}^k y_{(t)} \text{ for all } k \in \{1, \ldots, n\}$$

The generalized Lorenz SWR, denoted $\succsim_G$, is defined as follows: for all $x, y \in X$,

$$x \succsim_G y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim_G^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}. \tag{1}$$

We next introduce the leximin SWR. For all $n \in \mathbb{N}$, let $\succsim_L^n$ denote the finite-horizon leximin SWR

[9] Hammond Equity is originally formulated by Hammond (1976, 1979) in the finite population case.

[10] These conditions are generically referred to as two-person equity axioms in the finite population framework. On this, see d'Aspremont and Gevers (2002) and Bossert and Weymark (2004).

defined on $\mathbb{R}^n$: for all $x, y \in \mathbb{R}^n$,

$$x \succsim_L^n y \Leftrightarrow \begin{cases} x \text{ is a permutation of } y, \text{ or} \\ \text{there exists } m \in \{1, \ldots, n\} \text{ such that } x_{(t)} = y_{(t)} \text{ for all } t < m \text{ and } x_{(m)} > y_{(m)}. \end{cases}$$

The leximin SWR, denoted $\succsim_L$, is defined as: for all $x, y \in X$,

$$x \succsim_L y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim_L^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}. \tag{2}$$

We now formally state the characterizations by Bossert et al. (2007), which will be used to establish our main results later.

Proposition 1 (Bossert et al. (2007), Theorem 1)**.** *A SWR $\succsim$ on X satisfies **SP**, **FA**, and **PDE** if and only if $\succsim_G$ is a subrelation to $\succsim$.*

Proposition 2 (Bossert et al. (2007), Theorem 2)**.** *A SWR $\succsim$ on X satisfies **SP**, **FA**, and **HE** if and only if $\succsim_L$ is a subrelation to $\succsim$.*

Now, the natural question to ask is whether it is possible to realize the stronger notion of impartiality formalized as **QA** in a strongly Paretian and equitable (in the sense of **PDE** or **HE**) SWR. This paper provides a positive answer to this question. We define $\mathcal{Q}$-anonymous extensions of the generalized Lorenz and the leximin SWRs in the same way as Banerjee (2006a) does for the Suppes-Sen SWR and the utilitarian SWR (see our Section 4).[11] The $\mathcal{Q}$-generalized Lorenz criterion, denoted $\succsim_{QG}$, is defined as: for all $x, y \in X$,

$$x \succsim_{QG} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_G y. \tag{3}$$

The $\mathcal{Q}$-leximin principle $\succsim_{QL}$ is defined as: for all $x, y \in X$,

$$x \succsim_{QL} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_L y. \tag{4}$$

In what follows, we refer to Banerjee's (2006a) way of extension as $\mathcal{Q}$-extension. As will be shown later, each of the $\mathcal{Q}$-extensions of the generalized Lorenz SWR and the leximin SWR is also well-

[11] It should be noted that $\mathcal{Q}$-anonymous SWRs are also proposed in Lauwers (1997b) and by Fleurbaey and Michel (2003).

defined as a SWR on X, and the former satisfies all the axioms of **SP**, **QA**, and **PDE** and the latter satisfies **HE** as well as these three axioms. Thus, these two $\mathcal{Q}$-extensions not only provide the resolution to the distributional conflicts among generations but also realize the stronger notion of impartiality, **QA**, than **FA** in the intergenerational welfare evaluation, i.e. can be the resolution to both two problems we mentioned earlier.

Once the possibility of a strongly Paretian, $\mathcal{Q}$-anonymous and equitable SWR is established, we should proceed to specify the class of those SWRs in the next step. Our main theorems provide the characterizations of the classes of SWRs that satisfy **SP**, **QA**, and **PDE** or **HE**.

Theorem 1. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **PDE** if and only if $\succsim_{QG}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

Theorem 2. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **HE** if and only if $\succsim_{QL}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

As discussed by Basu and Mitra (2007) and Banerjee (2006a), the theorems 1 and 2 are interpreted as saying that $\succsim_{QG}$ (resp. $\succsim_{QL}$) is *the least restrictive* SWR among all the SWRs that satisfy **SP**, **QA**, and **PDE** (resp. **HE**). Formally, for all $x, y \in X$, we have

$$x \succsim_{QG} y \text{ if and only if } x \succsim y \text{ for all } \succsim \in \Xi_{QG}, \tag{5}$$

and

$$x \succsim_{QL} y \text{ if and only if } x \succsim y \text{ for all } \succsim \in \Xi_{QL}, \tag{6}$$

where Ξ_{QG} (resp. Ξ_{QL}) is the set of all SWRs that satisfy **SP**, **QA**, and **PDE** (resp. **HE**). The only if part of (5) (resp. (6)) is obvious from the only if statement of Theorem 1 (resp. Theorem 2). The if part of (5) (resp. (6)) is also straightforward from the fact that $\succsim_{QG} \in \Xi_{QG}$ (resp. $\succsim_{QL} \in \Xi_{QL}$).

4 An alternative characterization of $\mathcal{Q}$-utilitarian SWR

From Theorems 1 and 2, the difference between the $\mathcal{Q}$-generalized Lorenz and the $\mathcal{Q}$-leximin SWRs can be explained in terms of their consequentialist equity properties, **PDE** and **HE**. Another plausible SWR that satisfies both of **SP** and **QA** is the $\mathcal{Q}$-extension of the utilitarian SWR proposed by Banerjee (2006a) under the name $\mathcal{Q}$-utilitarian SWR. He characterizes it in terms of **SP**, **QA**, and an informational invariance axiom called *Partial Translation Scale Invariance* (**PTSI**).[12] In view of Banerjee's (2006a) result, our characterizations of the $\mathcal{Q}$-generalized Lorenz and the $\mathcal{Q}$-leximin SWRs are also interpreted as showing the possibility of adding the consequentialist equity axioms in the strongly Paretian and $\mathcal{Q}$-anonymous SWRs and also as saying that the admissible SWRs will considerably change if we impose the equity axiom, **PDE** or **HE**, instead of the invariance axiom **PTSI**. Now, if we obtain the characterization of the $\mathcal{Q}$-utilitarian SWR using a consequentialist equity axiom, it will be of much help to those who want to compare the three strongly Paretian and $\mathcal{Q}$-anonymous SWRs, the $\mathcal{Q}$-generalized Lorenz, the $\mathcal{Q}$-leximin, and the $\mathcal{Q}$-utilitarian SWRs, in terms of their consequentialist equity properties. In this section, we provide an alternative characterization of the $\mathcal{Q}$-utilitarian using an equity axiom.

We begin with the formal definitions of the utilitarian SWR and its $\mathcal{Q}$-extension. Let $\succsim_U^n$ denote the finite-horizon utilitarian SWR defined on $\mathbb{R}^n$: for all $x, y \in \mathbb{R}^n$,

$$x \succsim_U^n y \Leftrightarrow \textstyle\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i.$$

The utilitarian SWR $\succsim_U$ is defined by combining the finite-horizon utilitarian SWR and the Pareto criterion: for all $x, y \in X$,

$$x \succsim_U y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim_U^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}. \tag{7}$$

The $\mathcal{Q}$-utilitarian SWR $\succsim_{QU}$ is defined as follows: for all $x, y \in X$,

$$x \succsim_{QU} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_U y. \tag{8}$$

[12]**PTSI** is also called *Partial Unit Comparability* and is defined as: for all $x, y \in X$, $\alpha \in \mathbb{R}^{\mathbb{N}}$, and $n \in \mathbb{N}$, if $x^{+n} = y^{+n}$ and $x \succsim y$ then $x + \alpha \succsim y + \alpha$. For the detailed discussion about the invariance axioms, we refer the reader to Bossert and Weymark (2004) and d'Aspremont and Gevers (2002).

We now introduce an equity axiom which is stated for two-generation conflicting situations similar to those considered in **PDE** and **HE**.[13]

Incremental Equity (IE): For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $x_i - y_i = y_j - x_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \sim y$.

In contrast to **PDE**, this axiom asserts that, for any transfer among two generations, the initial utility stream and the post-transfer stream are considered to be equally good without any reference to the relative utility levels of conflicting two generations. Thus, **IE** is interpreted as requiring the intergenerational decision making to be neutral with respect to utility transfer among two generations. Since a transposition of two generations' utilities is a special case of utility transfer between two generations, **IE** implies **FA**.[14]

The following proposition tells that **IE** clearly distinguishes the SWRs including the utilitarian SWR as a subrelation from the other strongly Paretian SWRs.[15]

Proposition 3. *A SWR $\succsim$ on X satisfies **SP** and **IE** if and only if $\succsim_U$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

We now move to the characterization of the $\mathcal{Q}$-utilitarian SWR. As shown in the following theorem, when we strengthen the notion of impartiality by additionally imposing **QA**, the $\mathcal{Q}$-utilitarian SWR is the least restrictive SWR among the admissible ones.

Theorem 3. *A SWR $\succsim$ on X satisfies **SP**, **QA**, and **IE** if and only if $\succsim_{QU}$ is a subrelation to $\succsim$.*

Proof. See Appendix. ■

Theorems 1 to 3 together tell us how admissible SWRs will change in accordance with which of the equity axioms, **PDE**, **HE**, and **IE**, is additionally imposed on the strongly Paretian and $\mathcal{Q}$-anonymous SWRs. Table 1 summarizes the characterizations in this paper and compares them with those established by Banerjee (2006a), Basu and Mitra (2007), and Bossert et al. (2007). For each row in Table 1, the class of SWRs that satisfy the axioms indicated by $\oplus$ is characterized as the one composed of all SWRs

[13]**IE** was first proposed by Blackorby et al. (2002) in a finite population framework. See also d'Aspremont and Gevers (2002) and Blackorby et al. (2005).

[14]Note that any finite permutation is equivalently represented as finite composition of transpositions.

[15]In Basu and Mitra (2007), $\succsim_U$ is also characterized in terms of **SP**, **FA**, and **PTSI**.

including the SWR in the first column as a subrelation, and furthermore, each rule out of the class satisfies (resp. violates) the axioms indicated by + (resp. –). Compared to the characterizations in Bossert et al. (2007), our results are regarded as establishing (i) the possibility of reflecting the stronger notion of impartiality than **FA** in strongly Paretian and equitable SWRs, and compared to Banerjee's (2006a) work, (ii) the possibility of realizing the consequentialist equities in strongly Paretian and $\mathcal{Q}$-anonymous SWRs.

Table 1: Characterizations of $\mathcal{F}$-anonymous SWRs and their $\mathcal{Q}$-extensions

SWR	*efficiency*	*impartiality*		*equity*			*invariance*	
(least restrictive)	**SP**	**FA**	**QA**	**PDE**	**HE**	**IE**	**PTSI**	*characterized in ...*
$\mathcal{Q}$-G-Lorenz $\succsim_{QG}$	⊕	+	⊕	⊕		−	−	Theorem 1
G-Lorenz $\succsim_G$	⊕	⊕		⊕		−	−	Theorem 1 in Bossert et al. (2007)
$\mathcal{Q}$-Leximin $\succsim_{QL}$	⊕	+	⊕	+	⊕	−	−	Theorem 2
Leximin $\succsim_L$	⊕	⊕		+	⊕	−	−	Theorem 2 in Bossert et al. (2007)
$\mathcal{Q}$-Utilitarian $\succsim_{QU}$	⊕	+	⊕	−	−	⊕	+	Theorem 3
	⊕	+	⊕	−	−	+	⊕	Theorem 3 in Banerjee (2006a)
Utilitarian $\succsim_U$	⊕	+		−	−	⊕	+	Proposition 3
	⊕	⊕		−	−	+	⊕	Theorem 1 in Basu and Mitra (2007)

5 Conclusion

In this paper, we examined the possibility of adding the strong impartiality axiom formalized as **QA** in strongly Paretian and equitable SWRs, and we obtained positive results. In particular, we show that the $\mathcal{Q}$-extension of the generalized Lorenz SWR is characterized with **SP**, **QA**, and **PDE** and the $\mathcal{Q}$-extension of the Leximin SWR in terms of **SP**, **QA**, and **HE**. We also characterize the $\mathcal{Q}$-utilitarian SWR using the equity axiom **IE**. Our characterizations together show how the admissible SWRs will change in accordance with which of the equity axioms, **PDE**, **HE**, and **IE**, is additionally imposed on the strongly Paretian and $\mathcal{Q}$-anonymous SWRs. From Arrow's (1963) variant of Szpilrajn's (1930) theorem, we can conclude that there exists an ordering on X which satisfies **SP**, the axiom of efficiency, **QA**, the axiom of impartiality, and **PDE**, **HE**, or **IE**, the axioms of consequentialist equity. These results provide plausible resolutions to the existing impossibilities of an impartial or equitable social welfare ordering (SWO) obtained by Diamond (1965), Banerjee (2006b), and Hara et al. (forthcoming) and also strengthen the resolution in Bossert et al. (2007) established with the weaker impartiality requirement **FA**.[16]

We should also note that, compared to the results obtained with **FA**, incompleteness of the least restrictive SWR is improved in the class characterized with **QA**. The recent contribution by Mitra and Basu (2007) provides an important suggestion about further research to be carried out on this issue. As shown in their paper, a group of cyclic permutation matrices $\mathcal{Q}$ is the largest class of permutations in terms of which the **SP**-compatible anonymity axiom is well-defined. Consequently, in the presence of **SP**, it is impossible to strengthen the notion of impartiality further than **QA**. And moreover, if we think of **SP** and **QA** as indispensable requirements to be satisfied in the intergenerational welfare evaluation, more comparable social evaluation would be possible only by strengthening the axioms other than **QA** or employing additional axioms. The preference-continuity axiom considered in Asheim and Tungodden (2004) or the axiom of consistency proposed by Basu and Mitra (2007) seems to be a plausible candidate for the axiom to be additionally imposed. The issue of the possibility of additionally employing these axioms is left for future research.

Appendix

We begin with two important lemmata that shed light on properties of the $\mathcal{Q}$-extension of a $\mathcal{F}$-anonymous SWR. Each of the lemmata is stated for the class of SWRs that satisfy three basic properties common to $\succsim_G$, $\succsim_L$, and $\succsim_U$. Note that each of $\succsim_G$, $\succsim_L$, and $\succsim_U$ is defined in terms of the Pareto criterion and the sequence of finite-horizon SWRs, $(\succsim_G^n)_{n=1}^{\infty}$, $(\succsim_L^n)_{n=1}^{\infty}$, and $(\succsim_U^n)_{n=1}^{\infty}$ respectively. Each of the sequences satisfies the following properties:[17]

Property 1 (P1): For all $n \in \mathbb{N}$ and all $x, y \in \mathbb{R}^n$ with $x > y$, $x \succ^n y$;

Property 2 (P2): For all $n \in \mathbb{N}$, all $r \in \mathbb{R}$, and all $x, y \in \mathbb{R}^n$, $x \succsim^n y \Rightarrow (x, r) \succsim^{n+1} (y, r)$;

[16] In Banerjee (2006b), a SWO is required to be representable as the Bergson-Samuelson social welfare function, and in Diamond (1965) and Hara et al. (forthcoming) a certain continuity axiom is imposed on a SWO.

[17] **P1** is the finite-horizon version of **SP**. **P2** is a kind of separability requirement similar to *Extended Independence of the Utilities of Unconcerned Individuals* (or *Existence Independence*) introduced by Blackorby et al. (2002, 2005) in the framework of variable population social choice, which requires our evaluation to be independent of the addition of an unconcerned generation. **P3** is a well-known anonymity axiom in a finite-horizon framework.

Property 3 (P3): For all $n \in \mathbb{N}$ and all $x, y \in \mathbb{R}^n$, if x is a permutation of y, then $x \sim^n y$.

The lemmata are stated for the class of infinite-horizon SWRs, denoted Ξ, each of which is defined in terms of a sequence of finite-horizon SWRs satisfying **P1**, **P2**, and **P3** and of the Pareto crietrion. We now formally define the class Ξ. Let $(\succsim^n)_{n\in\mathbb{N}}$ be a sequence of finite-horizon SWRs which consists of one finite-horizon SWR $\succsim^n$ for each possible length of time horizon $n \in \mathbb{N}$. Let $\mathcal{S}$ be the set of all possible sequences of finite-horizon SWRs $(\succsim^n)_{n\in\mathbb{N}}$ that satisfies the properties **P1** to **P3**. For each $(\succsim^n)_{n\in\mathbb{N}} \in \mathcal{S}$, we can associate a binary relation $\succsim$ on X as follows: for any $x, y \in X$,

$$x \succsim y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}.^{18} \tag{9}$$

Let φ denote the mapping that associates to any sequence $(\succsim^n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ a binary relation on X defined in (9). We define Ξ as

$$\Xi = \varphi(\mathcal{S}),$$

i.e. the set of binary relations on X each of which is associated with a sequence of finite-horizon relation $(\succsim^n)_{n\in\mathbb{N}} \in \mathcal{S}$ in a way of (9). As will be shown in Claim 1, every binary relation in Ξ is well-defined as a SWR on X. Moreover, by **P3** and (9), each $\succsim$ in Ξ is $\mathcal{F}$-anonymous. For each $\succsim \in \Xi$, its $\mathcal{Q}$-extension, denoted $\succsim_{\mathcal{Q}}$, is defined as follows: for all $x, y \in X$,

$$x \succsim_{\mathcal{Q}} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim y. \tag{10}$$

We are ready to state the following lemmata. We owe a lot to Banerjee's (2006a) work in establishing the lemmata. Our results are regarded as the generalizations of his results established for the Suppes-Sen and the utilitarian SWRs.

Lemma 1. *For all $\succsim \in \Xi$, its $\mathcal{Q}$-extension $\succsim_{\mathcal{Q}}$ is also a SWR on X.*

[18] d'Aspremont (2007) refers to this type of binary relation as *simplified criterion*.

Lemma 2. *For all $\succsim \in \Xi$, its $\mathcal{Q}$-extension $\succsim_{\mathcal{Q}}$ has the following property: for any $x, y \in X$,*

$$\begin{cases} x \succ_{\mathcal{Q}} y \text{ if and only if there exists } P \in \mathcal{Q} \text{ such that } Px \succ y; & \text{(11a)} \\ x \sim_{\mathcal{Q}} y \text{ if and only if there exists } P \in \mathcal{Q} \text{ such that } Px \sim y. & \text{(11b)} \end{cases}$$

Proof of Lemma 1. The proof proceeds through two claims.

Claim 1. *For all $\succsim \in \Xi$, $\succsim$ is reflexive and transitive, i.e. well-defined as a SWR on X.*

Proof of Claim 1. Reflexivity is straightforward. To confirm $\succsim$ is transitive, consider any $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$. We will show $x \succsim z$ holds. By (9), there exist $n, n' \in \mathbb{N}$ such that (i) $x^{-n} \succsim^n y^{-n}$ and $x^{+n} \geqslant y^{+n}$ and (ii) $y^{-n'} \succsim^{n'} z^{-n'}$ and $y^{+n'} \geqslant z^{+n'}$. Let $\bar{n} = \max\{n, n'\}$. We demonstrate the proof for the case of $\bar{n} = n'$. The same argument can be applied to the case of $\bar{n} = n$. Notice that $x^{+\bar{n}} \geqslant z^{+\bar{n}}$. Thus, we are enough to show that $x^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$. By **P2**, $x^{-n} \succsim^n y^{-n}$ implies $(x^{-n}, y_{n+1}, \dots, y_{\bar{n}}) \succsim^{\bar{n}} y^{-\bar{n}}$. By **P1**, $x^{-\bar{n}} \succsim^{\bar{n}} (x^{-n}, y_{n+1}, \dots, y_{\bar{n}})$. Since $\succsim^{\bar{n}}$ is transitive, $x^{-\bar{n}} \succsim^{\bar{n}} y^{-\bar{n}}$ holds, which together with $y^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$ and the transitivity of $\succsim^{\bar{n}}$ give $x^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$.

Claim 2. *For any $\succsim \in \Xi$, we have the following: for any $P \in \mathcal{Q}$ and any $x, y \in X$,*

$$\begin{cases} x \succ y \text{ if and only if } Px \succ Py; & \text{(12a)} \\ x \sim y \text{ if and only if } Px \sim Py. & \text{(12b)} \end{cases}$$

Proof of Claim 2. We will show that $x \succsim y$ if and only if $Px \succsim Py$, from which the equivalence assertions in (12a) and (12b) immediately follow.

(*only if* part) Assume $x \succsim y$. By (9), there exists $n \in \mathbb{N}$ such that

$$x^{-n} \succsim^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}. \tag{13}$$

Take $P \in \mathcal{Q}$ arbitrarily. We want to show that $Px \succsim Py$. Without loss of generality, let P be a k-period cyclic permutation matrix. Then, we can find $\hat{n} \in \mathbb{N}$ such that (i) $k\hat{n} \geq n$ and (ii) $P(k\hat{n})$ is a finite dimensional permutation matrix. Let $\bar{n} = k\hat{n}$, and consider the following profiles $w, z \in X$:

$$w = (x^{-\bar{n}}, (Px)^{+\bar{n}}) \text{ and } z = (y^{-\bar{n}}, (Py)^{+\bar{n}}).$$

From the definitions of w and z and (13), it follows that

$$w^{-n} \succsim^n z^{-n} \text{ and } w^{+n} = (x_{n+1}, \dots, x_{\bar{n}}, (Px)^{+\bar{n}}) \geqslant (y_{n+1}, \dots, y_{\bar{n}}, (Py)^{+\bar{n}}) = z^{+n}.$$

Thus, by (9), $w \succsim z$. Note that, by (9) and **P3**, $\succsim$ satisfies **FA**. By **FA**, $w \sim Px$ and $z \sim Py$. Since, by Claim 1, $\succsim$ is transitive, $Px \succsim Py$ is obtained as desired.

(*if* part) Assume $Px \succsim Py$. Since $P' \in \mathcal{Q}$, we obtain $x = P'(Px) \succsim P'(Py) = y$ by the "only if" part of the lemma.

Proof of Lemma 1. Take $\succsim \in \Xi$ arbitrarily and let $\succsim_{\mathcal{Q}}$ be its $\mathcal{Q}$-extension. Reflexivity is straightforward from the fact that $I \in \mathcal{Q}$ and $\succsim$ is reflexive. We show that $\succsim_{\mathcal{Q}}$ is transitive. Assume that $x \succsim_{\mathcal{Q}} y$ and $y \succsim_{\mathcal{Q}} z$. Then, by (10), there exist $P, Q \in \mathcal{Q}$ such that $Px \succsim y$ and $Qy \succsim z$. By Claim 2, $Px \succsim y \Leftrightarrow Q(Px) \succsim Qy$. By Claim 1, $\succsim$ is transitive, and thus $Q(Px) \succsim z$ holds. Since $QP \in \mathcal{Q}$, $x \succsim_{\mathcal{Q}} z$ follows from (10). ■

Proof of Lemma 2. First, we prove the equivalence assertion in (11a). Note that, by the definiton of $\succsim_{\mathcal{Q}}$, $x \succ_{\mathcal{Q}} y$ is equivalent to

$$\begin{cases} \exists P \in \mathcal{Q} \text{ such that } Px \succsim y & \text{(14a)} \\ \quad \text{and} \\ \forall Q \in \mathcal{Q}, \neg(Qy \succsim x). & \text{(14b)} \end{cases}$$

(*only if* part of (11a)) Assume $x \succ_{\mathcal{Q}} y$. The proof is done by contradiction. Suppose that there is no $P \in \mathcal{Q}$ such that $Px \succ y$, or equivalently, such that $Px \succsim y$ and $\neg(y \succsim Px)$. From (14a), there exists $P \in \mathcal{Q}$ such that $Px \succsim y$. Thus, for this $P \in \mathcal{Q}$, we must have $Px \succsim y$ and $y \succsim Px$, i.e. $Px \sim y$. Then, by (12b), we have $x = P'(Px) \sim P'y$, which contradicts (14b).

(*if* part of (11a)) Assume that there exists $P \in \mathcal{Q}$ such that $Px \succ y$. By (10), $x \succsim_{\mathcal{Q}} y$. We want to show $\neg(y \succsim_{\mathcal{Q}} x)$. We prove this by contradiction. Suppose that $y \succsim_{\mathcal{Q}} x$ holds. Then, by (10), there exists $Q \in \mathcal{Q}$ such that $Qy \succsim x$. By Claim 2, $P(Qy) \succsim Px$. Let R denote the product PQ. Since $\succsim$ is transitive, $Px \succ y$ and $Ry \succsim Px$ give $Ry \succ y$. Then, by (9), there exists $n \in \mathbb{N}$ such that

$$(Ry)^{-n} \succsim^n y^{-n} \text{ and } (Ry)^{+n} \geqslant y^{+n}.$$

Note that R belongs to Q. Without loss of generality, let R be a k-period cyclic permutation matrix, i.e., for any $\hat{n} \in \mathbb{N}$, $R(k\hat{n})$ is a finite-dimensional permutation matrix. Consider $\bar{n} \in \mathbb{N}$ such that $\bar{n} \geq n$ and $\bar{n} = k\hat{n}$ for some $\hat{n} \in \mathbb{N}$. By **P3** and the fact that $\bar{n} \geq n$, we have

$$(Ry)^{-\bar{n}} \sim^{\bar{n}} y^{-\bar{n}} \text{ and } (Ry)^{+\bar{n}} \geqslant y^{+\bar{n}}.$$

If we have $(Ry)^{+\bar{n}} = y^{+\bar{n}}$, $Ry \sim y$ follows, and this contradicts $Ry \succ y$. But, in the case of $(Ry)^{+\bar{n}} > y^{+\bar{n}}$, we can find $m \in \mathbb{N}$ such that $(Ry_{\bar{n}+1}, \ldots, Ry_{k(\hat{n}+m)}) > (y_{\bar{n}+1}, \ldots, y_{k(\hat{n}+m)})$, which contradicts the fact that R is a k-period cyclic permutation matrix. Thus, in either case, we obtain a contradiction.

Next, we prove the equivalence assertion in (11b).

(*only if* part of (11b)) Assume $x \sim_{\mathcal{Q}} y$. Then, by (10), we can find $P \in \mathcal{Q}$ such that $Px \succsim y$. If we had $Px \succ y$, it would follow from (11a) that $x \succ_{\mathcal{Q}} y$ holds, which contradicts $x \sim_{\mathcal{Q}} y$. Thus, we must have $Px \sim y$.

(*if* part of (11b)) Assume that there exists $P \in \mathcal{Q}$ such that $Px \sim y$. Then, by (10), we have $x \succsim_{\mathcal{Q}} y$. Moreover, by (12b), $x = P'(Px) \sim P'y$ holds. Then, by (10) again, we also obtain $y \succsim_{\mathcal{Q}} x$. Thus, $x \sim_{\mathcal{Q}} y$ follows. ■

We now provide the proofs of Theorems 1 and 2.

Proof of Theorem 1. (*only if* part) Assume that a SWR $\succsim$ on X satisfies **SP**, **QA**, and **PDE**. Since **QA** implies **FA**, it follows from Proposition 1 that $\succsim_G$ is a subrelation to $\succsim$. First, we show that if $x \succ_{\mathcal{Q}G} y$ then $x \succ y$. Assume $x \succ_{\mathcal{Q}G} y$. By (11a), there exists $P \in \mathcal{Q}$ such that $Px \succ_G y$. Since $\succsim_G$ is a subrelation to $\succsim$, $Px \succ y$ holds. By **QA**, $x \sim Px$. By the transitivity of $\succsim$, $x \succ y$ follows. Next, assume $x \sim_{\mathcal{Q}G} y$. We want to show $x \sim y$. By (11b), there exists $P \in \mathcal{Q}$ such that $Px \sim_G y$. Since $\succsim_G$ is a subrelation to $\succsim$, $Px \sim y$. By **QA**, $x \sim Px$. The transitivity of $\succsim$ gives $x \sim y$.

(*if* part) Assume that $\succsim_{\mathcal{Q}G}$ is a subrelation to $\succsim$. From Lemma 2 and the fact that $I \in \mathcal{Q}$, $\succsim_G$ is a subrelation to $\succsim_{QG}$, which means that $\succsim_G$ is a subrelation to $\succsim$. Thus, from Proposition 1, $\succsim$ satisfies **SP** and **PDE**. As for **QA**, it is checked as follows. Since $\succsim_G$ is reflexive, $x = P'(Px) \sim_G x$ holds for any $x \in X$ and any $P \in \mathcal{Q}$. By (11b), $Px \sim_{QG} x$. Since $\succsim_{QG}$ is a subrelation to $\succsim$, $Px \sim x$. ■

Proof of Theorem 2. Using Proposition 2, the only if part is proved in the same way as in the proof of Theorem 1. The if part is also proved by the same argument as in the proof of Theorem 1. ■

Next, we prove Proposition 3 and Theorem 3.

Proof of Proposition 3. The if part is straightforward and we omit it. We prove the only if part in two steps.

Step 1.[19] We show that $x \sim y$ holds whenever there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $x^{+n} = y^{+n}$. The case of $n = 1$ is trivial. If $n \geq 2$, consider the following operation: fix $\bar{n} \in \{1, \dots, n-1\}$ arbitrarily and construct $\bar{x} \in X$ as follows: $\bar{x}_{\bar{n}} = y_{\bar{n}}$ for $\bar{n}$; $\bar{x}_n = x_n + x_{\bar{n}} - y_{\bar{n}}$ for n; $\bar{x}_k = x_k$ for all $k \in \mathbb{N} \setminus \{\bar{n}, n\}$. By **IE**, $x \sim \bar{x}$. Applying the above operation once for each $\bar{n} \in \{1, \dots, n-1\}$ repeatedly, we can construct the profile y from x. Then, the transitivity of $\succsim$ gives $x \sim y$.

Step 2. From Step 1 and (7), it is obvious that if $x \sim_U y$ then $x \sim y$. We show that if $x \succ_U y$ then $x \succ y$. Assume $x \succ_U y$. By (7), there exists $n \in \mathbb{N}$ such that $(\sum_{i=1}^{n} x_i, x^{+n}) > (\sum_{i=1}^{n} y_i, y^{+n})$. Then, we can find $\bar{n} \geq n$ such that

$$\textstyle\sum_{i=1}^{\bar{n}} x_i > \sum_{i=1}^{\bar{n}} y_i \text{ and } x^{+\bar{n}} \geqslant y^{+\bar{n}}.$$

Take $z \in X$ such that $z_1 = y_1 + \sum_{i=1}^{\bar{n}} (x_i - y_i)$, $z_i = y_i$ for all $i \in \{2, 3, \dots, \bar{n}\}$, and $z_j = x_j$ for all $j \in \{\bar{n}+1, \dots\}$. From Step 1, $x \sim z$ follows. By **SP**, $z \succ y$. The transitivity of $\succsim$ gives $x \succ y$. ■

Proof of Theorem 3. Using Proposition 3, the proof is done in the same way as in the proof of Theorem 1. ■

[19]The result demonstrated in this step is well-known fact. See, for example, Blackorby et al. (2002, 2005) for the finite population case, and also Asheim and Tungodden (2004) for the infinite-horizon case.

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