Potential, coalition formation and coalition structure*

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Abstract

The present paper studies a new potential function of a cooperative game with a coalition structure. When define a marginal contribution of each player, a notion of a coalition formation by Hart and Kurz (1983) is considered. Our potential is a real-valued function in contrast to Winter (1992)'s one which is vector-valued function whose dimension is the number of elements in the coalition structure. As well as Hart and MasColell (1989)'s approach, a marginal contribution of our function leads to a new solution concept. The solution concept is closely related to the weighted Shapley value and its weight is endogenously determined by the solution. In addition to the axiomatization by the usual additivity approach, it is characterized by the similar property to the balanced contributions of the Shapley value (Myerson (1980)).

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1 Introduction

Hart and MasColell (1989) offered a beautiful approach to characterize the Shapley value (Shapley (1953b)) by a notion of a potential for cooperative games. Their motivation is related to the marginalism that is familiar to economics. It means that each player should receive his payoff due to his marginal contribution, the difference between the worth of the total cooperation and that without him. This approach, however, causes a problem in the context of cooperative game theory because the sum of player's marginal contribution to the total cooperation over them may not equal the amount whose distribution now they talk about.

The resolution of this difficulty proposed by Hart and MasColell (1989) is that they introduce a function which associates with each game (N, v) a real number, define each player's marginal contribution in terms of this function, and require this function that the worth of the total cooperation $v(N)$ must be consistently distributed among the players according to the marginal contributions. They show that the function, which is called a potential for cooperative games, is uniquely determined and the marginal contribution in terms of it equals the Shapley value.

The present paper extends Hart and MasColell (1989)'s results to cooperative games with coalition structures. We consider a function which assigns a real number to each game with a coalition structure $(N, v, \mathcal{B})$ just like their approach. The difference lies in the way of defining a marginal contribution of each player. One aspect of the marginal contribution of a player is additional revenue by his joining the project and the other is reduction of revenue by his leaving. We would like to take a stand of the second interpretation and therefore consider the reactions of the others to his deviation. Inspired by Hart and Kurz (1983), we consider two types of the reaction to the deviation as the two extreme cases. One is that players in the coalition to which the deviant player belongs do not respond and the other is that they break their cooperative relationship. Our interesting result is that both two definitions lead to the same function and we call it a potential function for cooperative games with coalition structures.

The different definition from Winter (1992), who considers a vector-valued function, its dimension depending on the number of elements in the coalition structure, leads to the different results. While Winter's potential function induces the Owen's coalitional value (Owen (1977)) as the marginal contribution, we obtain a new solution concept of games with coalition structures from our potential. It is related to the two steps Shapley value like a solution proposed by Kamijo (2005) and the weighted Shapley value. The solution of Kamijo (2005) is defined by applying the Shapley value to both a game among coalitions and a game within a coalition. On the other hand, a solution defined in this paper uses the weighted Shapley value to the game among the coalitions where each weight is equal to the number of players in each coalition. Therefore in our solution, the weight is determined by the solution itself.

To use the number of players in a coalition as the basis of its weight is one of the natural ideas to choose the weights. (See, for example, Kalai (1977) and Thomson (1986)). The asymmetric treatment of each coalition in the coalition structure to define a solution is also considered by Levy and McLean (1989). They define the weighted coalition structure value, in which each coalition may have the different weight, and extend and/or unite the results of Owen's coalitional value by Owen (1977) and the weighted Shapley value by Kalai and Samet (1987) and Kalai and Samet (1988). Even though the sizes of coalitions are a natural candidate of the weight vector, there is an unacceptable aspect of the definition due to the weighted coalition structure value by Levy and McLean (1989). Following the definition of Levy and McLean (1989), not only an original coalition of the coalition structure (i.e., $B_k \in \mathcal{B}$, $\mathcal{B}$ is the coalition structure) but also any subcoalition of the coalition (i.e., $T \subsetneq B_k$) have the size-based weight $w_k = |B_k|$. However this is inconsistent to the views that the size of a coalition itself makes up the bargaining power. Recently, Vidal-Puga (2005b) resolves this inconsistency by simply re-defining a weight of subcoalition T as $|T|$ and defines a new solution concept of cooperative game with coalition structures. Our research belongs to the same line of the above studies. However we would like to emphasize that *(1) our solution itself determines the weights of coalitions in contrast to the weighted coalition structure value by Levy and McLean (1989) and (2) the size-relevant weight is obtained by our potential function and the views of coalition formation unlike Vidal-Puga (2005b), who starts from the definition of the solution itself.* The comparison of these solution concepts are presented in section 5.

This paper is organized as follows. In the next section, basic notations and definitions are presented and Hart and MasColell (1989)' results are summarized. In section 3, we state our main results: a potential function for cooperative games with coalition structures is characterized and a new solution concept is defined. Our solution is axiomatized by two ways in section 4. Section 5 contains concluding remarks and supplementary discussions including the topics of coalition formation games, the comparison of our solution and the others and implementability of the solution.

2 Potential and weighted potential function

We start with definitions for standard TU (transferable utility) games. A pair (N, v) is a *TU game* or a *cooperative game* where $N = \{1, \ldots, n\}$ is a set of finite players and v is a function which associates with each subset S of N a real number. A nonempty subset of S is called a *coalition.* We assume by convention that $v(\emptyset) = 0$. We denote by Γ the set of all games with finite players.

Let $\mathbb{R}^N$ be an n-dimensional Euclidean space in which each axis corresponds to each player. A vector $x \in \mathbb{R}^N$ is called a *payoff vector* and its i's element x_i denotes the payoff of player $i \in N$. It is called a *feasible payoff vector* if $\sum_{i \in N} x_i \leqq v(N)$ holds and is called *efficient* if an equality holds. A solution function of cooperative games is a function which assigns each feasible payoff vector to each game $(N, v) \in \Gamma$. If a solution assigns an efficient payoff vector to each game, it is called an *efficient solution function.*

Let σ be a permutation on N, that is, $\sigma : N \to N$ is a bijection on N. We denote by $\Sigma(N)$ the set of all permutations on N. Given (N, v), we define the marginal contribution of player i at order σ by $m_i^\sigma(N, v) = v(\{j \in N : \sigma(j) < \sigma(i)\} \cup \{i\}) - v(\{j \in N : \sigma(j) < \sigma(i)\})$. Then the *Shapley value* ϕ (Shapley (1953b)) is defined as follows. For any $i \in N$,

$$\phi_i(N, v) = \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} m_i^\sigma(N, v).$$

The interpretation of the above expression is that each player randomly comes into a room and they agree that the player who just enters the room receives his marginal contribution to the players who are already in the room. Then the Shapley value is just the expectation of the marginal contribution according to this probability distribution.

Hart and MasColell (1989) show that the Shapley value is closely connected to a potential for games. For a function $P : \Gamma \to \mathbb{R}$ such that $P(\emptyset, v) = 0$, we define operator D_i as follows.

$$D_i P(N, v) = P(N, v) - P(N \setminus \{i\}, v) \quad \text{for each } i \in N, \tag{1}$$

where $(N \setminus \{i\}, v)$ is a subgame of (N, v) on $N \setminus \{i\}$. An interpretation of operator D_i is that it represents the marginal contribution of player i to N with respect to function P. From the viewpoint of marginalism familiar to economics, it is natural that player i should obtain

$D_iP(N,v)$. Condition A enables the players to consistently distribute the total amount $v(N)$ according to this marginalism.

Condition A: $\sum_{i\in N} D_iP(N,v) = v(N)$ for all $(N,v) \in \Gamma$.

When function P satisfies condition A, we call it a *potential function for TU games.* Condition A is equivalent to the following equation:

$$P(N,v) = \frac{1}{n}\big(v(N) - \sum_{i\in N} P(N\setminus\{i\},v)\big). \tag{2}$$

Then, starting with $P(\emptyset,v)=0$, by this formula $P(\{i\},v) = \frac{1}{1}(v(\{i\})-0) = v(\{i\})$ holds for any $i \in N$. Next, for any two players coalition $\{i,j\} \subseteq N$, $P(\{i,j\},v) = \frac{1}{2}(v(\{i,j\}) - P(\{i\},v) - P(\{j\},v)) = \frac{1}{2}(v(\{i,j\}) - v(\{i\}) - v(\{j\}))$. By repeating the similar calculation, we obtain $P(N,v)$ from the above equation. Next theorem holds.

Theorem 1 (Hart and MasColell (1989)) *Let $(N,v)\in\Gamma$. (i) The potential function P is uniquely determined, and (ii) $D_iP(N,v) = \phi_i(N,v)$ for all $i \in N$.*

Non-symmetric generalization of the above approach is also considered in Hart and Mas-Colell (1989). It is closely related to the *weighted Shapley value.* It is first considered in the original paper of Shapley (1953a) and extended to a notion of a weight system in order to deal with zero weight of a player by Kalai and Samet (1987) and Kalai and Samet (1988).

Let $w = (w_i)_i$ be a positive weight vector, that is $w_i > 0$ for any i. Then the weighted Shapley value ϕ^w is defined as follows. For any $i \in N$,

$$\phi_i^w(N,v) = \sum_{\sigma\in\Sigma(N)} pr^w(\sigma)\, m_i^\sigma(N,v)$$

where, for $\sigma = (i_1,\ldots,i_n)$, $pr^w(\sigma) = \Pi_{j=1}^n \frac{w_{i_j}}{\sum_{k=1}^j w_{i_k}}$.

To obtain this probability, consider the situation that one player in N, say i_n, is randomly selected, due to a probability distribution such that the probability for a player to be selected is proportional to his weight and put in the last of the order. Next, another player i_{n-1} is selected by the same process for $n-1$ players and put in the second last of the order. Continuing the same process by $n-2$ times, we have a order $(i_1,\ldots,i_n)$ and the probability of occurrence of this order is just the above formula.

If function $P^w : \Gamma \to \mathbb{R}$ satisfies the following condition, we call it a *weighted potential function for TU games.*

Condition B: $\sum_{i\in N} w_i D_i P^w(N,v) = v(N)$ for all $(N,v) \in \Gamma$.

Condition B can be transformed into:

$$P^w(N,v) = \frac{1}{\sum_{i\in N} w_i}\big(v(N) - \sum_{i\in N} w_i P^w(N\setminus\{i\},v)\big). \tag{3}$$

Next theorem shows the parallel results to Theorem 1.

Theorem 2 (Hart and MasColell (1989)) *Let $(N,v) \in \Gamma$ and $w = (w_i)_i$ be a positive weight vector. Then, (i) the weighted potential function P^w is uniquely determined, and (ii) $w_i D_i P^w(N,v) = \phi_i^w(N,v)$ for all $i \in N$.*

3 A potential function for games with coalition structures

A *TU game with a coalition structure* or *a cooperative game with a coalition structure* is $(N,v,\mathcal{B})$ where $(N,v) \in \Gamma$ and $\mathcal{B} = \{B_1,\ldots,B_m\}$ is a collection of the subsets of N such that $B_k \cap B_h = \emptyset$ for $k \neq h$ and $\bigcup_{k=1}^m B_k = N$. We call $\mathcal{B}$ a *coalition structure*. We denote by Δ a set of all games with coalition structures. For set S, we denote by $[S]$ the coalition structure in which everyone is separated, that is, $[S] = \{\{i\}\}_{i\in S}$.

The purpose of this section is to extend the results about the potential function to games with coalition structures. Let Q be a function which associates with each game $(N,v,\mathcal{B}) \in \Delta$ a real number. We assume that $Q(\emptyset,v,\emptyset) = 0$. Following Hart and MasColell (1989)'s approach, next we have to define a marginal contribution of each player to function Q. The marginal contribution of a player is the difference of the function evaluated by the two different states: one is that he is there and the other is that he deviates and leaves from the cooperation. Therefore the *marginalism* means in one aspect that the deviant player from the situation obtains this difference. Even if we admit the deviant player to receive this difference, however it is reasonable for the other players to react against the deviation and reform their relationship of cooperation. So the calculation of the difference (marginal contribution) varies a bit from that of Hart and MasColell (1989).

The idea that players react against the deviation is inspired by the field of coalition formation. Following the seminal paper of Hart and Kurz (1983) in this field, we consider the two type of reaction against the deviation. These two are corresponding to the two extreme cases of the reactions. One is that players in the same coalition of a deviant player "fall apart," i.e., they reform a singleton coalition structure (γ type) and the other is that they "stick together," i.e., they do not react after a deviation (δ type).

Let $i \in N$ be a deviant player and $B_k \in B, i \in B_k$ be a deviant coalition, which means the coalition containing the deviant player. We consider the reaction of the other players in two levels: the reaction of the other coalitions to the deviant coalition and that of players in the deviant coalition to the deviant player. In the first level, the coalitions other than B_k break away with the coalition with keeping their own coalition structure. As a result, the original game is decomposed into two parts, a game for the players in the deviant coalition except for the deviant player (i.e., for players in $B_k \setminus \{i\}$) and a game for the players in coalitions other than the deviant coalition (i.e., for players in $N \setminus B_k$). In the second level, the remaining players in $B_k \setminus \{i\}$ react in either γ or δ type. For instance, if a coalition structure is $\{123|45|6\}$ and player 1 deviates, then the resulting situation is either $\{2|3\}$ and $\{45|6\}$ (γ type) or $\{23\}$ and $\{45|6\}$ (δ type). The technical reason that we adopt the two levels reaction against the deviation is discussed in concluding remarks.

As a result, we define two operators, $\bar{D}_i$ and $\hat{D}_i$, as follows. For $i \in B_k \in \mathcal{B}$,

$$\bar{D}_i Q(N,v,\mathcal{B}) = Q(N,v,\mathcal{B}) - \Big(Q(N \setminus B_k, v, \mathcal{B} \setminus \{B_k\}) + Q(B_k \setminus \{i\}, v, [B_k \setminus \{i\}])\Big), \quad (4)$$

$$\hat{D}_i Q(N,v,\mathcal{B}) = Q(N,v,\mathcal{B}) - \Big(Q(N \setminus B_k, v, \mathcal{B} \setminus \{B_k\}) + Q(B_k \setminus \{i\}, v, \{B_k \setminus \{i\}\}\Big). \quad (5)$$

Next we define condition C and condition D.

Condition C: For each $(N,v,\mathcal{B}) \in \Delta$, $\sum_{i \in N} \bar{D}_i Q(N,v,\mathcal{B}) = v(N)$.
Condition D: For each $(N,v,\mathcal{B}) \in \Delta$, $\sum_{i \in N} \hat{D}_i Q(N,v,\mathcal{B}) = v(N)$.

Suppose that there exist functions $\bar{Q}$ and $\hat{Q}$ which satisfy the condition C and condition D respectively. Then the following lemma shows that the connection between the potential function for cooperative games and functions $\bar{Q}$ and $\hat{Q}$.

Lemma 1 *If a coalition structure is either grand or singleton one, then both $\bar{Q}$ and $\hat{Q}$ coincide with potential function for cooperative games. That is, for all $(N,v) \in \Gamma$,*

$$\bar{Q}(N,v,\{N\}) = \hat{Q}(N,v,\{N\}) = P(N,v) \quad \textit{and} \quad \bar{Q}(N,v,[N]) = \hat{Q}(N,v,[N]) = P(N,v)$$

hold.

Proof. For $(N, v, [N]) \in \Delta$, (4) and (5) become

$$\bar{D}_i\bar{Q}(N, v, [N]) = \bar{Q}(N, v, [N]) - \bar{Q}(N \setminus \{i\}, v, [N \setminus \{i\}]),$$

$$\hat{D}_i\hat{Q}(N, v, [N]) = \hat{Q}(N, v, [N]) - \hat{Q}(N \setminus \{i\}, v, [N \setminus \{i\}])$$

because $Q(\emptyset, v, \emptyset) = 0$. The above two equations together with the conditions (C) and (D) entail

$$\bar{Q}(N, v, [N]) = \frac{1}{n}\big(v(N) - \sum_{i\in N} \bar{Q}(N \setminus \{i\}, v, [N \setminus \{i\}])\big).$$

$$\hat{Q}(N, v, [N]) = \frac{1}{n}\big(v(N) - \sum_{i\in N} \hat{Q}(N \setminus \{i\}, v, [N \setminus \{i\}])\big).$$

Therefore $\bar{Q}(N, v, [N]) = \hat{Q}(N, v, [N]) = P(N, v)$ holds because these equations are equivalent to equation (2). (Remind that stating from one player coalition, to apply equation (2) repeatedly, we obtain the unique potential function for cooperative games.)

Next we consider $(N, v, \{N\}) \in \Delta$. Then (5) becomes

$$\bar{D}_iQ(N, v, \{N\}) = \bar{Q}(N, v, \{N\}) - \bar{Q}(N \setminus \{i\}, v, \{N \setminus \{i\}\}).$$

and (4) becomes

$$\hat{D}_iQ(N, v, \{N\}) = \hat{Q}(N, v, \{N\}) - \hat{Q}(N \setminus \{i\}, v, [N \setminus \{i\}]) = \hat{Q}(N, v, \{N\}) - P(N \setminus \{i\}, v).$$

Therefore $\bar{Q}(N, v, \{N\}) = \hat{Q}(N, v, \{N\}) = P(N, v)$ also holds true. □

The only difference between (4) and (5) lies in whether the second term is $Q(B_k\setminus\{i\}, v, [B_k\setminus \{i\}])$ or $Q(B_k \setminus \{i\}, v, \{B_k \setminus \{i\}\})$. The above lemma shows, however, that these two terms are the same. Therefore condition C and condition D induce the same function Q ($= \bar{Q} = \hat{Q}$) and we call it a *potential function for cooperative games with coalition structures*. In what follows, we denote simply by D_i the marginal contribution operator. Next theorem shows the existence of the potential function for cooperative games with coalition structures and characterizes its correct formula.

Theorem 3 *For any* $(N, v, \mathcal{B}) \in \Delta$*,*

$$Q(N, v, \mathcal{B}) = P^w(M, u),$$

where $M = \{k : B_k \in B\}$*,* $w = (w_k)_{k\in M}$ *such that* $w_k = |B_k|$ *and*

$$u(H) = v(\bigcup_{k\in H} B_k) - \sum_{k\in H} v(B_k) + \sum_{k\in H} |B_k|P(B_k, v)$$

for each $H \subseteq M$*.*

Proof. Let $(N, v, \mathcal{B}) \in \Delta$ and $(N', v, \mathcal{B}')$ be a subgame of $(N, v, \mathcal{B})$, where $N' \subseteq N$ and $\mathcal{B}' \subseteq \mathcal{B}$. Put $M = \{k : B_k \in \mathcal{B}\}$ and $H = \{k : B_k \in \mathcal{B}'\} \subseteq M$. If we apply condition C to $(N', v, \mathcal{B}')$, we have

$$\begin{aligned}
v(\bigcup_{k\in H} B_k) &= \sum_{i\in N'} D_i Q(N', v, \mathcal{B}') \\
&= \sum_{k\in H}\sum_{i\in B_k} \big(Q(N', v, \mathcal{B}') - Q(N' \setminus B_k, v, \mathcal{B}' \setminus \{B_k\}) - Q(B_k \setminus \{i\}, v, \{B_k \setminus \{i\}\})\big) \\
&= \sum_{k\in H} |B_k| \big(Q(N', v, \mathcal{B}') - Q(N' \setminus B_k, v, \mathcal{B}' \setminus \{B_k\})\big) \\
&\qquad - \sum_{k\in H} |B_k| P(B_k, v) + \sum_{k\in H}\sum_{i\in B_k} (P(B_k, v) - P(B_k \setminus \{i\}, v)) \\
&= \sum_{k\in H} |B_k| \big(Q(N', v, \mathcal{B}') - Q(N' \setminus B_k, v, \mathcal{B}' \setminus \{B_k\})\big) \\
&\qquad - \sum_{k\in H} |B_k| P(B_k, v) + \sum_{k\in H}\sum_{i\in B_k} \phi_i(B_k, v) \\
&= \sum_{k\in H} |B_k| \big(Q(N', v, \mathcal{B}') - Q(N' \setminus B_k, v, \mathcal{B} \setminus \{B_k\})\big) - \sum_{k\in H} |B_k| P(B_k, v) + \sum_{k\in H} v(B_k).
\end{aligned}$$

The second last equality is by (ii) of Theorem 1 and the last equality is by efficiency of the Shapley value. Therefore we obtain

$$\sum_{k\in H} |B_k| \big(Q(N', v, \mathcal{B}') - Q(N' \setminus B_k, v, \mathcal{B}' \setminus \{B_k\})\big) = u(H) \tag{6}$$

We now prove this theorem by induction on the number of elements in $\mathcal{B}'$. When $|\mathcal{B}'| = 1$, that is $\mathcal{B}' = \{B_k\}$, then $P^w(\{k\}, u) = \frac{u(\{k\})}{|B_k|} = \frac{1}{|B_k|}(v(B_k) - v(B_k) + |B_k| P(B_k, v)) = P(B_k, v)$ and $Q(B_k, v, \{B_k\}) = P(B_k, v)$ by Lemma 1. Therefore the theorem holds for $|\mathcal{B}'| = 1$.

Assume that $Q(N', v, \mathcal{B}') = P^w(H, u)$ holds for any subgame $(N', v, \mathcal{B}')$ such that $|\mathcal{B}'| < m$. We consider the game $(N, v, \mathcal{B})$ with $|\mathcal{B}| = m$. Applying equation (6) to $(N, v, \mathcal{B})$, we obtain that

$$\sum_{k\in M} |B_k| \left(Q(N, v, \mathcal{B}) - P^w(M \setminus \{k\}, u)\right)) = u(M).$$

bt the hypothesis of the induction. By condition B and the uniqueness of the weighted potential function, we conclude that $Q(N, v, \mathcal{B}) = P^w(M, u)$. $\square$

Let $(N, v, \mathcal{B}) \in \Delta$ and $\mathcal{B} = \{B_1, \ldots, B_m\}$. A *game among coalitions* $(M, v_{\mathcal{B}})$ corresponding to $(N, v, \mathcal{B})$ is defined as follows: $M = \{1, \ldots, m\}$ is the set of coalitional indices of the elements in $\mathcal{B}$ and $v_{\mathcal{B}}(H) = v(\bigcup_{k\in H} B_k)$ for each $H \subseteq M$. (This game is referred as an intermediate game in Peleg and Sudhölter (2003) and as a quotient game in Owen (1977))

Theorem 3 leads to the following results.

Theorem 4 *The following two properties are satisfied.*

(i) $\sum_{i\in B_k} D_iQ(N,v,\mathcal{B}) = \phi^w_k(M,v_{\mathcal{B}})$ *for all* $B_k \in \mathcal{B}$, *and*

(ii) for all $B_k \in \mathcal{B}$ *and* $i \in B_k$,

$$D_iQ(N,v,\mathcal{B}) = \frac{\phi^w_k(M,v_{\mathcal{B}}) - v(B_k)}{|B_k|} + \phi_i(B_k,v),$$

where $w = (w_k)_{k\in M}$ *such that* $w_k = |B_k|$ *for each* $k \in M$.

Proof. Let $(N,v,\mathcal{B}) \in \Delta$. By Theorem 2 and Theorem 3,

$$Q(N,v,\mathcal{B}) - Q(N\setminus B_k, v, \mathcal{B}\setminus\{B_k\}) = P^w(M,u) - P^w(M\setminus\{k\},u) = \frac{1}{|B_k|}\phi^w_k(M,u).$$

Note that $u(H) = v_{\mathcal{B}}(H) + \sum_{h\in H}\big(|B_h|P(B_h,v) - v(B_h)\big)$ for each $H \subseteq M$. Because the weighted Shapley value satisfies covariance under strategic equivalence,

$$\phi^w_k(M,u) = \phi^w_k(M,v_{\mathcal{B}}) + |B_k|P(B_k,v) - v(B_k).$$

Therefore

$$\begin{aligned} D_iQ(N,v,\mathcal{B}) &= Q(N,v,\mathcal{B}) - Q(N\setminus B_k, v, \mathcal{B}\setminus\{B_k\}) - Q(B_k\setminus\{i\}, v, \{B_k\setminus\{i\}\}) \\ &= \frac{\phi^w_k(M,v_{\mathcal{B}})}{|B_k|} + P(B_k,v) - \frac{v(B_k)}{|B_k|} - P(B_k\setminus\{i\},v) \\ &= \frac{\phi^w_k(M,v_{\mathcal{B}}) - v(B_k)}{|B_k|} + \phi_i(B_k,v), \end{aligned}$$

and

$$\begin{aligned} \sum_{i\in B_k} D_iQ(N,v,\mathcal{B}) &= \sum_{i\in B_k}\left(\frac{\phi^w_k(M,v_{\mathcal{B}}) - v(B_k)}{|B_k|} + \phi_i(B_k,v)\right) \\ &= \phi^w_k(M,v_{\mathcal{B}}). \end{aligned}$$

□

We define a solution concept ψ^c as follows.

$$\psi^c_i(N,v,\mathcal{B}) = D_iQ(N,v,\mathcal{B}), \quad \forall i \in N.$$

We call ψ^c the *collective value*.

This solution is based on the idea of the two steps Shapley value: in the first step, the weighted Shapley value is applied to a game among coalitions, i.e., $(M, v_{\mathcal{B}})$ and in the second step, the Shapley value is applied to a game within a coalition, i.e., (B_k, v). In the first step, the weights are the sizes of each coalition, i.e., $w_k = |B_k|$ for each $B_k \in \mathcal{B}$. After the first step bargaining, each coalition, say B_k, receives $\phi_k^w(M, v_{\mathcal{B}})$ and in the second step, players in B_k talk about the distribution of $\phi_k^w(M, v_{\mathcal{B}})$. Then, they agree that a pure surplus of the first step bargaining, $\phi_k^w(M, v_{\mathcal{B}}) - v(B_k)$ is equally divided among them and $v(B_k)$ is distributed by the Shapley value.

We use the weighted Shapley value instead of the Shapley value to the game among the coalition in order to reflect the size of each group as the bargaining power. Kalai and Samet stated in section 7 in their paper (Kalai and Samet (1987)) as follows.

> It is important for applications in which the players themselves are, or are representing, groups of individuals. Such is the case for example when the players are parties, cities, or management boards. ... A natural candidate for a solution is the weighted Shapley value where the players are weighted by the size of the constituencies they stand for.

Thus, ψ^c respects the collective of players itself as one of the bargaining power. This is the reason that we name it the collective value.

As well as the Shapley value, the story of random arrival of the players gives us another formula of the collective value. It is also the story to construct and justify the size-relevant weights.

Consider the situation that the players randomly arrive at a hotel to participate in a meeting of their group. The hotel has m rooms indexed by $1, ..., m$. A player who first arrives at the hotel can reserve the last room m for his coalition and settles into the last seat in this room. A player who arrives second, if she is not a member of the first player's coalition, she can not take room m since the room is already reserved and she reserves the second last room $m-1$ for her coalition. If she is a member of the first player's coalition, she settles into the second last seat in room m. Continuing these arguments by $n-2$ times, we obtain a order σ^* on N from any initial order σ.

For instance, consider a six-person situation where the coalition structure is $\{123|45|6\}$. Let $\sigma = (1, 4, 2, 5, 6, 3)$. Then the first player is player 1 and he reserves room 3 for his coalition

$\{1,2,3\}$ and settles into the last seat in the room. The second player is player 4 and he reserves room 2 for $\{4,5\}$ and settles into the last seat in the room. The third player is player 2 and he settles into the second last seat in room 3 because player 1 has already reserved the room and sits himself down the last seat. Continuing these processes, we obtain the following:

$$(\,\cdot \mid \cdot \mid \cdot\,) \underset{1}{\rightarrow} (\,\cdot \mid \cdot \mid 1\,) \underset{4}{\rightarrow} (\,\cdot \mid 4 \mid 1\,) \underset{2}{\rightarrow} (\,\cdot \mid 4 \mid 2,1\,) \underset{5}{\rightarrow} (\,\cdot \mid 5,4 \mid 2,1\,)$$
$$\underset{6}{\rightarrow} (\,6 \mid 5,4 \mid 2,1\,) \underset{3}{\rightarrow} (\,6 \mid 5,4 \mid 3,2,1\,)$$

Therefore $\sigma^* = (6,5,4,3,2,1)$.

Let $(N, v, \mathcal{B}) \in \Delta$, $\mathcal{B} = \{B_1, \ldots, B_m\}$ and $M = \{1, \ldots, m\}$. Formally, for any $\sigma \in \Sigma(N)$, we define a order of coalitions $\pi(\sigma)$ on M such that $\pi(\sigma)(k) < \pi(\sigma)(h)$ if and only if there exits $i \in B_k$ and $\sigma(i) < \sigma(j)$ holds for any $j \in B_h$. Next we define a order $\sigma[k]$ on B_k by that for any $i, j \in B_k$, $\sigma[k](i) < \sigma[k](j)$ if and only if $\sigma(i) < \sigma(j)$. Define a order σ^{**} on N by $\sigma^{**} = (\sigma[\pi(\sigma)(1)], \ldots, \sigma[\pi(\sigma)(m)])$. Then σ^* is a reverse order of σ^{**}, that is, $\sigma^*(i) = m - \sigma^{**}(i) + 1$.

Finally, we calculate each player's marginal contribution according to σ^* and take its expectation over the probability distribution such that each order σ on N is equally occurred as well as the Shapley value. Here remind that the first player in coalition does the specific task to reserve the room for his coalition. Therefore we should count this fact to compute their marginal contributions. For $i \in B_k$, we define a *collective marginal contribution* of player i at order σ^* as follows.

$$cm_i(\sigma^*) = \begin{cases} m_i^{\sigma^*[k]}(B_k, v) & \text{if } i \text{ is not the last in } B_k \text{ at order } \sigma^*, \\ m_i^{\sigma^*[k]}(B_k, v) + m_k^{\pi(\sigma^*)}(M, v_{\mathcal{B}}) - v(B_k) & \text{if } i \text{ is the last in } B_k \text{ at order } \sigma^*. \end{cases}$$

Therefore the collective marginal contribution of player i is his marginal contribution within his coalition if he is not the last player in B_k at the order and is the sum of his marginal contribution within his coalition and his coalition's marginal contribution minus the worth of his coalition if he is the last in B_k. The term $m_k^{\pi(\sigma^*)}(M, v_{\mathcal{B}}) - v(B_k)$ is viewed as the reward for the reservation of the meeting room.

The following proposition holds.

Proposition 1 *For any $i \in N$, $i \in B_k \in \mathcal{B}$,*

$$\psi_i^c(N, v, \mathcal{B}) = \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} cm_i(\sigma^*).$$

Proof. We first show that $\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} m_i^{\sigma^*[k]}(B_k,v) = \phi_i(B_k,v)$. Given a order σ^k on B_k, then $\text{Prob}(\sigma^*[k]=\sigma^k) = \frac{1}{|B_k|!}$ because $\text{Prob}(\sigma^*[k]=\sigma^k)$ is irrelevant to the choice of σ^k. Therefore $\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} m_i^{\sigma^*[k]}(B_k,v) = \frac{1}{|\Sigma(B_k)|}\sum_{\sigma^k\in\Sigma(B_k)} m_i^{\sigma^k}(B_k,v) = \phi_i(B_k,v)$.

Next we show that $\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} m_k^{\pi(\sigma^*)}(M,v_{\mathcal{B}}) = \phi_k^w(M,v_{\mathcal{B}})$ where $w=(w_k)_{k\in M}$ and $w_k=|B_k|$. Given a order $\pi=(k_1,\ldots,k_m)$ on M, then $\text{Prob}(\pi(\sigma^*)(m)=k_m)$ equals the probability that a player in S_{k_m} comes first when every player has the equal probability to come first. Therefore $\text{Prob}(\pi(\sigma^*)(m)=k_m) = \frac{|B_{k_,}|}{n}$. By the same argument, we obtain that $\text{Prob}(\pi(\sigma^*)(m-1)=k_{m-1}|\pi(\sigma^*)(m)=k_m) = \frac{|B_{k_{m-1}}|}{n-|B_{k_m}|}$. Generally, we obtain that

$$\text{Prob}(\pi(\sigma^*)(i)=k_i|\pi(\sigma^*)(j)=k_j,\forall j>i) = \frac{|B_{k_i}|}{n-\sum_{j=i+1}^m |B_{k_j}|}.$$

Therefore

$$\begin{aligned}\text{Prob}(\pi(\sigma^*)=\pi) &= \Pi_{i=1}^m \text{Prob}(\pi(\sigma^*)(i)=k_i|\pi(\sigma^*)(j)=k_j,\forall j>i)\\ &= \Pi_{i=1}^m \frac{|B_{k_i}|}{n-\sum_{j=i+1}^m |B_{k_j}|}\\ &= \Pi_{i=1}^m \frac{|B_{k_i}|}{\sum_{j=1}^i |B_{k_j}|}\\ &= \Pi_{i=1}^m \frac{w_i}{\sum_{j=1}^i w_j} = pr^w(\pi)\end{aligned}$$

Finally, since each player i in coalition B_k has the equal probability to be the first player in B_k, each i has the equal probability to obtain the coalitional marginal contribution $m_k^{\pi(\sigma^*)}(M,v_{\mathcal{B}})$ minus the worth of their coalition $v(B_k)$. Therefore we have a desired result. □

4 Characterization

4.1 Group balanced contributions

The Shapley value satisfies the following nice property from the view point of fairness in terms of the marginal contributions and the balancedness of objections and counterobjections (See chp.14 in Osborne and Rubinstein (1994)). This property is referred in Myerson (1980) and is easily proved by the potential function of Hart and MasColell (1989).

Proposition 2 *The Shapley value satisfies the following property. For every $i\in N$ and $j\in N, j\neq i$, $\phi_i(N,v)-\phi_i(N\setminus\{j\},v) = \phi_j(N,v)-\phi_j(N\setminus\{i\},v)$*

The above condition is called the *balanced contributions property.* It means that for any two players, one player's marginal contribution to the other player's Shapley value is the same as the other player's marginal contribution to the first player's Shapley value. Therefore their contributions are balanced each other and this is the reason of its name. This condition with efficiency suffices to characterize the Shapley value.

Theorem 5 (Myerson (1980)) *The Shapley value ϕ is the unique efficient solution function on Γ satisfying the balanced contributions property.*

What is an extension of this property to games with coalition structure? If cooperation between the different groups requires the every relevant players' cooperation (e.g., every player in the same group is veto player to the cooperation with the other group or each player has the essential ability to cooperate with outside players), then the lack of one player's cooperation means the lack of his coalition's cooperation. Then, his contribution to the other player is equivalent to his coalition's contribution to her payoff. Therefore "balanced contributions" in this setting means that his coalition's contribution to her payoff must be equal to her coalition's contribution to his payoff. As shown in the following proposition, the collective value satisfies this kind of property. We refer the condition (ii) of the following proposition as *group balanced contributions.*

Proposition 3 *Let $(N, v, \mathcal{B}) \in \Delta$. The following two properties are satisfied.*

(i) If $|\mathcal{B}| = 1$, for every $i \in N$ and $j \in N, j \neq i$, $\psi^c_i(N, v, \{N\}) - \psi^c_i(N \setminus \{j\}, v, \{N \setminus \{j\}\}) = \psi^c_j(N, v, \{N\}) - \psi^c_j(N \setminus \{i\}, v, \{N \setminus \{i\}\})$ hold true, and

(ii) If $|\mathcal{B}| \geqq 2$, for every $i \in B_k \in \mathcal{B}$ and for every $j \in B_h \in \mathcal{B}$, $B_k \neq B_h$, $\psi^c_i(N, v, \mathcal{B}) - \psi^c_i(N \setminus B_h, v, \mathcal{B} \setminus \{B_h\}) = \psi^c_j(N, v, \mathcal{B}) - \psi^c_j(N \setminus B_k, v, \mathcal{B} \setminus \{B_k\})$ holds true.

Proof. We first consider the case that $|\mathcal{B}| = 1$. Then $\psi^c(N, v, \{N\}) = \phi(N, v)$ holds true by its definition. We obtain the desired result because of Proposition 2.

Next we consider the case $|\mathcal{B}| \geqq 2$. Let Q be the potential function for cooperative games with coalition structures. We denote $\psi^c(N, v, \mathcal{B})$ and $Q(N, v, \mathcal{B})$ simply by $\psi^c(N, \mathcal{B})$

and $Q(N, \mathcal{B})$ in this proof. Then by its definition,

$$
\begin{aligned}
\psi_i^c(N,\mathcal{B}) - \psi_i^c(N \setminus B_h, \mathcal{B} \setminus \{B_h\}) \\
&= Q(N,\mathcal{B}) - Q(N \setminus B_k, \mathcal{B} \setminus \{B_k\}) - Q(B_k \setminus \{i\}, \{B_k \setminus \{i\}\}) \\
&\quad - (Q(N \setminus B_h, \mathcal{B} \setminus \{B_h\}) - Q(N \setminus (B_k \cup B_h), \mathcal{B} \setminus \{B_k, B_h\}) - Q(B_k \setminus \{i\}, \{B_k \setminus \{i\})) \\
&= Q(N,\mathcal{B}) - Q(N \setminus B_k, B \setminus \{B_k\}) \\
&\quad - (Q(N \setminus B_h, \mathcal{B} \setminus \{B_h\}) - Q(N \setminus (B_k \cup B_h), \mathcal{B} \setminus \{B_k, B_h\})) \\
&= Q(N,\mathcal{B}) - Q(N \setminus B_h, \mathcal{B} \setminus \{B_h\}) \\
&\quad - (Q(N \setminus B_k, \mathcal{B} \setminus \{B_k\}) - Q(N \setminus (B_k \cup B_h), \mathcal{B} \setminus \{B_k, B_h\})) \\
&= Q(N,\mathcal{B}) - Q(N \setminus B_h, \mathcal{B} \setminus \{B_h\}) - Q(B_h \setminus \{j\}, \{B_h \setminus \{j\}\}) \\
&\quad - (Q(N \setminus B_k, \mathcal{B} \setminus \{B_k\}) - Q(N \setminus (B_k \cup B_h), \mathcal{B} \setminus \{B_k, B_h\}) - Q(B_h \setminus \{j\}, \{B_h \setminus \{j\}\})) \\
&= \psi_j^c(N,\mathcal{B}) - \psi_j^c(N \setminus B_k, \mathcal{B} \setminus \{B_h\}).
\end{aligned}
$$

So we have a desired result. $\square$

This proposition means that, by Theorem 3, for every $B_k \in \mathcal{B}$ and for every $B_h \in \mathcal{B}$,

$$\frac{\phi_k^w(M, v_{\mathcal{B}}) - \phi_k^w(M \setminus \{h\}, v_{\mathcal{B}})}{|B_k|} = \frac{\phi_h^w(M, v_{\mathcal{B}}) - \phi_h^w(M \setminus \{k\}, v_{\mathcal{B}})}{|B_h|}.$$

This is the special case of the properties of the weighted Shapley value that for $(N, v) \in \Gamma$, its weight $(w_i)_{i \in N}$, and for every $i, j \in N$,

$$\frac{\phi_i^w(N, v) - \phi_i^w(N \setminus \{j\}, v)}{w_i} = \frac{\phi_j^w(N, v) - \phi_j^w(N \setminus \{i\}, v)}{w_j}.$$

This property is shown in Hart and MasColell (1989).

Next theorem shows that the group balanced contribution is almost sufficient to characterize the collective value.

Theorem 6 *The collective value is the unique efficient solution function satisfying the following two properties:*

(i) Coincidence between the grand and the singleton coalition structure: For all $(N, v) \in \Gamma$, $\psi(N, v, \{N\}) = \psi(N, v, [N])$,

(ii) Group balanced contributions property.

Proof. We have shown that the collective value satisfies the efficiency and the group balanced contributions. It also satisfies the coincidence between the grand and the singleton coalition structure because it coincides with the Shapley value in both two cases. Therefore we will show the converse part.

Let ψ be a efficient solution function satisfying the group balanced contribution and the coincidence between the grand and the singleton. Fix $(N, v, \mathcal{B}) \in \Delta$. We first show that ψ coincides with the Shapley value when $|\mathcal{B}| = 1$ or n. When $|\mathcal{B}| = n$, the group balanced contribution coincides with the balanced contribution. Because of Theorem 5, $\psi(N, v, [N]) = \phi(N, v)$. Moreover, thanks to the Coincidence between the grand and the singleton coalition structure, we obtain that $\psi(N, v, \{N\}) = \psi(N, v, [N]) = \phi(N, v)$.

Next we show the following claims.

Claim 1: For all $B_k \in \mathcal{B}$,

$$\sum_{i \in B_k} \psi_i(N, v, \mathcal{B}) = |B_k|\, D_k P^w(M, v_{\mathcal{B}}) \tag{7}$$

where P^w is the weighted Potential function with weight vector $w = (w_k)_{k \in M}$ such that $w_k = |B_k|$ for each $k \in M$.

Let $(B_k, v, \{B_k\})$ be a subgame of $(N, v, \mathcal{B})$ to the coalition B_k. Then the left hand side of the condition (7) is

$$\sum_{i \in B_k} \psi_i(B_k, v, \{B_k\}) = v(B_k)$$

by efficiency of ψ. The right hand side of the condition (7) is

$$|B_k|\, D_k P^w(\{k\}, v_{\mathcal{B}}) = |B_k| \frac{v(B_k)}{|B_k|} = v(B_k).$$

by equation (3). Therefore the condition (7) holds true for any subgame $(B_k, v, \{B_k\})$ of $(N, v, \mathcal{B})$.

We assume that this condition is satisfied for any subgame $(N', v, \mathcal{B}')$ of $(N, v, \mathcal{B})$ such that $\mathcal{B}' \subsetneq \mathcal{B}, \mathcal{B}' \neq \emptyset$ and $N' = \cup_{B_k \in \mathcal{B}'} B_k$. We now show that it holds true for $(N, v, \mathcal{B})$.

Condition (7) is equivalent to

$$\sum_{i \in B_k} \psi_i(N, v, \mathcal{B}) = |B_k|(P^w(M, v_{\mathcal{B}}) - P^w(M \setminus \{k\}, v_{\mathcal{B}})).$$

Then we obtain that

$$P^w(M, v_{\mathcal{B}}) = \frac{\sum_{i \in B_k} \psi_i(N, v, \mathcal{B})}{|B_k|} + P^w(M \setminus \{k\}, v_{\mathcal{B}}).$$

We show that $\frac{\sum_{i\in B_k}\psi_i(N,v,\mathcal{B})}{|B_k|}+P^w(M\setminus\{k\},v_{\mathcal{B}})$ is constant for every $k\in M$. Take any $B_k\in\mathcal{B}$ and $B_h\in\mathcal{B}, B_k\neq B_h$. Then,

$$\frac{\sum_{i\in B_k}\psi_i(N,v,\mathcal{B})}{|B_k|}+P^w(M\setminus\{k\},v_{\mathcal{B}})-\left(\frac{\sum_{j\in B_h}\psi_j(N,v,\mathcal{B})}{|B_h|}+P^w(M\setminus\{h\},v_{\mathcal{B}})\right)$$

equals

$$\frac{\sum_{i\in B_k}\psi_i(N,v,\mathcal{B})}{|B_k|}-\frac{\sum_{j\in B_h}\psi_j(N,v,\mathcal{B})}{|B_h|}+\Big(P^w(M\setminus\{k\},v_{\mathcal{B}})-P^w(M\setminus\{h\},v_{\mathcal{B}})\Big). \qquad (8)$$

The bracketed terms of (8) equals

$$P^w(M\setminus\{k\},v_{\mathcal{B}})-P^w(M\setminus\{k,h\},v_{\mathcal{B}})-\big(P^w(M\setminus\{h\},v_{\mathcal{B}})-P^w(M\setminus\{k,h\},v_{\mathcal{B}})\big).$$

By the definition of operator D for the weighted potential function and the assumption,

$$\begin{aligned}&=D_hP^w(M\setminus\{k\},v_{\mathcal{B}})-D_kP^w(M\setminus\{h\},v_{\mathcal{B}})\\&=\frac{\sum_{j\in B_h}\psi_j(N\setminus B_k,v,\mathcal{B}\setminus\{B_k\})}{|B_h|}-\frac{\sum_{i\in B_k}\psi_i(N\setminus\{B_h\},v,\mathcal{B}\setminus\{B_h\})}{|B_k|}.\end{aligned}$$

Note that by the group balanced contributions, $\psi_i(N,v,\mathcal{B})-\psi_i(N\setminus\{B_h\},v,\mathcal{B}\setminus\{B_h\})=\psi_j(N,v,\mathcal{B})-\psi_j(N\setminus\{B_k\},v,\mathcal{B}\setminus\{B_k\})=$ constant for every $i\in B_k$ and for every $j\in B_h$. Substitute the above for the bracketed terms of (8), we conclude that (8) equals zero by group balanced contributions. Therefore for some real number K,

$$\frac{\sum_{i\in B_k}\psi_i(N,v,\mathcal{B})}{|B_k|}+P^w(M\setminus\{k\},v_{\mathcal{B}})=K$$

holds true for any $k\in M$.

Then by efficiency of ψ, we obtain that

$$v_{\mathcal{B}}(M)=\sum_{k\in M}\sum_{i\in B_k}\psi_i(N,v,\mathcal{B})=\sum_{k\in M}|B_k|(K-P^w(M\setminus\{k\},v_{\mathcal{B}}))$$

Therefore K is exactly the weighted potential function $P^w(M,v_{\mathcal{B}})$ because of its uniqueness (Theorem 2) and the claim is shown.

Next we show the following claim.

Claim 2: $\psi_i(N,v,\mathcal{B})=\bar{C}+\psi_i(B_k,v,\{B_k\})$ for every $i\in B_k$ where $\bar{C}$ is a constant real number.

We prove Claim 2 by the induction on the cardinality of $\mathcal{B}$. When $|\mathcal{B}|=1$, this is obvious because we simply put $\bar{C}=0$.

Assume that the claim holds true when the number of element in $\mathcal{B}$ is less that m ($m \geqq 2$). For $(N, v, \mathcal{B})$ such that $|\mathcal{B}| = m$, by group balanced contributions, given $B_h \in \mathcal{B}$,

$$\psi_i(N, v, \mathcal{B}) - \psi_i(N \setminus \mathcal{B}_h, v, \mathcal{B} \setminus \{B_h\}) = \bar{C}_1 \quad \text{for every } i \in B_k, B_k \neq B_h$$

By the assumption of the induction, the left hand side of the above equation is

$$\psi_i(N, v, \mathcal{B}) - (\bar{C}_2 + \psi_i(B_k, v, \{B_k\})).$$

Therefore we obtain that

$$\psi_i(N, v, \mathcal{B}) = \bar{C}_1 + \bar{C}_2 + \psi_i(B_k, v\{B_k\}) = \bar{C} + \psi_i(B_k, v, \{B_k\}).$$

This is desired result.

By Claim 1, we know that the summation of $\psi_i(N, v, \mathcal{B})$ over $i \in B_k$ is exactly $|B_k|\, D_k P^w(M, v_{\mathcal{B}}) = \phi_k^w(M, v_{\mathcal{B}})$. Then we conclude that

$$\bar{C} = \frac{\phi_k^w(M, v_{\mathcal{B}}) - \sum_{i \in B_k} \psi_i(B_k, v, \{B_k\})}{|B_k|} = \frac{\phi_k^w(M, v_{\mathcal{B}}) - v(B_k)}{|B_k|}$$

by efficiency of ψ. Therefore if $\psi_i(B_k, v, \{B_k\})$ is uniquely determined, $\psi(N, v, \mathcal{B})$ is also determined. However when $|\mathcal{B}| = 1$, we have shown that ψ equals the Shapley value ϕ. Hence we obtain

$$\psi_i(N, v, \mathcal{B}) = \frac{\phi_k^w(M, v_{\mathcal{B}}) - v(B_k)}{|B_k|} + \phi_i(B_k, v).$$

□

As in the proof of the above theorem, Coincidence between the Grand and the Singleton Coalition Structure is necessary in order to prove that the solution concept coincides with the Shapley value for $(N, v, \{N\})$. Therefore the following corollaries hold.

Corollary 1 *The collective value is the unique efficient solution function satisfying the following two properties:*

(i) $\psi(N, v, \{N\}) = \phi(N, v)$ *for all* $(N, v) \in \Gamma$.

(ii) Group balanced contributions property.

Corollary 2 *The collective value is the unique efficient solution function satisfying the following two properties:*

(i) For any $i, j \in N$*,* $\psi_i(N, v, \{N\}) - \psi_i(N\backslash\{j\}, v, \{N\backslash\{j\}) = \psi_j(N, v, \{N\}) - \psi_j(N\backslash\{i\}, v, \{N\backslash \{i\})$*.*
(ii) Group balanced contributions property.

4.2 Additivity

Shapley himself axiomatized the Shapley value by additivity axiom in his seminal paper (Shapley (1953b)). His idea of the proof is that each game can be decomposed into unanimity basis, the other axioms determines the value of the solution for each unanimity game and finally the additivity means that the summation of the value of the solution for each unanimity game multiplied by the corresponding coefficient is the same as the value of the solution for the original game. The collective value also satisfies the additivity and is axiomatized through this axiom. Before staging our second characterization, we have to define some notations and terminologies.

Let $(N, v, \mathcal{B})$. We say that $i \in N$ is a *null player* if $v(S \cup \{i\}) = v(S)$ for any $S \subseteq N \setminus \{i\}$ and $i \in N$ is a *dummy* if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for any $S \subseteq N \setminus \{i\}$. Clearly a null player is also dummy but the converse is not true. We say that $i \in N$ and $j \in$ are *symmetric* in N if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N \setminus \{i, j\}$ and $i \in N$ and $j \in$ are symmetric in T if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq T \setminus \{i, j\}$. For each $T \subseteq N$, we define a game u_T by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. A game (N, u_T) is called *T-unanimity game.* A coalition $T \subseteq N$ is called *partnership* in (N, v) if $v(S) = v(C \cup S)$ whenever $C \subsetneq T$ and $S \subseteq N \setminus T$. (Note that because S may be empty, the partnership of T implies that $v(C) = 0$ for all $C \subsetneq T$.)

Let ψ be a solution function on Δ and $(N, v, \mathcal{B}), (N, v', \mathcal{B}) \in \Delta$.

Theorem 7 *The collective value is the unique efficient solution function satisfying the following four.*

(i) Additivity: $\psi(N, v, \mathcal{B}) + \psi(N, v', \mathcal{B}) = \psi(N, v + v', \mathcal{B})$*.*

(ii) Equal Power of Partnership Members: If $T \subseteq N$ *is a partnership in* (N, v) *and* $M' = \{k \in M : B_k \cap T \neq \emptyset\}$ *is also a partnership in* $(M, v_{\mathcal{B}})$*, then* $\psi_i(N, v, \mathcal{B}) = \psi_j(N, v, \mathcal{B})$ *for any* $i, j \in T$*.*

(iii) Strong Restricted Equal Treatment (S-RETP): If $i \in B_k$ and $j \in B_k$ are symmetric in (B_k, v), then $\psi_i(N, v, \mathcal{B}) = \psi_j(N, v, \mathcal{B})$.

(iv) Coalition Structure-Null Player (CS-NP): If B_k is a dummy coalition (i.e., k is a dummy player in $(M, v_{\mathcal{B}})$) and $i \in B_k$ is a null player, then $\psi_i(N, v, \mathcal{B}) = 0$.

Before proving the theorem, we have to give some comments to the properties mentioned above. Equal Power of Partnership Members says that all the members of partnership T obtain the equal payoff if its projection on M is also a partnership in $(M, v_{\mathcal{B}})$. The Shapley value defined by $\psi^s(N, v, \mathcal{B}) = \phi(N, v)$ satisfies Equal Power of Partnership Members because all the players in T are symmetric in (N, v) and the Shapley value gives equal payoff to symmetric players.

Strong Restricted Equal Treatment and Coalition Structure-Null Player are the axioms introduced in Kamijo (2005) to characterize his two steps Shapley value. Strong Restricted Equal Treatment is stronger than Restricted Equal Treatment which both the Shapley value and the Owen's coalitional value satisfy. Coalition Structure-Null Player is weaker than the usual Null Player axiom. Therefore the Shapley value satisfies the all the properties except for Strong Restricted Equal Treatment and the Owen value does not satisfies Equal Power of Majority Members and Strong Restricted Equal.

First we show the following lemma.

Lemma 2 *Let $w \in \mathbb{R}^N_{++}$ be a weight vector of N. If T is a partnership in (N, v), then $\phi^w_i(N, v) : \phi^w_j(N, v) = w_i : w_j$ for all $i, j \in T$.*

Proof of Lemma 2. First we define a terminology used in this proof. Given a order $\sigma \in \Sigma(N)$, we say that coalition R is last at order σ if the set of last $|R|$ players at σ is R.

Given $\sigma \in \Sigma(N)$, the marginal contribution of player $i \in T$ at order σ, $m^\sigma_i(N, v)$ is 0 whenever i is not last in T at order σ because T is a partnership in (N, v). Fix $S \subseteq N \setminus T$ and put $R = N \setminus (T \cup S)$ and $r = |R|$. The marginal contribution of player $i \in T$ to coalition $S \cup T \setminus \{i\}$ equals to the marginal contribution of partnership T to coalition S, that is

$$v(S \cup T) - v(S \cup T \setminus \{i\}) = v(S \cup T) - v(S)$$

holds and this is irrelevant to the choice of player $i \in T$. Moreover

$$
\begin{aligned}
\text{Prob}(\{j \in N \,:\, \sigma(j) \leqq \sigma(i)\} = S \cup T) &= \text{Prob}(\text{ coalition } R \text{ is last at } \sigma) \times \frac{w_i}{\sum_{j \in S \cup T} w_j} \\
&= \sum_{\sigma \in \Sigma(R)} \Pi_{j=1}^{r} \frac{w_{\sigma(j)}}{\sum_{k \in T \cup S} w_k + \sum_{k=1}^{j} w_{\sigma(k)}} \\
&\quad \times \frac{w_i}{\sum_{j \in S \cup T} w_j} \\
&=: pr(S)\, w_i
\end{aligned}
$$

holds. Therefore for any $i \in T$,

$$
\phi_i^w(N, v) = \sum_{S \subseteq N \setminus T} pr(S)\, w_i \,(v(S \cup T) - v(S))
$$

holds and the lemma is shown. □

Proof of Theorem 7. It is easily shown that the collective value satisfies all the axioms except for Equal Power of Partnership Members.

Let $T \subseteq N$ be a partnership in (N, v) and $M' = \{k \in M \;:\; B_k \cap T \neq \emptyset\}$ be also a partnership in $(M, v_{\mathcal{B}})$. When $|M'| \geqq 2$, let $k \in M'$. Then $v(S \cup C) = v(S)$ for any $S \subseteq B_k \setminus T$ and $C \subseteq T \cap B_k \subsetneq T$. Therefore $\phi_i(B_k, v, \{B_k\}) = 0$ for any $i \in T \cap B_k$. Since $w_k = |B_k|$ for any $k \in M$, $\frac{\phi_k^w(M, v_{\mathcal{B}})}{|B_k|} = \frac{\phi_h^w(M, v_{\mathcal{B}})}{|B_h|}$ for any $k, h \in M'$ by Lemma 2. By the partnership of M' in $(M, v_{\mathcal{B}})$, $v_B(k) = v(B_k) = 0$ for each $k \in M'$. Thus $\psi_i^c(N, v, \mathcal{B}) = \psi_j(N, v, \mathcal{B})$ holds for any $i, j \in T$. When $|M'| = 1$, since $T \subseteq B_k, k \in M'$ is a partnership in (N, v), all the players in T are symmetric in (N, v) and of course they are symmetric in (B_k, v) Thus $\psi_i(B_k, v, \{B_k\})$ is constant over $i \in T$. Therefore the collective value satisfies Equal Power of Partnership Members.

Next we now show the converse part. Let ψ be a solution function on Δ which satisfies all the axioms. $(N, v, \mathcal{B}) \in \Delta$. Since ψ satisfies Additivity, it is sufficient to show that $\psi(N, cu_T, \mathcal{B})$ is uniquely determined, where $c \in \mathbb{R}$ and cu_T is a scalar multiple of u_T by c. Let $D = \{k \,:\, B_k \in \mathcal{B}, B_k \cap T \neq \emptyset\}$. Since $B_k \in \mathcal{B}$, $k \notin D$ is a dummy coalition and $i \in B_k$ is a null player, $\psi_i(N, cu_T, \mathcal{B}) = 0$ by CS-NP. Therefore efficiency means that $\sum_{k \in D} \sum_{i \in B_k} \psi_i(N, cu_T, \mathcal{B}) = c$.

Clearly T is a partnership in (N, cu_T) and D is also a partnership in $(M, (u_T)_{\mathcal{B}})$. Therefore $\psi_i(N, cu_T, \mathcal{B}) = \psi(N, cu_T, \mathcal{B})$ for all $i, j \in T$ by Equal Power of Partnership Members.

Case a: $|D| = 1$. Let $k \in D$. Since B_k is a dummy coalition and $i \in B_k \setminus T$ is a null player, $\psi_i(N, cu_T, \mathcal{B}) = 0$ by CS-NP. Therefore $\psi_i(N, cu_T, \mathcal{B}) = \frac{c}{|T|}$.

Case b: $|D| \geqq 2$. For each $B_k \in \mathcal{B}$, $k \in D$, $i \in B_k$ and $j \in B_k$ are symmetric in (B_k, v). Therefore $\psi_i(N, cu_T, \mathcal{B}) = \psi_j(N, cu_T, \mathcal{B})$ by S-RETP. Moreover $\psi_i(N cu_T, \mathcal{B}) = \psi_j(N, cu_T, \mathcal{B})$ for $i \in T \cap B_k$ and for $j \in T \cap B_h$. As a result, for any $i \in \cup_{k \in D} B_k$, $\psi_i(N, cu_T, \mathcal{B}) = \frac{c}{\sum_{h \in D} |B_h|}$.
□

We have to check the independence of each axiom from the others. Efficiency and additivity are obvious. The Shapley value ($\psi^s(N, v, \mathcal{B}) := \phi(N, v)$ for all $(N, v, \mathcal{B}) \in \Delta$) satisfies all the axioms except for Strong Restricted Equal Treatment. The solution defined in Kamijo (2005) ($\psi_i^{ss}(N, v, \mathcal{B}) = \frac{\phi_k(M, v_{B_k}) - v(B_k)}{|B_k|} + \phi_i(B_k, v)$ for $B_k \in \mathcal{B}$ and for $i \in B_k$) fulfills all the axioms other than Equal Power of Partnership Members. The egalitarian solution defined by $\psi^e(N, v, \mathcal{B}) := \frac{v(N)}{|N|}$ satisfies all the axioms except for Coalition Structure-Null Player.

Finally, by the proof of Theorem 7, we have the following proposition.

Proposition 4 *Let $T \subseteq N$, $c \in \mathbb{R}$ and $D = \{k \in M : B_k \in \mathcal{B}, B_k \cap T \neq \emptyset\}$. The collective value for $(N, cu_T, \mathcal{B})$ is: if $|D| \geqq 2$,*

$$\psi_i^c(N, cu_T, \mathcal{B}) = \begin{cases} \frac{c}{\sum_{h \in D} |B_h|} & i \in \bigcup_{h \in D} B_h \\ 0 & \text{otherwise} \end{cases}$$

and if $|D| = 1$,

$$\psi_i^c(N, cu_T, \mathcal{B}) = \begin{cases} \frac{c}{|T|} & i \in T \\ 0 & \text{otherwise} \end{cases}$$

where cu_T is defined by $(cu_T)(S) = cu_T(S)$ for all $S \subseteq N$.

5 Remarks and discussions

5.1 Two levels reactions

The technical reason of using two levels reaction of the other player to the deviation in defining our marginal contribution operator is that unless we consider the first level reaction, every finding is absorbed into the Hart and MasColell (1989)' result. The reason is following. If we define a marginal contribution operator D_i by

$$D_i Q(N, v, \mathcal{B}) = Q(N, v, \mathcal{B}) - Q(N \setminus \{i\}, v, \mathcal{C})$$

where $\mathcal{C}$ is any coalition structure on $N \setminus \{i\}$ and impose the condition that $\sum_{i \in N} D_i Q(N, v, \mathcal{B}) = v(N)$, $Q(N, v, \mathcal{B})$ must be the Hart and MasColell (1989)' potential function $P(N, v)$ (Please remind that equation (2) and the calculation procedure of the potential function). This result holds for any choice of the coalition structure C on $N \setminus \{i\}$. Therefore it is meaningless to consider the reaction of the other players against the deviation in this case.

Winter (1992) avoids this difficulty by using the vector-valued function as the candidate of the potential for cooperative games with coalition structures. Let $\mathcal{B} = \{B_1, \ldots, B_m\}$ be a coalition structure. Then Winter (1992)'s potential function $Q^W = (Q_1^W, \ldots, Q_m^W)$ is an m dimensional vector-valued function and the marginal contribution of player $i \in B_k \in \mathcal{B}$ is defined by:

$$D_i Q^W(N, v, \mathcal{B}) = Q_k^W(N, v, \mathcal{B}) - Q_k^W(N \setminus \{i\}, v, \mathcal{B} \setminus \{B_k\} \cup \{B_k \setminus i\}).$$

He shows that the function which satisfies the two condition that $\sum_{i \in N} D_i Q(N, v, \mathcal{B}) = v(N)$ and $\sum_{i \in B_k} D_i Q(N, v, B) = D_k Q(M, v_{\mathcal{B}}, \{M\})$ is uniquely determined and $D_i Q(N, v, \mathcal{B})$ is player i's Owen's coalitional value (Owen (1977)).

5.2 Other weighted coalition structure values

An extension of the weighted Shapley value to TU games with coalition structure is first considered by Levy and McLean (1989). In their approach, the weight for each coalition is exogenously given, in contrast to ours in which the weight is endogenously determined by the solution itself ($w_k = |B_k|$ for all $B_k \in \mathcal{B}$). Even if we choose size-relevant weight for each coalition, the weighted coalition structure value of Levy and McLean (1989) is different from the collective value.

Let $(N, v, \mathcal{B}) \in \Delta$. We define a game among coalition where $B_k \in \mathcal{B}$ is replaced by a subcoalition $T \subset B_k$ by $(M, v_{\mathcal{B},T})$, where for any $H \subset M$, $v_{B,T}(H) = v(\bigcup_{h \in H} B_h \cup T)$ if $k \in H$ and $v_{\mathcal{B},T}(H) = v(\bigcup_{h \in H} B_h)$ otherwise. Given the size-relevant weight vector $w \in \mathbb{R}^M_{++}$, where $w_k = |B_k|$ for any $B_k \in \mathcal{B}$, the weighted coalition structure value of Levy and McLean (1989) with size-relevant weight is defined as follows. For any $B_k \in \mathcal{B}$ and for any $i \in B_k$,

$$\psi_i^{LM,w}(N, v, \mathcal{B}) = \phi_i(B_k, v_k^*)$$

where $v_k^*(T) = \phi_k^w(M, v_{\mathcal{B},T})$ for any $T \subset B_k$. Therefore due to the above definition, not only the distribution for each coalition but also the power of the subcoalition are measured by the weighted Shapley value with the size relevant weight. However, there is an unreasonable figure in this definition. Why does subcoalition T have the same weight as the original coalition

B_k? To overcome this point, Vidal-Puga (2005b) defines other size-relevant weighted coalition structure value as follows. For any $B_k \in \mathcal{B}$ and for any $i \in B_k$,

$$\psi_i^V(N, v, \mathcal{B}) = \phi_i(B_k, v_k^{**})$$

where $v_k^{**}(T) = \phi_k^{w'}(M, v_{\mathcal{B},T})$ for any $T \subset B_k$ and $w'_k = |T|$ and $w'_h = |B_h|$ for any $h \in M, h \neq k$.

Following the above way to define the solutions, the collective value is expressed as follows. For any $B_k \in \mathcal{B}$ and for any $i \in B_k$,

$$\psi_i^c(N, v, \mathcal{B}) = \phi_i(B_k, v_k)$$

where $v_k(B_k) = \phi_k^w(M, v_{\mathcal{B}})$ and $v_k(T) = v(T)$ for any $T \subsetneq B_k$. The definition of v_k illustrates the setting considered by the collective value, where all the agents in B_k are key players to cooperate with outside members (See also Kamijo (2005)).

5.3 Coalition formation under unanimity games

It is worth to mention that what coalition structure seems to be occurred by a coalition formation game under the collective value. Since we consider the reaction of the outside members against the deviation when we define the marginal contribution operator and the resulting potential function leads to the collective value, what the stable coalition structure is under the payoff distribution of the collective value is an interesting question. We will answer this question in the unanimity games.

In a general interpretation of a coalition structure, players form a coalition to better off their bargaining power. Therefore given a solution of cooperative game with coalition structures ψ, we expect the following condition:

$$\sum_{i \in B_k} \psi_i(N, v, \mathcal{B}) + \sum_{i \in B_h} \psi_i(N, v, \mathcal{B}) \leqq \sum_{i \in B_k \cup B_h} \psi_i(N, v, \mathcal{B}') \tag{9}$$

where $B_k \in \mathcal{B}$ and $B_h \in \mathcal{B}$ and $\mathcal{B}' = \mathcal{B} \setminus \{B_k, B_h\} \cup \{B_k \cup B_h\}$. However solutions often fail to satisfy this condition. In fact, the Owen coalitional value and the value defined in Kamijo (2004) do not satisfy it. This problem is known as Harsanyi paradox (See Harsanyi (1977) and Vidal-Puga (2005b)). The collective value satisfies equation (9) because of Proposition 4. Therefore *to consider the reaction of the outside members against the deviation in defining a payoff allocation resolves the Harsanyi paradox.*

Again by Proposition 4, players in T obtain the highest payoff when anyone in T do not cooperate with members in $N \backslash T$. Therefore either myopic (*core*) or farsighted view (*equilibrium binding agreements* defined by Ray and Vohra (1997) and sequentially stable coalition structures proposed by Funaki and Yamato (2005)) induces the set $\{\mathcal{B} : |B_k \setminus T| = 0 \text{ for any } B_k \in \mathcal{B}\}$ as the candidates of stable coalition structures. Thus, all the players in T obtain the highest payoff $\frac{1}{|T|}$ in stable outcomes.

5.4 Implementability

A non-cooperative bargaining model which achieves the collective value in subgame perfect equilibrium is considered in Kamijo (2006). It is based on the bidding mechanism introduced by Perez-Castrillo and Wettstein (2001) which implements the Shapley values in monotonic games. The other coalition structure values are also implemented by the modified versions of the bidding mechanism (the Owen's coalitional value for Vidal-Puga and Bergantinos (2003), Vidal-Puga (2005a) and Kamijo (2006), the value defined in Kamijo (2005) for Kamijo (2004) and Kamijo (2006) and ψ^V for Vidal-Puga (2005b)).

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