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Balanced Contributions Properties and Values for Cooperative Games

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Abstract

The balanced contributions property introduced by Myerson (1980; International Journal of Game Theory 9, 169-182) characterizes the Shapley value as a unique efficient and one-point solution for cooperative games. By replacing the reduced game in the property with the other types of reduced games, variations of the property are obtained. The new balanced contributions properties characterize other efficient and one-point solutions for cooperative games such as the egalitarian value, the CIS value, and the ENSC value. The characterizations lead to non-cooperative implementations of those values. In addition, since all of those values are linear, any convex combination of the values, such as the α-egalitarian Shapley value and the α-consensus value, and a value in a class of equal surplus sharing solutions are characterized and implemented in the similar way.

JEL classification: C71, C72

Keywords: balanced contributions property, egalitarian value, CIS value, ENSC value, axiomatization, implementation,

1 Introduction

One of the important criteria in collective decision-making problems is fairness. In cooperative game theory, the widely used fairness criterion is the balanced contributions property introduced by Myerson (1980). The property asserts that "for any two players i and j, i's contribution to j should be

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equal to j's contribution to i," hence, the property is interpreted as a kind of the fairness property among players.

Player i's contribution to j is evaluated by the difference between the two values which j receives in the original game and in the reduced game induced from i's withdrawal. In the balanced contributions property, i's withdrawal induce the restricted game on the set of players except i. However, a restriction of the game on the set of remaining players is slightly different from the situation induced from a player's withdrawal. Consider the situation in which people discuss how to allocate the goods that have already been produced. If someone leaves the situation with obtaining some goods, the remaining players discuss how to allocate the remaining goods rather than the amount of goods remaining players can produce.

Kongo et al. (2007) consider such situations and define the *marginal games* in which players being withdrawn cooperate to remaining players with obtaining the value they can generate by themselves. By using the marginal games, they introduce a variation of the balanced contributions property and the new property characterizes the Shapley value. Since the original balanced contributions property also characterizes the Shapley value, the Shapley value is an efficient and one-point solution characterized by the two different fairness criteria.

In this paper, by considering three other reduced games, we introduce three variations of the balanced contributions properties. Each of the three properties characterizes three well-known values for cooperative games: the egalitarian value, the CIS value, and the ENSC value. Thus, all of those values are seem to be "fair" solutions as well as the Shapley value and the differences among them come from the differences in evaluation of player's contribution to the other.

The paper is organized as follows. Notations and definitions are presented in Section 2. The variations of the balanced contributions property and characterizations of the egalitarian value, the CIS value, and the ENSC value are given in Section 3. The non-cooperative implementations of the values are presented in Section 4. Further generalization are included in Section 5.

2 Preliminaries

A pair (N, v) is a *cooperative game (with transferable utility)* where $N \subseteq \mathbb{N}$ is a finite set of *players* and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a *characteristic function*. Let $|N| = n$ where $|N|$ represents the cardinality of N. A subset

S of N is called a *coalition*. For any $S \subseteq N$, $v(S)$ represents the worth of the coalition. For simplicity, each singleton is represented as i instead of $\{i\}$ when there exist no fear of confusion.

Let $\mathcal{G}$ be a set of all cooperative games. A *value* on $\mathcal{G}$ is a mapping that assigns to each $(N, v) \in \mathcal{G}$ an n-dimensional vector $(x_i)_{i\in N}$ that satisfies $\sum_{i\in N} x_i = v(N)$.

One of the well-known values on the class of cooperative games is the Shapley value introduced by Shapley (1953). Given $(N, v) \in \mathcal{G}$, the *Shapley value* $Sh(N, v) = (Sh_i(N, v))_{i\in N}$ is defined as follows: For each $i \in N$,

$$Sh_i(N, v) = \sum_{S \subseteq N\backslash i} \frac{|S|!(n-1-|S|)!}{n!} \left(v(S \cup i) - v(S)\right).$$

Other well-known values on the class of cooperative games is the egalitarian value, the CIS value, and the ENSC value (see Driessen and Funaki (1991)). Given $(N, v) \in \mathcal{G}$,

- the *egalitarian value* $EG(N, v) = (EG_i(N, v))_{i\in N}$ is defined as follows: For each $i \in N$,
$$EG_i(N, v) = \frac{v(N)}{n}.$$
- The *CIS value (Center-of-gravity of the Imputation-Set value)*, $CIS(N, v) = (CIS_i(N, v))_{i\in N}$ is defined as follows: For each $i \in N$,
$$CIS_i(N, v) = \frac{v(N) - \sum_{j\in N} v(j)}{n} + v(i).$$
- The *ENSC value (Egalitarian Non-Separable Contribution value)*, $ENSC(N, v) = (ENSC_i(N, v))_{i\in N}$ is defined as follows: For each $i \in N$,
$$ENSC_i(N, v) = \frac{v(N) - \sum_{j\in N}(v(N) - v(N\backslash j))}{n} + v(N) - v(N\backslash i).$$

Let $S \subseteq N$. The *S-marginal game* $(N\backslash S, v^S)$ of (N, v) is the game that assigns to each coalition $T \subseteq N\backslash S$ the worth of $T \cup S$ minus the worth of S, that is, for each $T \subseteq N\backslash S$,

$$v^S(T) = v(S \cup T) - v(S).$$

In the S-marginal game, any non-empty subset of $N\backslash S$ can win the cooperation of S by paying the value $v(S)$ to S

The S-*volunteer game* $(N\backslash S, \hat{v}^S)$ is the game that assigns to each coalition $T \subseteq N\backslash S$ with $T \neq \emptyset$ the worth of $T \cup S$ and to the empty set 0, that is, for each $T \subseteq N\backslash S$,

$$\hat{v}^S(T) = \begin{cases} v(S \cup T) & \text{if } T \neq \emptyset \\ 0 & \text{if } T = \emptyset. \end{cases}$$

In the S-volunteer game, any non-empty subset of $N\backslash S$ can win the cooperation of S by paying nothing to S.

The *projection* S-*marginal game* $(N\backslash S, \bar{v}^S)$ is the game that assigns to $N\backslash S$ the worth of N minus the worth of S and each coalition $T \subsetneq N\backslash S$ the worth of the coalition, that is, for each $T \subseteq N\backslash S$,

$$\bar{v}^S(T) = \begin{cases} v(N) - v(S) & \text{if } T = N\backslash S \\ v(T) & \text{if } T \neq N\backslash S. \end{cases}$$

In the projection S-marginal game, only the set $N\backslash S$ can win the cooperation of S by paying $v(S)$ to S and any proper subset of $N\backslash S$ cannot win any cooperation of S. This game is equivalent to the *projection reduced game* of (N, v) with respect to $x \in \mathbb{R}^n$ and $S \subseteq N$ if $\sum_{i \in S} x_i = v(S)$.

The *complement* S-*marginal game* $(N\backslash S, \tilde{v}^S)$ is the game that assigns to each coalition $T \subseteq N\backslash S$ with $T \neq \emptyset$ the worth of $T \cup S$ minus S's marginal contribution to N and to the empty set 0, that is, for each $T \subseteq N\backslash S$,

$$\tilde{v}^S(T) = \begin{cases} v(S \cup T) - (v(N) - v(N\backslash S)) & \text{if } T \neq \emptyset \\ 0 & \text{if } T = \emptyset. \end{cases}$$

In the complement S-marginal game, each non-empty subset of $N\backslash S$ can win the cooperation of S by paying $v(N) - v(N\backslash S)$ to S. This game is equivalent to the *complement reduced game* of (N, v) with respect to $x \in \mathbb{R}^n$ and $S \subseteq N$, introduced by Moulin (1985), if $\sum_{i \in S} x_i = v(N) - v(N\backslash S)$.

3 Balanced contributions properties

Let φ be a value for cooperative games. Myerson (1980) introduces the following *balanced contributions property.*

Balanced contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, v|_{N\backslash j}) = \varphi_j(N, v) - \varphi_j(N\backslash i, v|_{N\backslash i}),$$

where $v|_{N\backslash k} : 2^{N\backslash k} \to \mathbb{R}$ is defined as $v|_{N\backslash k}(S) = v(S)$ for $S \subseteq N\backslash k$ with $k = i, j$.

The above property asserts that "for any two players i and j, i's contribution to j should be equal to j's contribution to i," and i's contribution to j is evaluated by the difference between the two values that j receives in the original game and in the reduced game. As we mentioned in the previous section, there are varieties of the reduced games; hence there are variations of the evaluations of each player's contributions to the other.

By using the marginal games, Kongo et al. (2007) present the following variation of the balanced contributions property.

Balanced M-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, v^j) = \varphi_j(N, v) - \varphi_j(N\backslash i, v^i).$$

In the above property, the marginal games are used instead of the restriction of original games. Kongo et al. (2007) show that the above property characterize the Shapley value.

By using the other three games introduced in the previous section, three variations of the balanced contributions property is presented.

Balanced V-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, \hat{v}^j) = \varphi_j(N, v) - \varphi_j(N\backslash i, \hat{v}^i).$$

Balanced PM-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, \bar{v}^j) = \varphi_j(N, v) - \varphi_j(N\backslash i, \bar{v}^i).$$

Balanced CM-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, \tilde{v}^j) = \varphi_j(N, v) - \varphi_j(N\backslash i, \tilde{v}^i).$$

We obtain the following.

Lemma 1. *(i) The egalitarian value satisfies the balanced V-contributions property.*

(ii) The CIS value satisfies the balanced PM-contributions property.

(iii) The ENSC value satisfies the balanced CM-contributions property.

Proof. For each $(N,v) \in \mathcal{G}$, in the case of $|N| = 1$, they are obvious. Let $|N| \geq 2$ and for any $i, j \in N$ with $i \neq j$:

For (i);

$$EG_i(N,v) - EG_j(N,v) = 0,$$

and

$$EG_i(N\backslash j, \hat{v}^j) - EG_j(N\backslash i, \hat{v}^i) = \frac{\hat{v}(N\backslash j)}{n-1} - \frac{\hat{v}(N\backslash i)}{n-1} = \frac{v(N)}{n-1} - \frac{v(N)}{n-1} = 0.$$

For (ii);

$$CIS_i(N,v) - CIS_j(N,v) = v(i) - v(j),$$

and

$$\begin{aligned}
&CIS_i(N\backslash j, \bar{v}^j) - CIS_j(N\backslash i, \bar{v}^i) \\
&= \frac{\bar{v}^j(N\backslash j) - \sum_{k \in N\backslash j} \bar{v}^j(k)}{n-1} + \bar{v}^j(i) - \frac{\bar{v}^i(N\backslash i) - \sum_{k \in N\backslash i} \bar{v}^i(k)}{n-1} - \bar{v}^i(j) \\
&= \frac{v(N) - \sum_{k \in N} v(k)}{n-1} + v(i) - \frac{v(N) - \sum_{k \in N} v(k)}{n-1} - v(j) = v(i) - v(j).
\end{aligned}$$

For (iii);

$$ENSC_i(N,v) - ENSC_j(N,v) = -v(N\backslash i) + v(N\backslash j),$$

and

$$\begin{aligned}
&ENSC_i(N\backslash j, \tilde{v}^j) - ENSC_j(N\backslash i, \tilde{v}^i) \\
&= \frac{\tilde{v}^j(N\backslash j) - \sum_{k \in N\backslash j}(\tilde{v}^j(N\backslash j) - \tilde{v}^j(N\backslash \{j,k\}))}{n-1} + \tilde{v}^j(N\backslash j) - \tilde{v}^j(N\backslash \{j,i\}) \\
&\quad - \frac{\tilde{v}^i(N\backslash i) - \sum_{k \in N\backslash i}(\tilde{v}^i(N\backslash i) - \tilde{v}^i(N\backslash \{i,k\}))}{n-1} - \tilde{v}^i(N\backslash i) + \tilde{v}^i(N\backslash \{i,j\}) \\
&= \frac{v(N\backslash j) - \sum_{k \in N\backslash j}(v(N\backslash j) - v(N\backslash k) + v(N) - v(N\backslash j))}{n-1} \\
&\quad + v(N\backslash j) - v(N\backslash i) + v(N) - v(N\backslash j) \\
&\quad - \frac{v(N\backslash i) - \sum_{k \in N\backslash i}(v(N\backslash i) - v(N\backslash k) + v(N) - v(N\backslash i))}{n-1} \\
&\quad - v(N\backslash i) + v(N\backslash j) - v(N) + v(N\backslash i)
\end{aligned}$$

$$= \frac{\sum_{k\in N} v(N\backslash k) - (n-1)v(N)}{n-1} - v(N\backslash i) + v(N)$$
$$- \frac{\sum_{k\in N}(v(N\backslash k) - (n-1)v(N)}{n-1} + v(N\backslash j) - v(N)$$
$$= -v(N\backslash i) + v(N\backslash j). \quad \square$$

Moreover, each of the three properties characterizes the egalitarian value, the CIS value and the ENSC value, respectively.

Theorem 1. *(i) The egalitarian value is the unique value which satisfies the balanced V-contributions property.*

(ii) The CIS value is the unique value which satisfies the balanced PM-contributions property.

(iii) The ENSC value is the unique value which satisfies the balanced CM-contributions property.

Proof. We show only (ii), since (i) and (iii) are shown in the similar manner.

By Lemma 1, it is sufficient to show the uniqueness of the value satisfying the property. We use the induction with respect to the number of players. Let φ be a value on the class of cooperative games. In the case of $|N| = 1$, $\varphi_i(N, v) = v(i) = CIS_i(N, v)$ for $i \in N$. If $|N| = 2$, by the balanced PM-contributions property,

$$\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N\backslash j, \bar{v}^j) - \varphi_j(N\backslash i, \bar{v}^i) = -v(j) + v(i).$$

Together with $\varphi_i(N, v) + \varphi_j(N, v) = v(N)$, the above equality implies

$$\varphi_i(N, v) = \frac{v(N) - v(j) - v(i)}{2} + v(i) = CIS_i(N, v),$$

and

$$\varphi_j(N, v) = \frac{v(N) - v(i) - v(j)}{2} + v(j) = CIS_j(N, v).$$

Let $n \geq 2$ and suppose $\varphi = CIS$ in case of there are less than n players. Consider the case of n players. Fix $i \in N$; by the balanced PM-contributions property and the induction hypothesis, for any $j \in N\backslash i$,

$$\begin{aligned}\varphi_i(N, v) - \varphi_j(N, v) &= \varphi_i(N\backslash j, \bar{v}^j) - \varphi_j(N\backslash i, \bar{v}^i)\\ &= CIS_i(N\backslash j, \bar{v}^j) - CIS_j(N\backslash i, \bar{v}^i)\\ &= CIS_i(N, v) - CIS_j(N, v).\end{aligned}$$

Summing up the above equalities over $j \in N \backslash i$, we obtain

$$(n-1)\varphi_i(N,v) - \sum_{j \neq i} \varphi_j(N,v) = (n-1)CIS_i(N,v) - \sum_{j \neq i} CIS_j(N,v).$$

Together with $\sum_{k \in N} \varphi_k(N,v) = v(N) = \sum_{k \in N} CIS_k(N,v)$, the above equality implies

$$n\varphi_i(N,v) - v(N) = nCIS_i(N,v) - v(N).$$

Since $n \geq 2$, $\varphi_i(N,v) = CIS_i(N,v)$. For any $j \neq i$, $\varphi_j(N,v) = CIS_j(N,v)$ is shown in the same manner. Hence $\varphi = CIS$ in the case of there are n players. □

By the result, recursive representations of the egalitarian value, the CIS value, and the ENSC value are obtained as follows:

Proposition 1. *For each* $(N,v) \in \mathcal{G}$ *and any* $i \in N$*,*

(i) $EG_i(N,v) = \frac{1}{n} \sum_{j \neq i} EG_i(N \backslash j, \hat{v}^j)$,

(ii) $CIS_i(N,v) = \frac{1}{n} v(i) + \frac{1}{n} \sum_{j \neq i} CIS_i(N \backslash j, \bar{v}^j)$, *and*

(iii) $ENSC_i(N,v) = \frac{1}{n}(v(N) - v(N \backslash i)) + \frac{1}{n} \sum_{j \neq i} ENSC_i(N \backslash j, \tilde{v}^j)$,

Proof. We show only (iii), since (i) and (ii) are shown in the similar manner.

By the balanced CM-contributions property of the ENSC value, for any $i, j \in N$ with $i \neq j$,

$$ENSC_i(N \backslash j, \tilde{v}^j) = ENSC_i(N,v) - ENSC_j(N,v) + ENSC_j(N \backslash i, \tilde{v}^i).$$

Summing up the above equality over $j \in N \backslash i$ and divided by n, we obtain

$$\begin{aligned}
&\frac{1}{n} \sum_{j \neq i} ENSC_i(N \backslash j, \tilde{v}^j) \\
&\quad = \frac{n-1}{n} ENSC_i(N,v) - \frac{1}{n} \sum_{j \neq i} ENSC_j(N,v) + \frac{1}{n} \sum_{j \neq i} ENSC_j(N \backslash i, \tilde{v}^i) \\
&\quad = \frac{n-1}{n} ENSC_i(N,v) - \frac{1}{n} \left(v(N) - ENSC_i(N,v)\right) + \frac{1}{n} \sum_{j \neq i} ENSC_j(N \backslash i, \tilde{v}^i) \\
&\quad = ENSC_i(N,v) - \frac{1}{n} v(N) + \frac{1}{n} \tilde{v}^i(N \backslash i) \\
&\quad = ENSC_i(N,v) - \frac{1}{n} v(N) + \frac{1}{n} v(N \backslash i).
\end{aligned}$$

Rearranging the above equality, we obtain the desired result. □

4 Implementation

In this section, given a cooperative game, we consider three non-cooperative games each of which implements the egalitarian value, the CIS value, and the ENSC value of the cooperative game, respectively, as equilibrium payoffs. In those non-cooperative games, variations of marginal games and the recursive formulas we mentioned in the paper play important roles.

Given a cooperative game $(N, v) \in \mathcal{G}$, the non-cooperative game $\hat{\Gamma}(N, v)$ is defined in the following recursive manner.

In case $|N| = 1$, player $i \in N$ obtains $v(i)$ and the game is over.

Assume that the non-cooperative game is known when there are less than n players. We define the case where there are n players.

t=1 Each player $i \in N$ makes *bids* $b^i_j \in \mathbb{R}$ for every player $j \neq i$.

For each $i \in N$, the *net bid* B^i is the sum of the bids he made minus the sum of the bids the others made to him, that is, $B^i = \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i$. Let $\beta = argmax_i B^i$, where in the case of multiple maximizers, one of them is randomly chosen. The chosen player β pays b^β_j to every player $j \neq \beta$.

t=2 Player β makes an offer $x^\beta_j \in \mathbb{R}$ to every player $j \in N \backslash \beta$.

t=3 Players in $N \backslash \beta$ respond to the offer in a sequential manner, say $(j_1, \ldots, j_{n-1})$. An order of the players makes no matter. Response is either "accept it" or "reject it".

In case player j_h accepts the offer, the next player j_{h+1} responds to it. If every j_h accepts the offer, the players come to an agreement. If there is some rejection, an agreement is not reached.

When an agreement is reached, proposer β pays the proposed payoff x_j for any $j \in N \backslash \beta$ in return for obtaining the value of their total cooperation, $v(N)$. Thus, the payoff for responder j is

$$b^\beta_j + x^\beta_j$$

and the payoff for proposer β is

$$v(N) - \sum_{j \neq \beta} b^\beta_j - \sum_{j \neq \beta} x^\beta_j.$$

Then the game is over.

On the other hand, when an agreement is not reached, the proposer leaves the game with obtaining nothing and the remaining players $N \backslash \beta$ continue the non-cooperative game $\hat{\Gamma}(N \backslash \beta, \hat{v}^{\beta})$.

Non-cooperative games $\bar{\Gamma}(N, v)$ and $\tilde{\Gamma}(N, v)$ are defined almost the same as the game $\hat{\Gamma}(N, v)$. The difference among them is that, when an agreement is not reached at t=3, in $\bar{\Gamma}(N, v)$, the proposer β leaves the game with obtaining $v(\beta)$ and remaining players continue the non-cooperative game $\bar{\Gamma}(N \backslash \beta, \bar{v}^{\beta})$, and in $\tilde{\Gamma}(N, v)$, the proposer β leaves the game with obtaining $v(N) - v(N \backslash \beta)$ and remaining players continue the non-cooperative game $\tilde{\Gamma}(N \backslash \beta, \tilde{v}^{\beta})$. These non-cooperative games are variations of a non-cooperative game $\Gamma(N, v)$ in Kongo et al. (2007) and are inspired by *bidding mechanisms* presented in Pérez-Castrillo and Wettstein (2001).

Kongo et al. (2007) show that $\Gamma(N, v)$ implements the Shapley value for any $(N, v) \in \mathcal{G}$. Similarly, we obtain the following result.

Theorem 2. *For any* $(N, v) \in \mathcal{G}$,

(i) $\hat{\Gamma}(N, v)$ *produces the egalitarian value payoff in any subgame perfect equilibrium (henceforth, SPE),*

(ii) $\bar{\Gamma}(N, v)$ *produces the CIS value payoff in any SPE, and*

(iii) $\tilde{\Gamma}(N, v)$ *produces the ENSC value payoff in any SPE.*

Proof. We prove only (i), since (ii) and (iii) are shown in the similar way.

The proof proceeds by induction with respect to the number of players. If $|N| = 1$, the egalitarian value is equal to the value of stand-alone coalition; hence, the theorem holds. Assume that the theorem holds in case there are less than n players and consider the case when there are n players.

First, we show that there exists an SPE whose payoff coincides with the egalitarian value of the game (N, v). Consider the following strategy for each player.

t=1 Each player $i \in N$ announces $b^i_j = EG_j(N, v) - EG_j(N \backslash i, \hat{v}^i)$ for every $j \neq i$.

t=2 A proposer β offers $x^{\beta}_j = EG_j(N \backslash \beta, \hat{v}^{\beta})$ for every $j \in N \backslash \beta$.

t=3 A responder j accepts the offer if $x^{\beta}_j \geqq EG_j(N \backslash \beta, \hat{v}^{\beta})$ and rejects it otherwise.

If all players take the above strategies, an agreement is formed at t=3 and the game is over. It is clear that the above strategy profile yields the egalitarian value for any player who is not the proposer β since $b_j^\beta + x_j^\beta = EG_j(N, v)$ for any $j \neq \beta$. The proposer β obtains $v(N) - \sum_{j \neq \beta} b_j^\beta - \sum_{j \neq \beta} x_j^\beta = v(N) - \sum_{j \neq \beta} EG_j(N, v) = EG_\beta(N, v)$. Note that each player obtains his egalitarian value whether or not the player is a proposer. In other words, given the strategies, an outcome is the same regardless of who is chosen as a proposer.

To check whether the above strategies constitute an SPE, first, we show that the strategies at t=3 are best responses for each of the players. Let j_{n-1} be the last player who has to decide whether accept or reject the offer. If no other players reject an offer, player j_{n-1}'s best response is accept the offer if $x_{j_{n-1}}^\beta \geqq EG_{j_{n-1}}(N \backslash \beta, \hat{v}^\beta)$ and reject it otherwise. Knowing that the above mentioned reaction of the last player, the second last player j_{n-2}'s best response is accept the offer if $x_{j_{n-2}}^\beta \geqq EG_{j_{n-2}}(N \backslash \beta, \hat{v}^\beta)$ and reject it otherwise. Using the same argument to go backward, we can show that the strategies mentioned above constitute an SPE of the subgame starting from t=3.

Next, we prove that the strategies at t=2 are best responses for each of them. By the strategies, the proposer β obtains $v(N) - \sum_{j \neq \beta} EG_j(N \backslash \beta, \hat{v}^\beta) = 0$ in the subgame starting from t=2. If he offers some player j the value $\bar{x}_j^\beta$ less than $EG_j(N \backslash \beta, \hat{v}^\beta)$, the offer is rejected by the player and the proposer obtains 0 which is not strictly better off. If he offers some player j the value $\hat{x}_j^\beta$ larger than $EG_j(N \backslash \beta, \hat{v}^\beta)$ without lowering the offer to the other players, the offer is accepted but the share of the proposer is strictly worse off. Thus, the above mentioned strategies constitute a SPE of the subgame starting from t=2.

Then, we show that the strategies t=1 are best responses for each of them. Given the strategies, for any $i \in N$,

$$
\begin{aligned}
B^i &= \sum_{j \neq i} b_j^i - \sum_{j \neq i} b_i^j \\
&= \sum_{j \neq i} (EG_j(N, v) - EG_j(N \backslash i, \hat{v}^i)) - \sum_{j \neq i} (EG_i(N, v) - EG_i(N \backslash j, \hat{v}^j)) = 0,
\end{aligned}
$$

since EG satisfies the balanced V-contributions property. Hence, all players are chosen to be a proposer with probability $\frac{1}{n}$. As seen before, the outcome is the same regardless of who is chosen as a proposer. Given the above mentioned strategies, consider the case that player i changes his strategy to

$\bar{b}_j^i = b_j^i + a_j$ for each of $j \neq i$. If $\sum_{j\neq i} a_j < 0$, i is not chosen as a proposer; hence, his final payoff is unchanged. If $\sum_{j\neq i} a_j = 0$, i may be chosen to be a proposer. In the case that he is not chosen as a proposer, his final payoff is unchanged. In the case that he is chosen as a proposer, his final payoff is

$$v(N)-\sum_{j\neq i}\bar{b}_j^i-\sum_{j\neq i}EG_j(N\backslash i,\hat{v}^i) = v(N)-\sum_{j\neq i}b_j^i-\sum_{j\neq i}EG_j(N\backslash i,\hat{v}^i) = EG_i(N,v),$$

which means his final payoff is unchanged. If $\sum_{j\neq i} a_j > 0$, i must be chosen to be a proposer. However, by the previous result, he obtains

$$v(N)-\sum_{j\neq i}\bar{b}_j^i-\sum_{j\neq i}EG_j(N\backslash i,\hat{v}^i) < v(N)-\sum_{j\neq i}b_j^i-\sum_{j\neq i}EG_j(N\backslash i,\hat{v}^i) = EG_i(N,v).$$

Thus, his share is strictly worse off. Therefore, the above mentioned strategies constitute a SPE.

Next, we prove that any SPE implements the egalitarian value payoff as an equilibrium outcome by the following series of claims.

Claim 1: In any subgame starting from t=2, a proposer β obtains 0 and each of the other players obtains his egalitarian value of the game $(N\backslash\beta, \hat{v}^\beta)$ in any SPE.

Let β be a proposer. There are two types of SPEs: (a) SPEs in which someone rejects the offer at t=3 and (b) SPEs in which players reach an agreement at t=3.

In case (a), by the definition of the non-cooperative game $\hat{\Gamma}(N,v)$ and the induction hypothesis, β obtains 0 and each of the other players obtains his egalitarian value of the game $(N\backslash\beta, \hat{v}^\beta)$.

By the induction hypothesis, each player $j \neq \beta$ surely obtains $EG_j(N\backslash\beta, \hat{v}^\beta)$ by rejecting the offer. Hence, in case (b), each player $j \neq \beta$ obtains not less than $EG_j(N\backslash\beta, \hat{v}^\beta)$. Thus, the proposer β obtains at most 0 since $v(N) - \sum_{j\neq\beta} EG_j(N\backslash\beta, \hat{v}^\beta) = 0$. But, the proposer β surely obtains 0 when the offer is rejected. Hence, he must obtain 0 in case (b). Therefore, the claim also holds in this case.

Claim 2: In any SPE, $B^i = \sum_{j\neq i} b_j^i - \sum_{j\neq i} b_i^j = 0$ for any $i \in N$.

Claim 3: In any SPE, each player's payoff is the same regardless of who is chosen as a proposer.

The above two claims are the same as Claim (c) and (d) of Pérez-Castrillo and Wettstein (2001), and are shown in the same manner, respectively.

Claim 4: In any SPE, the final payoff coincides with the egalitarian value.

Let u_i^j be i's equilibrium payoff when j is the proposer at t=1. By Claim 1,

$$u_i^i = -\sum_{k\neq i} b_k^i$$

and for each $j \neq i$,

$$u_i^j = b_i^j + EG_i(N\backslash j, \hat{v}^j).$$

Thus,

$$\sum_{j\in N} u_i^j = -\sum_{k\neq i} b_k^i + \sum_{j\neq i} b_i^j + \sum_{j\neq i} EG_i(N\backslash j, \hat{v}^j).$$

By Claim 2, the above equality is equivalent to

$$\sum_{j\in N} u_i^j = \sum_{j\neq i} EG_i(N\backslash j, \hat{v}^j).$$

By Claim 3, $\sum_{j\in N} u_i^j = nu_i^k$ for each $k \in N$. Therefore, for each $k \in N$,

$$u_i^k = \frac{1}{n}\sum_{j\neq i} EG_i(N\backslash j, \hat{v}^j).$$

By Proposition 1, the right-hand side of the above equality coincides with $EG_i(N, v)$. □

The following table summarizes our results and those of Kongo et al. (2007).

Table 1: A summary of implementations

Games	Γ	$\hat{\Gamma}$	$\bar{\Gamma}$	$\tilde{\Gamma}$
rejection at t=3,				
proposer β obtains	$v(\beta)$	0	$v(\beta)$	$v(N) - v(N\backslash\beta)$
others play	$(N\backslash\beta, v^\beta)$	$(N\backslash\beta, \hat{v}^\beta)$	$(N\backslash\beta, \bar{v}^\beta)$	$(N\backslash\beta, \tilde{v}^\beta)$
implements	Sh	EG	CIS	ENSC

5 Concluding remarks

To conclude the paper, we discuss generalizations of our results.

A value φ is *linear* if for any $\lambda, \lambda' \in \mathbb{R}$ and any $(N, v), (N, v') \in \mathcal{G}$,

$$\varphi(N, \lambda v + \lambda' v') = \lambda \varphi(N, v) + \lambda' \varphi(N, v'),$$

where $(N, \lambda v + \lambda' v') \in \mathcal{G}$ is defined as $(\lambda v + \lambda' v')(S) = \lambda v(S) + \lambda' v'(S)$ for any $S \subseteq N$. Since all of the Shapley value, the egalitarian value, the CIS value, and the ENSC value are linear, any convex combination of them is characterized and implemented in the same manner as we did in this paper.

Convex combinations of those values are studied in Joosten (1996), van den Brink and Funaki (2004), and Ju et al. (2007). Joosten (1996) introduced the α-egalitarian Shapley value that is a convex combination of the egalitarian value and the Shapley value, van den Brink and Funaki (2004) studied convex combinations of the egalitarian value, the CIS value, and the ENSC value, and Ju et al. (2007) introduced the α-consensus value that is a convex combination of the CIS value and the Shapley value. Here, we mention only the α-egalitarian Shapley value in detail, but the following discussions are applicable to the other convex combinations of the four values.

Let $\alpha \in [0, 1]$. The *α-egalitarian Shapley value* ϕ^α of the game (N, v) is

$$\phi^\alpha(N, v) = \alpha EG(N, v) + (1 - \alpha) Sh(N, v).$$

For any $\alpha \in [0, 1]$, the reduced games for a characterization of the α-egalitarian Shapley value are convex combinations of the volunteer games and the marginal games. Given $(N, v) \in \mathcal{G}$, $\alpha \in [0, 1]$, and $S \subseteq N$, the game $(N \backslash S, v^{S,\alpha})$ is defined as, for each $T \subseteq N \backslash S$,

$$v^{S,\alpha}(T) = \alpha \hat{v}^S(T) + (1 - \alpha) v^S(T) = \begin{cases} v(S \cup T) - (1 - \alpha) v(S) & \text{if } T \neq \emptyset \\ 0 & \text{if } T = \emptyset. \end{cases}$$

In the above game, any non-empty subset of $N \backslash S$ can win the cooperation of S by paying the value $(1 - \alpha) v(S)$ to S. The α-egalitarian Shapley value is characterized by the following property:

For each $(N, v) \in \mathcal{G}$, each $\alpha \in [0, 1]$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N \backslash j, v^{j,\alpha}) = \varphi_j(N, v) - \varphi_j(N \backslash i, v^{i,\alpha}).$$

For an implementation, we construct the non-cooperative game $\Gamma^{\alpha}(N, v)$ in which almost the same as the game $\hat{\Gamma}(N, v)$. The difference between the two is that, when an agreement is not reached at t=3, the proposer β leaves the game with obtaining 0 and remaining players continue the non-cooperative game $\Gamma^{\alpha}(N\backslash\beta, \hat{v}^{\beta})$ with probability α, and the proposer β leaves the game with obtaining $v(\beta)$ and remaining players continue the non-cooperative game $\Gamma^{\alpha}(N\backslash\beta, v^{\beta})$ with probability $1 - \alpha$.

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