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# A two-step Shapley value in a cooperative game with a coalition structure

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## **Abstract**

In this paper, we study cooperative games with coalition structures. We show that a solution concept which applies the Shapley value to games inter- and intra-coalitions and in which the pure surplus that the coalition obtains is allocated among the intra-coalition members in egalitarian way, is axiomatized by modified axioms on null players and symmetric players and the usual three axioms: Efficiency, Additivity and Coalitional Symmetry. In addition to the original definition, we give two expressions of this solution concept. One is an average of modified marginal contributions and the other is the weighted Shapley value of a game with restricted communication.

JEL Classification Numbers: C70; C71

Keywords: coalition structure; Shapley value; two-step approach; egalitarian intra-group members

# 1 Introduction

The purpose of this short paper is to revisit a distributive analysis of a cooperative surplus among players when they already partition themselves into ‘coalitions’ before realizing cooperation. Two traditional works of Aumann and Dreze (1974) and Owen (1977) respectively propose distribution rules, solution concepts in a framework of a cooperative game with a coalition structure, which are different from each other but each of which is considered to be an extension of the Shapley value to the case. Because both studies implicitly or explicitly assume that players forms coalitions in order to affect their bargaining positions, they share a common presumption that a participant in a coalition who does not make an effective contribution to his fellows receive nothing. Thus, these studies lack a perspective that a (formed) coalition often has a tendency to a generous reallocation of the surplus within the members of the coalition as if it is a system of mutual assistance among the internal members, even though it is not established for such an end in the beginning.<sup>1</sup>

To reflect such a point of view, we introduce two new axioms in solution theory of a cooperative game with a coalition structure, both of which are slightly different from the ones considered in existing studies. The first axiom represents something like mutual aid of the formed coalitions. This is expressed by the statement on a *null player*: even a null player could obtain some portion of a bargaining surplus if a coalition that he belongs to generates it. The other is related to an *equity* criterion applied to members in the coalitions. This requires that two distinct players should be treated equally, *i.e.*, receive the same amount, if these two are judged to be in an equal position in their *internal* coalition. We show that these two axioms with the usual three axioms (Efficiency, Additivity and Coalitional Symmetry) lead to a unique solution concept which is also considered to be an extension of the Shapley value in cooperative games with coalition structures. This solution is interpreted as an allocation of the cooperative surplus by using the Shapley value in two-step bargaining process: a bargaining inter-coalitions and a bargaining intra-coalitions. Moreover, the bargaining surplus of the coalition is allocated among the intra-coalition members in egalitarian way. In addition to this definition, we give two expressions of this solution concept. One is an average of modified marginal contributions and the other is the weighted Shapley value of games with restricted communication derived from a coalition structure.

The rest of the paper is organized as follows. In the next section, we prepare the basic notations and definitions used in this paper. The correct statement about our new axioms is introduced in Section 3. In the section, we present our main result on an axiomatic characterization of the solution concept. Section 4 gives some remarks on the new solution concept.

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<sup>1</sup>Such tendency of formed coalitions is examined and explained in various contexts. Kropotkin (1972), from a view of evolutionary economics, argues that generous reallocation scheme intra-group members arises from a human evolution in the struggle for life. Noh (1999) examines a rent-seeking problem among two groups and show that the individuals within a group reach an agreement of egalitarian-like intra-group sharing rule in the presence of inter-group competition. Researchers in community psychology consider that the members of mutual assistance organizations can suitably respond divergent events of life such as hospitalization for illness, victim of a crime, debt, and loss of a spouse and such augmenting stressful situations around ourselves lead to a recent development of a number of mutual assistance organizations (Levine 1988). Further, in the study of labor-management, profit sharing including stock-ownership schemes, which can be seen as generous reallocation not only within workers but also among owners and employees, leads to helping-on-the-job (Drago and Turnbull 1988) and mutual monitoring (Kandel and Lazear 1992), and it is shown that profit sharing has a positive relation with factor productivity (FitzRoy and Kraft 1987) and profitability (FitzRoy and Kraft 1986) of the firm.

## 2 Preliminary

A pair  $(N, v)$  is a *cooperative game* or simply a *game* where  $N = \{1, \dots, n\}$  is a set of finite players and  $v$  is a real valued function defined on all subsets of  $N$  satisfying  $v(\emptyset) = 0$ . A nonempty subset  $S$  of  $N$  is called a *coalition*. A set of games with finite players is denoted by  $\Gamma$ .

Given a game  $(N, v) \in \Gamma$ , let  $\mathcal{C} = \{C_1, \dots, C_m\}$  be a partition of  $N$ , that is,  $C_k \cap C_h = \emptyset$  for  $k \neq h$  and  $\bigcup_{k=1}^m C_k = N$ . Then  $\mathcal{C}$  is called a *coalition structure* of  $N$ . A triple  $(N, v, \mathcal{C})$  is called a *game with a coalition structure*. A set of games with coalition structures with finite players is denoted by  $\Delta$ .

A vector  $x \in \mathbb{R}^N$  is called a *payoff vector* and its  $i$ -th element  $x_i$  denotes the payoff of player  $i$ . A payoff vector  $x \in \mathbb{R}^N$  is called *feasible* in  $(N, v)$  if  $\sum_{i \in N} x_i \leq v(N)$  holds and *efficient* if the equality holds. A solution of games is a function which assigns every game  $(N, v)$  to a feasible payoff vector of the game. A solution of cooperative games with coalition structures is defined in the same manner.

Let  $(N, v) \in \Gamma$ . Let  $\theta$  be an order on  $N$ , that is,  $\theta$  is a bijection on  $N$ . A set of all the orders on  $N$  is denoted by  $\Theta(N)$ . A set of players preceding to  $i$  at order  $\theta$  is  $A_i^\theta = \{j \in N : \theta(j) < \theta(i)\}$ . A marginal contribution of player  $i$  at order  $\theta$  in  $(N, v)$  is defined by  $m_i^\theta(N, v) = v(A_i^\theta \cup \{i\}) - v(A_i^\theta)$ . Then the Shapley value  $\text{Sh}$  of  $(N, v)$  is defined as follows:

$$\text{Sh}_i(N, v) = \frac{1}{|\Theta(N)|} \sum_{\theta \in \Theta(N)} m_i^\theta(N, v), \text{ for all } i \in N,$$

where  $|\cdot|$  represents the cardinality of the set. Thus, the Shapley value is an average of marginal contribution vectors where each order  $\theta \in \Theta(N)$  occurs in an equal probability, that is,  $1/|\Theta(N)|$ .

Let  $(N, v, \mathcal{C}) \in \Delta$ . Owen (1977) extends a notion of the Shapley value to a game in which players are already partitioned into ‘‘unions,’’ that is, a game with a coalition structure. An order  $\theta \in \Theta(N)$  is *consistent* with  $\mathcal{C}$  if the following condition holds: for any  $i, j, k \in N$  with  $\theta(i) < \theta(k) < \theta(j)$ ,  $i, j \in C_h \in \mathcal{C}$  implies that player  $k$  also belongs to coalition  $C_h$ . Thus, in the consistent order, players line up in a way that players in the same coalition are side-by-side. A set of all the orders on  $N$  consistent with  $\mathcal{C}$  is denoted by  $\Theta(N, \mathcal{C})$ . Then coalitional value CV introduced by Owen (1977) is defined as follows:

$$\text{CV}_i(N, v, \mathcal{C}) = \frac{1}{|\Theta(N, \mathcal{C})|} \sum_{\theta \in \Theta(N, \mathcal{C})} m_i^\theta(N, v), \text{ for all } i \in N.$$

A player  $i \in N$  is called a *null player* if  $v(S) = v(S \cup \{i\})$  for any  $S \subseteq N \setminus \{i\}$  and a *dummy player* if  $v(S) + v(\{i\}) = v(S \cup \{i\})$  for any  $S \subseteq N \setminus \{i\}$ . Two players  $i, j \in N$  are called *symmetric* in  $(N, v)$  if  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$ .

For  $(N, v, \mathcal{C}) \in \Delta$ ,  $(M, v_\mathcal{C})$  is called a *game among coalitions* where  $M = \{1, \dots, m\}$  is a set of coalitional indices of the elements in  $\mathcal{C}$  and  $v_\mathcal{C}(H) = v(\bigcup_{k \in H} C_k)$  for each  $H \subseteq M$ .<sup>2</sup> This is interpreted as a game played by the (representatives of the) coalitions. For  $S \subseteq N$ , a subgame of  $(N, v)$  to  $S$  is simply denoted by  $(S, v|_S)$  where  $v|_S(T) = v(T)$  for all  $T \subseteq S$ .

<sup>2</sup>This game is referred to as an intermediate game in Peleg and Sudhölter (2003) and as a quotient game in Owen (1977).

### 3 Results

First, we define new axioms about null players and symmetric players. Let  $(N, v, \mathcal{C}) \in \Delta$  and let  $\psi$  be a solution of cooperative games with coalition structures. Then,

**Coalitional Null Player:** If  $i \in C_k$  is a null player in  $(N, v)$  and  $k \in M$  is a dummy player in  $(M, v_{\mathcal{C}})$  (that is,  $C_k$  is a dummy coalition), then  $\psi_i(N, v, \mathcal{C}) = 0$ .

**Internal Equity:** If  $i \in C_k \in \mathcal{C}$  and  $j \in C_k$  are symmetric in subgame  $(C_k, v|_{C_k})$ , then  $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$ .

Thus, in the statement of Coalitional Null Player, the usual requirement on a null player (null player axiom in existing studies) that a null player obtain nothing in any situation is weakened so that he could obtain more than his own contributions because of the strong position of his coalition or mutual assistance between the internal members in the coalition. Internal Equity requires that two distinct players who are judged to be in an equal position in the internal situation (*i.e.*, subgame  $(C_k, v|_{C_k})$ ) should be equally treated and thus receive the same amount of the surplus.

On the one hand, it is easily shown that the coalitional value satisfies Coalitional Null Player since it always gives nothing to the null player. On the other hand, it does not satisfy Internal Equity. In fact, consider a three-person game  $(N, v, \mathcal{C})$  where  $N = \{1, 2, 3\}$ ,  $\mathcal{C} = \{\{1\}, \{2, 3\}\}$ , and  $v(\{1, 2\}) = v(\{1, 2, 3\}) = 1$  and  $v(S) = 0$  otherwise. Then the coalitional value gives  $(1/2, 1/2, 0)$  for the players. However players 2 and 3 are symmetric in  $(\{2, 3\}, v|_{\{2, 3\}})$ .

Next theorem shows that there exists a unique solution on  $\Delta$  different from the coalitional value, satisfying these two axioms and usual three axioms (Efficiency, Additivity and Coalitional Symmetry).

**Theorem 1.** *There exists a unique solution of cooperative games with coalition structures on  $\Delta$  satisfying Coalitional Null Player, Internal Equity and the following three:*

- (i) *Efficiency:*  $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = v(N)$ .
  - (ii) *Additivity:*  $\psi(N, v, \mathcal{C}) + \psi(N, v', \mathcal{C}) = \psi(N, v + v', \mathcal{C})$ , where  $(v + v')(S) = v(S) + v'(S)$  for any  $S \subseteq N$ .
  - (iii) *Coalitional Symmetry:* If  $k \in M$  and  $h \in M$  are symmetric in  $(M, v_{\mathcal{C}})$ , then  $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = \sum_{i \in C_h} \psi_i(N, v, \mathcal{C})$ .
- This solution is defined by the following formula:*

$$\psi_i^{\delta}(N, v, \mathcal{C}) = \frac{\text{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \text{Sh}_i(C_k, v|_{C_k}), \quad \text{for } i \in C_k \in \mathcal{C}. \quad (1)$$

**Proof.** First, we show that  $\psi^{\delta}$  satisfies all the five axioms. Efficiency and Additivity are obvious by the definition of  $\psi^{\delta}$  since the Shapley value satisfies these two axioms. Next,  $\psi^{\delta}$  satisfies Coalitional Symmetry because the summation of  $\psi_i^{\delta}(N, v, \mathcal{C})$  over  $C_k \in \mathcal{C}$  is player  $k$ 's Shapley value of a game among coalitions  $(M, v_{\mathcal{C}})$  and the Shapley value gives the equal payoffs to the symmetric players (symmetry axiom of the Shapley value, see Shapley 1953b). Since the first term of the definition of  $\psi^{\delta}$  is the same for all the players in coalition  $C_k \in \mathcal{C}$ , the different payoffs among the players in  $C_k$  are caused by the subgame  $(C_k, v|_{C_k})$ . Thus  $\psi^{\delta}$  fulfills Internal Equity because the Shapley value satisfies symmetry axiom. It also satisfies Coalitional Null Player axiom because the Shapley value gives dummy player  $i$  his stand-alone value  $v(\{i\})$ .

Next we will show the converse part. Let  $\psi$  be a solution on  $\Delta$  satisfying the five axioms. Given  $T \subseteq N$ , let  $(N, u_T)$  be a  $T$  unanimity game where  $u_T(S) = 1$  if  $S \supseteq T$  and  $u_T(S) = 0$  otherwise. Given  $c \in \mathbb{R}$ , let  $cu_T$  be a  $T$  unanimity game  $u_T$  multiplied by a scalar  $c$ . Then, by Additivity axiom, it is enough to show that  $\psi(N, cu_T, \mathcal{C})$  is uniquely determined by the five axioms.

For  $T \subseteq N$ , define  $D \subseteq M$  by  $\{k \in M : C_k \in \mathcal{C}, C_k \cap T \neq \emptyset\}$ . Then  $(M, (cu_T)_{\mathcal{C}})$  is a  $D$ -unanimity game multiplied by  $c$ , i.e.,  $(M, cu_D)$ . Then Coalitional Symmetry together with Coalitional Null Player axiom implies that for  $k \in M \setminus D$  and for  $i \in C_k$ ,  $\psi_i(N, cu_T, \mathcal{C}) = 0$  and for  $k \in D$ ,  $\sum_{i \in C_k} \psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|D|}$ .

For an internal distribution of the members in  $C_k, k \in D$ , we consider two cases.

**Case (a):**  $|D| \geq 2$ . Then, for any  $k \in D$ ,  $i \in C_k$  and  $j \in C_k$  are symmetric in  $(C_k, (cu_T)|_{C_k})$ . Therefore  $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|C_k| \cdot |D|}$  for any  $i \in C_k, k \in D$ .

**Case (b):**  $|D| = 1$ . Then put  $\mathcal{C} = \{C_1\}$ . For  $i \in C_1 \setminus T$ ,  $\psi_i(N, cu_T, \mathcal{C}) = 0$  by Coalitional Null Player since player 1 is dummy in  $(M, (cu_T)_{\mathcal{C}})$  and player  $i$  is null in  $(N, v)$ . Moreover any  $i, j \in T$  are symmetric in  $(C_1, (cu_T)|_{C_1})$ . Therefore by Internal Equity,  $\psi_i(N, cu_T, \mathcal{C}) = \frac{c}{|T|}$  for any  $i \in T$ . ■

We can interpret  $\psi^\delta$  as a *two-step Shapley value* in the following sense. In the first step, each coalition  $C_k \in \mathcal{C}$  acts like a single player and obtains player  $k$ 's Shapley value of a game among coalitions,  $(M, v_{\mathcal{C}})$ . Thus, an allotment of first step for coalition  $C_k$  is  $\text{Sh}_k(M, v_{\mathcal{C}})$ . In the second step, all the players in  $C_k$  agree with the following two things. First, they agree that  $\text{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)$  is a pure surplus (it is non-negative if the game is superadditive<sup>3</sup>) of the first step and therefore is split equally among the members in  $C_k$ . Second, they agree that remaining part  $v(C_k)$  is distributed by the rule of the Shapley value for their subgame  $(C_k, v|_{C_k})$ . Thus, the pure bargaining surplus of the first stage is distributed within the members in the egalitarian way, which seems to reflect a generous reallocation or an aspect of mutual aid among the members embedded in our two axioms.

Before checking the independence of each axiom from the others, the next remark is worth mentioning.

**Remark 1.** *It is worth mentioning that we can omit Efficiency from Theorem 1. In fact, the other axioms with non-emptiness of a solution which we implicitly assume, imply Efficiency. The reason why we add Efficiency in Theorem 1 is to easily compare our result with Owen's (1977) one in the next section.*

*The main logic is similar to Theorem 8.1.3 of Peleg and Sudhölter (2003). Let  $(N, v^0)$  be zero-game such that  $v^0(S) = 0$  for any  $S \subseteq N$  and  $\mathcal{C}$  be a coalition structure on  $N$ . Then,  $\psi(N, v^0, \mathcal{C})$  must be  $0^N = (0, \dots, 0) \in \mathbb{R}^N$  by Coalitional Null Player. Let  $(N, v, \mathcal{C}) \in \Delta$ . By Additivity,  $\psi(N, v, \mathcal{C}) + \psi(N, -v, \mathcal{C}) = \psi(N, v - v, \mathcal{C}) = \psi(N, v^0, \mathcal{C}) = 0^N$ . So  $\psi(N, v, \mathcal{C}) = -\psi(N, -v, \mathcal{C})$  holds. By the definition of a solution,  $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) \leq v(N)$  and  $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = -\sum_{i \in N} \psi_i(N, -v, \mathcal{C}) \geq -(-v(N))$ . Thus,  $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = v(N)$  holds.*

**Example 1.** *We define the following solutions in order to check the independence of each axiom from the others. Let  $\theta$  be an order on the set of all the integers  $\mathbb{N}\{1, 2, 3, \dots\}$ . For any set  $S \subset \mathbb{N}$ , let  $\theta[S]$  denote an order on  $S$  induced from  $\theta$  such that for any  $i, j \in S$ ,  $\theta[S](i) < \theta[S](j)$  exactly if  $\theta(i) < \theta(j)$ .*

**1.** *For  $(N, v) \in \Gamma$ , let  $\text{Nu}(N, v)$  denote the nucleolus of  $(N, v)$  proposed by Schmeidler (1969). For any  $(N, v, \mathcal{C}) \in \Delta$ , we define a solution  $\psi^{(i)}$  by  $\psi_i^{(i)}(N, v, \mathcal{C}) = \frac{\text{Nu}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \text{Nu}_i(C_k, v|_{C_k})$  for any  $i \in C_k \in \mathcal{C}$ . Then,  $\psi^{(i)}$  satisfies Coalitional Symmetry, Internal Equity and Coalitional Null Player but not Additivity.*

<sup>3</sup>For any  $S, T \subseteq N$  with  $S \cap T = \emptyset$ ,  $v(S \cup T) \geq v(S) + v(T)$ .

2. For any  $i \in C_k \in \mathcal{C}$ , define  $\psi_i^{(ii)}(N, v, \mathcal{C}) = \frac{m_k^{\theta[M]}(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + \text{Sh}_i(C_k, v|_{C_k})$ . Then,  $\psi^{(ii)}$  satisfies Additivity, Internal Equity and Coalitional Null Player but not Coalitional Symmetry.
3. Then for any  $i \in C_k \in \mathcal{C}$ , define  $\psi_i^{(iii)}(N, v, \mathcal{C}) = \frac{\text{Sh}_k(M, v_{\mathcal{C}}) - v(C_k)}{|C_k|} + m_i^{\theta[C_k]}(C_k, v|_{C_k})$ . Then,  $\psi^{(iii)}$  satisfies Additivity, Coalitional Symmetry and Coalitional Null Player but not Internal Equity.
4. For any  $i \in C_k \in \mathcal{C}$ ,  $\psi_i^{(iv)}(N, v, \mathcal{C}) = \psi_i^e(N, v, \mathcal{C}) = \frac{\text{Sh}_k(M, v_{\mathcal{C}})}{|C_k|}$ . Then,  $\psi^e$  satisfies Additivity, Coalitional Symmetry and Internal Equity but not Coalitional Null Player.

Among several solutions described in the above example, solution  $\psi^e$ , which uses the Shapley value for inter-coalitions and the egalitarian solution for intra-coalitions, also prepares the similar requirement of coalitions as generous reallocation system or mutual assistance considered in this paper, but slightly give much weight to an egalitarian aspect within the internal members. In fact, this solution is axiomatized as follows.

**Theorem 2.**  $\psi^e$  is a unique solution of cooperative games with coalition structures on  $\Delta$  satisfying Efficiency, Additivity, Coalitional Symmetry and the following two:

**Null Coalition:** If  $k \in M$  is a null player in  $(M, v_{\mathcal{C}})$ , then  $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = 0$ .

**Internal Egalitarianism:** For any  $i \in C_k \in \mathcal{C}$  and for any  $j \in C_k \in \mathcal{C}$ ,  $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$ .

We omit the proof of Theorem 2 because this is similarly constructed to the proof of Theorem 1. For the independence of the axioms in the above theorem, consider solutions, for  $i \in C_k \in \mathcal{C}$ ,

5.  $\psi_i^{(v)}(N, v, \mathcal{C}) = \frac{\text{Nu}_k(M, v_{\mathcal{C}})}{|C_k|}$ ,
6.  $\psi_i^{(vi)}(N, v, \mathcal{C}) = \frac{m_k^{\theta[M]}(M, v_{\mathcal{C}})}{|C_k|}$ , and
7.  $\psi_i^{(vii)}(N, v, \mathcal{C}) = \frac{v(N)}{|N|}$ .

Then,  $\psi^{(v)}$ ,  $\psi^{(vi)}$ ,  $\psi^\delta$ , and  $\psi^{(vii)}$  respectively show the independence of Additivity, Coalitional Symmetry, Internal Egalitarianism and Null Coalition from the other three.

## 4 Remarks

### 4.1 Comparison with the Owen's coalitional value

The coalitional value is characterized by Efficiency, Additivity, Coalitional Symmetry and the following two axioms. (See Owen 1977. However Peleg and Sudhölter 2003 show that by the same reason of Remark 1, we can conduct the axiomatization of the coalitional value without Efficiency.)

**Null Player:** If  $i \in N$  is a null player in  $(N, v)$ , then  $\psi_i(N, v, \mathcal{C}) = 0$ .

**Restricted Equal Treatment:** If  $i, j \in C_k \in \mathcal{C}$  are symmetric in  $(N, v)$ , then  $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$ .

It is clear that Restricted Equal Treatment is weaker than Internal Equity and Null Player requires more than Coalitional Null Player. Thus we understand that the essential difference between the two solutions lies in the treatment of null players and symmetric players.

## 4.2 Random arrival interpretation

It seems that the definition of  $\psi^\delta$  is too much based on “two step bargaining process.” We are, however, able to express  $\psi^\delta$  as an average of the *modified* version of the marginal contributions as well as Sh and CV. Let  $\theta \in \Theta(N, \mathcal{C})$ . Let  $\theta_M$  be an order on  $M$  derived from  $\theta$  such that for  $k, h \in M$ ,  $\theta_M(k) < \theta_M(h)$  if and only if  $\theta(i) < \theta(j)$  for all  $i \in C_k$  and for all  $j \in C_h$ .  $\theta_M$  is well-defined if  $\theta$  is consistent with  $\mathcal{C}$ . For  $i \in C_k \in \mathcal{C}$ , we define modified marginal contribution of player  $i$  at order  $\theta$ ,  $\bar{m}_i^\theta$  by

$$\bar{m}_i^\theta(N, v, \mathcal{C}) = \begin{cases} m_i^{\theta[C_k]}(C_k, v|_{C_k}) & \text{if } i \text{ is not last at order } \theta[C_k], \\ m_i^{\theta[C_k]}(C_k, v|_{C_k}) + m_k^{\theta_M}(M, v_{\mathcal{C}}) - v(C_k) & \text{if } i \text{ is last at order } \theta[C_k]. \end{cases}$$

The following theorem holds.

**Theorem 3.**  $\psi^\delta$  is expressed as follows: for  $i \in N$ ,

$$\psi_i^\delta(N, v, \mathcal{C}) = \frac{1}{|\Theta(N, \mathcal{C})|} \sum_{\theta \in \Theta(N, \mathcal{C})} \bar{m}_i^\theta(N, v, \mathcal{C}). \quad (2)$$

**Proof.** It is easily verified from this formula by Eq.(1) and the definition of the modified marginal contribution. ■

## 4.3 Restricted communication

Myerson (1977) considers a situation that a communication between players is restricted on an undirected graph of  $N$  (see also Myerson 1980). Along this line of research, Aumann and Dreze’s (1974) value, which is defined by  $AD_i(N, v, \mathcal{C}) = Sh_i(C_k, v)$  for all  $i \in C_k \in \mathcal{C}$ , is considered to represent a situation that a coalition structure describes a communication restriction such that players in the same coalition communicate with each other, but each coalition is physically separated. This situation is also described as the graph such that each maximal component of the graph corresponds to a coalition in  $\mathcal{C}$  and each subgraph on the component is a complete graph. Thus, Aumann and Dreze’s value coincides with the Myerson value for such a graph situation. However, this interpretation of coalition structure does not fit the view that players form coalitions for the division of  $v(N)$  since Aumann and Dreze’s value does not satisfy the efficiency but the relative efficiency ( $\sum_{i \in N} \psi_i(N, v, \mathcal{C}) = \sum_{C_k \in \mathcal{C}} v(C_k)$ ). This motivates another view of communication restriction by a coalition structure in the following sense:

- (i) players in the same coalition  $C_k \in \mathcal{C}$  can freely communicate with each other, and
- (ii) players in  $C_k$  can communicate with players in the other coalitions if there is a permission of all the players in  $C_k$ .

Condition (i) means that players in any sub coalition  $S \subseteq C_k \in \mathcal{C}$  can communicate with each other and thus obtain their worth of coalition,  $v(S)$ . In addition to (i), (ii) implies that there is a possibility of cooperation among players in the different coalitions. This is possible only if all the players in the relevant coalitions agree. Let  $i \in C_k$  and  $C_h \in \mathcal{C}, C_h \neq C_k$ . While  $C_k$  and  $C_h$  obtain their worth  $v(C_k \cup C_h)$ ,  $C_k - i$  and  $C_h$  obtain the sum of  $v(C_k - i)$  and  $v(C_h)$  because there is no permission by player  $i$  or there is no permission of the party which the coalition represents and which requires the unanimous agreement.<sup>4</sup>

<sup>4</sup>Carreras (1992) refers the similar restriction of coalition as “voting discipline” in the context of simple games.

**Definition 1.** Let  $(N, v, \mathcal{C}) \in \Delta$ .  $\mathcal{C}$ -communication restricted game  $(N, v^{\mathcal{C}})$  is defined as follows. For all  $S \subseteq N$ ,

$$v^{\mathcal{C}}(S) = v\left(\bigcup_{C_k \in \mathcal{C}(S)} C_k\right) + \sum_{T \in \mathcal{C}^0(S)} v(T)$$

where  $\mathcal{C}(S) = \{C_k \in \mathcal{C} : C_k \subseteq S\}$  and  $\mathcal{C}^0(S) = \{C_k \cap S : C_k \cap S \neq C_k, C_k \in \mathcal{C}\}$ .

Then, we obtain the following relationship between  $\psi^{\delta}$  and the weighted Shapley value  $\text{Sh}^w$  (see, Shapley 1953a and Kalai and Samet 1987). Let  $w = (w_i)_{i \in N}$  be a positive weight vector of  $N$ . The weighted Shapley value  $\text{Sh}^w$  is defined as follows:

$$\text{Sh}_i^w(N, v) = \sum_{\theta \in \Theta(N)} \mu^w(\theta) m_i^{\theta}(N, v)$$

for all  $i \in N$  where for  $\theta = (i_1, \dots, i_n)$ ,  $\mu^w(\theta) = \prod_{j=1}^n \frac{w_{i_j}}{\sum_{k=1}^j w_{i_k}}$ . The following theorem holds.

**Theorem 4.** Let  $w = (w_i)_{i \in N}$  be a weight vector such that  $w_i = \frac{1}{|C_k|}$  for all  $i \in C_k \in \mathcal{C}$ . Then,

$$\psi^{\delta}(N, v, \mathcal{C}) = \text{Sh}^w(N, v^{\mathcal{C}}). \quad (3)$$

**Proof.** Eq.(3) is obtained from Eq.(2) and the following fact. For  $\theta \in \Theta(N)$ , we define  $\theta^* \in \Theta(M)$  by the condition that  $\theta^*(k) > \theta^*(h)$  if and only if there exists  $i \in C_k$  such that  $\theta(i) > \theta(j)$  for all  $j \in C_h$ . Then, the probability that  $\theta^*$  coincides with some order  $\pi \in \Theta(M)$  when each  $\theta$  appears according to the probability distribution  $\mu^w(\cdot)$  is just  $\frac{1}{|M|!} = \frac{1}{|\Theta(M)|}$ . ■

Thus, Eq.(3) shows that a two-step Shapley value  $\psi^{\delta}$  is a weighted Shapley value for  $\mathcal{C}$ -communication restricted game. However, using reciprocal of the cardinality of a coalition as the weight of the member of the coalition does not seem to have much justification; rather usual non-weighted Shapley value appears to be more acceptable. This motivates the following definition of a solution  $\psi^{\gamma}$  in a game with a coalition structure.

**Definition 2.** Let  $(N, v, \mathcal{C}) \in \Delta$ . A solution  $\psi^{\gamma}$  of a game with a coalition structure is defined as follows:

$$\psi^{\gamma}(N, v, \mathcal{C}) = \text{Sh}(N, v^{\mathcal{C}}).$$

An unintuitive result holds for this solution. Interestingly, this solution is other version of two-step Shapley value: here, in the first step, the weighted Shapley value with coalition-size based weight is applied for a game among coalition, and then the Shapley value is applied for the subgame. Thus, the following holds:

$$\psi_i^{\gamma}(N, v, \mathcal{C}) = \frac{\text{Sh}_k^w(M, v^{\mathcal{C}}) - v(C_k)}{|C_k|} + \text{Sh}_i(C_k, v|_{C_k}), \quad \text{for } i \in C_k \in \mathcal{C},$$

where  $w_i = |C_k|$  for all  $i \in C_k \in \mathcal{C}$ . The properties of this solution are extensively studied in our companion paper (Kamijo 2007) including axiomatizations and a potential function of the solution.

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