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Q-anonymous social welfare relations on infinite utility streams

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Abstract

We explore a social welfare relation (SWR) defined on infinite utility streams which satisfies $\mathcal{Q}$-Anonymity, a stronger anonymity condition than Finite Anonymity. Following d'Aspremont (unpublished manuscript), we define a specific type of SWR called simplified criterion in a general form. Then, we show that any extension à la Banerjee (Soc Choice Welfare 27: 327-339) of a finitely anonymous simplified criterion is characterized by only replacing Finite Anonymity with $\mathcal{Q}$-Anonymity in the set of conditions characterizing the simplified criterion in question. The characterizations of the extended versions of the generalized Lorenz and the leximin SWRs, called $\mathcal{Q}$-generalized Lorenz SWR and $\mathcal{Q}$-leximin SWR respectively, and the alternative characterization of the extended utilitarian SWR, called $\mathcal{Q}$-utilitarian SWR, are also established as corollaries to our general characterization result.

JEL Classification Numbers: D63, D71

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1 Introduction

Consider a situation where we face an urgency of choosing one among several alternative policies which will affect infinitely many future generations as well as the present generation (e.g. environmental policies). To deal with such a social decision problem, it is plausible to appeal a certain social evaluation ordering which satisfies some reasonable conditions every future generation as well as the present one would readily accept. If we are concerned only with each generation's welfare measured in terms of utility, such an evaluation ordering will be a social welfare ordering (SWO) defined on the set of infinite-horizon utility profiles (one for each attainable utility profile of an alternative policy).

A theoretical analysis on an ordering on infinite-horizon utility streams has been carried out particularly in social choice theory. In the literature, it is known that there is a serious trade-off between the condition of efficiency formalized as *Strong Pareto* and that of impartial treatment of generations called *Strong Anonymity*. Strong Anonymity is defined in terms of all logically possible permutations on (countably) infinite generations and asserts that two utility streams related by a permutation on generations are socially indifferent. Lauwers (1997a) shows that this anonymity postulate is logically incompatible with Strong Pareto.[1]

To avoid the problem of Pareto-Anonymity dilemma, a weakened anonymity condition called *Finite Anonymity* (or *Weak Anonymity*) has been considered to explore a strongly Paretian and (weakly) anonymous SWO on infinite utility streams. Finite Anonymity is defined by confining admissible permutations to those which interchange *finitely* many generations (usually called finite permutations). It is known that Finite Anonymity is compatible with Strong Pareto. Svensson (1980) is the first to provide a possibility result that there exists an ordering on infinite utility streams that satisfies Strong Pareto and Finite Anonymity.[2] His result is established by explicitly presenting the infinite-horizon extension of the Suppes-Sen grading principle, called Suppes-Sen social welfare relation (SWR), and showing that it satisfies both two conditions.[3] Recently, the characterizations of several other SWRs have been also

[1]On this, see also the discussions by Van Liedekerke and Lauwers (1997) and in Asheim and Tungodden (2004).

[2]See also Asheim et al. (2001). Svensson established his result originally as a resolution to another substantive dilemma known as Diamond's (1965) impossibility that if an ordering on infinite utility streams is required to be strongly Paretian and continuous with respect to the sup topology, then it will inevitably violate Finite Anonymity.

[3]The Suppes-Sen grading principle was first proposed in Suppes (1966) in a two-person setting and was generalized into

established: in Asheim and Tungodden (2004), the catching up and overtaking criteria and the leximin versions of these two criteria; by Basu and Mitra (2007), the infinite-horizon extension of utilitarian principle and the catching up and overtaking criteria; in Bossert et al. (2007), the infinite-horizon extensions of the generalized Lorenz criterion and the leximin principle. While every rule we mentioned here is a quasi-ordering, i.e. reflexive and transitive but not complete binary relation on utility streams, Arrow's (1963) variant of Szpilrajn's (1930) theorem ensures that there exists an ordering that respects the ranking determined by the rule in question.

Although these existing results firmly suggest the possibility of making a SWO of infinite utility streams in a not only efficient but also impartial way in the sense that both Strong Pareto and Finite Anonymity are satisfied, in the literature it has often been argued that Finite Anonymity is too weak to reflect the impartial treatment of infinitely many generations and a stronger notion of anonymity is needed. In the recent paper of Mitra and Basu (2005), they characterize the strongest notion of anonymity which is compatible with Strong Pareto. They call this strongest notion of Pareto-compatible anonymity *$\mathcal{Q}$-Anonymity*. $\mathcal{Q}$-Anonymity is defined in terms of a *group of cyclic permutations* (see the next section for details), and it prescribes social indifference relation for a broader class of pairs of utility streams than Finite Anonymity does.[4]

In the literature, not so many efforts have been carried out in exploring a $\mathcal{Q}$-anonymous social welfare ordering on infinite utility streams yet. Exceptions are Lauwers (1997b), Fleurbaey and Michel (2003), Mitra and Basu (2005), and Banerjee (2006). Among them, it is only Banerjee (2006) that establishes the characterizations of $\mathcal{Q}$-anonymous SWRs.[5] In his paper, Banerjee formulates the extensions of the Suppes-Sen and the utilitarian infinite-horizon SWRs, called *$\mathcal{Q}$-Suppes-Sen* SWR and *$\mathcal{Q}$-utilitarian* SWR respectively, each of which is defined by using a group of cyclic permutations, and moreover, he shows that each of his new rules is characterized by replacing Finite Anonymity with $\mathcal{Q}$-Anonymity in the set of axioms that characterizes the finite-anonymous version of the rule, the infinite-

n-person case by Sen (1970).

[4] $\mathcal{Q}$-Anonymity was first introduced in Lauwers (1997b) by the name of *Fixed Step Anonymity*. See also Fleurbaey and Michel (2003).

[5] It should be noted, however, that Fleurbaey and Michel (2003), adopting a less restricted definition of subrelation than usually considered in the literature, provides the axiomatic characterization of the overtaking criterion based on the extended utilitarian welfare relation proposed by Lauwers (1997b).

horizon Suppes-Sen and the infinite-horizon utilitarian SWRs respectively.

The purpose of this paper is to explore other $\mathcal{Q}$-anonymous SWRs and to characterize them. Banerjee's (2006) characterizations of his extended versions of the infinite-horizon Suppes-Sen and the infinite-horizon utilitarian SWRs suggests that for a certain class of finitely anonymous infinite-horizon SWRs, the extension à la Banerjee of a rule can be characterized by only replacing Finite Anonymity with $\mathcal{Q}$ -Anonymity. Indeed, we will show that this is true for a class of some specific types of SWRs that includes well-established infinite-horizon SWRs such as the infinite-horizon leximin principle and the infinite-horizon generalized Lorenz criterion as well as the infinite-horizon Suppes-Sen and the infinite-horizon utilitarian SWRs.

What the four infinite-horizon SWRs we just mentioned above have in common is that each of these rules determines a social ranking of utility streams by applying well-established *finite-horizon* criterion, such as utilitarianism and the leximin principle, to the first n generations in combination with the Pareto criterion for the *infinite* future generations (see later sections for the formal definitions). In other words, these rules are defined by the well-established finite-horizon welfare relations and the Pareto criterion. Moreover, each of them is completely distinguished in terms of the adopted finite-horizon welfare relation. Therefore, for a given finite-horizon welfare relation, we can define an infinite-horizon welfare relation that applies the finite-horizon relation to the first n generations and the Pareto criterion to the infinite future generations. d'Aspremont (2005) refers to this type of infinite-horizon relation as *simplified criterion generated by a (sequence of) finite-horizon relation(s)* and gives an analysis on the logical relationship between a simplified criterion and his overtaking criterion defined in a general form.

In this paper, we will examine the Banerjee-like extension of a simplified criterion. First, we will show that under a few moderate conditions on a finite-horizon welfare relation, the generated simplified criterion will surely be reflexive and transitive binary relation on infinite utility streams, i.e. a quasi-ordering. Our main result is stated for an arbitrary simplified criterion which satisfies reflexivity and transitivity. It shows that any Banerjee-like extension of a reflexive and transitive simplified criterion generated by a finite-horizon welfare relation is characterized by only replacing Finite Anonymity with $\mathcal{Q}$-Anonymity in the set of axioms which characterizes the simplified criterion in question. This result can be seen as a generalization of those obtained by Banerjee (2006) and implies some important corollaries that the extensions à la Banerjee of the infinite-horizon leximin and the generalized Lorenz welfare relations can be characterized by replacing Finite Anonymity with $\mathcal{Q}$-Anonymity in the existing characterization results.

The rest of the paper is organized as follows. Section 2 introduces notation and definitions. In Section 3, we briefly review Banerjee's (2006) extensions of the Suppes-Sen and the utilitarian SWRs. In Section 4, we present a formal definition of a simplified criterion and establish our main theorem. Section 5 provides the characterizations of the Banerjee-like extensions of the leximin and the generalized Lorenz welfare relations, and also an alternative characterization of the $\mathcal{Q}$-utilitarian SWR, as corollaries to our main theorem. Section 6 concludes.

2 Notation and definitions

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ the set of all positive integers $\{1, 2, \dots\}$. We let $X \equiv \mathbb{R}^{\mathbb{N}}$ be the domain of infinite utility streams. An infinite-dimensional vector $x = (x_1, x_2, \dots)$ is a typical element of X. For all $x \in X$ and all $n \in \mathbb{N}$, we denote $(x_1, x_2, \dots, x_n)$ by x^{-n} and $(x_{n+1}, x_{n+2}, \dots)$ by x^{+n}. Thus, given any $x \in X$ and $n \in \mathbb{N}$, we can write $x = (x^{-n}, x^{+n})$.

A SWR is a binary relation, $\succsim$, on X which is reflexive and transitive, i.e. a quasi-ordering. We use, as usual, $\succ$ to denote an asymmetric component of $\succsim$ and $\sim$ a symmetric component of $\succsim$, respectively, i.e. $x \succ y$ if and only if $x \succsim y$ holds but $y \succsim x$ does not, and $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. A SWR $\succsim_A$ is a subrelation to a SWR $\succsim_B$ if for $x, y \in X$, (i) $x \sim_A y$ implies $x \sim_B y$ and (ii) $x \succ_A y$ implies $x \succ_B y$.

Following Mitra and Basu (2005) and Banerjee (2006), we represent any permutation by a permutation matrix. A permutation matrix is an infinite matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$ satisfying the following properties

(i) For each $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij(i)} = 0$ for all $j \neq j(i)$

(ii) For each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{i(j)j} = 0$ for all $i \neq i(j)$.

Given any permutation matrix P, we denote by P' its unique inverse which satisfies $P'P = PP' = I$, where I denotes the infinite identity matrix. We denote the set of all permutation matrices by $\mathcal{P}$. Given a permutation matrix $P \in \mathcal{P}$ and $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j\in\{1,2,\dots,n\}}$ by $P(n)$. A finite permutation matrix is a permutation matrix P such that there exists $n \in \mathbb{N}$ such that $(Px)^{+n} = x^{+n}$. The set of all finite permutation matrices is denoted by $\mathcal{F}$.

As in the papers of Mitra and Basu (2005) and Banerjee (2006), we focus on a particular class of *cyclic* permutations which defines a *group* under the usual matrix multiplication.[6] A permutation matrix $P \in \mathcal{P}$ is said to be cyclic if, for any unit vector $e = (0, \dots, 0, 1, 0, \dots) \in X$, there exists $k \in \mathbb{N}$ such that k-times repeated applications of P to e generate e again, i.e. $P^k e = e$. Throughout the paper, we let $\mathcal{Q} = \{P \in \mathcal{P}$: there exists $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $P(nk)$ is a finite-dimensional permutation matrix$\}$.[7] It is easily checked that $\mathcal{Q}$ is the class of cyclic permutations and defines a group (with respect to matrix multiplication), and also that $\mathcal{F} \subseteq \mathcal{Q}$.

Negation of a statement is indicated by the logical quantifier $\neg$. Our notation for vector inequalities on X is as follows. For all $x, y \in X$, we write (i) $x \geqslant y$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$ and (ii) $x > y$ if $x \geqslant y$ and $x \neq y$.

3 $\mathcal{F}$-Anonymous SWRs and Banerjee's (2006) $\mathcal{Q}$-anonymous extensions

In this section, we briefly review Banerjee's (2006) extensions of finitely anonymous SWRs. We should begin with presenting the formal definition of the condition of Finite Anonymity. In the current framework, Finite Anonymity is defined as the condition that asserts social indifference relation for the pair of utility streams which are related by a finite permutation matrix, $P \in \mathcal{F}$. In this paper, we call it $\mathcal{F}$-Anonymity.

$\mathcal{F}$-Anonymity: For all $x \in X$ and all $P \in \mathcal{F}$, $Px \sim x$.

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[6] Let $\mathcal{G}$ be a set of permutation matrices. $\mathcal{G}$ is said to define a group under the usual matrix multiplication if it satisfies the following four properties: (i) for all $P, Q \in \mathcal{G}$, $PQ \in \mathcal{G}$, (ii) there exists $I \in \mathcal{G}$ such that for all $P \in \mathcal{G}$, $IP = PI = P$, (iii) for all $P \in \mathcal{G}$, there exists $P' \in \mathcal{G}$ such that $P'P = PP' = I$, and (iv) for all $P, Q, R \in \mathcal{G}$, $(PQ)R = P(QR)$.

[7] This class of permutations was first introduced in Lauwers (1997b) and called *fixed step transformations*. See also Fleurbaey and Michel (2003).

$\mathcal{F}$-Anonymity formalizes an idea of impartial treatment of generations by using finite permutations.

A stronger impartiality requirement will be considered as follows.

$\mathcal{P}$-Anonymity: For all $x \in X$ and all $P \in \mathcal{P}$, $Px \sim x$.

Apparently, $\mathcal{P}$-Anonymity reflects an idea of impartial treatment of generations in the most strongest form in our formalization using permutations. One reason why $\mathcal{F}$-Anonymity, a weaker condition, is usually adopted in the literature is that $\mathcal{P}$-Anonymity is incompatible with the following usual efficiency condition.[8]

Strong Pareto: For all $x, y \in X$ with $x > y$, $x \succ y$.

$\mathcal{F}$-Anonymity is surely compatible with Strong Pareto. In the literature, several $\mathcal{F}$-anonymous and strongly Paretian SWRs have been proposed, and moreover, characterization results have also been established (see, for example, Svensson (1980), Asheim et al. (2001), Basu and Mitra (2007), and Bossert et al. (2007)).[9] Examples include the Suppes-Sen and the utilitarian SWRs.

The Suppes-Sen SWR, denoted $\succsim_S$, is defined as: for all $x, y \in X$,

$$x \succsim_S y \Leftrightarrow \text{ there exists } P \in \mathcal{F} \text{ such that } Px \geqslant y.$$

The utilitarian SWR, $\succsim_U$, is defined as: for all $x, y \in X$,

$$x \succsim_U y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } \textstyle\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i \text{ and } x^{+n} \geqslant y^{+n}.$$

The former adopts Sen's (1970) n-person generalization of Suppes' (1966) grading principle in combination with the Pareto criterion for the infinite future generations, and the latter combines the finite-horizon utilitarian welfare relation with the Pareto criterion.

In the literature, it has often been argued that a stronger notion of anonymity than $\mathcal{F}$-Anonymity is needed to realize impartial treatment of all generations. The following two utility streams are usually

[8]On this, see Lemma 1 in Lauwers (1997a).

[9]In the literature on infinite-horizon SWRs, the term "characterization" (of a certain SWR) is used to mean the characterization of all SWRs that include the SWR in question as a subrelation.

considered to demonstrate that $\mathcal{F}$-Anonymity is too weak to operationalize equal treatment of infinitely many generations:

$$x = (1, 0, 1, 0, 1, 0, \ldots) \text{ and } y = (0, 1, 0, 1, 0, 1, \ldots).$$

$\mathcal{F}$-Anonymity, which is defined in terms of finite permutations, cannot declare that x and y are equally good, whereas such a prescription is intuitively appealing.

One plausible candidate which will provide such an conclusion is the following anonymity condition.

$\mathcal{Q}$-Anonymity: For all $x \in X$ and all $P \in \mathcal{Q}$, $Px \sim x$.

$\mathcal{Q}$-Anonymity was first introduced by Lauwers (1997b) by the name *Fixted Step Anonymity*. This anonymity condition is defined in terms of a group of cyclic permutations. In the above example, y is obtained through a 2-period cyclic permutation. Thus, $\mathcal{Q}$-Anonymity declares that x and y are socially indifferent. In their recent paper, Mitra and Basu (2005) established a striking characterization that $\mathcal{Q}$ is the largest class of permutations in terms of which we can define the anonymity condition which is compatible with Strong Pareto.[10]

Using the profiles x and y considered above, it is easily checked that neither of the Suppes-Sen and the utilitarian SWRs satisfies $\mathcal{Q}$-Anonymity. Thus, we need to explore new SWRs which satisfy $\mathcal{Q}$-Anonymity. Banerjee (2006) is the first to formulate $\mathcal{Q}$-anonymous SWRs and to establish the characterizations of them.[11] He provided the following extensions of the Suppes-Sen and the utilitarian SWRs.

The $\mathcal{Q}$-Suppes-Sen SWR, denoted $\succsim_{QS}$, is defined as follows: for all $x, y \in X$,

$$x \succsim_{QS} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_S y.$$

[10] For other alternative stronger postulates of anonymity than $\mathcal{F}$-Anonymity, see Lauwers (1995, 1998), Van Liedekerke and Lauwers (1997), and Fleurbaey and Michel (2003).

[11] It should be noted that $\mathcal{Q}$-anonymous SWRs are also proposed in Lauwers (1997b) and by Fleurbaey and Michel (2003).

The $\mathcal{Q}$-utilitarian SWR $\succsim_{QU}$ is defined as follows: for all $x, y \in X$,

$$x \succsim_{QU} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_U y.$$

We refer to this type of extension of a $\mathcal{F}$-anonymous SWR proposed by Banerjee (2006) as $\mathcal{Q}$-*extension*. He showed that the characterization of the $\mathcal{Q}$-Suppes-Sen SWR $\succsim_{QS}$ and the $\mathcal{Q}$-utilitarian SWR $\succsim_{QU}$, respectively, can be established by replacing $\mathcal{F}$-Anonymity with $\mathcal{Q}$-Anonymity in the set of axioms which characterizes the Suppes-Sen SWR $\succsim_S$ and the utilitarian SWR $\succsim_U$, respectively. We refer the reader to Propositions 2 and 3 in Banerjee (2006) for details, and also to Svensson (1980) and Asheim et al. (2001) for the axiomatization of the Suppes-Sen SWR and to Basu and Mitra (2007) for that of the utilitarian SWR. We limit ourselves to note that the Suppes-Sen SWR is characterized in terms of $\mathcal{F}$-Anonymity and Strong Pareto and the utilitarian SWR is of $\mathcal{F}$-Anonymity, Strong Pareto, and an invariance condition called Partial Unit Comparability.

The purpose of this paper is to explore other new $\mathcal{Q}$-anonymous SWRs and to provide the characterizations of them. The rest of the paper is devoted to this task.

4 Characterization of $\mathcal{Q}$-extension of simplified criterion

Banerjee's (2006) characterizations of the $\mathcal{Q}$-extensions of the Suppes-Sen and the utilitarian SWRs suggest that a $\mathcal{Q}$-extension of a certain kind of $\mathcal{F}$-anonymous SWR would be characterized if we require $\mathcal{Q}$-Anonymity, a stronger anonymity condition, to be satisfied instead of $\mathcal{F}$-Anonymity. We will show that this is true for the class of SWRs, called *simplified criterion*, that includes several well-established $\mathcal{F}$-anonymous SWRs such as the generalized Lorenz and the leximin SWRs as well as the Suppes-Sen and the utilitarian SWRs.

The simplified criterion was first categorized by d'Aspremont (2005). Let $\succsim^n$ denote a finite-horizon SWR defined on $\mathbb{R}^n$ and $(\succsim^n)_{n=1}^{\infty}$ be a sequence of this finite-horizon SWR. The simplified criterion is defined in terms of a sequence of a finite-horizon SWR and the Pareto criterion applied to infinite future generations. Formally, a simplified criterion generated by a sequence $(\succsim^n)_{n=1}^{\infty}$, denoted $\succsim_{(\succsim^n)}$, is defined as the following binary relation on X: for any $x, y \in X$,

$$x \succsim_{(\succsim^n)} y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}. \tag{1}$$

Each different simplified criterion $\succsim_{(\succsim^n)}$ is distinguished in accordance with the adopted finite-horizon SWR $\succsim^n$. A simplified criterion $\succsim_{(\succsim^n)}$ is a ranking rule that applies the finite-horizon SWR $\succsim^n$ to the first n generations and the Pareto criterion to the infinite future generations starting from $n + 1$ period. Obviously, the Suppes-Sen SWR $\succsim_S$ and the utilitarian SWR $\succsim_U$, respectively is the simplified criterion generated by the sequence of the finite-horizon Suppes-Sen SWR and the finite-horizon utilitarian SWR, respectively.

We provide a few remarks on a simplified criterion. First, there is no guarantee that a simplified criterion defined above in a general form will be a SWR even though the finite-horizon relation $\succsim^n$ is a SWR for each $n \in \mathbb{N}$. Indeed, there are some examples of finite-horizon SWR by which the generated simplified criterion fails to be SWR, particularly leads to a violation of transitivity (see Footnote 13).

Since our interest lies on a reflexive and transitive simplified criterion, we limit our analysis to those simplified criteria which are generated by a sequence of a finite-horizon SWR satisfying the following two properties to assure the generated simplified criterion to be a SWR: one is

Property 1: For all $n \in \mathbb{N}$ and all $x^{-n}, y^{-n} \in \mathbb{R}^n$ with $x^{-n} > y^{-n}$, $x^{-n} \succ^n y^{-n}$;

and the other is

Property 2: For all $n \in \mathbb{N}$, all $r \in \mathbb{R}$, and all $x^{-n}, y^{-n} \in \mathbb{R}^n$, if $x^{-n} \succsim^n y^{-n}$, then $(x^{-n}, r) \succsim^{n+1} (y^{-n}, r)$.

Property 1 is the well-known strong Pareto condition defined on $\mathbb{R}^n$, and Property 2 is a kind of separability condition, which requires that adding an unconcerned generation has no relevant impact on our evaluation and the ranking of the original pair of n-dimensional utility vectors must be respected in that of the new $(n + 1)$-dimensional vectors.[12] These two conditions together ensure that a generated

[12]Property 2 is similar to *Existence Independence* introduced by Blackorby et al. (2005) in the framework of variable population social choice.

simplified criterion be certainly a SWR.[13] We now state the following remark.

Remark 1. For any sequence of a finite-horizon SWR $(\succsim^n)_{n=1}^{\infty}$ which satisfies Properties 1 and 2, the generated simplified criterion $\succsim_{(\succsim^n)}$ is a SWR. This can be easily checked as follows. Reflexivity is straightforward. To confirm $\succsim_{(\succsim^n)}$ is transitive, consider any $x, y, z \in X$ such that $x \succsim_{(\succsim^n)} y$ and $y \succsim_{(\succsim^n)} z$. We will show $x \succsim_{(\succsim^n)} z$ holds. By definition, there exist $n, n' \in \mathbb{N}$ such that (i) $x^{-n} \succsim^n y^{-n}$ and $x^{+n} \geqslant y^{+n}$ and (ii) $y^{-n'} \succsim^{n'} z^{-n'}$ and $y^{+n'} \geqslant z^{+n'}$. Let $\bar{n} = \max\{n, n'\}$. Notice that $x^{+\bar{n}} \geqslant z^{+\bar{n}}$. Thus, we are enough to show that $x^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$. By Property 2, $x^{-n} \succsim^n y^{-n}$ implies that $(x^{-n}, y_{n+1}, \ldots, y_{\bar{n}}) \succsim^{\bar{n}} y^{-\bar{n}}$. By Property 1, $x^{-\bar{n}} \succsim^{\bar{n}} (x^{-n}, y_{n+1}, \ldots, y_{\bar{n}})$. Thus, transitivity of $\succsim^{\bar{n}}$ gives $x^{-\bar{n}} \succsim^{\bar{n}} y^{-\bar{n}}$. By the same argument, we also obtain $y^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$. Hence, by the transitivity of $\succsim^{\bar{n}}$, $x^{-\bar{n}} \succsim^{\bar{n}} z^{-\bar{n}}$ follows. ■

In the rest of the paper, we analyze a reflexive and transitive simplified criterion and its $\mathcal{Q}$-extension. To distinguish an arbitrary simplified criterion (which may not be SWR) and reflexive and transitive one, we hereafter use the term "simplified SWR" to mean a reflexive and transitive simplified criterion.

Next, we move to another remark on a simplified criterion. The following result is straightforward from the definition of a simplified criterion and we omit the easy proof.

Remark 2. A simplified criterion generated by a sequence $(\succsim^n)_{n=1}^{\infty}$ satisfies $\mathcal{F}$-Anonymity if and only if, for each case of $n \in \mathbb{N}$, the adopted finite-horizon SWR $\succsim^n$ satisfies the usual anonymity property defined on a finite domain $\mathbb{R}^n$: for all $x^{-n}, y^{-n} \in \mathbb{R}^n$, $x^{-n} \sim^n y^{-n}$ whenever x^{-n} is a permutation of y^{-n}. ■

We now introduce the definition of a $\mathcal{Q}$-extension of a simplified SWR generated by $(\succsim^n)_{n=1}^{\infty}$. A $\mathcal{Q}$-

[13] The following examples of simplified criteria show that neither of Properties 1 and 2 can be dropped. The following simplified criterion $\succsim^*$: $x \succsim^* y \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $\min_i x_i^{-n} + \sum_{i=1}^n x_i^{-n} \geq \min_i y_i^{-n} + \sum_{i=1}^n y_i^{-n}$ and $x^{+n} \geqslant y^{+n}$, satisfies Property 1 but not Property 2. $\succsim^*$ fails to be transitive. This can be checked as follows. For $x = (3, 3, 0, 0, \ldots), y = (0, 8, 0, 0, \ldots), z = (0, 7, 0, 0, \ldots)$, we have $x \sim^* y$; $y \succ^* z$; but $x \sim^* z$. Next, consider the following simplified criterion $\succsim^{**}$: $x \succsim^{**} y \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^n x_i^{-n} \leq \sum_{i=1}^n y_i^{-n}$ and $x^{+n} \geqslant y^{+n}$. This simplified criterion satisfies Property 2 but not Property 1. Then, it fails to be transitive: for $x = (0, 3, 0, 0, \ldots), y = (1, 1, 0, 0, \ldots), z = (2, 1, 0, 0, \ldots)$, we have $x \sim^{**} y$; $y \succ^{**} z$; but $x \sim^{**} z$.

extension of a simplified SWR $\succsim_{(\succsim^n)}$, denoted $\succsim_{\mathcal{Q}(\succsim^n)}$, is defined as follows: for all $x, y \in X$,

$$x \succsim_{\mathcal{Q}(\succsim^n)} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_{(\succsim^n)} y. \tag{2}$$

As will be shown later, $\succsim_{\mathcal{Q}(\succsim^n)}$ is a SWR.

The main result established in this paper is that any $\mathcal{Q}$-extension of a $\mathcal{F}$-anonymous simplified SWR is characterized by only replacing $\mathcal{F}$-Anonymity with $\mathcal{Q}$-Anonymity in the set of the conditions that characterizes the $\mathcal{F}$-anonymous simplified SWR in question. To state the result formally, let $\mathcal{A}(\succsim_{(\succsim^n)})$ denote the set of conditions which characterizes the simplified SWR $\succsim_{(\succsim^n)}$. We are now ready to state our main result.

Theorem 1. *Suppose that a simplified SWR $\succsim_{(\succsim^n)}$ on X satisfies $\mathcal{F}$-Anonymity. Then, a SWR $\succsim$ on X satisfies $\mathcal{Q}$-Anonymity and all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$ if and only if $\succsim_{\mathcal{Q}(\succsim^n)}$ is a subrelation to $\succsim$.*[14]

The proof proceeds through the following two lemmata.

Lemma 1. *For any $P \in \mathcal{Q}$ and any $x, y \in X$, we have*

$$x \succ_{(\succsim^n)} y \textit{ if and only if } Px \succ_{(\succsim^n)} Py \tag{3}$$

and

$$x \sim_{(\succsim^n)} y \textit{ if and only if } Px \sim_{(\succsim^n)} Py. \tag{4}$$

Proof. We will show that $x \succsim_{(\succsim^n)} y$ if and only if $Px \succsim_{(\succsim^n)} Py$, from which the equivalence assertions in (3) and (4) immediately follow.

(only if part) Assume $x \succsim_{(\succsim^n)} y$. Then, there exists $n \in \mathbb{N}$ such that

$$x^{-n} \succsim^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}.$$

[14]Recall that $\mathcal{Q}$-Anonymity logically implies $\mathcal{F}$-Anonymity. Thus, $\mathcal{F}$-Anonymity is now redundant and we can drop it if it is included in the list $\mathcal{A}(\succsim_{(\succsim^n)})$.

Without loss of generality, let P be a k-period cyclic permutation matrix. We want to show that $Px \succsim_{(\succsim^n)} Py$. Since $P \in \mathcal{Q} \subseteq \mathcal{S}$, we can find $\hat{n} \in \mathbb{N}$ such that $k\hat{n} \geq n$ and $P(k\hat{n})$ is a finite dimensional permutation matrix. Let $n' = k\hat{n}$. We consider the following profiles $x', y' \in X$:

$$x' = (x^{-n'}, (Px)^{+n'}) \text{ and } y' = (y^{-n'}, (Py)^{+n'}).$$

Since $x^{-n} = x'^{-n}$ and $y^{-n} = y'^{-n}$, we obtain $x'^{-n} \succsim^n y'^{-n}$. Moreover, from $x^{+n} \geqslant y^{+n}$, we have $(x_{n+1}, \ldots, x_{n'}) \geqslant (y_{n+1}, \ldots, y_{n'})$ and $(Px)^{+n'} \geqslant (Py)^{+n'}$, i.e. $x'^{+n} \geqslant y'^{+n}$ holds. Thus, $x' \succsim_{(\succsim^n)} y'$ follows. Since $\succsim_{(\succsim^n)}$ is $\mathcal{F}$-anonymous, $x' \sim_{(\succsim^n)} Px$ and $y' \sim_{(\succsim^n)} Py$. From transitivity of $\succsim_{(\succsim^n)}$, $Px \succsim_{(\succsim^n)} Py$ is obtained as desired.

(if part) Assume $Px \succsim_{(\succsim^n)} Py$. Since $P' \in \mathcal{Q}$, using the "only if" part of the lemma, we obtain $x \succsim_{(\succsim^n)} y$. ■

Lemma 2. *For any $x, y \in X$,*

$$x \succ_{\mathcal{Q}(\succsim^n)} y \textit{ if and only if there exists } P \in \mathcal{Q} \textit{ such that } Px \succ_{(\succsim^n)} y \tag{5}$$

and

$$x \sim_{\mathcal{Q}(\succsim^n)} y \textit{ if and only if there exists } P \in \mathcal{Q} \textit{ such that } Px \sim_{(\succsim^n)} y. \tag{6}$$

Proof. We prove the equivalence assertion in (5). The proof of (6) is straightforward from (5) and the definition of $\succsim_{\mathcal{Q}(\succsim^n)}$, and we omit easy proof.

Note that, by the definiton of $\succsim_{\mathcal{Q}(\succsim^n)}$, $x \succ_{\mathcal{Q}(\succsim^n)} y$ is equivalent to

$$\exists P \in \mathcal{Q} \text{ such that } Px \succsim_{(\succsim^n)} y \tag{7}$$

and

$$\forall Q \in \mathcal{Q}, \neg(Qy \succsim_{(\succsim^n)} x). \tag{8}$$

(only if part) Assume $x \succ_{\mathcal{Q}(\succsim^n)} y$. The proof is done by contradiction. Suppose that there exists $P \in \mathcal{Q}$ such that $Px \sim_{(\succsim^n)} y$. By (4), $x = P'(Px) \sim_{(\succsim^n)} P'y$. This implies $P'y \succsim_{(\succsim^n)} x$, which contradicts (8). By (7), there exists $P \in \mathcal{Q}$ such that $Px \succ_{(\succsim^n)} y$.

(if part) Assume that there exists $P \in \mathcal{Q}$ such that $Px \succ_{(\succsim^n)} y$. From the definition of $\succsim_{\mathcal{Q}(\succsim^n)}$, $x \succsim_{\mathcal{Q}(\succsim^n)} y$. We want to show $\neg(y \succsim_{\mathcal{Q}(\succsim^n)} x)$. The proof is done by contradiction. Suppose $y \succsim_{\mathcal{Q}(\succsim^n)} x$. Then, there exists $Q \in \mathcal{Q}$ such that $Qy \succsim_{(\succsim^n)} x$. By Lemma 1, $P(Qy) \succsim_{(\succsim^n)} Px$. From transitivity of $\succsim_{(\succsim^n)}$ and $Px \succsim_{(\succsim^n)} y$, $P(Qy) \succ_{(\succsim^n)} y$. Let R denote the product PQ. Note that $R \in \mathcal{Q}$. Without loss of generality, let R be a k-period cyclic permutation matrix. By the definition of $\succsim_{(\succsim^n)}$, there exists $n \in \mathbb{N}$ such that $(Ry)^{-n} \succsim^n y^{-n}$ and $(Ry)^{+n} \geqslant y^{+n}$. Since $R \in \mathcal{Q} \subseteq S$, we can find $n' \in \mathbb{N}$ such that $n' \geq n$ and $n' = k\hat{n}$ for some $\hat{n} \in \mathbb{N}$. The anonymous $\succsim^{n'}$ declares $(Ry)^{-n'} \sim^{n'} y^{-n'}$. If we have $(Ry)^{+n'} = y^{+n'}$, $Ry \sim_{(\succsim^n)} y$ follows, and a contradiction is obtained. Thus, $(Ry)^{+n'} > y^{+n'}$ holds. Now, we distinguish two cases: (i) the vector inequality $(Ry)^{+n'} > y^{+n'}$ contains finite strict inequalities, or (ii) it contains infinite strict inequalities. First, we consider the case (i). In this case, we can find $n'' \in \mathbb{N}$ such that $n'' \geq n'$, $n'' = k\hat{\hat{n}}$ for some $\hat{\hat{n}} \geq \hat{n}$, and $(Ry)^{+n''} = y^{+n''}$. Since $\succsim^{n''}$ is anonymous, $(Ry)^{-n''} \sim^{n''} y^{-n''}$. Therefore, by the definition of $\succsim_{(\succsim^n)}$, we have $Ry \sim_{(\succsim^n)} y$ which contradicts $Ry \succ_{(\succsim^n)} y$. Next, we examine the case (ii). In this case, we can find some $\tilde{n} > \hat{n}$ such that $(Ry_{k\tilde{n}+1}, \ldots, Ry_{k(\tilde{n}+1)}) > (y_{k\tilde{n}+1}, \ldots, y_{k(\tilde{n}+1)})$. This contradicts the fact that R is a k-period cyclic permutation matrix. ■

Proof of Theorem 1. (only if part) Assume that a SWR $\succsim$ on X satisfies $\mathcal{Q}$-Anonymity and all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. Recall that the unique inverse of $P \in \mathcal{Q}$ is denoted by P'. Assume $x \succ_{\mathcal{Q}(\succsim^n)} y$. We want to show $x \succ y$. By (5), there exists $P \in \mathcal{Q}$ such that $Px \succ_{(\succsim^n)} y$. Note that $\mathcal{Q}$-Anonymity implies $\mathcal{F}$-Anonymity. Thus, $\succsim$ now satisfies all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. Consequently, $\succsim$ includes $\succsim_{(\succsim^n)}$ as a subrelation. This together with $Px \succ_{(\succsim^n)} y$ implies $Px \succ y$. Since $\succsim$ is $\mathcal{Q}$-anonymous,

$x = P'(Px) \sim Px$. From transitivity of $\succsim$, $x \succ y$ is obtained as desired. Next, assume $x \sim_{\mathcal{Q}(\succsim^n)} y$. We want to show $x \sim y$. By (6), there exists $P \in \mathcal{Q}$ such that $Px \sim_{(\succsim^n)} y$. Since $\succsim_{(\succsim^n)}$ is a subrelation to $\succsim$, $Px \sim y$. By $\mathcal{Q}$-Anonymoity, $x = P'(Px) \sim Px$. Transitivity of $\succsim$ implies $x \sim y$. This shows that $\succsim_{\mathcal{Q}(\succsim^n)}$ is a subrelation to $\succsim$.

(if part) Assume that $\succsim_{\mathcal{Q}(\succsim^n)}$ is a subrelation to $\succsim$. We are enough to show that $\succsim_{\mathcal{Q}(\succsim^n)}$ satisfies $\mathcal{Q}$-Anonymity and all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. By Lemma 2 and the fact that $I \in \mathcal{Q}$, $\succsim_{(\succsim^n)}$ is a subrelation to $\succsim_{\mathcal{Q}(\succsim^n)}$, which in turn implies that $\succsim_{\mathcal{Q}(\succsim^n)}$ satisfies all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. As to $\mathcal{Q}$-Anonymity, it is obvious by the definition of $\succsim_{\mathcal{Q}(\succsim^n)}$ and the fact that $\succsim_{(\succsim^n)}$ is reflexive. ■

By Lemma 1, we can confirm that $\succsim_{\mathcal{Q}(\succsim^n)}$ is a SWR. Reflexivity is straightforward from the fact that $I \in \mathcal{Q}$ and $\succsim_{(\succsim^n)}$ is reflexive. We show that $\succsim_{\mathcal{Q}(\succsim^n)}$ is transitive. Assume that $x \succsim_{\mathcal{Q}(\succsim^n)} y$ and $y \succsim_{\mathcal{Q}(\succsim^n)} z$. Then, by definition, there exist $P, Q \in \mathcal{Q}$ such that $Px \succsim_{(\succsim^n)} y$ and $Qy \succsim_{(\succsim^n)} z$. By Lemma 1, $Px \succsim_{(\succsim^n)} y \Leftrightarrow Q(Px) \succsim_{(\succsim^n)} Qy$. Thus, the transitivity of $\succsim_{(\succsim^n)}$ gives $Q(Px) \succsim_{(\succsim^n)} z$, and $x \succsim_{\mathcal{Q}(\succsim^n)} z$ follows from the fact that $QP \in \mathcal{Q}$.

As discussed in Basu and Mitra (2007) and Banerjee (2006), our theorem 1 can be interpreted as saying that $\succsim_{\mathcal{Q}(\succsim^n)}$ is the least restrictive SWR among all the SWRs that satisfy $\mathcal{Q}$-Anonymity and all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. Formally, for all $x, y \in X$, we have

$$x \succsim_{\mathcal{Q}(\succsim^n)} y \text{ if and only if } x \succsim y \text{ for all } \succsim \in \Xi_{\mathcal{Q}(\succsim^n)}, \tag{9}$$

where $\Xi_{\mathcal{Q}(\succsim^n)}$ is the set of all SWRs that satisfy $\mathcal{Q}$-Anonymity and all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$. The only if part of (9) is obvious from the only if statement of the theorem. The if part of (9) is also straightforward from the fact that $\succsim_{\mathcal{Q}(\succsim^n)} \in \Xi_{\mathcal{Q}(\succsim^n)}$ (this fact is easily checked: $\mathcal{Q}$-Anonymity is obvious by the definition of $\succsim_{\mathcal{Q}(\succsim^n)}$; for the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$, recall that $I \in \mathcal{Q}$).

Our characterization of a $\mathcal{Q}$-extension of a simplified SWR is established in a general form, and it can be applicable to any $\mathcal{F}$-anonymous simplified criterion. The next section provides some characterizations of $\mathcal{Q}$-extensions of $\mathcal{F}$-anonymous simplified SWRs as corollaries.

5 Corollaries

5.1 Characterizations of $\mathcal{Q}$-extensions of the generalized Lorenz and the leximin SWRs

In this section, we provide some corollaries to Theorem 1. We start with introducing an additional notation. For $n \in \mathbb{N}$ and $x \in X$, let $(x^{-n}_{(1)}, x^{-n}_{(2)}, \ldots, x^{-n}_{(n)})$ denote a rank-ordered permutation of x^{-n} such that $x^{-n}_{(1)} \le x^{-n}_{(2)} \le \cdots \le x^{-n}_{(n)}$, ties being broken arbitrarily.

In their recent paper, Bossert et al. (2007) proposed the two simplified SWRs called the generalized Lorenz SWR and the leximin SWR, and also they established the characterizations of them.

The generalized Lorenz SWR $\succsim_G$ is defined as follows: for all $x, y \in X$,

$$x \succsim_G y \Leftrightarrow \begin{cases} \text{there exists } n \in \mathbb{N} \text{ such that } \sum_{t=1}^{k} x^{-n}_{(t)} \ge \sum_{t=1}^{k} y^{-n}_{(t)} \text{ for all } k \in \{1, \ldots, n\}, \text{ and} \\ x^{+n} \geqslant y^{+n}. \end{cases}$$

For all $n \in \mathbb{N}$, let $\succsim^n_L$ denote the leximin social welfare relation defined on $\mathbb{R}^n$ as follows: for all $x^{-n}, y^{-n} \in \mathbb{R}^n$,

$$x^{-n} \succsim^n_L y^{-n} \Leftrightarrow \begin{cases} x^{-n} \text{ is a permutation of } y^{-n}, \text{ or} \\ \text{there exists } m \in \{1, \ldots, n\} \text{ such that } x^{-n}_{(t)} = y^{-n}_{(t)} \text{ for all } t < m \text{ and } x^{-n}_{(m)} > y^{-n}_{(m)}. \end{cases}$$

The leximin SWR $\succsim_L$ is defined as: for all $x, y \in X$,

$$x \succsim_L y \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim^n_L y^{-n} \text{ and } x^{+n} \geqslant y^{+n}.$$

Bossert et al. (2007) characterizes the generalized Lorenz and the leximin SWRs using the following consequentialist equity conditions.

Pigou-Dalton Equity Principle: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \le x_j < y_j$ and $x_i - y_i = y_j - x_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Hammond Equity Principle: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $y_i < x_i \le x_j < y_j$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Both two axioms are stated in a conflicting situation between only two generations.[15] Pigou-Dalton Equity Principle asserts that an order-preserving transfer from a relatively better-off generation to a relatively worse-off generation, i.e. the well-known Pigou-Dalton transfer, is strictly socially preferable. Hammond Equity Principle is an infinite-horizon extension of the equity condition formulated by Hammond (1976, 1979) in the finite population case. This condition is logically stronger than Pigou-Dalton Equity Principle. It asserts that an order-preserving change which diminishes inequality of utilities between conflicting two generations is strictly socially improving, whereas Pigou-Dalton Equity Principle agrees to such a value judgment as long as utility differences of conflicting two generations are equal.

Bossert et al. (2007) established the following characterizations of the generalized Lorenz and the leximin SWRs.

Proposition 1 (Bossert et al. (2007), Theorems 1 and 2)**.** (i) *A SWR $\succsim$ on X satisfies Strong Pareto, $\mathcal{F}$-Anonymity, and Pigou-Dalton Equity Principle if and only if $\succsim_G$ is a subrelation to $\succsim$.*
(ii) *A SWR $\succsim$ on X satisfies Strong Pareto, $\mathcal{F}$-Anonymity, and Hammond Equity Principle if and only if $\succsim_L$ is a subrelation to $\succsim$.*

Since the generalized Lorenz and the leximin SWRs, respectively, is the simplified SWRs generated by the sequence of the finite-horizon generalized Lorenz SWR and of the finite-horizon leximin SWR respectively, we can apply our Theorem 1 to establish the characterizations of the $\mathcal{Q}$-extensions of these two simplified SWRs. In view of the definition of the $\mathcal{Q}$-extension of a simplified SWR in (2), the definitions of their $\mathcal{Q}$-extensions are straightforward.

$\mathcal{Q}$-generalized Lorenz relation, denoted $\succsim_{QG}$, is defined as follows: for all $x, y \in X$,

$$x \succsim_{QG} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_G y.$$

[15] The weak versions of the conditions are also considered in the literrature. On this, we refer the reader to d'Aspremont and Gevers (2002) for a finite population case and to Bossert et al. (2007) for an infinite case.

$\mathcal{Q}$-leximin relation $\succsim_{QL}$ is defined as follows: for all $x, y \in X$,

$$x \succsim_{QL} y \Leftrightarrow \text{there exists } P \in \mathcal{Q} \text{ such that } Px \succsim_L y.$$

We now state the following characterizations as corollaries to our Theorem 1.

Corollary 1. (i) *A SWR $\succsim$ on X satisfies Strong Pareto, $\mathcal{Q}$-Anonymity, and Pigou-Dalton Equity Principle if and only if $\succsim_{QG}$ is a subrelation to $\succsim$.*
(ii) *A SWR $\succsim$ on X satisfies Strong Pareto, $\mathcal{Q}$-Anonymity, and Hammond Equity Principle if and only if $\succsim_{QL}$ is a subrelation to $\succsim$.*

5.2 Alternative characterization of $\mathcal{Q}$-utilitarian SWR

Finally, we provide an alternative characterization of the $\mathcal{Q}$-extension of the utilitarian SWR. We now introduce an infinite-horizon extension of the *Incremental Equity* which was first proposed by Blackorby et al. (2002) in a finite population setting.

Incremental Equity Principle: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that (i) $x_i - y_i = y_j - x_j > 0$, and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \sim y$.

It is easily checked that Incremental Equity Principle logically implies $\mathcal{F}$-Anonymity. In contrast to Pigou-Dalton Equity Principle, this condition asserts much stronger and somewhat controversial value judgment that, for any transfer among two generations, the initial utility stream and the post-transfer stream are considered to be equally good without any reference to the relative utility levels of conflicting two generations.

The following proposition tells that this condition clearly distinguishes the utilitarian SWR from the other strongly Paretian SWRs.

Proposition 2. *A SWR $\succsim$ on X satisfies Strong Pareto and Incremental Equity Principle if and only if $\succsim_U$ is a subrelation to $\succsim$.*

Proof. Step 1.[16] We show that $x \sim y$ holds whenever there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $x^{+n} = y^{+n}$. The case of $n = 1$ is trivial. If $n \geq 2$, consider the following operation: fix $n' \in \{1, \ldots, n-1\}$ arbitrarily and construct $x' \in X$ as follows: $x'_{n'} = y_{n'}$ for n'; $x'_n = x_n + x'_{n'} - y_{n'}$ for n; $x'_k = x_k$ for all $k \in \mathbb{N} \setminus \{n', n\}$. Applying the above operation once for each $n' \in \{1, \ldots, n-1\}$ repeatedly, we can construct the profile y. Incremental Equity Principle and transitivity of $\succsim$ together conclude $x \sim y$.

Step 2. From Step 1, it is obvious that $x \sim_U y \Rightarrow x \sim y$. We show $x \succ_U y \Rightarrow x \succ y$. Assume $x \succ_U y$. Then, there exists $n \in \mathbb{N}$ such that $(\sum_{i=1}^{n} x_i, x^{+n}) > (\sum_{i=1}^{n} y_i, y^{+n})$. We can find $n' \geq n$ such that $\sum_{i=1}^{n'} x_i > \sum_{i=1}^{n'} y_i$ and $x^{+n'} \geqslant y^{+n'}$. Define $z \in X$ as $z_1 = y_1 + \sum_{i=1}^{n'}(x_i - y_i)$, $z_i = y_i$ for all $i \in \{2, 3, \ldots, n'\}$, and $z_j = x_j$ for all $j \in \{n'+1, \ldots\}$. From the result in Step 1, $x \sim z$. By Strong Pareto, $z \succ y$. From transitivity of $\succsim$, $x \succ y$. ■

In view of Proposition 2, we immediately obtain the following characterization of the $\mathcal{Q}$-utilitarian SWR as a corollary to Theorem 1.

Corollary 2. *A SWR $\succsim$ on X satisfies Strong Pareto, $\mathcal{Q}$-Anonymity, and Incremental Equity Principle if and only if $\succsim_{QU}$ is a subrelation to $\succsim$.*

6 Concluding Remarks

In this paper, we established the characterization of a $\mathcal{Q}$-extension of a simplified SWR in a general form. Our characterization makes clear the logical relationship between a simplified SWR and its $\mathcal{Q}$-extension. From Theorem 1, the difference between a simplified SWR and its $\mathcal{Q}$-extension is solely ascribed to the extent of impartial treatment of infinitely many generations: in a $\mathcal{Q}$-anonymous way or only in a $\mathcal{F}$-anonymous way. In other words, if we require the stronger notion of impartiality formalized as $\mathcal{Q}$-Anonymity to be reflected in addition to the conditions characterizing a certain $\mathcal{F}$-anonymous simplified SWR, we must respect the ranking determined by the $\mathcal{Q}$-extension of the simplified SWR in question. As we briefly mentioned in the introduction, from Arrow's (1963) variant of Szpilrajn's (1930) theorem,

[16]The result we demonstrate in this step is well-known fact. See, for example, Blackorby et al. (2002, 2005) for the finite society case, and also Asheim and Tungodden (2004) for the infinite horizon case.

we can conclude that, for any $\mathcal{F}$-anonymous simplified SWR $\succsim_{(\succsim^n)}$, there exists an ordering on infinite utility streams which satisfies all the conditions in $\mathcal{A}(\succsim_{(\succsim^n)})$ and $\mathcal{Q}$-Anonymity. Therefore, our result can be regarded as a stronger existence theorem of strongly Paretian and anonymous ordering than the existing characterizations of $\mathcal{F}$-anonymous simplified SWRs, which could be seen as an alternative and stronger resolution to Diamond's (1965) impossibility result as well.

A $\mathcal{Q}$-extension of a simplified SWR is one plausible way to improve the completeness of the simplified SWR. A $\mathcal{Q}$-extension of a simplified SWR is, however, still incomplete welfare relation. Thus, there still be room for improvement to the completeness of social evaluation. As we noted earlier, Mitra and Basu (2005) showed that the largest class of infinite permutation matrices compatible with Strong Pareto is a group of cyclic permutation matrices, i.e. $\mathcal{Q}$. Therefore, if a social planner thinks of Strong Pareto and $\mathcal{Q}$-Anonymity as indispensable condition to be satisfied in the social evaluation of intergenerational issues, more complete social evaluation would be possible only through strengthening or additionally employing conditions other than Strong Pareto and $\mathcal{Q}$-Anonymity. This task is left for future work.

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