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Bidding for social alternatives: a simple one-stage mechanism and its extension

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Abstract

We examine two alternative group decision mechanisms in a standard quasi-linear environment: a simple one-stage bidding mechanism and a multi-stage bidding mechanism. First, we show that the simple one-stage bidding mechanism works only in limited situations and may fail to realize an efficient social alternative. Then, we alternatively propose a multi-stage mechanism and show that there always exists a subgame perfect equilibrium in our multi-stage bidding mechanism, i.e. it does always work, and also that in any subgame perfect equilibrium of the mechanism a socially efficient alternative is realized.

JEL Classification: C72; D71

Keywords: Bidding mechanism; Social choice; Implementation

1 Introduction

In the recent literature on cooperative game theory, the bidding-based mechanisms where individuals simultaneously reveal their willingness to pay to each of the other individuals have been intensively explored to implement cooperative solutions in the cost/surplus-sharing problems. In the pioneering work of Pérez-Castrillo and Wettstein [7], they formulated the mechanism involving a bidding stage where each individual is required to reveal a payable bid to each of the other individuals in exchange for obtaining the right to be a proposer in the subsequent stages, and they showed that in its subgame perfect equilibrium the Shapley value is realized as the final payoff to each individual. Vidal-Puga and Bergantiños [11] examined the extension of the mechanism considered in [7] and succeeded in implementing the Owen value. The bidding-based mechanism is also considered in the paper of Mutuswami et al. [5] in the context of a local public goods provision and in Pérez-Castrillo and Wettstein [9] in the context of an efficient network formation, respectively. In each of the papers [5] and [9], it has been shown that their mechanism realizes the Shapley value as the equilibrium payoff in an appropriately defined cooperative game. Macho-Stadler et al. [2] considered the case where the surplus generated by individuals' cooperation partly depends on the externality by the individuals who does not participate in the cooperation. They devised two mechanism, one for each case of positive externality or negative one, and showed that the payoffs in the subgame perfect equilibrium of each of the mechanisms coincide with the extension of the Shapley value formulated by the average approach established in their companion paper [3]. Finally, in the recent paper of Slikker [10], the bidding mechanism is applied to implement network allocation rules such as the Myerson value, the position value, and the component-wise egalitarian solution.

The purpose of this paper is to examine possible applications of the bidding-based mechanism in the context of collective choice problem of social alternatives. Examples include a location of a public facility such as a public school, a disposal center, and a nuclear-related facility. We consider a standard quasi-linear environment with finite social alternatives each of which affects all of individuals in a society. The two companion papers of Pérez-Castrillo and Wettstein [6, 7] are the first to address such a collective choice problem. In their papers, they analyzed how to choose a single alternative among all the individuals' most favorite ones. They formulated a one-stage bidding mechanism and proved that their mechanism always realizes the most efficient alternative among all the individuals' proposals. In the present paper, we consider a slightly generalized group decision problem. We consider the case where the feasible social alternatives are not restricted to the individuals' most favorite ones. In a quasi-linear environment, the concept of efficiency can be given as the maximization of sum of individuals' utilities generated by the alternative. Therefore, assuming that the individuals are self-regarding utility maximizers, there is no guarantee that the individuals' most favorite alternatives are socially efficient ones, and it will be the case that there is an alternative that none of the individuals most prefers but in which the highest total utility is achieved. In this paper, we examine two alternative bidding mechanisms to realize a socially efficient alternative: one is a simple one-stage mechanism, and the other is a multi-stage mechanism.

In Section 2, we first examine a simple one-stage bidding mechanism. In our one-stage mechanism, each individual is asked to make her/his bid to each of the other individuals, and one individual, called a proposer, is chosen based on the bids and s/he obtains the right to determine which of the alternatives will be realized in the society. The proposer is determined in the following way. For each individual, the net bid is calculated as the sum of her/his bids to the other individuals minus the sum of the bids by the other individuals to her/him. Then, the individual with the highest net bid becomes the proposer (if there exist more than one, then randomly chosen). Our first main result is that this simple mechanism works only in limited situations where the individuals' most favorite alternatives generate the same sums of individuals' utilities. Moreover, even in the cases where the mechanism can successfully work, it may fail to realize socially efficient alternative as an equilibrium outcome.

In order to realize a socially efficient alternative, we examine an alternative bidding mechanism in Section 3. We modify the simple one-stage mechanism in Section 2 and formulate a three-stage mechanism. The first stage of the mechanism is the same as in the one-stage mechanism. In contrast to the one-stage mechanism, after a proposer is determined, the modified mechanism moves to the second stage where the proposer makes an offer composed of a social alternative that s/he wants to realize and monetary transfers to the other individuals. Then, in the final stage, the individuals other than the proposer sequentially reply "to accept the offer" or "to reject it." If the offer is unanimously accepted, the monetary transfers are carried out and the proposed social alternative is realized. In the case where the offer is rejected by at least one individual, the monetary transfers are not carried out and the proposer's most favorite alternative is realized. We show that this multi-stage bidding mechanism works in any situation and always realizes a socially efficient alternative in any of its subgame perfect equilibria.

Finally, we discuss some applications of the bidding mechanism to the implementation of solutions established in cooperative game theory and also suggest how we can rearrange these mechanisms in the framework of collective choice problem we consider in the paper.

2 Bidding mechanism

Let $N = \{1, \ldots, n\}$ be the set of n individuals. X is the finite set of social alternatives. We assume that each individual's utility function U_i is linearly separable with respect to money, i.e. there exists $u_i : X \to \mathbb{R}$ such that, for all $(x, m) \in X \times \mathbb{R}$,

$$U_i(x, m) = u_i(x) + m. \tag{1}$$

For any $i \in N$, let x^i denote the individual i's most favorite alternative, i.e. $x^i \in \arg\max_{x \in X} \{u_i(x)\}$. We define a subset $X^f \subseteq X$ as $X^f = \{x^i : i \in N\}$, i.e. the set of individuals' most favorite alternatives. We consider the social decision problem where we have to choose a single alternative among the feasible alternatives X. In order to reconcile individuals' conflicting interests and to determine a single socially best alternative, we consider the following one-stage *bidding mechanism*:

i). Each individual i makes bids $b^i = (b^i_1, \ldots, b^i_{i-1}, b^i_{i+1}, \ldots, b^i_n) \in \mathbb{R}^{n-1}$, one for each $j \neq i$.

ii). For each i, calculate a net bid $B^i := \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i$.

iii). An individual with the highest net bid becomes a proposer and obtain the right to determine an alternative x^* which is realized in the society in return for the actual payment of her/his bids to the other individuals. If we have more than one individuals with the highest net bid, the proposer is randomly chosen among them.

The proposer is denoted by α, i.e. $\alpha \in \arg\max_{i \in N}\{B^i\}$. Obviously, in an equilibrium of this simple mechanism, if exists, the proposer α chooses her/his most favorite alternative, i.e. $x^* = x^\alpha$. Thus, the final payment to the proposer α is given as $u_\alpha(x^\alpha) - \sum_{j \neq \alpha} b_j^\alpha$ and the one to each $j \neq \alpha$ as $u_j(x^\alpha) + b_j^\alpha$.

The one-stage bidding mechanism defined above can be seen as a normal form game. We denote this normal form game by $B(N, X, (U_i)_{i \in N})$. We now examine a Nash equilibrium of this mechanism. The following lemmata 1 and 2 show that we still have the same equilibrium properties as obtained in the mechanism of Pérez-Castrillo and Wettstein [7].

Lemma 1. *For any equilibrium strategy profile $(b^i)_{i \in N}$, every individual's net bid must be zero, i.e. $B^i = 0$ for any $i \in N$.*

Proof. Let $\Omega := \{i : B^i \geq B^j \ \forall j \in N\}$. If $\Omega = N$, the fact that $\sum_{i \in N} B^i = 0$ trivially implies $B^i = 0$ for each $i \in N$. We now show that, for any equilibrium strategy profile $(b^i)_{i \in N}$, $\Omega = N$ follows. We prove this by contradiction. Suppose that $\Omega \subseteq N$ and $\Omega \neq N$. Then, we can find two distinct individuals $i \in \Omega$ and $k \in N \backslash \Omega$. Let $\delta > 0$, and consider the following profile $(\hat{b}^i)_{i \in N}$ such that $\hat{b}^j = b^j \ \forall j \neq i$, and $\hat{b}^i_j = b^i_j + \delta/|\Omega|$ if $j \in \Omega \backslash \{i\}$; $\hat{b}^i_j = b^i_j - \delta$ if $j = k$; $\hat{b}^i_j = b^i_j$ otherwise. The new net bids are $\hat{B}^j = B^j - \delta/|\Omega| \ \forall j \in \Omega$; $\hat{B}^k = B^k + \delta$; $\hat{B}^j = B^j \ \forall j \in N \backslash (\Omega \cup \{k\})$. Since $B^j > B^l$ holds for any $j \in \Omega$ and any $l \in N \backslash \Omega$, we still obtain $\hat{B}^j > \hat{B}^l$ for sufficiently small δ. Thus, $\hat{\Omega} := \{i : i = \hat{B}^i \geq \hat{B}^j \ \forall j \in N\}$ completely coincides with Ω. However, for the individual i, we have $\sum_{j \neq i} \hat{b}^i_j < \sum_{j \neq i} b^i_j$, and thus, her/his new strategy $\hat{b}^i$ increases her/his expected final payoff. ■

Lemma 2. *For any equilibrium strategy profile $(b^i)_{i \in N}$, each player receives the same final payoff regardless of who becomes a proposer.*

Proof. From Lemma 1, each individual's net bid coincides with each other in the equilibrium. Thus, every individual could become a proposer with the same probability. We prove the contrapositive of the lemma. Suppose that some individual i could get the highest payoff in the case where s/he is a proposer than in the case where some other individual is a proposer. Then, sufficiently small increases in her/his bids to the other individuals improve her/his final payoff so that s/he will deviate from the equilibrium strategy. Similarly, if the individual i could obtain the highest payoff when some other individual j is a proposer than in the other cases, s/he has an incentive to decrease her/his bid to the individual j. ■

The two lemmata have an important and somewhat undesirable implication about the existence of an equilibrium of the one-stage mechanism. Let $(b^i)_{i\in N}$ be any equilibrium strategy profile. We now demonstrate that a strong condition on the individuals' proposals is needed to ensure the existence of an equilibrium in the mechanism. From Claim 1, B^i must be zero for each $i \in N$, or equivalently, we have

$$\sum_{j\neq i} b^i_j = \sum_{j\neq i} b^j_i, \quad \forall i \in N, \tag{2}$$

and thus every player could become a proposer with a strictly positive probability. Moreover, from Lemma 2, each player receives the same final payoff regardless of who becomes a proposer. Therefore, for any two distinct individuals $j,k \in N$, the final payoff to an individual $i \neq j,k$ must be the same regardless of which of the individuals, j and k, becomes a proposer:

$$u_i(x^k) + b^k_i = u_i(x^j) + b^j_i. \tag{3}$$

And moreover, each player i is indifferent between the case where i her/himself becomes a proposer and the case where some other player k is a proposer. Thus, by (2) and (3), we have the following:

$$\begin{aligned} u_i(x^k) + b^k_i &= u_i(x^i) - \sum_{j\neq i} b^i_j = u_i(x^i) - \sum_{j\neq i} b^j_i = u_i(x^i) - \left[b^k_i + \sum_{j\neq i,k} \left(b^k_i + u_i(x^k) - u_i(x^j) \right) \right] \\ &= u_i(x^i) - (n-1)b^k_i - (n-2)u_i(x^k) + \sum_{j\neq i,k} u_i(x^j). \end{aligned} \tag{4}$$

Hence, we obtain $nb^k_i = \sum_{j\in N} u_i(x^j) - nu_i(x^k)$, and then the equilibrium bid by k to i must be uniquely determined as follows:

$$b^k_i = \frac{1}{n} \sum_{j\in N} u_i(x^j) - u_i(x^k). \tag{5}$$

On the other hand, Lemma 1 tells that we have $\sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i = 0$, $\forall i \in N$. Consequently, by (5), the following two must be the same value:

$$\sum_{j\neq i} b^i_j = \sum_{j\neq i}\left(\frac{1}{n}\sum_{k\in N} u_j(x^k) - u_j(x^i)\right), \tag{6}$$

and

$$\sum_{j\neq i} b^j_i = \sum_{j\neq i}\left(\frac{1}{n}\sum_{k\in N} u_i(x^k) - u_i(x^j)\right). \tag{7}$$

It is obvious that (6) and (7) are not always compatible with each other. From this observation, we obtain the following proposition.

Proposition 1. *If a bidding game $B(N, X, (U_i)_{i\in N})$ has a Nash equilibrium, then the following condition must hold:*

$$\sum_{k\in N} u_k(x^i) = \sum_{k\in N} u_k(x^j), \quad \forall i, j \in N. \tag{8}$$

Proof. From (6) and (7), respectively, we have

$$\sum_{j\neq i} b^i_j = \sum_{j\neq i}\left(\frac{1}{n}\sum_{k\in N} u_j(x^k) - u_j(x^i)\right) = \frac{1}{n}\sum_{j\in N}\sum_{k\in N} u_j(x^k) - \frac{1}{n}\sum_{k\in N} u_i(x^k) - \sum_{j\neq i} u_j(x^i), \tag{9}$$

and

$$\sum_{j\neq i} b^j_i = \sum_{j\neq i}\left(\frac{1}{n}\sum_{k\in N} u_i(x^k) - u_i(x^j)\right) = \frac{n-1}{n}\sum_{k\in N} u_i(x^k) - \sum_{j\neq i} u_i(x^j). \tag{10}$$

Substituting (9) and (10) into (2), we obtain the following:

$$\begin{aligned}\sum_{i\neq j} b^i_j = \sum_{j\neq i} b^j_i &\Leftrightarrow \frac{1}{n}\sum_{j\in N}\sum_{k\in N} u_j(x^k) = \sum_{k\in N} u_i(x^k) + \sum_{j\neq i} u_j(x^i) - \sum_{j\neq i} u_i(x^j) \\ &\Leftrightarrow \frac{1}{n}\sum_{j\in N}\sum_{k\in N} u_j(x^k) = \sum_{j\in N} u_j(x^i). \end{aligned} \tag{11}$$

The LHS of the last equation is the constant determined independently of the individual i. Thus, the proof is completed. ■

Proposition 1 provides the necessary condition of the existence of an equilibrium of the bidding mechanism $B(N,X,(U_i)_{i\in N})$. It shows that there exists an equilibrium of the bidding mechanism $B(N,X,(U_i)_{i\in N})$ only in the case where each individual's best alternative x^i gives rise to the same sum of the individuals' utilities, i.e. the condition (8). The next proposition shows that the converse assertion is also true, i.e. if the bidding mechanism satisfies this condition then there exists an equilibrium in the mechanism.

Proposition 2. *For any bidding game $B(N,X,(U_i)_{i\in N})$ that satisfies the following condition:*

$$\sum_{k\in N} u_k(x^i) = \sum_{k\in N} u_k(x^j), \quad \forall i,j\in N,$$

there exists a Nash equilibrium. Moreover, under the above condition, the equilibrium bids are determined by the equation (5) and the equilibrium payoff to $i\in N$ is given as:

$$\frac{1}{n}\sum_{k\in N} u_i(x^k).$$

Proof. Let $(b^i)_{i\in N}$ be the profile of the bids defined in (5), i.e. $b_i^k = \frac{1}{n}\sum_{j\in N} u_i(x^j) - u_i(x^k)\ \forall i,k\in N$. We will show that $(b^i)_{i\in N}$ is a Nash equilibrium. It is easy to verify that $\sum_{j\neq i} b_j^i = \sum_{j\neq i} b_i^j$ follows for any $i,j\in N$, i.e. the net bid B^i is equal 0 for each $i\in N$, because $\sum_{k\in N} u_k(x^i) = \sum_{k\in N} u_k(x^j)$ holds for all $\forall i,j\in N$. Therefore, every individual could become a proposer with the same probability. We now show that each individual's final payoff is the same regardless of who becomes a proposer. Fix an individual i arbitrarily. If some other individual j becomes a proposer, the final payoff to the individual i is

$$u_i(x^j) + b_i^j = u_i(x^j) + \frac{1}{n}\sum_{k\in N} u_i(x^k) - u_i(x^j) = \frac{1}{n}\sum_{k\in N} u_i(x^k). \quad (12)$$

On the other hand, if the individual i her/himself becomes a proposer, s/he will gain

$$
\begin{aligned}
u_i(x^i) - \sum_{j\neq i} b^i_j &= u_i(x^i) - \sum_{j\neq i}\left(\frac{1}{n}\sum_{k\in N} u_j(x^k) - u_j(x^i)\right) \\
&= \sum_{j\in N} u_j(x^i) - \frac{1}{n}\sum_{k\in N}\left(\sum_{j\neq i} u_j(x^k) + u_i(x^k)\right) + \frac{1}{n}\sum_{k\in N} u_i(x^k) \\
&= \sum_{j\in N} u_j(x^i) - \frac{1}{n}\left(n\sum_{j\in N} u_j(x^k)\right) + \frac{1}{n}\sum_{k\in N} u_i(x^k) = \frac{1}{n}\sum_{k\in N} u_i(x^k) \quad \text{by (8).} \qquad (13)
\end{aligned}
$$

By the same argument as in the proof of Lemma 2, every individual has no incentive to deviate from the strategy profile $(b^i)_{i\in N}$.

As we have demonstrated, in any Nash equilibrium, the equilibrium bids are uniquely determined by (5). By (12) and (13), the final payoff to an individual i is given as $\frac{1}{n}\sum_{k\in N} u_i(x^k)$ for each $i \in N$. ■

From Propositions 1 and 2, we obtain the following characterization of the existence of an equilibrium in the one-stage bidding mechanism.

Theorem 1. *The bidding mechanism defined as the normal form game $B(N, X, (U_i)_{i\in N})$ has a Nash equilibrium if and only if the following condition holds:*

$$\sum_{k\in N} u_k(x^i) = \sum_{k\in N} u_k(x^j), \quad \forall i, j \in N.$$

Under the assumption of quasi-linearity of individuals' utility functions, we can define the following efficiency property.

X^f-efficiency: An alternative $x \in X^f$ is said to be *X^f-efficient* if

$$\sum_{k\in N} u_k(x) \geq \sum_{k\in N} u_k(x^j), \quad \forall j \in N.$$

From Theorem 1, we immediately obtain the following result.

Corollary 1. *In an equilibrium of the bidding mechanism $B(N,X,(U_i)_{i\in N})$, the realized social alternative $x^* = x^\alpha$ is X^f-efficient.*

More generalized welfare property is also considered as follows.

Social efficiency: An alternative $x \in X$ is said to be *socially efficient* if, for any $y \in X$,

$$\sum_{k\in N} u_k(x) \geq \sum_{k\in N} u_k(y).$$

As we have noted in the introduction, the social alternative realized in an equilibrium of the bidding mechanism may fail to be socially efficient. To make sure of this point, we give the following example. Suppose that $N = \{1,2\}$, $X = \{x,y,z\}$, and

$$u_1(x) = 4, \quad u_1(y) = 1, \quad u_1(z) = 3,$$
$$u_2(x) = 1, \quad u_2(y) = 4, \quad u_2(z) = 3.$$

In this case, $x^1 = x$ and $x^2 = y$, and $\sum_{i\in N} u_i(x^1) = \sum_{i\in N} u_i(x^2)$ holds. Thus, either x or y is realized as an equilibrium outcome of the mechanism. However, neither of them are socially efficient because z gives rise to $\sum_{i\in N} u_i(z) = 6 > 5$.

In this section, we examined the simple one-stage bidding mechanism and found that there are two problems concerning this simple mechanism: one is that an equilibrium may fail to exist in general, and the other is that an equilibrium outcome may not be socially efficient even when an equilibrium of the mechanism does exist. This motivates us to explore the mechanism which always works in any situation and realizes a socially efficient alternative. In the next section, we modify the one-stage bidding mechanism and show that our modified mechanism works in any situation and realizes a socially efficient alternative as an equilibrium outcome.

3 Modified bidding mechanism

In the preceding section, we found that our simple one-stage mechanism can not always lead to an equilibrium. Moreover, this mechanism may fail to realize a socially efficient alternative. In their paper, Pérez-Castrillo and Wettstein [6] have successfully avoided the former problem, i.e. the non-existence of an equilibrium, by the use of an extended strategy space.[1] However, since their one-stage mechanism is designed in the framework aimed at choosing a single proposal from all the individuals' ones, not from all feasible alternatives, the direct application of their one-stage mechanism in our model still fails to realize a socially efficient alternative in the case of the example we considered in the preceding section. In order to resolve each of the two problems of our one-stage bidding mechanism, we modify the one-stage mechanism. Our modified bidding mechanism proceeds in three stages:

Stage 1. The first stage is similar to our one-stage mechanism. Each individual i makes a bid $b^i_j \in \mathbb{R}$ for each $j \neq i$, i.e. $(b^i_j)_{j\neq i} \in \mathbb{R}^{n-1}$. For each i, calculate a net bid $B^i := \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i$. A proposer α is randomly chosen among those with the highest net bid. The proposer α pays her/his bid b^α_j to each $j \neq \alpha$.

Stage 2. The proposer α makes an offer composed of a social alternative x s/he wants to realize and the payments $(t_j)_{j\neq\alpha}$, i.e. $(x,(t_j)_{j\neq\alpha}) \in X \times \mathbb{R}^{n-1}$. The payment t_j is interpreted as a transfer (resp. claim) to an individual j if it is positive (resp. negative).

Stage 3. Every individual other than α sequentially replies "*to accept the offer*" or "*to reject it*." If we have a rejection by at least one individual, the offer is rejected. Otherwise, the offer is accepted. In the case of acceptance, the proposed alternative x is realized and the payment t_j is carried out between the proposer α and each $j \neq \alpha$. If the offer is rejected, the payments offered in Stage 2 are not carried out and x^α is realized.

The only difference between the one-stage mechanism in the last section and the three-stage mechanism defined above is the monetary transfers offered in Stage 2 of the three-stage mechanism. In this three-stage

[1] In their one-stage bidding mechanism, each individual announces either of 1 or 0, interpreted as "really want" or "not," as well as her/his bids to the other individuals. Consequently, the strategy space of each individual becomes $\mathbb{R}^{n-1} \times \{0,1\}$. A proposer is randomly chosen among the individuals who have the highest net bid and announces 1. If the set of such individuals is empty, a proposer is randomly chosen among the individuals with the highest net bid.

mechanism, each individual receives the following final payment:

$$\text{an offer is accepted} \quad \Rightarrow \quad \begin{cases} \alpha & : \ u_\alpha(x) - \sum_{j\neq\alpha} b_j^\alpha - \sum_{j\neq\alpha} t_j \\ j \neq \alpha & : \ u_j(x) + b_j^\alpha + t_j, \end{cases} \tag{14}$$

$$\text{an offer is rejected} \quad \Rightarrow \quad \begin{cases} \alpha & : \ u_\alpha(x^\alpha) - \sum_{j\neq\alpha} b_j^\alpha \\ j \neq \alpha & : \ u_j(x^\alpha) + b_j^\alpha. \end{cases} \tag{15}$$

For the modified bidding mechanism, we obtain the following result.

Theorem 2. *There exists a subgame perfect equiribrium in the modified bidding mechanism. Moreover, in any subgame perfect equilibrium, a socially efficient alternative is realized.*

Proof. First, we show that, in any subgame which starts at Stage 2, a socially efficient alternative is realized and each individual $j \neq \alpha$ receives a payoff equal to $u_j(x^\alpha)$ in any subgame perfect equilibrium. We consider an optimal offer of a proposer α at Stage 2 who anticipates the actions of the other individuals at Stage 3. It is obvious that each individual $j \neq \alpha$ accepts (resp. rejects) an offer (x, t_i) if $u_i(x) + t_i > u_i(x^\alpha)$ (resp. $u_i(x) + t_i < u_i(x^\alpha)$) in Stage 3. By the usual argument, each $j \neq \alpha$ is assumed to accept the offer (x, t_i) such that $u_i(x) + t_i = u_i(x^\alpha)$.[2] We distinguish two cases: x^α is (i) socially efficient, or (ii) not. In the case of (ii), an optimal offer $(x^*, (t_i^*)_{i\neq\alpha}) \in X \times \mathbb{R}^{n-1}$ can be obtained as a solution of the following maximization problem:

$$\max_{(x,(t_i)_{i\neq\alpha})} u_\alpha(x) - \sum_{i\neq\alpha} t_i, \quad s.t. \ u_i(x) + t_i \geq u_i(x^\alpha), \ \forall i \neq \alpha. \tag{16}$$

[2] To show that such an assumption is formally justified, it will be sufficient to provide the following example. Consider the following ultimatum offer game: first, player 1 makes an offer $(x_1, 1 - x_1)$ of the distribution of the feasible payoff which amounts to 1. Then, player 2 replies "accept it" or "not." In the case of acceptance, the offer is realized. If the offer is rejected, both receive zero, $(0,0)$. It is obvious that the offer $(1,0)$ and "accept any offer which gives $1 - x_1 \geq 0$ to player 2" constitute a subgame perfect equilibrium. On the other hand, in the case of the rejection of the offer which gives $1 - x_1 = 0$ to player 2, i.e. the rejection of $(1,0)$, such a strategy of player 2 never leads to a subgame perfect equilibrium because, provided that player 2 rejects the offer $(1,0)$, player 1 has an incentive to change the offer $(1,0)$ to $(x_1, 1 - x_1)$ with $x_1 \in (0,1)$ but s/he also wants to change the offer $(x_1, 1 - x_1)$ to $(x_1', 1 - x_1')$ with $x_1 < x_1' < 1$, and thus any of the offers by player 1 can never be the best response to the strategy of player 2. We can easily apply this observation to the case of sequential replies by multiple players (see, for example, Claim (b) in Theorem 1 by Pérez-Castrillo and Wettstein [7]).

Since an optimal offer $(x^*, (t_i^*)_{i\neq\alpha})$ must satisfy the constraint in (16) with equality for each $i \neq \alpha$, the optimal offer can be obtained as follows:

$$t_i^* = u_i(x^\alpha) - u_i(x^*),\ \forall i \neq \alpha, \tag{17}$$

$$x^* \in \arg\max_{x\in X} \sum_{i\in N} u_i(x) - \sum_{i\neq\alpha} u_i(x^\alpha). \tag{18}$$

By definition, such an offer is accepted by every other individual $i \neq \alpha$. It is easily verified that this offer is optimal for the proposer α. From (18), x^* is a socially efficient alternative. Because x^α is socially inefficient, we have

$$\sum_{i\in N} u_i(x^*) - \sum_{i\in N} u_i(x^\alpha) > 0 \ \Rightarrow\ \sum_{i\in N} u_i(x^*) - \sum_{i\neq\alpha} u_i(x^\alpha) > u_\alpha(x^\alpha). \tag{19}$$

From (17), (19) can be rewritten as

$$u_i(x^*) - \sum_{i\neq\alpha} t_i^* > u_\alpha(x^\alpha). \tag{20}$$

Therefore, from (20) and the fact that $(x^*, (t_i^*)_{i\neq\alpha})$ solves the problem (16), this offer is optimal for α. If we have more than one solutions to (16), any of them is a optimal offer. In the case of (i), any of the optimal offers defined in (17) and (18) is still optimal for the proposer α. Note that, in this case, the inequality in (19), thus also the one in (20), is replaced with equality. Thus, the payoff received by the proposer in the subgame that starts at Stage 2 is equal to $u_\alpha(x^\alpha)$. In addition to these offers, it is also optimal for the proposer to announce an offer that gives the payoff strictly less than $u_i(x^\alpha)$ to some individual i. Such an offer and the rejection by i at Stage 3 together also constitute a subgame perfect equilibrium in the subgame that starts at Stage 2.

As seen in the above argument, in any subgame perfect equilibrium of the game that starts at Stage 2, the social alternative realized as an equilibrium outcome of the game is always socially efficient. Given the subgame perfect equilibria in the subgame that starts at Stage 2, it follows from Theorem 1 that there exists a Nash equilibrium in the truncated game at Stage 1. ■

Our multi-stage mechanism can always realize a socially efficient alternative in its subgame perfect equilibrium. The key is the monetary transfer offered in Stage 2, which leads a proposer α to offer a socially efficient alternative which may not be the most favorable one for the proposer. As shown in the proof of Theorem 2, the surplus by such a conciliatory proposal is gained by the proposer through the monetary transfers.

4 Applications and extensions

We now discuss some applications of the bidding mechanism to the implementation of solutions established in cooperative game theory and also suggest how we can rearrange these implementation mechanisms in the collective choice context we have considered in the preceding sections.

4.1 Weighted bidding game

Our modified bidding mechanism in the preceding section consists of the bidding stage and the subsequent extensive form games where the individuals other than the proposer sequentially reveal "*accept the offer*" or "*reject it*." We now reformulate the mechanism in a more general form. For each $i \in N$, let Δ_i be an arbitrary extensive form game which will be played in the case where the individual i becomes a winner of the bidding stage. We write $\Delta = (\Delta_i)_{i \in N}$. We denote the payoff to an individual $j \in N$ in a subgame perfect equilibrium of Δ_i by u^i_j. Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{++}$ be a positive weight on the individuals N. The multi-stage mechanism considered in Section 3 can be generalized as the mechanism that consists of a *weighted* bidding game and the subsequent extensive form game Δ_α corresponding to the winner α determined in the weighted bidding game. The weighted bidding mechanism $\Gamma(N, w, \Delta)$ is defined as follows:

i). Each individual i makes a bid $b^i_j \in \mathbb{R}$ for each $j \neq i$, i.e. $(b^i_j)_{j \neq i} \in \mathbb{R}^{n-1}$.

ii). For each i, calculate a weighted net bid $B^i(w) := \sum_{j \neq i} w_i b^i_j - \sum_{j \neq i} w_j b^j_i$.

iii). An individual with the highest weighted net bid (if there exist more than one, randomly chosen among them) is the winner of the bidding stage and only the winner actually pays her/his bids to the other players.

iv). The extensive form game Δ_α corresponding to the winner α is played.

The final payoff to an individual $i \neq \alpha$ in $\Gamma(N,w,\Delta)$ is the sum of the bid paid to i in the weighted bidding stage and the payoff obtained in the subsequent game Δ_α. On the other hand, the winner α receives the payoff obtained in the game Δ_α minus the total bids paid to the other individuals.

By the similar argument used to derive the equilibrium bid in (5) and Theorem 1 in Section 2, we immediately obtain the following result.

Theorem 3. *Suppose that the sum of the equilibrium payoffs $(u_1^i,\ldots,u_n^i)$ in Δ_i is the same for each $i \in N$, i.e.*

$$\sum_{k\in N} u_k^i = \sum_{k\in N} u_k^j, \quad \forall i,j \in N.$$

Then, the equilibrium bids $(b^{i*})_{i\in N}$ in the weighted bidding stage are uniquely determined for the payoff vector $u = \big((u_j^1)_{j\in N},\ldots,(u_j^n)_{j\in N}\big) \in \mathbb{R}^{n^2}$ as follows:

$$b_j^i = \sum_{k\in N} \frac{w_k}{\sum_{l\in N} w_l} u_j^k - u_j^i, \quad \forall i \in N,\ \forall j \neq i,$$

and the equilibrium payoff to an individual i in $\Gamma(N,w,\Delta)$ is given as:[3]

$$\sum_{k\in N} \frac{w_k}{\sum_{l\in N} w_l} u_i^k.$$

Theorem 3 shows that, under the condition stated in the theorem, we can use the weighted bidding game as a random device which selects a subsequent game Δ_i. In the literature on the implementation of cooperative solutions, random devices are used in many mechanisms. As stated in the theorem and also observed in the preceding sections, the (weighted) bidding game not only randomizes the winner of the game but also actually generates the payoffs equal to the *expected* ones in its equilibrium. Thus, the (weighted) bidding game can be applied to implement cooperative solutions not in expected payoffs but

[3] In the case where we have other subgame paerfect equilibria generating another payoff vector v, we obtain the equilibrium bids and the final payoffs in the same way as for the case of u if v satisfies the condition of the theorem.

in *actually realized* ones. In this respect, the bidding approach to the implementation mechanism has an advantage to the other usual random device approaches.

4.2 Applications to implementing cooperative solutions

We now discuss some applications of the bidding mechanisms to the implementation of cooperative solutions. The cooperative game theory mainly explores the allocation of the total amount of transferable utilities which can be attained by individuals' cooperation. A game in coalitional form with transferable utility, or cooperative game, is a pair (N,v) where $v: 2^N \to \mathbb{R}$ is a characteristic function that measures the worth of a coalition $S \subseteq N$ and $v(S)$ represents the total amount of utility that the coalition S can obtain without any cooperation by the members outside of S. It is usually assumed that $v(\emptyset) = 0$. A solution ϕ for a cooperative game (N,v) is a function that associates with each game (N,v) a payoff vector $\phi(N,v) = (\phi_i(N,v))_{i\in N} \in \mathbb{R}^n$ under the feasibility constraint $\sum_{i\in N} \phi_i(N,v) \leqq v(N)$. A solution ϕ which always satisfies the feasibility constraint with equality is said to be *efficient*.

In what follows, we provide some examples each of which shows how we can apply the bidding mechanism to implement a cooperative solution. The key is that, under a moderate condition such that $v(S) + v(\{i\}) \leqq v(S \cup \{i\})$ for all $S \subseteq N$ with $i \notin S$,[4] the condition stated in Theorem 3 is most likely to hold in the form that $\sum_{k\in N} u^i_k = v(N)$ for each $i \in N$. The first one is the implementation of the Shapley value established by Pérez-Castrillo and Wettstein [7].

Example 1. *Consider the following mechanism defined in a recursive manner.*

(i) If N is singleton, i.e. $N = \{i\}$, the individual i obtains her/his value of stand-alone coalition, $v(\{i\})$;

(ii) If $|N| \geqq 2$, given that the mechanism is defined for any k-person case with $k < |N|$, the following mechanism is played by N:

> *Stage 1. The bidding game is played by N and an individual with the highest net bid (randomly chosen if there exist more than one), say α, becomes a proposer and pays her/his bids to the other players.*

[4]This condition is usually called *zero-monotonicity*.

Stage 2. The proposer α makes an offer of transfer $p_j \in \mathbb{R}$ to each of the other individuals.

Stage 3. Every individual other than α sequentially replies "to accept the offer" or "to reject it." If the offer is rejected by some individual, the offer is rejected. Otherwise, the offer is accepted. In the case of acceptance, the transfer p_j is carried out between the proposer α and each of $j \neq \alpha$ and the proposer obtains the value of their cooperation $v(N)$. If the offer is rejected, the proposer α gains $v(\{\alpha\})$ and the individuals other than α, i.e. $N \setminus \{\alpha\}$, play the mechanism again.

This mechanism implements the Shapley value in its subgame perfect equilibrium.

To provide the outline of the proof will help understanding how we can apply the bidding mechanism to the implementation of cooperative solutions. For any subset $S \subseteq N$, we define the *S-unanimity game* denoted by u_S as: $u_S(T) = 1$ if $T \supseteq S$; and $u_S(T) = 0$ otherwise. It is well known that any characteristic function v is represented as a linear combination of the unanimity games such that $v = \sum_{S \subseteq N; S \neq \emptyset} d_v(S) u_S$, where $d_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ for each $S \subseteq N$. The Shapley value Sh of a game (N, v) is defined as:

$$\mathrm{Sh}_i(N, v) = \sum_{S \subseteq N; S \ni i} \frac{d_v(S)}{|S|},$$

for any $i \in N$. The proof proceeds by the induction on the number of the individuals. If $N = \{i\}$, by the definition of the mechanism, s/he obtains $v(\{i\}) = \mathrm{Sh}_i(N, v)$. To prove the n-person case, suppose that a payoff vector in any of the subgame perfect equilibria of the mechanism coincides with the Shapley value for each case of $|N| = t$ with $1 \leqq t \leqq n-1$, and consider the case of $|N| = n$. Let α be a proposer determined in Stage 1. In Stage 3, if some individual $i \neq \alpha$ rejects the offer made by α in Stage 2, by the induction hypothesis, each individual $j \neq \alpha$ obtains her/his Shapley value of the game $(N \setminus \{\alpha\}, v)$. Then, by the similar arguments to the proof of Theorem 2 and some weak condition on the characteristic function v (e.g. zero-monotonicity), it can be shown that the subgame which starts from Stage 2 generates a unique subgame perfect equilibrium payoffs such that each $i \neq \alpha$ obtains

$$\mathrm{Sh}_i(N \setminus \{\alpha\}, v)$$

and the proposer α receives

$$v(N) - \sum_{i \neq \alpha} \mathrm{Sh}_i(N \setminus \{\alpha\}, v) = v(N) - v(N \setminus \{\alpha\}).$$

Therefore, letting $u_j^\alpha = \mathrm{Sh}_i(N \setminus \{i\}, v)$ for each $j \neq \alpha$ and $u_\alpha^\alpha = v(N) - v(N \setminus \{\alpha\})$ and also applying Theorem 3, the equilibrium payoff to each individual i is given as:

$$\frac{v(N) - v(N \setminus \{i\})}{n} + \frac{1}{n} \sum_{i \neq j} \mathrm{Sh}_i(N \setminus \{j\}, v). \tag{21}$$

The value provided in (21) is the recursively represented Shapley value introduced by Maschler and Owen [4].

Pérez-Castrillo and Wettstein [7] also showed that their mechanism with a slight extension implements the weighted Shapley value.

Example 2. *If we replace the bidding stage in the mechanism in Example 1 by the weighted bidding stage, this new mechanism implements the weighted Shapley value* Sh^w *defined as: for each* $i \in N$*,*

$$\mathrm{Sh}_i^w(N, v) = \sum_{S \subseteq N; S \ni i} \frac{w_i}{\sum_{j \in S} w_j} d_v(S).$$

We next provide the example of implementation of the weighted CIS (center of the imputation set) value.[5] The weighted CIS value is defined as: for each $i \in N$,

$$\mathrm{CIS}_i^w(N, v) = \frac{w_i}{\sum_{k \in N} w_k} \left(v(N) - \sum_{j \in N} v(\{j\}) \right) + v(\{i\}).$$

Example 3. *We modify the mechanism in Example 1 in the following way: (i) the weighted bidding game is played in Stage 1; and (ii) in the case of rejection in Stage 3, each* $i \in N$ *receives* $v(\{i\})$ *and the mechanism ends. This modified mechanism implements the weighted CIS value in its subgame perfect equilibrium.*

By the same argument as in the proof of Theorem 2, under the appropriate condition such as $v(N) \geq$

[5] The CIS value has been originally defined in Driessen and Funaki [1].

$\sum_{j\in N} v(\{j\})$, the subgame which starts from Stage 2 generates a unique equilibrium payoff vector: for the proposer α, $v(N)-\sum_{j\neq\alpha} v(\{j\})$; and for each $i\neq\alpha$, $v(\{i\})$. Thus, applying Theorem 3, the final payoff vector must coincides with the weighted CIS value of (N,v).

Finally, we suggest the way to rearrange the mechanism of the above examples in the framework of the collective choice problem we have discussed in the paper. Given a finite set of alternatives X, we define a characteristic function v^X associated with X as follows: for all $S\subseteq N$,

$$v^X(S)=\max_{x\in X}\sum_{i\in S} u_i(x).^6 \tag{22}$$

By the definition of the characteristic function v^X, the cooperative game defined as (N,v^X) may fail to be zero-monotonic. Thus, to implement the cooperative solutions considered above, we need some slightly ad hoc modification in the mechanisms considered in the examples. To implement the (weighted) Shapley value of (N,v^X), the mechanisms in Examples 1 and 2 must be changed in the following way: (i) the proposer α offers a social alternative x and a transfer t_j to each $j\neq\alpha$ in Stage 2; (ii) in the case of acceptance, the social alternative x is realized and transfer t_j is carried out between α and each $j\neq\alpha$; and (iii) in the case of rejection, the proposer α faces with the worst outcome (e.g. expelled from the society). With these modification, each of the mechanisms in Examples 1 and 2 generates the subgame perfect equilibrium payoffs $\mathrm{Sh}(N,v^X)$ and $\mathrm{Sh}^w(N,v^X)$, respectively. To implement the weighed CIS value, in addition to the modifications (i) and (ii) above, we need to change (iii) as follows: in Stage 3, if an individual j rejects the offer made in Stage 2, the individuals other than j face with their worst outcomes and the individual j realizes her/his most favorite alternative x^j.

5 Conclusion

We examined two alternative bidding mechanisms in the framework of collective choice of social alternatives: one is the simple one-stage bidding mechanism, and the other is the multi-stage bidding mechanism.

[6] The characteristic function, v^X, defined in (22) is slightly different from those usually considered in the literature. $v^X(S)$ is interpreted as the amount of total payoff the individuals in S can attain by themselves when the individuals outside of S leave the society.

There are two serious defects in the simple one-stage mechanism. There does not always exist an equilibrium in the mechanism, and moreover, it may fail to generate a socially efficient alternative even if an equilibrium does exist. Our multi-stage bidding mechanism overcomes these two problems and always realizes a socially efficient alternative as an equilibrium outcome.

As seen in the present paper and also in Pérez-Castrillo and Wettstein [6], the bidding game actually realizes the payoffs equal to the expected ones in its equilibrium. Thus, the bidding game can be said to be an equitable random device in both *ex ante* and *ex post* criteria: an equal probability to be a proposer, and the actually received payoffs equal to the expected ones, respectively. This property allows us to implement cooperative solutions not in expected payoffs but in actually realized ones. Moreover, this equity property, particularly the ex post equity, of the bidding mechanism is noteworthy in the context of collective choice problem.

Finally, as we have seen in the last section, in the collective choice framework considered in the paper, it is not easy to define the characteristic function that satisfies a moderate condition such as zero-monotonicity. Consequently, in the context of the collective decision making of social alternatives, it is quite difficult to apply the existing mechanisms that implements the cooperative solutions satisfying an efficiency property such as the Pareto efficiency. In this respect, the relevance of our multi-stage mechanism that always realizes a socially efficient alternative should be emphasized.

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