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Weight Monotonic Allocation Rules for Communication Situations with Asymmetry

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Abstract

This paper studies the relations between the weight of each player and his allocation in communication situations with asymmetry. In some important classes of games such as convex or superadditive games, the weighted Myerson value which is an extension of the Myerson value for communication situations with asymmetry is *not* weight monotonic, that is, the relatively increase of one player's weight may not increase his allocation. By extending the position value and the component-wise egalitarian value, we define and axiomatize new allocation rules both of which is weight monotonic in much wider classes of games, specifically, in superadditive and zero-monotonic games respectively.

Keywords: communication situations; weights; monotonicity;
JEL classification: C71

1 Introduction

One of the most useful and attractive solution concepts of the cooperative games with transferable utility is the Shapley value which is introduced by Shapley (1953b). The Shapley value is originally defined as each player's expected marginal contributions among all permutations of the player set. This definition is closely related to fair treatment of all players, that is, the probability of one player follows the other player is equal to that of the other player follows the player. By this property, in the Shapley value, the influence of cooperation of a set of players is equally divided among them. In real economic or social situations, however, the gain (or loss) generated by players cooperation may not be divided equally among them. For instance, consider the case in which one big firm and one small firm cooperates in a joint project. Suppose both of them cannot generate any profit by their own, the gain by cooperation is divided equally in the Shapley value. Yet if one firm need a greater effort in the project than the other, the equal division may be unfair in some sense. In that case, the gain should be divided proportionally to each of their effort. Thus, the modification of the Shapley value to satisfy more fair sense like the above is needed when we consider the application.

The Shapley value is characterized by four axioms, efficiency, the null player property, symmetry and additivity. The discussion above corresponds to weaken symmetry. Weakening symmetry is first considered by Shapley (1953a). He used weights of players and defined the weighted Shapley value.[1] These weights were introduced for the sake of reflection of players' bargaining power, however, Owen (1968) noticed that the weights are interpret as players' slowness to reach the game rather than players' bargaining power since the relatively increase of one player's weight may decrease his allocation. Then, the weighted Shapley value is not an appropriate allocation rule of the situation in the above.

The interpretation of weights is closely related to how to use the weights in the definition of the allocation rule. Thus, if we consider another allocation rule, weights may be interpreted as bargaining

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[1] Kalai and Samet (1987) also considered the weighted Shapley value but, in their definition, they used the weight system which is a generalization of the weights.

power.[2] By generalizing the position value which is introduced by Borm, Owen, and Tijs (1992) and the component-wise egalitarian value which is introduced by Slikker (2007), we define two allocation rules for the communication situations with asymmetry. The communication situations with asymmetry are more general settings than the settings in which the weighted Shapley value is defined. The allocation rule for the communication situations with asymmetry which corresponds to the weighted Shapley value is the weighted Myerson value introduced by Haeringer (1999), Thus we use the weighted Myerson value to compare our allocation rules and the weighted Shapley value.

The paper is constructed as follows. In Section 2, basic notations and definitions are given. In Section 3, the new allocation rule on communication situations with asymmetry is given and it is called weighted position value. In Section 4, another allocation rule is defined and it is called the weighted component-wise egalitarian value. In Section 5, each allocation rules are reconsidered with respect to the relation between weights and bargaining power which is called weight monotonicity. In Section 6, examples of the allocation rules are given.

2 Preliminaries

A finite set N is given and it denotes the set of all players. Let $|N| = n$ where $|\cdot|$ represents the cardinality of the set. A function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *cooperative game with transferable utility* or simply, a *game*. A game v is *zero-normalized* if for any $i \in N$, $v(\{i\}) = 0$. Throughout this paper, we consider only zero-normalized games. A set of all zero-normalized games on N is denoted by $\mathcal{V}^N$.

For any $S \subseteq N$, let $u_S : 2^N \to \mathbb{R}$ be a S-*unanimity game* which is defined as follows: for any $T \subseteq N$,

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}.$$

It is well known that any $v \in \mathcal{V}^N$ is represented as a linear combination of unanimity games, that is,

$$v = \sum_{S \subseteq N; S \neq \emptyset} \Delta_v(S) u_S$$

where $\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ is *dividend* of S.

The *Shapley value* (Shapley (1953b)) of a game v is a function $\phi : \mathcal{V}^N \to \mathbb{R}^n$ which is defined as:

$$\phi_i(v) = \sum_{S \subseteq N; S \ni i} \frac{\Delta_v(S)}{|S|}$$

for any $i \in N$.

To each player in N, we associate a weight $w_i \in \mathbb{R}_{++}$ and denotes $w = (w_i)_{i \in N}$. The weights of players add some information to the games, for instance, bargaining ability of each player or the size of each player if each player is a group of individuals. The *weighted Shapley value* ϕ^w (Shapley (1953b)) is defined as follows: for any $i \in N$,

$$\phi_i^w(v) = \sum_{S \subseteq N; S \ni i} \Delta_v(S) \frac{w_i}{\sum_{j \in S} w_j}.$$

Next, we consider communication relation between players of the games. Given a player set N, the bilateral communication channels between the players in N are described by a graph $g \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$. A set of all graphs on N is denoted by G^N. Each communication channel in a graph is called a *link* and it is represented as ij instead of $\{i, j\}$. Given a graph g, if there exists a finite sequence of players $i_1, \ldots, i_H$ such that $i_1 = i, i_H = j$ and $i_h i_{h+1} \in g$ for any $h = 1, \ldots, H-1$, then i *is connected to* j in the graph. Given a graph, any players can communicate freely with each other iff they are connected with each other. Let

$$N/g = \{\{j \in N | i \text{ is connected to } j \text{ in } g\} \cup \{i\} | i \in N\}.$$

[2]Haeringer (2006) used the weight schemes which is introduced from weight and enabled us to interpret them as bargaining power.

N/g represents the collection of communicable players in g. If a link ij is deleted from a graph g, we write $g - ij$. For any $S \subseteq N$, let $g(S) = \{ij \in g | i, j \in S\}$ which is a restriction of g on S. By $g(S)$, S/g is defined in the same manner as N/g, that is,

$$S/g = \{\{j \in S | i \text{ is connected to } j \text{ in } g(S)\} \cup \{i\} | i \in S\}.$$

A pair (v, g) is called a *communication situation*. An allocation rule on communication situations with asymmetry is a n-dimensional vector value function on $\mathcal{V}^N \times G^N \times \mathbb{R}^n_{++}$. By extending the Myerson value which is introduced by Myerson (1977), Haeringer (1999) defined an allocation rule μ^w on communication situations with asymmetry as the following way: for any $i \in N$,

$$\mu^w(v, g) = \phi^w(v^g),$$

where $v^g(S) = \sum_{C \in S/g} v(C)$ for any $S \subseteq N$.[3] It is called the *weighted Myerson value*. The weighted Myerson value is characterized by two axioms each of which relates to efficiency and balance of contributions respectively.

3 Weighted position value

In this section, we define another allocation rule on communication situations with asymmetry. First, we give two properties which allocation rules should satisfy. Let ψ be an allocation rule on communication situations with asymmetry.

Component efficiency (CE): ψ satisfies component efficiency iff for any $v \in \mathcal{V}^N$, any $g \in G^N$, any $w \in \mathbb{R}^n_{++}$ and any $C \in N/g$,

$$\sum_{i \in C} \psi_i(v, g, w) = v(C).$$

Weighted balanced link contributions (WBLC): ψ satisfies weighted balanced link contributions iff for any $v \in \mathcal{V}^N$, any $g \in G^N$, any $w \in \mathbb{R}^n_{++}$ and any $i, j \in N$,

$$\sum_{jk \in g_j} \frac{w_j}{w_j + w_k} \Big[\psi_i(v, g, w) - \psi_i(v, g - jk, w)\Big] = \sum_{ih \in g_i} \frac{w_i}{w_i + w_h} \Big[\psi_j(v, g, w) - \psi_j(v, g - ih, w)\Big],$$

where $g_k = \{kh \in g | h \in N\}$ for any $k = i, j$.

CE is straightforward, that is, all of the value generated by each set of communicable players must divide among them. While WBLC is rather complicated. First, for any $ih \in g$, $\frac{w_i}{w_i + w_h}$ is considered as i's bargaining power in the link. Suppose that each player can cut each of his link with probability which is equal to his bargaining power of the link, then $\frac{w_i}{w_i + w_h}(\psi_j(v, g, w) - \psi_j(v, g - ih, w))$ is interpreted as j's expected influence from i by deletion of link ih. Thus WBLC implies that for any two players, the sum of the expected influence from one to the other among all of one's link need to be balanced between them. If all players have the same weight in w, that is, $w_i = w_j$ for any $i, j \in N$, WBLC coincides with balanced link contributions in Slikker (2005).

Then, the followings holds.

Theorem 1. *There exists a unique allocation rule π^w which satisfies CE and WBLC. The allocation rule is defined as follows: for any $i \in N$*

$$\pi^w(v, g) = \sum_{ij \in g_i} \frac{w_i}{w_i + w_j} \phi_{ij}(r)$$

where $r : 2^g \to \mathbb{R}$ is called a link game such that for any $g' \subseteq g$, $r(g') = \sum_{C \in N/g'} v(C)$.

[3] Slikker and van den Nouweland (2000) considered more general settings where asymmetry is represented by *weight system* introduced by Kalai and Samet (1987).

Proof. The following proof is a modification of the proof of Theorem 3.1 of Slikker (2005).

First, we identify that π^w satisfies CE and WBLC. For CE, for any $C \in N/g$,

$$\sum_{k\in C} \pi^w_k(v,g) = \sum_{ij\in g(C)} \Big(\frac{w_i}{w_i+w_j} + \frac{w_j}{w_i+w_j}\Big)\phi_{ij}(r)$$
$$= \sum_{ij\in g(C)} \phi_{ij}(r|_{g(C)}) = r(g(C)) = \sum_{T\in N/g(C)} v(T) = v(C).$$

In the above equation, the second equality holds since for any $ij \in g(C)$ and any $g' \subseteq g - ij$, the marginal contributions of ij to g' are equal to those of ij to $g' \cap g(C)$ and the last equality holds since v is zero-normalized.

For WBLC, for any $i, j \in N$,

$$\sum_{jk\in g_j} \frac{w_j}{w_j+w_k}(\pi^w_i(v,g) - \pi^w_i(v,g-jk))$$
$$= \sum_{jk\in g_j} \frac{w_j}{w_j+w_k}\Big(\sum_{ih\in g_i}\frac{w_i}{w_i+w_h}\sum_{\substack{g'\subseteq g\\ g'\ni ih}}\frac{\Delta_r(g')}{|g'|} - \sum_{ih\in (g-jk)_i}\frac{w_i}{w_i+w_h}\sum_{\substack{g'\subseteq g-jk\\ g'\ni ih}}\frac{\Delta_{r|_{g-jk}}(g')}{|g'|}\Big)$$
$$= \sum_{jk\in g_j} \frac{w_j}{w_j+w_k}\Big(\sum_{g'\subseteq g}\frac{\Delta_r(g')}{|g'|}\sum_{ih\in g'_i}\frac{w_i}{w_i+w_h} - \sum_{g'\subseteq g-jk}\frac{\Delta_{r|_{g-jk}}(g')}{|g'|}\sum_{ih\in g'_i}\frac{w_i}{w_i+w_h}\Big)$$
$$= \sum_{jk\in g_j} \frac{w_j}{w_j+w_k}\sum_{\substack{g'\subseteq g\\ g'\ni jk}}\frac{\Delta_r(g')}{|g'|}\sum_{ih\in g'_i}\frac{w_i}{w_i+w_h}$$
$$= \sum_{g'\subseteq g}\sum_{jk\in g'_j} \frac{w_j}{w_j+w_k}\frac{\Delta_r(g')}{|g'|}\sum_{ih\in g'_i}\frac{w_i}{w_i+w_h}$$
$$= \sum_{g'\subseteq g}\sum_{ih\in g'_i} \frac{w_i}{w_i+w_h}\frac{\Delta_r(g')}{|g'|}\sum_{jk\in g'_j}\frac{w_j}{w_j+w_k}$$
$$= \sum_{ih\in g_i} \frac{w_i}{w_i+w_h}(\pi^w_j(v,g) - \pi^w_j(v,g-ih)).$$

To prove the uniqueness, let ψ be an allocation rule which satisfies CE and WBLC. The proof is by induction of the number of links in g. If $g = \emptyset$, CE implies $\psi(v,g,w) = v(\{i\}) = \pi^w(v,g)$ for any $i \in N$ thus, $\psi = \pi^w$. Let $m \geq 1$. Suppose that $\psi = \pi^w$ holds for any graph which contains less than $m-1$ links and consider the case g contains m links. Fix $C \in N/g$. If C is singleton, CE implies $\psi(v,g,w) = v(\{i\}) = \pi^w(v,g)$ for $i \in C$. If $|C| \geq 2$, without loss of generality, let $C = \{1,2,\ldots,c\}$. Applying WBLC to pairs $\{1,2\},\{1,3\},\ldots,\{1,c\}$, we obtain

$$\sum_{2k\in g_2}\frac{w_2}{w_2+w_k}\psi_1(v,g,w) - \sum_{1h\in g_1}\frac{w_1}{w_1+w_h}\psi_2(v,g,w)$$
$$= \sum_{2k\in g_2}\frac{w_2}{w_2+w_k}\psi_1(v,g-2k,w) - \sum_{1h\in g_1}\frac{w_1}{w_1+w_h}\psi_2(v,g-1h,w)$$
$$= \sum_{2k\in g_2}\frac{w_2}{w_2+w_k}\pi^w_1(v,g-2k) - \sum_{1h\in g_1}\frac{w_1}{w_1+w_h}\pi^w_2(v,g-1h);$$
$$\vdots$$
$$\sum_{ck\in g_c}\frac{w_c}{w_c+w_k}\psi_1(v,g,w) - \sum_{1h\in g_1}\frac{w_1}{w_1+w_h}\psi_c(v,g,w)$$
$$= \sum_{ck\in g_c}\frac{w_c}{w_c+w_k}\psi_1(v,g-ck,w) - \sum_{1h\in g_1}\frac{w_1}{w_1+w_h}\psi_c(v,g-1h,w)$$

$$= \sum_{ck \in g_c} \frac{w_c}{w_c + w_k} \pi_1^w(v, g - ck) - \sum_{1h \in g_1} \frac{w_1}{w_1 + w_h} \pi_c^w(v, g - 1h);$$

Also, by CE,

$$\sum_{i \in C} \psi_i(v, g, w) = v(C).$$

These c equalities form a regular system of linear equations in c variables and it has a unique solution which is the weighted position value. Hence for any $i \in C$, ψ_i coincides with π_i^w. For any $i \in C' \in N/g$ with $C' \neq C$ we can prove the coincidence between ψ and π^w in the same way. By induction of m, the proof is completed. □

If all players have the same weight in w, π^w coincides with the position value introduced by Borm, Owen, and Tijs (1992). Thus, we call π^w the *weighted position value*.

4 Weighted component-wise egalitarian value

In this section, replacing WBLC with the following property, we define another allocation rule on the communication situations with asymmetry.

Weighted balanced component contributions (WBCC): ψ satisfies weighted balanced component contributions iff for any $v \in \mathcal{V}^N$, any $g \in G^N$, any $w \in \mathbb{R}^n_{++}$ and any $i, j \in N$,

$$w_j(\psi_i(v, g, w) - \psi_i(v, g \backslash g(C_j), w)) = w_i(\psi_j(v, g, w) - \psi_j(v, g \backslash g(C_i), w))$$

where $C_k \in N/g$ with $C_k \ni k$ for any $k = i, j$.

Theorem 2. *There exists a unique allocation rule γ^w which satisfies CE and WBCC. The allocation rule is defined as follows: for any $i \in N$ with $i \in C \in N/g$,*

$$\gamma_i^w(v, g) = \frac{w_i}{\sum_{j \in C} w_j} v(C).$$

Proof. First, we identify that γ^w satisfies CE and WBCC. For CE, for any $C \in N/g$,

$$\sum_{i \in C} \gamma_i^w(v, g) = \sum_{i \in C} \frac{w_i}{\sum_{j \in C} w_j} v(C) = v(C).$$

For WBCC, for any $i, j \in N$, if $C_i = C_j = C$, then

$$\begin{aligned} w_j(\gamma_i^w(v, g) - \gamma_i^w(v, g \backslash g(C_j))) &= w_j\Big(\frac{w_i}{\sum_{k \in C} w_k} v(C) - 0\Big) = \frac{w_j \cdot w_i}{\sum_{k \in C} w_k} v(C) \\ &= w_i\Big(\frac{w_j}{\sum_{k \in C} w_k} v(C) - 0\Big) = w_i(\gamma_j^w(v, g) - \gamma_j^w(v, g \backslash g(C_i))), \end{aligned}$$

and if $C_i \neq C_j$,

$$w_j(\gamma_i^w(v, g) - \gamma_i^w(v, g \backslash g(C_j))) = 0 = w_i(\gamma_j^w(v, g) - \gamma_j^w(v, g \backslash g(C_i))).$$

To prove the uniqueness, let ψ be an allocation rule which satisfies CE and WBCC. Let $g \in G^N$ and $C \in N/g$. If $|C| = 1$, then CE implies $\psi_i(g, v, w) = v(\{i\}) = \gamma_i^w(g, v)$ for $i \in C$. Suppose $|C| \geq 2$ and fix $i \in C$. By applying WBCC to pairs i and any $j \in C \backslash \{i\}$, and the fact that $\psi_k(v, g \backslash g(C), w) = 0$ for any $k \in C$, we obtain

$$w_j \psi_i(v, g, w) = w_i \psi_j(v, g, w)$$

for any $j \in C \backslash \{i\}$. Summing up the above equation with respect to $j \in C \backslash \{i\}$, we have

$$\sum_{j \in C \backslash \{i\}} w_j \psi_i(v, g, w) = w_i \sum_{j \in C \backslash \{i\}} \psi_j(v, g, w)$$

By CE, $\sum_{j\in C\backslash\{i\}}\psi_i(v,g,w)=v(C)-\psi_i(v,g,w)$. Thus, the above equation is equal to

$$\psi_i(v,g,w)=\frac{w_i}{\sum_{j\in C}w_j}v(C)=\gamma_i^w(v,g).$$

For any $j\in C\backslash\{i\}$, we can prove the coincidence between $\psi_j(v,g,w)$ and $\gamma_j^w(v,g)$ in the same way. Hence $\psi=\gamma^w$. □

If all players have the same weight in w, γ^w coincides with the component-wise egalitarian value introduced by Slikker (2007). Thus, we call γ^w *weighted component-wise egalitarian value.*

5 Weight monotonicity

In this section, we reconsider the meaning of the weights of players. The weights are first introduced to games by Shapley (1953a) in order to represent bargaining power of each player. However, Owen (1968) noticed that, in the weighted Shapley value, the weights are interpreted as each player's slowness to reach the game rather than bargaining power since relatively increasing of one player's weight may decreases his allocation. (The 3-person majority game given in the next section illustrates this property.)

The interpretation of the weights is closely related to how to use the weights in the definition of a allocation rule. For the weighted Shapley value, the weights does not imply their bargaining power but in other allocation rule, the weights may be interpreted as their bargaining power. Also if we consider some specific class games, the weights may be interpreted as the bargaining power. In order to consider the relation between the weights and bargaining power, we use the following property:

Weight monotonicity: An allocation rule ψ satisfies weight monotonicity in $\bar{\mathcal{V}}^N\subseteq\mathcal{V}^N$ iff for any $w,w'\in\mathbb{R}^n_{++}$ which satisfies $w_i=w'_i$ for any $i\in N\backslash\{j\}$ and $w_j<w'_j$, any $v\in\bar{\mathcal{V}}^N$ and any $g\in G^N$,

$$\psi_j(v,g,w)<\psi_j(v,g,w').$$

When we consider weight monotonicity, it is important that we consider what class of games. We consider the following classes of games. A game v is *zero-monotonic* if for any $i\in N$ and $S\subseteq N\backslash\{i\}$, $v(S\cup\{i\})\geq v(S)+v(\{i\})$. A game v is *superadditive* if for any $S,T\subseteq N$ with $S\cap T=\emptyset$, $v(S\cup T)\geq v(S)+v(T)$. A game is *convex* if for any $i\in N$ and for any $S\subseteq T\subseteq N\backslash\{i\}$, $v(S\cup\{i\})-v(S)\leq v(T\cup\{i\})-v(T)$. By definition, convex games are superadditive and superadditive games are zero-monotonic.

The first result needs restriction on graph. Given a graph, a sequence of players $(i_1,i_2,\dots,i_K)$ with $K\geq 3$ called a *cycle* if $i_ki_{k+1}\in g$ for all $k=1,\dots K-1$ and $i_K=i_1$. A graph is *cycle-complete* if there exists a cycle in a graph then all pairs of players in the cycle has link in the graph. Then, the following holds.

Theorem 3. *If g is cycle-complete, the weighted Myerson value satisfies weight monotonicity in the class of convex games.*

Proof. By van den Nouweland and Borm (1991), if v is convex and g is cycle-complete, v^g is convex. Monderer, Samet, and Shapley (1992) showed that if the game is convex, the weighted Shapley value satisfies weight monotonicity. Hence the theorem holds. □

By strengthening the condition of game, we will drop the restriction on graph.

Definition 1 (k-convexity). *Let $k\geq 1$. A game v is k-convex if for any $i\in N$ and for any $S\subseteq T\subseteq N\backslash\{i\}$, $v(S\cup\{i\})-kv(S)\leq v(T\cup\{i\})-kv(T)$.*

For k-convexity, the followings hold.

Lemma 1. *If v is k-convex, $v(S)\leq v(T)$ holds for any $S\subseteq T\subseteq N$.*

Proof. Since we consider only zero-normalized game, when $S=\emptyset$, the definition of k-convexity implies $v(T\cup\{i\})\geq kv(T)$ for any $T\subseteq N\backslash\{i\}$. For any $S\subseteq T\subseteq N$, let $T\backslash S=\{i_1,i_2,\dots,i_r\}$. Then,

$$v(T)\geq kv(T\backslash\{i_1\})\geq k^2v(T\backslash\{i_1,i_2\})\geq\cdots\geq k^rv(T\backslash\{i_1,i_2,\dots,i_r\})=k^rv(S)\geq v(S)$$

where the last inequality holds since $k\geq 1$. □

Lemma 2. *If v is k-convex, then v is convex.*

Proof. Rearranging the equation in the definition of the k-convexity, for any $i \in N$ and for any $S \subseteq T \subseteq N \backslash \{i\}$ we have

$$v(T \cup \{i\}) - v(S \cup \{i\}) \geq k(v(T) - v(S)).$$

Lemma 1 and $k \geq 1$ implies $k(v(T) - v(S)) \geq v(T) - v(S)$ which completes the proof. □

By Lemma 2, k-convexity is the stronger condition than convexity. If $k \geq 1 + \sqrt{2}$, we can drop the restriction on graph in Theorem 3 by k-convexity. The next lemma is needed to obtain the result.

Lemma 3. *If v is k-convex and $k \geq 1 + \sqrt{2}$, then for any $i \in N$ and any $S \subseteq T \subseteq N \backslash \{i\}$, any partition $\mathcal{S}$ of S and any partition $\mathcal{T}$ of T,*

$$v(T \cup \{i\}) - \sum_{T_h \in \mathcal{T}} v(T_h) \geq v(S \cup \{i\}) - \sum_{S_k \in \mathcal{S}} v(S_k).$$

Proof. By k-convexity,

$$v(T \cup \{i\}) - v(S \cup \{i\}) \geq k(v(T) - v(S)) \geq v(T) + v(S) + (k-1)v(T) - (k+1)v(S).$$

By Lemma 2, k-convexity implies superadditivity. Hence,

$$v(T) + v(S) + (k-1)v(T) - (k+1)v(S) \geq \sum_{T_h \in \mathcal{T}} v(T_h) + \sum_{S_k \in \mathcal{S}} v(S_k) + (k-1)v(T) - (k+1)v(S).$$

Let $T \backslash S = \{i_1 \dots, i_m\}$. k-convexity implies $v(T) \geq k^m v(S)$. Thus,

$$(k-1)v(T) - (k+1)v(S) \geq (k^m(k-1) - (k+1))v(S).$$

Since $k \geq 1 + \sqrt{2}$,

$$k^m(k-1) - (k+1) \geq k(k-1) - (k+1) = (k - (1+\sqrt{2}))(k - (1-\sqrt{2})) \geq 0$$

holds for any $m \geq 0$ which completes the proof. □

Theorem 4. *If $k \geq 1 + \sqrt{2}$, the weighted Myerson value satisfies weight monotonicity in the class of k-convex games.*

Proof. By proof pf Theorem 3, it is sufficient to prove that if v is k-convex and $k \geq 1+\sqrt{2}$, v^g is convex for any graph.

Let $i \in N$, $S \subseteq T \subseteq N \backslash \{i\}$ and $g \in G^N$. Let

$$\mathcal{C} = \{C \in S/g | \text{ there exists } j \in C \text{ such that } ij \in g(S \cup \{i\})\}.$$

By definition, each elements of $\mathcal{C}$ is an element of S/g. Moreover, $\bigcup_{C \in \mathcal{C}} C \in (S \cup \{i\})/g$ that is, all players in an element $\mathcal{C}$ is connected with each other through adding i to S. Similarly, let

$$\mathcal{D} = \{D \in T/g | \text{ there exists } j \in D \text{ such that } ij \in g(T \cup \{i\})\}.$$

Then,

$$v^g(T \cup \{i\}) - v^g(T) - (v^g(S \cup \{i\}) - v^g(S)) = v\Big(\{i\} \cup \bigcup_{D \in \mathcal{D}} v(D)\Big) - \sum_{D \in \mathcal{D}} v(D) - \Big(v\Big(\{i\} \cup \bigcup_{C \in \mathcal{C}} C\Big) - \sum_{C \in \mathcal{C}} v(C)\Big)$$

Any players who are connected with each other in S/g is also connected with each other in T/g. Thus, $\bigcup_{D \in \mathcal{D}} D \supseteq \bigcup_{C \in \mathcal{C}} C$. By Lemma 3, the above equation is greater than zero which implies v^g is convex for any $g \in G^N$. □

For the weighted position value and the weighted component-wise egalitarian value, the followings hold.

Theorem 5. *The weighted position value satisfies weight monotonicity in the class of superadditive games.*

Proof. Since v is superadditive, for any $ij \in g$ and any $g' \subseteq g - ij$,

$$r(g' \cup \{ij\}) = \sum_{C \in N/(g' \cup \{ij\})} v(C) \geq \sum_{C \in N/g'} v(C) = r(g').$$

This implies for any $ij \in g$, marginal contributions to any $g' \subseteq g - ij$ is positive thus, the Shapley value of any ij is greater than 0.[4] Since $w'_i = w_i$, for any $i \in N$ and $w'_j > w_j$,

$$\pi_j^{w'}(v,g) - \pi_j^{w}(v,g) = \sum_{ji \in g_j} \frac{w'_j}{w'_j + w'_i} \phi_{ji}(r) - \sum_{ji \in g_j} \frac{w_j}{w_j + w_i} \phi_{ji}(r)$$

$$= \sum_{ji \in g_j} \frac{w_i(w'_j - w_j)}{(w'_j + w'_i)(w_j + w_i)} \phi_{ji}(r) \geq 0. \quad \square$$

Theorem 6. *The weighted component-wise egalitarian value satisfies weight monotonicity in the class of zero-monotonic games.*

Proof. Now we consider only zero-normalized game, zero monotonicity implies $v(S) \geq 0$ for any $S \subseteq N$. Since $w'_i = w_i$, for any $i \in N$ and $w'_j > w_j$,

$$\gamma_j^{w'}(v,g) - \gamma_j^{w}(v,g) = \Big(\frac{w'_j}{\sum_{i \in C} w'_i} - \frac{w_j}{\sum_{i \in C} w_i}\Big) v(C) = \frac{(w'_j - w_j)(\sum_{i \in C \setminus \{j\}} w_i)}{\sum_{i \in C} w'_i \cdot \sum_{i \in C} w_i} v(C) \geq 0. \quad \square$$

Thus, if an allocation rule is appropriate, the weights can be interpreted as players bargaining power.

6 Examples

In this section two examples are given to illustrate the results in the previous section.

Example 1. *Let* $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$, $g = \{12, 13, 23\}$, $w = (1,1,1)$ *and* $w' = (1,3,1)$.

In Example 1, v is superadditive but not convex. $\mu^w(v,g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mu^{w'}(v,g) = (\frac{7}{20}, \frac{6}{20}, \frac{7}{20})$ implies $\mu_2^w(v,g) > \mu_2^{w'}(v,g)$ though $w_2 < w'_2$. While $\pi^w(v,g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\pi^{w'}(v,g) = (\frac{1}{4}, \frac{2}{4}, \frac{1}{4})$ implies $\pi_2^w(v,g) < \pi_2^{w'}(v,g)$.

Example 2. *Let* $N = \{1,2,3,4\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0$, $v(\{2,3\}) = 1$, $v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = v(\{2,4\}) = v(\{3,4\}) = v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(N) = 3$, $g = \{12, 23, 34\}$, $w = (10,1,1,1)$ *and* $w' = (10,2,1,1)$.

In Example 2, v is zero-monotonic but not superadditive. $\pi^w(v,g) = (\frac{20}{11}, -\frac{7}{22}, \frac{1}{2}, 1)$ and $\pi^{w'}(v,g) = (\frac{5}{3}, -\frac{1}{3}, \frac{2}{3}, 1)$ implies $\pi_2^w(v,g) > \pi_2^{w'}(v,g)$ though $w_2 < w'_2$. While $\gamma^w(v,g) = (\frac{30}{13}, \frac{3}{13}, \frac{3}{13}, \frac{3}{13})$ and $\gamma^{w'}(v,g) = (\frac{30}{14}, \frac{6}{14}, \frac{3}{14}, \frac{3}{14})$ implies $\gamma_2^w(v,g) < \gamma_2^{w'}(v,g)$.

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[4]The original definition of the Shapley value is an expected marginal contributions of each player (see Shapley (1953b)).

References

BORM, P., G. OWEN, AND S. TIJS (1992): "On the Position Value for Communication Situations," *SIAM Journal of Discrete Mathematics*, 5, 305–320.

HAERINGER, G. (1999): "Weighted Myerson Value," *International Game Theory Review*, 1(2), 187–192.

——— (2006): "A New Weight Scheme for the Shapley Value," *Mathematical Social Science*, 52, 88–98.

KALAI, E., AND D. SAMET (1987): "On Weighted Shapley Value," *International Journal of Game Theory*, 16(3), 205–222.

MONDERER, D., D. SAMET, AND L. S. SHAPLEY (1992): "Weighted Values and the Core," *International Journal of Game Theory*, 21, 27–39.

MYERSON, R. B. (1977): "Graphs and Cooperation in Games," *Mathematics of Operations Research*, 2, 225–229.

OWEN, G. (1968): "A Note on the Shapley Value," *Management Science*, 14(11), 731–732.

SHAPLEY, L. S. (1953a): "Additive and Non-Additive Set Functions," Ph.D. thesis, Department of Mathematics, Princeton University.

——— (1953b): "A Value for n-Person Games," in *The Shapley Value*, ed. by A. E. Roth, pp. 41–48. Cambridge University Press.

SLIKKER, M. (2005): "A Characterization of the Position Value," *International Journal of Game Theory*, 33, 505–514.

——— (2007): "Bidding for surplus in network allocation problems," *Journal of Economic Theory*, forthcoming.

SLIKKER, M., AND A. VAN DEN NOUWELAND (2000): "Communcation Situations with Asymmetric Players," *Mathematical Methods of Operations Research*, 52, 39–56.

VAN DEN NOUWELAND, A., AND P. BORM (1991): "On the Convexity of Communication Games," *International Journal of Game Theory*, 19, 421–430.