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On the Supermodularity of Oligopoly Games

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Abstract

The main purpose is to prove the supermodularity (convexity) property of a cooperative game arising from an economical situation. The underlying oligopoly situation is based on a linear inverse demand function as well as linear cost functions for the participating firms. The characteristic function of the so-called oligopoly game is determined by maximizing, for any cartel of firms, the net profit function over the feasible production levels of the firms in the cartel, taking into account their individual capacities of production and production technologies. The (rather effective) proof of the supermodularity of the characteristic function of the oligopoly game relies on the use of maximizers for the relevant maximization problems. A similar proof technique will be reviewed for a related cooperative oligopoly game arising from a slightly modified oligopoly situation where the production technology of the cartel is determined by the most efficient member firm.

Keywords: Cooperative Game, Oligopoly TU-Game, Convexity, Supermodularity, Cartel

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1 Introduction

Supermodularity is a relatively new concept in oligopoly theory that has been introduced in the economic literature at the beginning of the nineties of the last century by Milgrom & Roberts in (1990, 1991), whereas in cooperative game theory the concept of supermodularity found much earlier attention by scientist through the work by Shapley (1971) and Maschler et al. (1972). In the context of cooperative game theory the term "convex" is used to indicate that the real-valued set-function defined on the power set is supermodular. While in noncooperative game theory a game is called supermodular if the set of feasible joint strategies is a sublattice and the payoff function of each player is supermodular and has increasing differences[1]. In noncooperative game theory a characterization of equilibrium points in non-zero-sum

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[1] In the sequel we follow the usual convention to speak in the context of noncooperative game theory from a supermodular game and in the context of cooperative game theory from a convex game. A discussion why it is justified to use the notion of convexity in the context of cooperative games can be found in Driessen (1988, p. 114).

supermodular games was given by Topkis (1979), where Vives (1990) has worked out general properties of supermodular games. Vives presented conditions for equilibrium points to increase with respect to exogenous parameter by relying on a lattice-theoretical fixpoint theorem due to Tarski (1955) and on results of Topkis (1978, 1979). Using lattice-theoretical methods in noncooperative game theory where the payoff functions are supermodular exploit certain order and monotonicity regularities that allow for instance the ranking of equilibrium points. The set of maximizers (equilibrium points) of a supermodular function (supermodular game) is a nonempty complete sublattice, and therefore a greatest and a least maximizer (equilibrium point) exist under some regular conditions. Moreover, supermodularity is preserved under the maximization operation (cf. Topkis (1978, 1979, 1998)).

In economic models very often the situation arises that the marginal payoff of a particular action of an agent is increasing in another action or activities, that means that the actions exhibit strategic complementarities, a notion that is related to increasing differences. According to the equivalence between supermodularity and increasing differences that arise under certain conditions supermodular games provide the appropriate framework to study strategic complementarities. Typical examples with strategic complementarities in oligopoly theory are price competition in markets with a homogeneous good and quantity competition in markets with complementary goods[2]. Economic models pointing out the property of monotonic comparative statics by relying on complementarities and supermodular functions have been studied first by Milgrom & Roberts (1990) and Bagwell & Ramey (1994) without noticing sufficient conditions for complementarities (cf. Topkis (1995, p.370-371)). A general model of the firm that gives sufficient conditions for complementarities to hold and that reflects monotonicity in the optimal decision as parameters vary was formulated by Topkis (1995). Moreover, it was shown what specific models of the firm satisfy these conditions.

In cooperative game theory with transferable utility the convexity property expresses that the incentives of a particular coalition for joining another coalition increases as the coalition size expands, or equivalently, that the marginal contribution of an arbitrary player increases as the coalition size grows, that means that the real-valued set-function has increasing differences. This can be interpreted as saying that such games also exhibit a kind of complementarity among the players. Similar to supermodular normal form games convex games possess many nice regular properties. Especially, the core is always nonempty Shapley (1971), and large Sharkey (1982); a fortiori Moulin (1990) has shown that it is even totally large. It coincides with the bargaining set for the grand coalition Maschler et al. (1972) and it is the unique stable set Shapley (1971). The kernel belongs to the core and consists of a single point, and it coincides with the nucleolus Maschler et al. (1972). Furthermore, the Shapley value is the center of gravity of the extreme points of the core and all marginal worth vectors are the extreme points of the core Shapley (1971). For convex games we can as well establish monotone comparative statics results by treating the real-valued set-function or the player set as parameters. For instance, the concept of population monotonic allocation scheme (PMAS) requires that the payoff for each player being member of an arbitrary coalition increases as the coalition becomes larger. Sprumont (1990) has shown that a sufficient condition for the existence of a PMAS is the convexity property and that in the case that convexity holds the extended marginal worth vectors yield a PMAS, and the extended Shapley value is a PMAS, too.

Although the lattice approach in connection with supermodularity is a widely accepted method by economists to analyze the underlying strategic conflict in an economic situation by studying the associated normal form game, the convexity approach hasn't found so much attention in economics to scrutinize situations where agents have strong incentives to operate together instead of operating independently. In the literature of cooperative game theory many examples of economic situations have been presented where

[2]To get an overview to what extent supermodularity have found its way into oligopoly theory see Vives (1999).

the corresponding cooperative TU-game is convex, and agents have strong incentives for mutual cooperation. The first convexity results related to a cooperative pollution game have been reported by Shapley & Shubik (1969). Champsaur (1975) showed that the cooperative game deduced from an economy with one private and one public good is convex. Closely related to the game model of the type presented by Champsaur are the activity optimization games with complementarity studied by Topkis (1987). Activity optimization games with complementarity are games where each coalition maximizes a common return function over the level of private activities of its members and over the levels of public activities that are available to any coalition. Every game in the class is convex as has been shown by Topkis (1987). Granot & Hojati (1990) provided sufficient conditions for convexity in cooperative games on cost allocation in communication networks. More recent convexity results have been derived by Breton et al. (2002) for different definitions of the real-valued set functions to investigate coalition stability and free riding in a game of pollution control. A first convexity result derived from an example of industrial organization was presented by Panzar & Willig (1977). The corresponding cooperative cost game of a natural monopoly that produces several goods that are consumed by many consumers was shown to be convex if the cost function exhibits cost complementarity or that each pair of distinct consumers consume different positive amounts and the cost function exhibits weak cost complementarity. Although, the classical Cournot situation is an example in oligopoly theory (industrial organization) where firms can be better off through cooperation than by acting alone, a convexity result for such games was given foremost recently by Zhao (1999). He established a necessary and sufficient condition for convexity of oligopoly market games with transferable technologies. Norde et al. (2002) proved that linear oligopoly games without transferable technologies are in general convex. While Driessen & Meinhardt (2005) gave for the most general case sufficient conditions for convexity to hold for oligopoly games without transferable technologies. Moreover, according to the fact that oligopoly TU-games may fail to be convex the authors have also investigated a relaxation of the convexity property called average-convexity[3] by providing sufficient conditions. For the class of common pool games with identical cost functions sufficient conditions of convexity are given by Driessen & Meinhardt (2001) and Meinhardt (1999*a*, 1999*b*, 2002). This class of games is a subclass of oligopoly games as have been shown in Driessen & Meinhardt (2001) and Meinhardt (2002).

The main purpose of this paper is to highlight an uniform approach to establish a new proof of the convexity property for cooperative linear oligopoly games with a homogenous good similar to the already existing proof of the convexity property for the so-called common pool games (cf. Driessen & Meinhardt (2001)). We outline that our uniform approach to prove the convexity property for classes of cooperative oligopoly games arises from maximization problems. Of course, the maximizers of any maximization problem are not necessarily unique, but nevertheless we can derive some monotonicity properties for the sum of maximizers and prices. These monotonicity property induces for linear oligopoly without transferable technologies supermodularity on the corresponding payoff functions similar to non-cooperative oligopoly games which arise from linear Cournot models. The essential part of the forthcoming proof technique is based on the *interchangeability* of both players i and j concerning the convexity condition. That is, this convexity condition (in terms of marginal contributions of two players in the TU game) does not change whenever players i and j are replaced by each other. One appealing consequence of this new proof technique is that we are able to generalize the convexity result first given by Zhao (1999) for the class of cooperative oligopoly with transferable technologies.

The remainder of the paper is organized as follows: In section 2 we introduce the notation and the definitions of a linear oligopoly situation without transferable technologies to establish the convexity con-

[3]The notion of average-convexity was introduced in the literature by Sprumont (1990). Properties of this class of games have been worked out in Iñarra & Usategui (1993).

dition for its corresponding cooperative oligopoly game. The section 3 provides more general sufficient conditions to derive convexity compared to the sufficient conditions obtained by Zhao (1999) in the case of linear cooperative oligopoly games with transferable technologies. The paper closes with some remarks.

2 The Supermodularity of an oligopoly game

Norde et al. (2002) established a convexity proof for the class of linear oligopoly games without transferable technologies that was based on the knowledge how to compute the various values of the coalitions to proceed their proof by the induction technique. The drawback of such a strategy was that we neither obtained any informations which mathematical properties of the model caused the convexity result nor meaningful economical interpretations of the derived properties to conclude which economical forces behind the model drives convexity. In opposite to the proof presented by Norde et al. (2002) our alternative proof technique to investigate the convexity property is based on maximizers and the interchangeability (symmetry) assumption about each pair of players. This alternative proof technique was very effective to derive sufficient conditions for convexity and average-convexity in non-linear oligopoly situations without transferable technologies as it was demonstrated in Driessen & Meinhardt (2005). Moreover, by relying on this uniform approach to prove convexity we benefit from very compact mathematical expressions which clearly highlights the economical forces in the model which are of particular importance to determine the convexity property. What we will observe in the sequel of our analysis is that we have to work out certain regular and monotonicity properties for the maximizers and prices to establish the convexity result. Alike to the linear Cournot model in non-cooperative game theory where the payoff functions exhibit supermodularity that imposes regular and monotonicity properties on the best reply functions, we observe similar conditions for the maximizers. Moreover, what is immediately clear from the exhibited price monotonicity result is that decreasing price functions are crucial for the convexity result and that we can exclude to consider price functions where an income effect plays a major rule to derive convexity. The mathematical properties are translated in a clear and obvious way in economical expressions, and therefore we gain insight into the economical forces which are causal for mutual cooperation in oligopoly games without transferable technologies. Since, convexity implies strong incentives for cooperative decision making.

A linear oligopoly situation $\langle N, (w_k)_{k\in N}, (c_k)_{k\in N}, a\rangle$ is determined by the capacities $w_k > 0,\ k \in N$ of the firms, their marginal costs $c_k > 0,\ k \in N$, and the prohibitive price (the intercept) $a > 0$ of the inverse demand function $p : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $p(q) := \max[0, a - q]$ for all $q \geq 0$. With a linear oligopoly situation $\langle N, (w_k)_{k\in N}, (c_k)_{k\in N}, a\rangle$, there is associated an oligopoly game in normal form $\Gamma = \langle N, (X_k, u_k)_{k\in N}\rangle$, the strategy sets $X_k,\ k \in N$, of which are given by the interval $X_k := [0, w_k]$ and the payoff functions $u_k,\ k \in N$, are given by

$$u_k((x_m)_{m\in N}) := \left[\max\left[0, a - \sum_{m\in N} x_m\right]\right] \cdot x_k - c_k \cdot x_k. \tag{2.1}$$

The strategy $x_k \in X_k$ for any firm $k \in N$ represents the quantity produced by firm k. Recall that in a Cournot market all firms producing a homogeneous good and the market price is determined by the total quantity of output $q := \sum_{k\in N} x_k$. The individual production decision of a firm k imposes externalities to the other firms transmitted by an induced change in the market price. The Formula (2.1) captures the typical structure of a payoff function in a Cournot market. The payoff to the firm k depends on its individual output decision x_k and on the total output of its opponents $q_{-k} := \sum_{m\in N\setminus\{k\}} x_m$, where the expression $p(q) \cdot x_k$ represents the revenue of firm k.

From the normal form game Γ with the payoff functions given by (2.1), we derive its associated cooperative game, that is formally specified by a pair $\langle N, v\rangle$. N is an nonempty finite set and $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function defined on the power set of N satisfying the usual convention that $v(\emptyset) = 0$. An element k of the nonempty finite set N is called a player while an element of the power set $S \in 2^N$ is called a coalition. The real number $v(S) \in \mathbb{R}$ is denoted as the value or worth of a coalition S. It is the maximal amount a coalition S can guarantee to itself. Furthermore, given a coalition S and a vector $\vec{z} \in \mathbb{R}^n$ we identify with the $|S|$-coordinates of the vector $\vec{z}$ the corresponding measure on S, such that $z(S) = \sum_{k \in S} z_k$.

The corresponding linear oligopoly game (without transferable technologies) $\langle N, v\rangle$, defined by $v := (v(T))_{T \subseteq N} \in \mathbb{R}^{2^{|N|}}$, is described as follows: $v(\emptyset) = 0$ and for all $T \subseteq N, T \neq \emptyset$,

$$v(T) = \max_{\vec{x}_T \in X_T} \left[\left[a - w(N \backslash T) - x(T) \right] \cdot x(T) - \sum_{k \in T} c_k \cdot x_k \right]. \tag{2.2}$$

Note that for every coalition $T \subseteq N$ we denote its strategy set by $X_T := \prod_{k \in T} X_k = \prod_{k \in T} [0, w_k]$. A possible payoff distribution of the value $v(T)$ for all $T \subseteq N$ is described by the projection of a vector $\vec{u} \in \mathbb{R}^n$ on its $|T|$-coordinates such that $u(T) \leq v(T)$ for all $T \subseteq N$.

In Driessen & Meinhardt (2005) it was shown that for oligopoly games of the type (2.2) the α- and β-values coincide, that is, for all $T \subseteq N, T \neq \emptyset$, it holds that

$$\begin{aligned} v_\alpha(T) &:= \max_{\vec{x}_T \in X_T} \min_{\vec{y}_{N\backslash T} \in X_{N\backslash T}} \left[p(x(T) + y(N \backslash T)) \right] \cdot x(T) - \sum_{k \in T} c_k \cdot x_k \\ &= \min_{\vec{y}_{N\backslash T} \in X_{N\backslash T}} \max_{\vec{x}_T \in X_T} \left[p(x(T) + y(N \backslash T)) \right] \cdot x(T) - \sum_{k \in T} c_k \cdot x_k =: v_\beta(S), \end{aligned} \tag{2.3}$$

Thus, $v := v_\alpha = v_\beta$. In general, the β-value is equal to or greater than the α-value. This can be interpreted as indicating that there exists weak incentive to react passively by awaiting the joint action of the opponents, i.e. waiting or reaction in a bargaining process does pay extra. Moreover, the most prominent solution concept in cooperative game theory the so-called core will be different. This has some negative side effects on stabilizing an agreement point inside of the α-core that doesn't belong to the β-core. In this case we can expect that some bargaining difficulties will appear.

Note that the α- and β-characteristic functions have been introduced by von Neumann & Morgenstern (1944) and were further developed by Aumann (1961). In the literature was also a third type of arguing discussed, the so-called s-types introduced by Moulin (1981, 1988) for two person games and generalized for $n \geq 2$ in the γ-value by Ostmann (1984, 1988, 1994). For these games, the opposition does not rely on the complete strategy set to stabilize proposals as in the α- and β-game, it relies instead of the best reply set. Strategies that hurt a coalition are not selected by its members. A more thorough discussion of the various characteristic function concepts can be found in Ostmann (1984).

The reader should be aware that the α- and β-characteristic functions represent two different kinds of perceptions on which principle the arguments of the opponents are based on to obtain an agreement in a stylized bargain process. That means, the α-value (β-value) represents the payoff the members of a coalition can guarantee to themselves (can not be prevented from) if the opposition is relying in a bargaining process on the α-argument, respectively the β-argument. Hence, the α- or β-value characterizes the potential bargaining power of a coalition under these specific kinds of arguing. Although, the α- and β-value coincides in oligopoly games, and are therefore based on the same perception that the opponents will flood the market to stabilize a certain outcome in a bargain process nothing will be carried out at this stage. In the case that the negotiation will break down and no agreement can be reached, a coalition can be sure that under the extrem circumstance that the opposition will carry out its threat and will flood the market, their members can get at least the α- or β-value (cf. Meinhardt (2002, Chapter 4)). Moreover, in reality a bargaining process will last only over a limited period, it is, therefore, not unrealistic to assume that the negotiating parties have neither the opportunity to improve their cost structure nor to extend their capacity beyond every limit to strengthen their bargaining power during the duration of the negotiation. Such unrealistic assumptions would make no sense, especially, in an environment with complete informations and no uncertainty, as in our case. Thus, the essential assumption about unlimited constraints in a Cournot setting is not applicable in a cooperative environment with this specific kind of underlying bargaining process.

Throughout the remainder of the section, we use the following notation. For any $T \subseteq N$, we write $\alpha_T := a - w(N \backslash T)$. Note that $\alpha_{T \cup k} = \alpha_T + w_k$ whenever $k \in N \backslash T$. Further, for any $T \subseteq N, T \neq \emptyset$, let $\vec{x}^T$ be a maximizer for the maximization problem of (2.2) with respect to the coalition T, i.e. $v(T) = \max_{\vec{x}_T \in X_T} d_T(\vec{x}_T)$ for all $T \subseteq N$, where the function $d_T : X_T \rightarrow \mathbb{R}$ is given by

$$d_T(\vec{x}_T) := \Big[a - w(N \backslash T) - x(T)\Big] \cdot x(T) - \sum_{k \in T} c_k \cdot x_k \qquad \text{for all } \vec{x}_T \in X_T. \tag{2.4}$$

Note that its partial derivative is denoted by $\frac{\partial d_T}{\partial x_k}(\vec{x}_T) = \alpha_T - 2 \cdot x(T) - c_k$ for all $k \in T$, for all $\vec{x}_T \in X_T$. Moreover, given a vector of maximizers $\vec{x}^T \in X_T$, we denote by q^T, the induced total production in the oligopoly market, i.e. $q^T := x^T(T) + w(N \backslash T)$. For the induced total production q^T the market price becomes $p(q^T) = a - q^T = a - w(N \backslash T) - x^T(T) = \alpha_T - x^T(T)$.

The first result states that the induced price increase $w_i - x_i^{S \cup i} \geq 0$ of a firm i by joining a coalition S is larger or equal to the induced price decrease due to an expansion of the production of the old member firms in S. Note that the opponents are already producing at full capacity, when a firm i leaves the coalition with the opponents and joins a coalition S. In such a situation the opponents cannot counter anymore the induced price increase of $w_i - x_i^{S \cup i} \geq 0$. Therefore, the price increase can only be offset by an expansion of the production of the old member firms, because the overall production opportunity has improved and it is worthwhile for the individual firms to expand the production level, whenever it is possible to do so.

Lemma 2.1. *Consider an oligopoly situation $\langle N, (w_i)_{i \in N}, (c_i)_{i \in N}, a \rangle$ without transferable technologies. Let $i \in N$ and for every $S \subseteq N \backslash \{i\}, S \neq \emptyset$, it holds the following price monotonicity relationship derived from the total induced production levels q^S and $q^{S \cup i}$*

$$p(q^S) \leq p(q^{S \cup i}) \qquad \Longleftrightarrow \qquad \Big[x^{S \cup i}(S) - x^S(S)\Big] \leq w_i - x_i^{S \cup i}. \tag{2.5}$$

Moreover, for every $i, j \in N, i \neq j$ and $S \subseteq N \backslash \{i, j\}, S \neq \emptyset$, if $p(q^{S \cup i}) \leq p(q^{S \cup j})$, then it holds that

$$x^{S \cup j}(S) - x^{S \cup i}(S) \leq w_j - x_j^{S \cup j}. \tag{2.6}$$

Proof. (i) Assume first that all members of S producing in S and $S \cup \{i\}$ at full capacity, i.e. $x_k^S = x_k^{S \cup i} = w_k$ for all $k \in S$ and for firm i we have $x_i^{S \cup i} \geq 0$. This leads to $p(q^S) = \alpha_S - w(S) \leq \alpha_S + w_i - x_i^{S \cup i} - w(S) = p(q^{S \cup i})$, since $w_i - x_i^{S \cup i} \geq 0$. This result holds true as well for any proportion $1 > \tau > 0$ for which firms in S produce at full capacity and the remaining proportion $(1 - \tau)$ produces nothing. Then we get $p(q^S) = \alpha_S - \tau \cdot w(S) \leq \alpha_S + w_i - x_i^{S \cup i} - \tau \cdot w(S) = p(q^{S \cup i})$. (ii) Now assume that all members of S producing nothing in coalition $S \cup \{i\}$, this implies the same production level as in coalition S that is $x_k^S = 0$ for each $k \in S$, since the price increase of at most $w_i - x_i^{S\cup i} \geq 0$ has improved the overall production opportunity in $S \cup \{i\}$ compared to S. We obtain $p(q^S) = \alpha_S \leq \alpha_S + w_i - x_i^{S\cup i} = p(q^{S\cup i})$. (iii) In order to investigate the final case observe that due to the increase in price $w_i - x_i^{S\cup i} \geq 0$ and if it holds for a firm $l \in S$ such that $\frac{\partial d_S}{\partial x_l}(\vec{x}^S) > 0$, we get $\frac{\partial d_{S\cup i}}{\partial x_l}(\vec{x}^{S\cup i}) > 0$, thus $x_l^S = x_l^{S\cup i} = w_l$. But then $l \in S$ is not an appropriate selection. We need to select a firm $l \in S$ such that $\frac{\partial d_S}{\partial x_l}(\vec{x}^S) \leq 0$ and $\frac{\partial d_{S\cup i}}{\partial x_l}(\vec{x}^{S\cup i}) \geq 0$ is given. If it holds for all firms $k \in S$ such that $x_k^S = x_k^{S\cup i} = w_k$, then no firm $k \in S$ exists that produces at an interior solution neither in S nor in $S \cup \{i\}$, thus the subcase (i) applies once again. According to these arguments we just consider a firm $l \in S$ that produces at an interior solution or at its capacity level in coalition $S \cup \{i\}$, this yields to $\alpha_S + w_i - 2 \cdot x^{S\cup i}(S \cup \{i\}) \geq c_l$. The overall production opportunity has improved in $S \cup \{i\}$ compared to S we can conclude by the foregoing argumentation that for firm $l \in S$ we get the following relationship between marginal revenue and marginal cost $\alpha_S - 2 \cdot x^S(S) \leq c_l$, and this implies

$$\alpha_S - 2 \cdot x^S(S) \leq c_l \leq \alpha_S + w_i - 2 \cdot x^{S\cup i}(S \cup \{i\}) \quad \Longleftrightarrow \quad \left[x^{S\cup i}(S \cup \{i\}) - x^S(S)\right] \leq \frac{w_i}{2} \leq w_i.$$

Hence, we conclude that we obtain the following price monotonicity $p(q^S) \leq p(q^{S\cup i})$ which is equivalent to $[x^{S\cup i}(S) - x^S(S)] \leq w_i - x_i^{S\cup i}$. This proves (2.5).
To prove the second part of the Lemma, observe that $p(q^{S\cup i}) \leq p(q^{S\cup j})$ is equivalent to

$$x^{S\cup j}(S) - x^{S\cup i}(S) \leq w_j - x_j^{S\cup j} - (w_i - x_i^{S\cup i}).$$

Since $w_i - x_i^{S\cup i} \geq 0$, this yields to

$$x^{S\cup j}(S) - x^{S\cup i}(S) \leq w_j - x_j^{S\cup j}.$$

With this argument we are done. □

REMARK 2.1.
Notice, we claim that due to the interchangeability of both players i and j we can put forward the essential assumption $p(q^{S\cup j}) \geq p(q^{S\cup i})$ without loss of generality in the forthcoming convexity proof, since if the assumptions is not satisfied player i can take the role of player j and vice versa. In summary, our approach to investigate convexity of oligopoly games is strongly based on an appropriately chosen interchangeability assumption about two maximizers (cf. Driessen & Meinhardt (2005)). ◇

Lemma 2.2. *Consider a linear oligopoly situation $\langle N, (w_k)_{k\in N}, (c_k)_{k\in N}, a\rangle$ without transferable technologies. For any $S \subseteq N\backslash\{i, j\}, S \neq \emptyset$ with $(x_k^{S\cup i})_{k\in S}$ and $p(q^{S\cup j}) \geq p(q^{S\cup i})$, then*

$$x^{S\cup j}(S \cup \{j\}) \geq x^{S\cup i}(S). \tag{2.7}$$

Proof. Note first that the joining of firm i and j to coalition S imposes a price increase compared to the initial situation without the participation of firm i and j, since $\frac{\partial p(\vec{x}_S)}{\partial w_k} = -1$ for all $k \in N\backslash S$. The cost structure remains unchanged under an increase in prices, thus the overall production situation has been improved for all old member firms in $S \cup \{m\}$ compared to S with $m = i, j$. Notice in addition that we have to consider two trivial cases. That is either $x_k^{S\cup j} = x_k^{S\cup i} = 0$ or $x_k^S = w_k$ for all $k \in S$. (i) In the former case, we get $x^{S\cup j}(S\cup\{j\}) = x_j^{S\cup j} \geq 0 = x^{S\cup i}(S)$, whereas in the latter case (ii) – the full capacity case – firms will never reduce their production in the new situation due to the overall improvement in their production condition. According to $(x_k^{S\cup i})_{k\in S}$, $x_j^{S\cup j} \geq 0$, we get $x^{S\cup j}(S\cup\{j\}) = x^{S\cup j}(S) + x_j^{S\cup j} = w(S) + x_j^{S\cup j} \geq w(S) = x^{S\cup i}(S)$. In order to conclude we have still to consider two non-trivial subcases for interior solutions. (iii) For this purpose selected a firm l in S for $S\cup\{m\}$ with $m = i, j$, and let us first assume without loss of generality that $w_l \geq x_l^{S\cup j} \geq x_l^{S\cup i} > 0$ is given. Notice that it is sufficient to select a firm l in S for $S\cup\{m\}$ with $m = i, j$ with strictly positive production to compare the production condition of coalition $S\cup\{j\}$ with coalition $S\cup\{i\}$. Whenever we can not find a firm l with strictly positive production or that $x_k^S = w_k$ for all $k \in S$, then one of the above trivial cases applies once again. Under the assumption $w_l \geq x_l^{S\cup j} \geq x_l^{S\cup i} > 0$ the overall production opportunity is better under $S\cup\{j\}$ than under $S\cup\{i\}$. This implies that $p(q^{S\cup j}) - x^{S\cup j}(S\cup\{j\}) \geq c_l$ and $c_l = p(q^{S\cup i}) - x^{S\cup i}(S\cup\{i\})$ and, thus we get

$$x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S\cup\{i\}) \leq p(q^{S\cup j}) - p(q^{S\cup i}).$$

Therefore, we need some estimation from below, but due to $\frac{\partial p(\vec{x}_S)}{\partial w_k} = -1$ for all $k \in N\backslash S$ and $(x_k^{S\cup i})_{k\in S}$ $x_j^{S\cup j} \geq 0$, we conclude that $x^{S\cup j}(S\cup\{j\}) \geq x^{S\cup i}(S)$. (iv) Now let us consider the subcase that we have $0 < x_l^{S\cup j} \leq x_l^{S\cup i} \leq w_l$. Thus the overall production situation is now better under the coalition $S\cup\{i\}$ than under $S\cup\{j\}$, we can conclude that $p(q^{S\cup j}) - x^{S\cup j}(S\cup\{j\}) = c_l$ and $c_l \leq p(q^{S\cup i}) - x^{S\cup i}(S\cup\{i\})$, hence

$$0 \leq p(q^{S\cup j}) - p(q^{S\cup i}) \leq x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S\cup\{i\}),$$

According to $x_j^{S\cup j} \geq 0$, $(x_k^{S\cup i})_{k\in S}$ and applying the fundamental interchangeability assumption $p(q^{S\cup j}) \geq p(q^{S\cup i})$, we obtain now $x^{S\cup j}(S\cup\{j\}) \geq x^{S\cup i}(S)$. With this argument we are done. □

Example 2.1. To see that the interchangeability assumption $p(q^{S\cup j}) \geq p(q^{S\cup i})$ is crucial to derive the result $x^{S\cup j}(S\cup\{j\}) \geq x^{S\cup i}(S)$ given by Lemma 2.2 consider an oligopoly situation with three firms $N = \{1,2,3\}$, an inverse demand function given by $p(\vec{x}) := 20 - (x_1 + x_2 + x_3)$, where the vector of marginal costs is specified by $\vec{c} = \{2,2,12\}$, and the vector of capacities is given by $\vec{w} = \{4,4,12\}$. The optimal production schemes for the various coalitions are given by

$$\begin{array}{lll}
x_1^{\{1\}} = 1, & x_2^{\{2\}} = 1, & x_3^{\{3\}} = 0, \\
x_1^{\{1,2\}} = 0, & x_2^{\{1,2\}} = 3, & \\
x_1^{\{1,3\}} = 4, & x_3^{\{1,3\}} = 0, & \\
x_2^{\{2,3\}} = 4, & x_3^{\{2,3\}} = 0, & \\
x_1^{N} = 4, & x_2^{N} = 4, & x_3^{N} = 0.
\end{array}$$

Exemplary, let $S = \{1\}$, $i = 2, j = 3$, then we get $p(q^{S\cup j}) = p(q^{\{1,3\}}) = 12 > 5 = p(q^{\{1,2\}}) = p(q^{S\cup i})$. By restricting the optimal production scheme of coalition $\{1,2\}$ on $x_1^{\{1,2\}}$, we get $x^{\{1,2\}}(\{1\}) = 0$. In addition, observe that $x^{\{1,3\}}(\{1,3\}) = 4$, thus we have $x^{\{1,3\}}(\{1,3\}) = 4 > 0 = x^{\{1,2\}}(\{1\})$ as required. Now, observe that by interchanging the role of the players, that is $i = 3, j = 2$, the total production of coalition $\{1,3\}$ remains unchanged on $x^{\{1,3\}}(\{1,3\}) = 4$, whereas the restriction of the optimal production vector gives us now $x^{\{1,3\}}(\{1\}) = 4$, and for coalition $\{1,2\}$ we have a total production of $x^{\{1,2\}}(\{1,2\}) = 3$. Hence, we get $x^{\{1,2\}}(\{1,2\}) = 3 < 4 = x^{\{1,3\}}(\{1\})$. But now we have $p(q^{S\cup i}) = p(q^{\{1,3\}}) > p(q^{\{1,2\}}) = p(q^{S\cup j})$ that violates clearly the condition $p(q^{S\cup j}) \geq p(q^{S\cup i})$. It is left to the reader to check the remaining cases. ◇

The next result states that if the sum of maximizers in coalition $S\cup\{j\}$ is larger than in $S\cup\{i\}$, i.e. $x^{S\cup j}(S\cup\{j\}) \geq x^{S\cup i}(S\cup\{i\})$, then the total induced price effect by exchanging player j by player i in a particular coalition is larger than the total induced price change if player i is joining the opposition and player j is leaving. To see that notice that by joining a particular coalition S a certain player will produce at most at full capacity even less, but if he belongs to the opposition, he is producing at full capacity. That means that the impact on the price level must be weaker in the former than in the latter case. According to the assumption the induced total production in $S \cup \{i\}$ must be less than in $S \cup \{j\}$. This yields to the observed price effect mentioned in Proposition 2.1 below.

Proposition 2.1. *Let $i, j \in N, i \neq j$ and $S \subseteq N\backslash\{i,j\}, S \neq \emptyset$. If $x^{S\cup j}(S \cup \{j\}) \geq x^{S\cup i}(S \cup \{i\})$, then*

$$w_i - w_j \leq p(q^{S\cup i}) - p(q^{S\cup j}). \tag{2.8}$$

Moreover, for every $i, j \in N, i \neq j$ and $S \subseteq N\backslash\{i,j\}, S \neq \emptyset$, if $x^{S\cup i}(S) \geq x^S(S)$, then it holds that

$$p(q^{S\cup i}) - p(q^{S\cup j}) \leq w_i - x_i^{S\cup i} + \Big[x^S(S) - x^{S\cup i}(S)\Big] \leq w_i - x_i^{S\cup i}. \tag{2.9}$$

Proof. Recall that $\alpha_{S\cup k} = \alpha_S + w_k$ and $\alpha_S = a - w(N\backslash S)$. By Lemma 2.1 we get the following monotonicity relationship $p(q^S) \leq p(q^{S\cup j})$ of the inverse demand function p and by the monotonicity of the sum of maximizers $x^S(S) \leq x^{S\cup i}(S)$, we get

$$\begin{aligned}
&w_i - w_j \leq w_i - w_j + x^{S\cup j}(S \cup \{j\}) - x^{S\cup i}(S) - x_i^{S\cup i} \\
&= \alpha_{S\cup i} - x^{S\cup i}(S \cup \{i\}) - \alpha_{S\cup j} + x^{S\cup j}(S \cup \{j\}) \\
&= p(q^{S\cup i}) - p(q^{S\cup j}) = p(q^{S\cup i}) - p(q^S) + p(q^S) - p(q^{S\cup j}) \\
&\leq p(q^{S\cup i}) - p(q^S) = \alpha_{S\cup i} - x^{S\cup i}(S \cup \{i\}) - \alpha_S + x^S(S) \\
&= w_i - x_i^{S\cup i} + \Big[x^S(S) - x^{S\cup i}(S)\Big] \leq w_i - x_i^{S\cup i}.
\end{aligned} \tag{2.10}$$

The first inequality holds true due to our assumption, whereas the second and third inequality is satisfied by the monotonicity properties. □

Example 2.2. To see that the sufficient condition $x^{S\cup i}(S) \geq x^S(S)$ mentioned in Proposition 2.1 does not hold in general consider an oligopoly situation with three firms $N = \{1, 2, 3\}$, an inverse demand function given by $p(\vec{x}) := 20 - (x_1 + x_2 + x_3)$, where the vector of marginal costs is specified by $\vec{c} = \{2, 13, 12\}$, and the vector of capacities is given by $\vec{w} = \{2, 2, 4\}$.

Let $S = \{3\}$, $i = 1, j = 2$ and observe that we obtain the following optimal production plans $x_1^{\{1,3\}} = 2, x_3^{\{1,3\}} = 1$ for the cartel $\{1, 3\}$. Whereas for cartel $\{2, 3\}$ we get $x_2^{\{2,3\}} = 0, x_3^{\{2,3\}} = 3$ and note that in case that firm 3 remains independent it produces $x_3^{\{3\}} = 2$. Thus, $x^{\{2,3\}}(\{2, 3\}) = 3 = 3 = x^{\{1,3\}}(\{1, 3\})$ but $x^{\{3\}}(\{3\}) = 2 > x^{\{1,3\}}(\{3\}) = 1$. ◇

In order to determine the economical factors which are crucial in obtaining the convexity condition (2.16) in oligopoly games without transferable technologies we need to consider the revenue function of a coalition T, denoted by $r_T(\vec{x}^T) := [\alpha_T - x^T(T)] \cdot x^T(T)$. Inequality (2.14) below states that the revenues evaluated at the appropriately chosen maximizers $\vec{x}^{S\cup i}$ and $\vec{x}^{S\cup j}$ exhibit supermodularity. This can be interpreted as indicating that the marginal revenue $r_{S\cup i}(\vec{x}^{S\cup i}) - r_S((x_k^{S\cup i})_{k\in S})$ of a firm i increases to

$r_{S\cup ij}((\vec{x}^{\,S\cup j}, x_i^{S\cup i})) - r_{S\cup j}(\vec{x}^{\,S\cup j})$ whenever the size of the capacity controlled by a cartel expands from $w(S)$ to $w(S \cup \{j\})$ and the underlying production schedule is changed from $\vec{x}^{\,S\cup i}$ to $\vec{x}^{\,S\cup j}$ while firm i has not adjusted its production plan to the new situation. In the forthcoming convexity proof we will notice that this kind of supermodularity of the revenues is the crucial property to derive the convexity result and that the associated costs are not important. We conclude that as long as the underlying production technologies is described by linear cost functions without synergy effects more general revenue functions have to satisfy this kind of supermodularity to obtain convexity of the game $\langle N, v\rangle$ of type (2.1), since from the cost side exists no potential damaging factors that might destroy convexity and therefore mutual cooperation in the cartel.

Lemma 2.3.

1. *Let $i \in N$ and for any $T \subseteq N\backslash\{i\}$ the following inequality holds*

$$\Big[x^T(T) - w(T)\Big] \cdot x_i^i \leq \Big[w_i - x_i^i\Big] \cdot x^T(T), \tag{2.11}$$

that is equivalent to

$$p(q^i) \cdot x_i^i + p(q^T) \cdot x^T(T) \leq \Big[p(q^T) + w_i - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big]. \tag{2.12}$$

2. *Let $i, j \in N, i \neq j$. For any $S \subseteq N\backslash\{i,j\}, S \neq \emptyset$ the following inequality is satisfied*

$$\Big[2 \cdot x_i^{S\cup i} - w_i\Big] \cdot \Big[x^{S\cup j}(S \cup \{j\}) - x^{S\cup i}(S)\Big] \leq x_i^{S\cup i} \cdot w_j, \tag{2.13}$$

that is equivalent to

$$r_{S\cup i}(\vec{x}^{\,S\cup i}) + r_{S\cup j}(\vec{x}^{\,S\cup j}) \leq r_S((x_k^{S\cup i})_{k\in S}) + r_{S\cup ij}((\vec{x}^{\,S\cup j}, x_i^{S\cup i})). \tag{2.14}$$

Proof. (1) Let us first observe that (2.11) is equivalent to (2.12). For this purpose we start with (2.12).

$$p(q^i) \cdot x_i^i + p(q^T) \cdot x^T(T) \leq \Big[p(q^T) + w_i - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big], \tag{2.12}$$

$$\Longleftrightarrow$$

$$\Big[\alpha_i - x_i^i\Big] \cdot x_i^i + \Big[\alpha_T - x^T(T)\Big] \cdot x^T(T) \leq \Big[\alpha_{T\cup i} - x^T(T) - x_i^i\Big] \cdot \cdot \Big[x^T(T) + x_i^i\Big]$$

Rearranging the maximizers x_i^i and $\vec{x}^{\,T}$, we obtain now

$$\Big[\alpha_i - x_i^i - \alpha_{T\cup i} + x^T(T) + x_i^i\Big] \cdot x_i^i \leq \Big[\alpha_{T\cup i} - x^T(T) - x_i^i - \alpha_T + x^T(T)\Big] \cdot x^T(T),$$

that is

$$\Big[x^T(T) + \alpha_i - \alpha_{T\cup i}\Big] \cdot x_i^i \leq \Big[w_i - x_i^i\Big] \cdot x^T(T).$$

Now consider that $\alpha_i - \alpha_{T\cup i} = -w(T)$, hence, the above inequality simplifies to

$$\Big[x^T(T) - w(T)\Big] \cdot x_i^i \leq \Big[w_i - x_i^i\Big] \cdot x^T(T). \tag{2.11}$$

Due to $x_i^i \leq w_i$ the right hand side (rhs) is non-negative and the left hand sight (lhs) is non-positive according to $x^T(T) \leq w(T)$. This proves (2.11).

(2) To start with the second part of the proof let us consider first the expression (2.14) given by

$$\begin{aligned}
&p(q^{S\cup i}) \cdot x^{S\cup i}(S\cup\{i\}) + p(q^{S\cup j}) \cdot x^{S\cup j}(S\cup\{j\}) \\
&\leq \left[p(q^{S\cup i}) - w_i + x_i^{S\cup i}\right] \cdot x^{S\cup i}(S) + \left[p(q^{S\cup j}) + w_i - x_i^{S\cup i}\right] \cdot \left[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\right] \\
&\Longleftrightarrow \\
&\left[\alpha_{S\cup i} - x^{S\cup i}(S\cup\{i\})\right] \cdot x^{S\cup i}(S\cup\{i\}) + \left[\alpha_{S\cup j} - x^{S\cup j}(S\cup\{j\})\right] \cdot x^{S\cup j}(S\cup\{j\}) \\
&\leq \left[\alpha_S - x^{S\cup i}(S)\right] \cdot x^{S\cup i}(S) + \left[\alpha_{S\cup ij} - x^{S\cup j}(S\cup\{j\}) - x_i^{S\cup i}\right] \cdot \left[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\right].
\end{aligned}$$

Recall that $\alpha_{T\cup k} = \alpha_T + w_k$ and $\alpha_T = a - w(N\backslash T)$. Now observe that the above inequality reduces to the simplified expression given by (2.13)

$$\begin{aligned}
&\left[w_i - x_i^{S\cup i}\right] \cdot x^{S\cup i}(S) \leq \left[w_i - x_i^{S\cup i}\right] \cdot x^{S\cup j}(S\cup\{j\}) + \left[w_j - x^{S\cup j}(S\cup\{j\}) + x^{S\cup i}(S)\right] \cdot x_i^{S\cup i} \\
&\Longleftrightarrow \\
&\left[w_i - x_i^{S\cup i}\right] \cdot \left[x^{S\cup i}(S) - x^{S\cup j}(S\cup\{j\})\right] \leq x_i^{S\cup i} \cdot w_j + x_i^{S\cup i} \cdot \left[x^{S\cup i}(S) - x^{S\cup j}(S\cup\{j\})\right] \\
&\Longleftrightarrow \\
&\left[2 \cdot x_i^{S\cup i} - w_i\right] \cdot \left[x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S)\right] \leq x_i^{S\cup i} \cdot w_j. \qquad (2.13)
\end{aligned}$$

Consider now that $2 \cdot x_i^{S\cup i} - w_i \leq x_i^{S\cup i}$ according to $x_i^{S\cup i} \leq w_i$. To conclude, notice first that we can put forward the interchangeability condition of both players i and j, that is given by $p(q^{S\cup j}) \geq p(q^{S\cup i})$, without loss of generality. Then by Lemma 2.2 that satisfies the fundamental interchangeability assumption $p(q^{S\cup j}) \geq p(q^{S\cup i})$, we have $x^{S\cup j}(S\cup\{j\}) \geq x^{S\cup i}(S)$. Hence, we conclude that the second term of (2.13) is non-negative. If it holds $2 \cdot x_i^{S\cup i} - w_i \leq 0$, then (2.13) is clearly satisfied, since the lhs of (2.13) is non-positive whereas the rhs of (2.13) is non-negative. To finish the proof we need to estimate

$$x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S) \leq w_j \qquad \text{if } x_i^{S\cup i} > \frac{w_i}{2}.$$

Which is clearly satisfied by part (2) of Lemma 2.1 given by inequality (2.6). This establishes (2.14). □

Example 2.3. In order to gain some familiarity with the interchangeability (symmetry) condition, let us resume Example 2.1. For the first case $S = \{1\},\ i = 2,\ j = 3$ and $p(q^{S\cup j}) \geq p(q^{S\cup i})$ for any $S \subseteq N\backslash\{i,j\}$, we obtain by applying formula (2.13)

$$[2 \cdot x_i^{S\cup i} - w_i] \cdot [x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S)] = [6-4] \cdot [4-0] = 8 < 3 \cdot 12 = 36 = x_i^{S\cup i} \cdot w_j.$$

It should be apparent that whenever we apply the interchangeability condition, we have also to interchange the labels in formula (2.13) to get a correct statement. That means for the case of $S = \{1\},\ i = 3,\ j = 2$ and $p(q^{S\cup j}) \leq p(q^{S\cup i})$, where we have interchanged the role of player i and j, that we get now for inequality (2.13) the following reformulation

$$[2 \cdot x_j^{S\cup j} - w_j] \cdot [x^{S\cup i}(S\cup\{i\}) - x^{S\cup j}(S)] \leq x_j^{S\cup j} \cdot w_i,$$

which is by using the numerical values of Example 2.1

$$[6-4]\cdot[4-0] = 8 < 3\cdot 12 = 36,$$

that is the desired result. According to the symmetry condition of the convexity property, this must be clearly the same result as in the former case, since we have only interchanged the role of both players, nothing more. $\diamond$

Proposition 2.2. *The oligopoly game without transferable technologies $\langle N, v\rangle$ of the form (2.2) is monotone, i.e. $v(T) \geq v(S)$ for all $S \subseteq T \subseteq N$.*

Proof. Let $T := S \cup \{i,j\}$ and let $S \subseteq N\backslash\{i,j\}, S \neq \emptyset, i \neq j$,. Consider the feasible production combination $(\vec{x}^{S\cup j}, 0) \in X_{S\cup j} \times X_i = X_{S\cup ij}$ to underestimate the value of coalition $S \cup \{i,j\}$ to derive

$$\begin{aligned} v(S\cup\{i,j\}) &\geq \left[\alpha_{S\cup ij} - x^{S\cup j}(S\cup\{j\}) + w_i\right]\cdot x^{S\cup j}(S\cup\{j\}) - \sum_{k\in S\cup j} c_k \cdot x_k^{S\cup j} \\ &= v(S\cup\{j\}) + w_i \cdot x^{S\cup j}(S\cup\{j\}) \\ \Longrightarrow \qquad & v(S\cup\{i,j\}) - v(S\cup\{j\}) \geq w_i \cdot x^{S\cup j}(S\cup\{j\}) \geq 0. \end{aligned}$$

This proves monotonicity of the oligopoly game of the form (2.2). $\square$

The oligopoly TU-game $\langle N, v\rangle$ is called to be *convex* if its characteristic function $v : 2^N \to \mathbb{R}$, as given by (2.2), satisfies one of the following two equivalent conditions (cf. Shapley (1971)):

$$v(S) + v(T) \leq v(S\cup T) + v(S\cap T) \qquad \text{for all } S, T \subseteq N \tag{2.15}$$

or equivalently,

$$v(S\cup\{i\}) - v(S) \leq v(S\cup\{i,j\}) - v(S\cup\{j\}) \qquad \text{if } S \subseteq N\backslash\{i,j\}. \tag{2.16}$$

Remind that for the linear case a first convexity proof was given by Norde et al. (2002). We present now an alternative proof that is based on the use of maximizers for the relevant maximization problems of the form (2.2) and on an interchangeability (symmetry) condition of players i and j given by $p(q^{S\cup j}) \geq p(q^{S\cup i})$. If this condition is not valid (or equivalently, $p(q^{S\cup j}) \leq p(q^{S\cup i})$), then an identical proof applies in which player i takes the role of player j and vice versa.

Theorem 2.1. *The oligopoly game without transferable technologies $\langle N, v\rangle$ of the form (2.2) is a convex game.*

Proof. In order to establish the convexity condition (2.16) we have to consider two cases $S = \emptyset$ and $S \neq \emptyset$.
Case one. We suppose that $S = \emptyset$. Now let $T \subseteq N\backslash\{i\}, T \neq \emptyset$. Instead of proving $v(\{i\}) + v(\{j\}) \leq v(\{i,j\})$ we prove the convexity condition by the extended version $v(\{i\}) + v(T) \leq v(T\cup\{i\})$, if $T \subseteq N\backslash\{i\}, T \neq \emptyset$. Concerning the maximization problem (2.2) we can describe $v(\{i\})$ and $v(T)$ by maximizers x_i^i and $\vec{x}^T$ such that

$$v(\{i\}) = \left[\alpha_i - x_i^i\right]\cdot x_i^i - c_i \cdot x_i^i \tag{2.17}$$

$$v(T) = \left[\alpha_T - x^T(T)\right]\cdot x^T(T) - \sum_{k\in T} c_k \cdot x_k^T. \tag{2.18}$$

By choosing the feasible production plan $((x_k^T)_{k\in T}, x_i^i) \in X_T \times X_i = X_{T\cup i}$ for the coalition $T \cup \{i\}$ we get

$$v(T \cup \{i\}) \geq \Big[\alpha_{T\cup i} - x^T(T) - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big] - \sum_{k\in T} c_k \cdot x_k^T - c_i \cdot x_i^i. \tag{2.19}$$

The cost terms cancel out and and we obtain formula (2.12) from the first part of Lemma 2.3

$$\Big[\alpha_i - x_i^i\Big] \cdot x_i^i + \Big[\alpha_T - x^T(T)\Big] \cdot x^T(T) \leq \Big[\alpha_{T\cup i} - x^T(T) - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big]. \tag{2.12}$$

That is equivalent to expression (2.11) by Lemma 2.3, hence

$$\Big[x^T(T) - w(T)\Big] \cdot x_i^i \leq \Big[w_i - x_i^i\Big] \cdot x^T(T). \tag{2.11}$$

The inequality is satisfied due to Lemma 2.3 and the convexity condition (2.16) is given for $S = \emptyset$.

Case two. Suppose that $S \neq \emptyset$. Now, let $i, j \in N, i \neq j$, and $S \subseteq N\backslash\{i,j\}$ and rewrite for convenience'sake the convexity condition (2.16) as

$$v(S \cup \{i\}) + v(S \cup \{j\}) \leq v(S) + v(S \cup \{i,j\}) \qquad \text{if } S \subseteq N\backslash\{i,j\}. \tag{2.20}$$

Concerning the maximization problems (2.2) with respect to the coalitions $S\cup\{j\}$ and $S\cup\{i\}$ respectively, we consider their maximizers $\vec{x}^{\,S\cup j}$ and $\vec{x}^{\,S\cup i}$ respectively in order to derive the following equalities:

$$v(S \cup \{j\}) = \Big[\alpha_{S\cup j} - x^{S\cup j}(S \cup \{j\})\Big] \cdot x^{S\cup j}(S \cup \{j\}) - \sum_{k\in S\cup\{j\}} c_k \cdot x_k^{S\cup j}, \tag{2.21}$$

$$v(S \cup \{i\}) = \Big[\alpha_{S\cup i} - x^{S\cup i}(S \cup \{i\})\Big] \cdot x^{S\cup i}(S \cup \{i\}) - \sum_{k\in S\cup\{i\}} c_k \cdot x_k^{S\cup i}. \tag{2.22}$$

We choose now the feasible production plan $(x_k^{S\cup i})_{k\in S} \in X_S$ for coalition S to obtain

$$v(S) \geq \Big[\alpha_S - x^{S\cup i}(S)\Big] \cdot x^{S\cup i}(S) - \sum_{k\in S} c_k \cdot x_k^{S\cup i}. \tag{2.23}$$

Similar, for coalition $S\cup\{i,j\}$, we choose a feasible production plan $(\vec{x}^{\,S\cup j}, x_i^{S\cup i}) \in X_{S\cup j} \times X_i = X_{S\cup ij}$ to get

$$\begin{aligned} v(S \cup \{ij\}) \geq {}& \Big[\alpha_{S\cup ij} - x^{S\cup j}(S \cup \{j\}) - x_i^{S\cup i}\Big] \cdot \Big[x^{S\cup j}(S \cup \{j\}) + x_i^{S\cup i}\Big] \\ & - \sum_{k\in S\cup j} c_k \cdot x_k^{S\cup j} - c_i \cdot x_i^{S\cup i}. \end{aligned} \tag{2.24}$$

The cost terms will cancel out and we arrive at

$$\begin{aligned} & \Big[\alpha_{S\cup i} - x^{S\cup i}(S \cup \{i\})\Big] \cdot x^{S\cup i}(S \cup \{i\}) + \Big[\alpha_{S\cup j} - x^{S\cup j}(S \cup \{j\})\Big] \cdot x^{S\cup j}(S \cup \{j\}) \\ \leq {}& \Big[\alpha_S - x^{S\cup i}(S)\Big] \cdot x^{S\cup i}(S) + \Big[\alpha_{S\cup ij} - x^{S\cup j}(S \cup \{j\}) - x_i^{S\cup i}\Big] \cdot \Big[x^{S\cup j}(S \cup \{j\}) + x_i^{S\cup i}\Big]. \end{aligned}$$

To establish the convexity condition (2.16) we have to show that the above inequality is satisfied. But from Lemma 2.3 we know that this expression is equivalent to

$$\left[2 \cdot x_i^{S\cup i} - w_i\right] \cdot \left[x^{S\cup j}(S \cup \{j\}) - x^{S\cup i}(S)\right] \leq x_i^{S\cup i} \cdot w_j.$$

This argument establishes convexity and we are done. □

REMARK 2.2.
The uniform approach to investigate convexity is strongly based on an appropriately chosen interchangeability assumption about two maximizers. Concerning the oligopoly games of (2.2) with a linear inverse demand function and linear cost functions, an alternative proof of its convexity property was established in Norde et al. (2002). Opposite to our uniform approach to investigate convexity, their alternative convexity proof neither refers to maximizers, nor to any essential interchangeability assumption about maximizers. In fact, their alternative convexity proof is extremely tedious, requires a lot of mathematical notation, and its proof technique is based on some induction arguments to be applied to a slightly adapted oligopoly game. Moreover, the worth of any coalition S in the initial oligopoly game $\langle N, v\rangle$ is treated as the following rather complicated expression:

$$v(S) = \sum_{k\in S} f_{w_k}(\max\left[0,\ a - w(N\backslash S) - c_k - 2 \cdot \sum_{\ell\in S,\ \ell<k} w_\ell\right]). \tag{2.25}$$

Here, for any $x > 0$, the associated function $f_x : \mathbb{R}_+ \rightarrow \mathbb{R}$ is given by $f_x(y) := \frac{y^2}{4}$ for all $0 \leq y \leq 2 \cdot x$, whereas $f_x(y) := x \cdot (y - x)$ whenever $y > 2 \cdot x$. All together, Norde et al.'s proof of convexity for the oligopoly games of (2.2) is fully based on mathematical arguments without offering any meaningful economical interpretation, whereas our uniform approach to investigate convexity establish as a first result some regular and monotonicity properties for the maximizers and prices. ◇

3 SUPERMODULARITY WITH TRANSFERABLE TECHNOLOGIES

In the previous section we have investigated the convexity property for oligopoly games without transferable technologies, that is, the production opportunity of a cartel is completely determined by all member firms. The most efficient production process cannot be adopted by the member firms. But this assumption might be realistic in the short run. In the long run one can expect that firms can adopt the most efficient technology by direct technology transfers or partly by the transformation of publicly available informations. In this section we study the convexity property in the case that synergy effects among firms are possible by assuming that firms in a cartel can costlessly acquire the technology of the most efficient firm in the cartel. Formally, we consider the case that the production opportunity of a coalition T will be described by $c_T := \min_{k\in T} c_k$ for every coalition $T \subseteq N$. We call such a situation an oligopoly problem with transferable technologies.

Zhao (1999) provided under very restrictive assumptions a necessary and sufficient condition under which a linear oligopoly game with transferable technologies is convex. Zhao assumed that the capacity level of each coalition is sufficiently large to produce at an interior solution and that its average variable cost is always less than or equal to the market price, that is, there is no shut down price. For the symmetric case these restrictive assumptions yield to the result that the capacity is above its Cournot equilibrium supply but bounded from above by its monopoly supply. A first paper that dispensed from these assumptions for linear oligopoly games were provided by Norde et al. (2002). Their major finding was that although these game types are not convex they are nevertheless totally balanced. Hence, a core solution exists and it exists a weak incentive to merge economic activities in larger units.

The purpose of this section is to reconsider the type of oligopoly games first studied by Zhao (1999) with our alternative proof technique that relies on maximizers and the symmetry assumption for each pair of players, instead of introducing artificial assumptions, to observe if we can reproduce the sufficient and necessary condition provided by Zhao (1999). In the sequel of this section we will observe that the assumptions imposed by Zhao were crucial to derive the necessary condition for convexity for these game types. Although we can reproduce the first result on sufficiency our second sufficiency result is more general than the one derived by Zhao that even coincides under very restrictive assumptions.

The corresponding linear oligopoly game with transferable technologies $\langle N, v \rangle$ is derived from the same normal form game with payoff functions of type (2.1). But in addition we have to care about the synergy effects among the firms, that is $c_T := \min_{k \in T} \; c_k$. Furthermore, the partial derivative changes slightly to $\frac{\partial d_T}{\partial x_k} = \alpha_T - 2 \cdot x(T) - c_T$ for all $k \in T$ and for all $\vec{x}_T \in X_T$. The cooperative oligopoly game with transferable technologies $\langle N, v \rangle$ is described as follows: for all $T \subseteq N, T \neq \emptyset$,

$$v(T) = \max_{\vec{x}_T \in X_T} \left[\Big[a - w(N \backslash T) - x(T) \Big] \cdot x(T) - c_T \cdot x(T) \right]. \tag{3.1}$$

As has been proved in Zhao (1999) the α- and β-values coincide again, hence $v := v_\alpha = v_\beta$.

Example 3.1. Now observe that in oligopoly games with transferable technologies we lose the price regularity $p(q^S) \leq p(q^{S \cup i})$ we obtained in the previous case. The induced price increase $w_i - x_i^{S \cup i}$ of firm i is outweighed by the expansion of the production $x^{S \cup i}(S) - x^S(S)$ of the old member firms in S. To see this consider an oligopoly situation with three firms $N = \{1, 2, 3\}$, an inverse demand function given by $p(\vec{x}) := 20 - (x_1 + x_2 + x_3)$, where the vector of marginal costs is specified by $\vec{c} = \{2, 2, 12\}$, and the vector of capacities is given by $\vec{w} = \{2, 2, 4\}$.

Let $S = \{3\}, i = 2, j = 1$ and observe that the optimal production plan of firm 3 is $x_3^{\{3\}} = 2$ that leads to a price level of $p(q^{\{3\}}) = 14$. The optimal production plans of cartel $\{1, 3\}$ and $\{2, 3\}$ respectively, are $x_1^{\{1,3\}} = 2, x_3^{\{1,3\}} = 4$ and $x_2^{\{2,3\}} = 2, x_3^{\{2,3\}} = 4$. This leads to the following price levels of $p(q^{\{1,3\}}) = 12$ and $p(q^{\{2,3\}}) = 12$ respectively. The reader may check that the corresponding oligopoly game $\langle N, v \rangle$ of the form (3.1) is not convex. $\diamond$

In contrast to the case of oligopoly situations without transferable technologies we can now derive some regular conditions on the sum of maximizers between $x^S(S)$ and $x^{S \cup i}(S)$ if we impose an additional assumption on the production process. The monotonicity result stated in Proposition 3.1 is valid, if we suppose that the reorganization of the production process between firms is not costless. Notice that due to synergy effects all firms in a coalition are identical efficient and therefore the optimal production vector will in general not be unique. The joining of a new firm i to a coalition S induces an improvement of technology either for all firms in S or just for the joining firm i. Since there is no difference in efficiency between firms there is no need to reorganize the production process by shifting production to the newcomer if the total sum of production can also be optimal produced by the old member firms.

Proposition 3.1. *Consider an oligopoly situation $\langle N, (w_i)_{i \in N}, (c_i)_{i \in N}, a \rangle$ with transferable technologies where the reorganization of the production process is not costless. For each $i \in N$ and for all $S \subseteq N \backslash \{i\}, S \neq \emptyset$, the maximizers $\vec{x}^S$ of the maximization problem (3.1) w.r.t. coalition S satisfy the following monotonicity property:*

$$x^{S \cup i}(S) \geq x^S(S),$$

and if $x_i^{S\cup i} > 0$ *then we get*

$$x^{S\cup i}(S \cup \{i\}) > x^S(S).$$

Proof. First observe that the joining of an additional firm i to coalition S imposes due to $\alpha_S + w_i$ an increase in price and therefore an improvement in the overall production opportunity. In addition, we have $c_{S\cup i} = \min\{c_S, c_i\}$, thus $c_{S\cup i} \leq c_S$. But then under the underlying assumption we get $x_k^{S\cup i} \geq x_k^S$ for all $k \in S$, that leads to $x^{S\cup i}(S) \geq x^S(S)$. If $x_i^{S\cup i} > 0$, then we get a fortiori $x^{S\cup i}(S \cup \{i\}) > x^S(S)$. □

Proposition 3.2. *The oligopoly game with transferable technologies* $\langle N, v\rangle$ *of the form* (3.1) *is monotone, i.e.* $v(T) \geq v(S)$ *for all* $S \subseteq T \subseteq N$.

Proof. To prove monotonicity of an oligopoly game $\langle N, v\rangle$ of the form (3.1) write $T := S \cup \{i\}$ with $S \subseteq N\backslash\{i\}, S \neq \emptyset$. Consider the feasible production combination $(\vec{x}^{\,S}, 0) \in X_S \times X_i = X_{S\cup i}$ to underestimate the value of coalition $S \cup \{i\}$ to derive

$$v(S \cup \{i\}) \geq \Big[\alpha_{S\cup i} - x^S(S) - c_{S\cup i}\Big] \cdot x^S(S) \geq \Big[\alpha_S - x^S(S) - c_{S\cup i}\Big] \cdot x^S(S),$$

then we get

$$\Big[p(q^S) - c_{S\cup i}\Big] \cdot x^S(S) \geq \Big[p(q^S) - c_S\Big] \cdot x^S(S) = v(S),$$

since $c_{S\cup i} = \min\{c_S, c_i\}$. This argument completes the proof. □

As in Zhao (1999) we introduce a set that consists of those firms and their associated coalitions whose marginal cost are strictly supermodular.

$$\Omega := \Big\{(S, i, j) \mid S \subseteq N\backslash\{i, j\} \quad \text{and} \quad c_S - c_{S\cup i} > c_{S\cup j} - c_{S\cup ij}\Big\} \tag{3.2}$$

Let $S \subseteq N\backslash\{i, j\}$ and define now

$$\delta_S := \alpha_S - c_S - x^{S\cup i}(S) \qquad \delta_{S\cup ij} := \alpha_{S\cup ij} - c_{S\cup ij} - x^{S\cup j}(S \cup \{j\}) - x_i^{S\cup i}, \tag{3.3}$$

$$\Delta_{S\cup k} := \frac{\alpha_{S\cup k} - c_{S\cup k}}{2} - x^{S\cup k}(S \cup \{k\}) \qquad k = i, j. \tag{3.4}$$

Similar, we suppose $\Omega \neq \emptyset$ and for any $(S, i, j) \in \Omega$, we let

$$\begin{aligned} f(S, i, j) := &\Big[c_{S\cup ij}^2 + c_S^2 - c_{S\cup i}^2 - c_{S\cup j}^2\Big] + 2 \cdot w(N\backslash S) \cdot \Big[c_S + c_{S\cup ij} - c_{S\cup i} - c_{S\cup j}\Big] \\ &+ 2 \cdot w_j \cdot \Big[w_i + c_{S\cup j} - c_{S\cup ij}\Big] + 2 \cdot w_i \cdot \Big[c_{S\cup i} - c_{S\cup ij}\Big] \end{aligned} \tag{3.5}$$

$$\mathcal{C}(S, i, j) := \Big[c_{S\cup ij}^2 + c_S^2 - c_{S\cup i}^2 - c_{S\cup j}^2\Big] - \Big[c_{S\cup i}^2 + c_{S\cup j}^2 - c_{S\cup i} \cdot \alpha_{S\cup i} - c_{S\cup j} \cdot \alpha_{S\cup j}\Big] \tag{3.6}$$

$$h(S, i, j) := -2 \cdot w_j \cdot x_i^{S\cup i} + 2 \cdot \Big[c_{S\cup ij} \cdot \delta_{S\cup ij} + c_S \cdot \delta_S - \Delta_{S\cup j} \cdot c_{S\cup j} - \Delta_{S\cup i} \cdot c_{S\cup i}\Big] \tag{3.7}$$

$$H(S, i, j) := \frac{f(S, i, j) + \mathcal{C}(S, i, j) + h(S, i, j)}{2 \cdot [c_S + c_{S\cup ij} - c_{S\cup i} - c_{S\cup j}]} \tag{3.8}$$

$$F(S,i,j) := \frac{f(S,i,j)}{2 \cdot [c_S + c_{S\cup ij} - c_{S\cup i} - c_{S\cup j}]} \tag{3.9}$$

$$\underline{\omega} := \min_{(S,i,j)\in\Omega} H(S,i,j) \qquad \omega := \min_{(S,i,j)\in\Omega} F(S,i,j). \tag{3.10}$$

REMARK 3.1.
Notice that instead of using the notations of (S,T,i), $S \subset T$, $i \in T$, and $S \neq \emptyset$ as in Zhao (1999), we use (S,i,j). One consequence of our notation is that Zhao's Theorem and our generalized Theorem are not anymore directly comparable. We have to rewrite the necessary and sufficient condition into our notation. The second consequence is that by defining $T = S \cup \{j\}$ the ω-value in (3.10) might be larger than that given in Zhao (1999), because $S \cup \{j\}$ covers only a smaller portion than the T's. But the problem with the notation used in Zhao (1999) is that the interchangeability assumption of player i and j is not anymore applicable. According to its proven effectiveness and ease to work out the convexity property for different classes of games as for instance in Driessen & Meinhardt (2001, 2005), Meinhardt (2002), we think that is more than justified to deviate from the notation introduced by Zhao. Although, it might cause some confusing at the first moment. $\diamond$

Now let us resume first the assumptions and the theorem given by Zhao (1999) that provides a necessary and sufficient condition for convexity in linear oligopoly TU-games with transferable technologies to notify the difference to our more general result which is derived by relying on appropriately chosen maximizers and the symmetry assumption for each pair of players. Zhao imposed the assumptions given below for an oligopoly game of type (3.1) to obtain his convexity result.

A1. The capacity level of each coalition S is sufficiently large to produce at an interior solution.

A2. For each coalition S, its average variable cost is always less than or equal to the market price, hence, there is no shut down price.

Theorem 3.1 (Zhao (1999)). *Let $\langle N, v\rangle$ be an oligopoly game with transferable technologies of the form (3.1) that satisfies the assumptions (A1) and (A2).*

1. *The game $\langle N, v\rangle$ is convex, if $\Omega = \emptyset$.*
2. *Let now $\Omega \neq \emptyset$, the game $\langle N, v\rangle$ is convex, if and only if $a \leq \omega$,*

Now, observe that our alternative proof technique allows us to dispense from the assumptions (A1) and (A2) to obtain a modified convexity result. Especially, the second result is weaker than the result worked out by Zhao (1999), since we can not reproduce his necessary condition, we get only sufficiency.

Theorem 3.2. *Let $\langle N, v\rangle$ be an oligopoly game with transferable technologies of the form (3.1).*

1. *The game $\langle N, v\rangle$ is convex, if $\Omega = \emptyset$ and $x^{S\cup j}(S \cup \{j\}) \geq x^{S\cup i}(S \cup \{i\})$ for every $i, j \in N, i \neq j$ and all $S \subseteq N\backslash\{i,j\},\ S \neq \emptyset$.*
2. *Let now $\Omega \neq \emptyset$, the game $\langle N, v\rangle$ is convex, if $a \leq \underline{\omega}$.*

Proof. (1) To prove the convexity condition (2.16) we have to distinguish two cases, namely $S = \emptyset$ and $S \neq \emptyset$.
Case one. Assume $S = \emptyset$. Now let $T \subseteq N\backslash\{i\}, T \neq \emptyset$. To derive the convexity property (2.16) we

concern about the maximization problem (3.1) to describe $v(\{i\})$ and $v(T)$ by maximizers x_i^i and $\vec{x}^T$ such that

$$v(\{i\}) = \Big[\alpha_i - x_i^i\Big] \cdot x_i^i - c_i \cdot x_i^i \tag{3.11}$$

$$v(T) = \Big[\alpha_T - x^T(T)\Big] \cdot x^T(T) - \; c_T \cdot x^T(T). \tag{3.12}$$

Similar to the proof for Theorem 2.1 we try to underestimate the worth of coalition $T \cup \{i\}$ by choosing the feasible production plan $((x_k^T)_{k \in T}, x_i^i) \in X_T \times X_i = X_{T \cup i}$ to get

$$v(T \cup \{i\}) \geq \Big[\alpha_{T \cup i} - x^T(T) - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big] - \; c_{T \cup i} \cdot \Big[x^T(T) + x_i^i\Big]. \tag{3.13}$$

Combining terms we get

$$\begin{aligned} &\Big[\alpha_i - x_i^i\Big] \cdot x_i^i + \Big[\alpha_T - x^T(T)\Big] \cdot x^T(T) \leq \Big[\alpha_{T \cup i} - x^T(T) - x_i^i\Big] \cdot \Big[x^T(T) + x_i^i\Big] \\ &+ c_i \cdot x_i^i + c_T \cdot x^T(T) - c_{T \cup i} \cdot \Big[x^T(T) + x_i^i\Big]. \end{aligned} \tag{3.14}$$

The first part is again the same expression of (2.11) already discussed in Lemma 2.3, we simplify to

$$\Big[x^T(T) - w(T)\Big] \cdot x_i^i \leq \Big[w_i - x_i^i\Big] \cdot x^T(T) + \Big[c_i - c_{T \cup i}\Big] \cdot x_i^i + \Big[c_T - c_{T \cup i}\Big] \cdot x^T(T). \tag{3.15}$$

Observe that $c_i \geq c_{T \cup i}$ and $c_T \geq c_{T \cup i}$ due to $c_{T \cup i} = \min\{c_i, c_T\}$. Thus, the cost terms are non-negative, while from the first part of Lemma 2.3 we know that the lhs is non-positive and the first term of the rhs of (3.15) is non-negative. We conclude that the inequality is satisfied and the convexity condition (2.16) of an oligopoly game $\langle N, v \rangle$ of the form (3.1) is given for $S = \emptyset$.

Case two. We suppose that $S \neq \emptyset$. Now, let $i, j \in N, i \neq j$, and $S \subseteq N \backslash \{i, j\}$. Concerning the maximization problems (3.1) with respect to the coalitions $S \cup \{j\}$ and $S \cup \{i\}$ respectively, we consider their maximizers $\vec{x}^{S \cup j}$ and $\vec{x}^{S \cup i}$ respectively in order to derive the following equalities:

$$v(S \cup \{j\}) = \Big[\alpha_{S \cup j} - x^{S \cup j}(S \cup \{j\})\Big] \cdot x^{S \cup j}(S \cup \{j\}) - \; c_{S \cup j} \cdot x^{S \cup j}(S \cup \{j\}) \tag{3.16}$$

$$v(S \cup \{i\}) = \Big[\alpha_{S \cup i} - x^{S \cup i}(S \cup \{i\})\Big] \cdot x^{S \cup i}(S \cup \{i\}) - \; c_{S \cup i} \cdot x^{S \cup i}(S \cup \{i\}). \tag{3.17}$$

We choose now the feasible production plan $(x_k^{S \cup i})_{k \in S} \in X_S$ for coalition S to obtain

$$v(S) \geq \Big[\alpha_S - x^{S \cup i}(S)\Big] \cdot x^{S \cup i}(S) - \; c_S \cdot x^{S \cup i}(S). \tag{3.18}$$

Similar, for coalition $S \cup \{i, j\}$, we choose a feasible production plan $(\vec{x}^{S \cup j}, x_i^{S \cup i}) \in X_{S \cup j} \times X_i = X_{S \cup ij}$ to get

$$v(S \cup \{ij\}) \geq \Big[\alpha_{S \cup ij} - x^{S \cup j}(S \cup \{j\}) - x_i^{S \cup i} - c_{S \cup ij}\Big] \cdot \Big[x^{S \cup j}(S \cup \{j\}) + x_i^{S \cup i}\Big]. \tag{3.19}$$

To establish the convexity condition (2.16) we have to show that the following inequality is satisfied

$$\begin{aligned}
&\Big[\alpha_{S\cup i} - x^{S\cup i}(S\cup\{i\})\Big]\cdot x^{S\cup i}(S\cup\{i\}) + \Big[\alpha_{S\cup j} - x^{S\cup j}(S\cup\{j\})\Big]\cdot x^{S\cup j}(S\cup\{j\}) \\
&\leq \Big[\alpha_{S} - x^{S\cup i}(S)\Big]\cdot x^{S\cup i}(S) + \Big[\alpha_{S\cup ij} - x^{S\cup j}(S\cup\{j\}) - x_i^{S\cup i}\Big]\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\Big] \\
&+ c_{S\cup i}\cdot x^{S\cup i}(S\cup\{i\}) + c_{S\cup j}\cdot x^{S\cup j}(S\cup\{j\}) - c_S\cdot x^{S\cup i}(S) - c_{S\cup ij}\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\Big]
\end{aligned} \tag{3.20}$$

But from Lemma 2.3 the expression can be reduced to

$$\begin{aligned}
&\Big[2\cdot x_i^{S\cup i} - w_i\Big]\cdot\Big[x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S)\Big] \leq x_i^{S\cup i}\cdot w_j \;\leq\; x_i^{S\cup i}\cdot w_j \\
&+ c_{S\cup i}\cdot x^{S\cup i}(S\cup\{i\}) + c_{S\cup j}\cdot x^{S\cup j}(S\cup\{j\}) - c_S\cdot x^{S\cup i}(S) - c_{S\cup ij}\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\Big].
\end{aligned} \tag{3.21}$$

It suffices to show that

$$c_{S\cup i}\cdot x^{S\cup i}(S\cup\{i\}) + c_{S\cup j}\cdot x^{S\cup j}(S\cup\{j\}) - c_S\cdot x^{S\cup i}(S) - c_{S\cup ij}\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\Big] > 0,$$

which is equivalent to

$$\begin{aligned}
&c_{S\cup i}\cdot x^{S\cup i}(S\cup\{i\}) + c_{S\cup j}\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x^{S\cup i}(S\cup\{i\}) - x^{S\cup i}(S\cup\{i\})\Big] - c_S\cdot\Big[x_i^{S\cup i} - x_i^{S\cup i}\Big] \\
&- c_S\cdot x^{S\cup i}(S) - c_{S\cup ij}\cdot\Big[x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S\cup\{i\}) + x^{S\cup i}(S\cup\{i\}) + x_i^{S\cup i}\Big] > 0 \\
&\Longleftrightarrow \\
&\Big[c_{S\cup i} + c_{S\cup j} - c_S - c_{S\cup ij}\Big]\cdot x^{S\cup i}(S\cup\{i\}) \\
&+ \Big[c_{S\cup j} - c_{S\cup ij}\Big]\cdot\Big[x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S\cup\{i\})\Big] + \Big[c_S - c_{S\cup ij}\Big]\cdot x_i^{S\cup i} \geq 0.
\end{aligned} \tag{3.22}$$

Observe first that $c_{S\cup j} \geq c_{S\cup ij}$ and $c_S \geq c_{S\cup ij}$, since it holds for $c_{S\cup ij} = \min\{c_i, c_{S\cup j}\}$ and for $c_{S\cup ij} = \min\{c_i, c_j, c_S\}$, therefore the above requirement holds, if $[c_{S\cup i} + c_{S\cup j} - c_S - c_{S\cup ij}] \geq 0$. Thus convexity is established if $\Omega = \emptyset$ is satisfied.

(2) Assume that $\Omega \neq \emptyset$. It remains to prove that the sufficient condition $a \leq \underline{\omega}$ must be satisfied to obtain the convexity of an oligopoly game $\langle N, v\rangle$ of the form (3.1). We need just to consider the case $S \neq \emptyset$. The proof for the case $S = \emptyset$ is identical to the proof given in the first part of (1). To proceed in the proof note first that

$$\Big[2\cdot x_i^{S\cup i} - w_i\Big]\cdot\Big[x^{S\cup j}(S\cup\{j\}) - x^{S\cup i}(S)\Big] \leq \Big[2\cdot x_i^{S\cup i} - w_i\Big]\cdot w_j \leq\; x_i^{S\cup i}\cdot w_j,$$

and use the definition introduced in (3.3) and (3.4) to rewrite the formula of (3.21) to obtain

$$\begin{aligned}
&c_{S\cup i}\cdot x^{S\cup i}(S\cup\{i\}) + c_{S\cup j}\cdot x^{S\cup j}(S\cup\{j\}) - c_S\cdot x^{S\cup i}(S) \\
&- c_{S\cup ij}\cdot\Big[x^{S\cup j}(S\cup\{j\}) + x_i^{S\cup i}\Big] + w_j\Big[w_i - x_i^{S\cup i}\Big] \geq 0
\end{aligned}$$

$$\Longleftrightarrow$$
$$c_{S\cup i}\cdot\left[\frac{\alpha_{S\cup i}-c_{S\cup i}}{2}-\Delta_{S\cup i}\right]+c_{S\cup j}\cdot\left[\frac{\alpha_{S\cup j}-c_{S\cup j}}{2}-\Delta_{S\cup j}\right]-c_S\cdot\left[\alpha_S-c_S-\delta_S\right]$$
$$-c_{S\cup ij}\cdot\left[\alpha_{S\cup ij}-c_{S\cup ij}-\delta_{S\cup ij}\right]+w_j\left[w_i-x_i^{S\cup i}\right]\geq 0,$$

which must be non-negative to establish the convexity condition. Now add the terms $(1/2)\cdot c_{S\cup k}\cdot((\alpha_{S\cup k}-c_{S\cup k})-(\alpha_{S\cup k}-c_{S\cup k}))$ for $k=i,j$ to the expression above, then we get

$$\left[c^2_{S\cup ij}+c^2_S-c^2_{S\cup i}-c^2_{S\cup j}\right]-\frac{1}{2}\cdot\left[c^2_{S\cup i}+c^2_{S\cup j}-c_{S\cup i}\cdot\alpha_{S\cup i}-c_{S\cup j}\cdot\alpha_{S\cup j}\right]$$
$$+\alpha_S\cdot\left[c_{S\cup i}+c_{S\cup j}-c_S-c_{S\cup ij}\right]+w_j\cdot\left[w_i+c_{S\cup j}-c_{S\cup ij}\right]+w_i\cdot\left[c_{S\cup i}-c_{S\cup ij}\right]$$
$$-w_j\cdot x_i^{S\cup i}+\left[c_{S\cup ij}\cdot\delta_{S\cup ij}+c_S\cdot\delta_S-\Delta_{S\cup j}\cdot c_{S\cup j}-\Delta_{S\cup i}\cdot c_{S\cup i}\right]\geq 0.$$

In order to have the convexity condition satisfied, it must hold for any $(S,i,j)\in\Omega$ that

$$2\cdot\left[c^2_{S\cup ij}+c^2_S-c^2_{S\cup i}-c^2_{S\cup j}\right]-\left[c^2_{S\cup i}+c^2_{S\cup j}-c_{S\cup i}\cdot\alpha_{S\cup i}-c_{S\cup j}\cdot\alpha_{S\cup j}\right]$$
$$+2\cdot w(N\backslash S)\cdot\left[c_S+c_{S\cup ij}-c_{S\cup i}-c_{S\cup j}\right]+2\cdot w_j\cdot\left[w_i+c_{S\cup j}-c_{S\cup ij}\right]+2\cdot w_i\cdot\left[c_{S\cup i}-c_{S\cup ij}\right]$$
$$-2\cdot w_j\cdot x_i^{S\cup i}+2\cdot\left[c_{S\cup ij}\cdot\delta_{S\cup ij}+c_S\cdot\delta_S-\Delta_{S\cup j}\cdot c_{S\cup j}-\Delta_{S\cup i}\cdot c_{S\cup i}\right]\geq 2\cdot a\cdot\left[c_S+c_{S\cup ij}-c_{S\cup i}-c_{S\cup j}\right].$$

Recall the formulas (3.5), (3.6), (3.7) and (3.8). We can simplify the above expression to

$$\frac{f(S,i,j)+\mathcal{C}(S,i,j)+h(S,i,j)}{2\cdot[c_S+c_{S\cup ij}-c_{S\cup i}-c_{S\cup j}]}=H(S,i,j)\geq a,$$

which must hold for any $(S,i,j)\in\Omega$ to establish convexity, hence we require that

$$\min_{(S,i,j)\in\Omega}H(S,i,j)=\underline{\omega}\geq a, \tag{3.23}$$

which is property we are looking for and we are done. □

REMARK 3.2.
Let us now reconsider the sufficient and necessary condition presented by Zhao (1999). One of the crucial assumption made by Zhao was the interior solution assumption. For this purpose suppose that the interior solution assumption is satisfied, then we get for the total sum of maximizers of each coalition S the following expression $x^S(S)=(\alpha_S-c_S)/2$ that yields to $v(S)=(\alpha_S-c_S)^2/4$. The marginal contributions of player i become

$$v(S\cup\{i\})-v(S)=\frac{1}{4}\cdot\left[c^2_{S\cup i}-c^2_S+2\cdot\alpha_S\cdot\Big(w_i+c_S-c_{S\cup i}\Big)+w_i\cdot\Big(w_i-2\cdot c_{S\cup i}\Big)\right],$$

and

$$v(S\cup\{i,j\})-v(S\cup\{j\})$$

$$= \frac{1}{4} \cdot \left[c^2_{S \cup ij} - c^2_{S \cup j} + 2 \cdot \alpha_{S \cup ij} \cdot \left(w_i + c_{S \cup j} - c_{S \cup ij} \right) + w_i \cdot \left(w_i - 2 \cdot c_{S \cup ij} \right) \right].$$

Plugging in these results in the convexity condition (2.16) and collecting terms we get the equivalent expression

$$\left[c^2_{S \cup ij} + c^2_S - c^2_{S \cup i} - c^2_{S \cup j} \right] + 2 \cdot w(N \backslash S) \cdot \left[c_S + c_{S \cup ij} - c_{S \cup i} - c_{S \cup j} \right]$$
$$+ 2 \cdot w_j \cdot \left[w_i + c_{S \cup j} - c_{S \cup ij} \right] + 2 \cdot w_i \cdot \left[c_{S \cup i} - c_{S \cup ij} \right] \geq 2 \cdot a \cdot \left[c_S + c_{S \cup ij} - c_{S \cup i} - c_{S \cup j} \right].$$

Using definition (3.5) we simplify to

$$\frac{f(S,i,j)}{2 \cdot [c_S + c_{S \cup ij} - c_{S \cup i} - c_{S \cup j}]} =: F(S,i,j) \geq a.$$

Again this must hold for each $(S,i,j) \in \Omega$, we reproduce the Zhao sufficient and necessary condition given by

$$\omega := \min_{(S,i,j) \in \Omega} F(S,i,j) \geq a. \tag{3.24}$$

It should be obvious that (3.24) is equivalent to (3.23), i.e. $F(S,i,j) = H(S,i,j)$ if and only if for each $(S,i,j) \in \Omega$ we have $h(S,i,j) = \mathcal{C}(S,i,j) = 0$. ◇

Example 3.2. Consider now an example where the interior solution assumption (A1) given in Zhao (1999) is not satisfied. For this purpose assume that the linear oligopoly situation with transferable technologies is described by $N = \{1,2,3\}$, an inverse demand function given by $p(\vec{x}) := 22 - (x_1 + x_2 + x_3)$, the vector of marginal costs is specified by $\vec{c} = \{2,4,6\}$, and the vector of capacities is given by $\vec{w} = \{4,3,2\}$. Note that we have set the parameter a to 22. Recall that we have to solve for each coalition the optimization problem of the form (3.1) to obtain the corresponding cooperative oligopoly game. It turns out that each coalition produces at its capacity level, hence the interior solution assumption (A1) is violated. The oligopoly TU-game is given by

$$v(\emptyset) = 0, \quad v(\{1\}) = 44, \quad v(\{2\}) = 27, \quad v(\{3\}) = 14,$$
$$v(\{1,2\}) = 77, \quad v(\{1,3\}) = 66, \quad v(\{2,3\}) = 45, \quad v(N) = 99.$$

This game is convex.

In a first step we check if the sufficient and necessary condition $a \leq \omega$ of Zhao's Theorem 3.1 is satisfied. Observe that $\Omega = \{(\{3\},1,2),(\{3\},2,1)\} \neq \emptyset$, hence we have to apply the second case of Theorem 3.1 and Theorem 3.2. This implies that we get $f(\{3\},1,2) = f(\{3\},2,1) = 84$, and therefore

$$\omega = F(\{3\},2,1) = F(\{3\},1,2) = \frac{f(\{3\},1,2)}{2 \cdot [c_{\{3\}} + c_N - c_{\{1,3\}} - c_{\{2,3\}}]} = \frac{84}{4} = 21 < 22 = a.$$

Thus, the sufficient and necessary condition $a \leq \omega$ is violated despite the fact that the game is convex.

Finally, let us verify if the sufficient condition $a \leq \underline{\omega}$ of Theorem 3.2 is fulfilled. By some calculation we get $\delta_N = 11, \delta_{\{3\}} = 7, \Delta_{\{1,3\}} = 5/2, \Delta_{\{2,3\}} = 2, \alpha_{\{1,3\}} = 19$ and for $\alpha_{\{2,3\}} = 18$. Plugging in these values in $C(\{3\},1,2),\ h(\{3\},2,1)$ or $C(\{3\},2,1),\ h(\{3\},1,2)$ respectively, then we get $C(\{3\},1,2) = C(\{3\},2,1) = 110$ and $h(\{3\},1,2) = h(\{3\},2,1) = 78$. Hence, we obtain

$$\underline{\omega} = H(\{3\},1,2) = \frac{f(\{3\},1,2) + C(\{3\},1,2) + h(\{3\},1,2)}{2 \cdot [c_{\{3\}} + c_N - c_{\{1,3\}} - c_{\{2,3\}}]} = \frac{84 + 110 + 78}{4} = 68 > 22 = a.$$

To summarize: the game is convex as desired and the condition $\underline{\omega} \geq a$ of Theorem 3.2 that implies convexity is valid. This example demonstrates the usefulness of the generalized sufficient condition for cases where the interior solution assumption of Zhao's Theorem 3.1 is not satisfied and the sufficient and necessary condition $\omega \geq a$ is not anymore applicable. Of course, in case that the interior solution assumption is satisfied the first choice to check convexity for linear oligopoly TU-games remains by verifying Zhao's condition $\omega \geq a$. ◇

4 Concluding Remarks

In this paper we studied the convexity (supermodularity) property of linear oligopoly situations with and without transferable technologies. Due to our rather effective proof technique to scrutinize the convexity of the characteristic function of the corresponding oligopoly game by relying on a suitable chosen interchangeability (symmetry) assumption about maximizers, we were able to derive for both game types in a first step certain regular and monotonicity properties concerning the maximizers and the prices to establish finally our convexity results. The proof of the main result for oligopoly situations without transferable technologies revealed that the revenues evaluated at appropriately chosen maximizers have to exhibit supermodularity and that the associated costs are not important to obtain convexity. In addition, we reanalyzed the linear oligopoly situation with transferable technologies first studied by Zhao (1999) to reestablish his sufficient and necessary condition for convexity without relying on his very restrictive assumptions. Especially, for the case where for some associated coalitions the marginal costs exhibit supermodularity, we were able to derive a more general sufficient condition than the one derived by Zhao, which only coincide under special assumptions. Since our approach was more general than the one chosen by Zhao we could not reproduce his necessary condition for convexity. Nevertheless, we could show that the interior solution assumption made by Zhao was crucial to derive his necessary condition.

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