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Non-dictatorial extensive social choice: a further observation

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Non-dictatorial extensive social choice: a further observation*

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Abstract

This paper examines social decision making involving individuals' interpersonal comparisons about the well-being of each member of them. Such a framework is usually called extensive social choice and is regarded as the extension of the Arrovian social choice. In the literature, Kevin Roberts (Rev Econ Stud 47: 409-420) obtained the Arrow-like impossibility result, i.e. the existence of a dictatorial individual, under some moderate conditions. His result involves two serious impossibilities: one is the impossibility of non-dictatorial, or anonymous, social decision making, and the other is that of utilizing increased informational basis, i.e. individuals' sympathetic evaluation on the well-being of the others. In order to resolve this impossibility result, we relax the rationality requirement of social preferences from transitivity, which was assumed in Roberts' analysis, into quasi-transitivity. We show that, under moderate conditions, although it is still hardly possible to utilize the increased informational basis, a new anonymous collective choice rule, called *lexical Pareto extension rule*, can be established by virtue of the enriched informational basis. However, we also prove that, under slightly strengthened conditions, an admissible anonymous rule is solely the direct extension of the one obtained in the Arrovian framework.

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1 Introduction

Extensive social choice considers the aggregation problem of individuals' interpersonal comparisons. In contrast to the Arrovian framework (Arrow 1963), in extensive social choice a social preference of alternatives is determined by aggregating individuals' opinions not only about their own well-being but also about the other individuals' well-being in the alternatives. This extended framework can be traced back to the classical work of Suppes (1966). In the literature, some impossibility results have been obtained (e.g. Sen (1970), Roberts (1980, 1995), and Suzumura (1996)). Among these impossibilities, the most serious one is the Arrow-like impossibility in Roberts (1980), which shows that the extension of the informational basis of social decision making does not help us to avoid Arrow's negative conclusion. To resolve Roberts' impossibility result, three alternative routes were explored and some non-dictatprial possibilities have been established; Ooghe and Lauwers (2005) and Kamaga (2007) considered cardinal or more cardinal utility information such as translation-scale measurable or ratio-scale measurable utility; Ooghe (2004) and also Ooghe and Lauwers (2005) analyzed social quasi-orderings, i.e. reflexive and transitive social preferences; Gaertner (1992) dropped binary independence condition.

This paper explores an alternative resolution to Roberts' impossibility in a different approach. In contrast to the three approaches above, we relax the rationality requirement of social preferences from transitivity into quasi-transitivity, and examine the possibility of non-dictatorial extensive social choice. Our interest does not simply lie on a resolution to Roberts' impossibility because, in the case of quasi-transitive social preferences, *the Pareto extension rule*, which was established by Sen (1970) in the Arrovian framework, can be directly reformulated as the non-dictatorial aggregation rule in this extended framework. The main purpose of this paper is to give an answer to the following question; in this extended framework, is it possible to find a resolution to Roberts' impossibility other than the direct reformulation of Sen's Pareto extension rule?

The paper is organized as follows. The next section presents notation and definitions. In Section 3, we list some basic axioms. In addition to the extensions of usual axioms considered in the Arrovian framework, we also provide an axiom which is based on the idea of anti-paternalism. Before proceeding to the case of quasi-transitive social preferences, we discuss transitive social preferences and provide some impossibility results in Section 4. These impossibility results will motivate us to consider quasi-transitive social preferences. In Section 5, we examine the case of quasi-transitive social preferences. We especially focus on the amount of preference information utilized to determine the social ranking and explore a resolution to Roberts' impossibility result. Section 6 concludes.

2 Notation and definitions

Let N be the finite set of n individuals. We assume $n \geq 2$. The typical element of N is i, j, k, or l. X is a finite set of social alternatives. We assume that it contains at least three alternatives, i.e. $\sharp X \geq 3$. For each $x \in X$ and each $i \in N$, the pair (x,i) denotes i's objective circumstance in x. For each $i \in N$, let $\tilde{R}_i$ denote i's extended preference ordering defined on $X \times N$. We write $(x,j)\tilde{R}_i(y,k)$ to mean that to be the individual j in the social state x is at least as good as to be the individual k in the state y according to i's opinion. In each individual's extended preference ordering, preference information concerning the individual's own position, e.g. $(x,i)\tilde{R}_i(y,i)$, is referred to as *individualistic preference information*, and similarly, preference information about some other individual, e.g. $(x,j)\tilde{R}_i(y,j)$ where $i \neq j$, is mentioned as *sympathetic preference information*. Let $\tilde{\mathbf{R}}$ be a profile of n-tuple of extended preference orderings such that $\tilde{\mathbf{R}} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n)$. $\tilde{\mathscr{R}}$ collects all logically possible extended preference orderings. $\mathscr{B}$ is the set of all logically possible binary relations of X.

An aggregation rule, denoted f, is defined as a mapping from the admissible set of profiles $\mathscr{D} \subseteq \tilde{\mathscr{R}}^n$ to a set of binary relations of X, i.e.

$$f : \mathscr{D} \longrightarrow \mathscr{B}.$$

We call f *collective choice rule*. We write $R_{f(\tilde{\mathbf{R}})}$ as $f(\tilde{\mathbf{R}}) = R_{f(\tilde{\mathbf{R}})}$, and $P_{f(\tilde{\mathbf{R}})}$ (resp. $I_{f(\tilde{\mathbf{R}})}$) as an asymmetric (resp. a symmetric) part of $R_{f(\tilde{\mathbf{R}})}$. In Section 4, $\mathscr{B}$ is assumed to be the set of *reflexive*, *complete*, and *transitive* binary relations of X, i.e. orderings of X, and in Section 5, transitivity is replaced with *quasi-transitivity*.[1] Note that quasi-transitivity is logically weaker than transitivity. Thus, in Section 5 we consider a broader class of collective choice rules than in Section 4.

In order to make comparisons between the results in this extended framework with those in the Arrovian framework, we distinguish two types of collective choice rule according to the amount of preference information utilized to determine the social ranking. We define $\mathscr{A}^f$ as the set of collective choice rules that satisfy the following property: $\forall x,y \in X$, $\forall \tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \mathscr{D}$,

$$[(w,i)\tilde{R}_i(z,i) \Leftrightarrow (w,i)\tilde{R}'_i(z,i)\ \forall i \in N\ \forall w,z \in X] \Rightarrow [xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y].$$

This property can be called *independence of sympathetic preference information*. By definition, any rule in $\mathscr{A}^f$ generates the same social rankings for any two profiles which contain the same individualistic preference information, and thus, eliminates the possible influence of sympathetic preference information altogether. Since it might be possible to interpret that the Arrovian social choice is the framework

[1] Let R be a binary relation defined on X. R satisfies; (i) reflexivity if and only if, $\forall x \in X$, xRx; (ii) completeness if and only if, $\forall x,y \in X$ with $x \neq y$, xRy or yRx; (iii) transitivity if and only if, $\forall x,y,z \in X$, $[xRy$ and $yRz] \Rightarrow xRz$; (iv) quasi-transitivity if and only if, $\forall x,y,z \in X$, $[xPy$ and $yPz] \Rightarrow xPz$.

which aims at aggregating the individualistic preference information, we call $\mathscr{A}^f$ *Arrovian subclass*. Let $\mathscr{E}^f$ denote the set of all logically possible collective choice rules. Obviously, $\mathscr{A}^f$ is a subclass of $\mathscr{E}^f$. Every non-dictatorial rule established in the Arrovian framework can be directly reformulated as a collective choice rules of the Arrovian subclass. Therefore, our interest especially lies on the possibility of establishing a new non-dictatorial collective choice rule that belongs to $\mathscr{E}^f \backslash \mathscr{A}^f$.

3 Basic axioms

We now introduce basic axioms. In extensive social choice, the extensions of the Arrovian axioms are usually considered. We start with the following domain condition.

Unrestricted Domain (UD)
The domain of a collective choice rule, f, consists of all logically possible profiles, i.e. $\mathscr{D} = \tilde{\mathscr{R}}^n$.

The collective choice rule satisfying UD is applicable in any society no matter how diverse the citizens' opinions are. Note that, for any rule f in $\mathscr{A}^f$, this axiom prescribes the same requirement as the axiom of unrestricted domain in the Arrovian social choice.

Next, we move to a pairwise independence axiom. The following axiom is the natural extension of the independence axiom considered in the Arrovian social choice.

Binary Independence of Irrelevant Alternatives (BIIA)
$\forall \tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \mathscr{D}$, $\forall x, y \in X$, if $(x, j)\tilde{R}_i(y,k) \Leftrightarrow (x, j)\tilde{R}'_i(y,k)$ $\forall i, j, k \in N$, then $xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y$.

BIIA asserts that the social ranking of any two alternatives must be independent of the preference information about the other alternatives. In other words, to determine the social ranking of any two alternatives, we only have to consider the preference information about these two alternatives. Consequently, assuming that a collective choice rule satisfies BIIA, a social preference relation on X can be constructed by applying each pairwise social decision making. Although BIIA is sometimes questioned in its parsimonious attitude toward preference information, this axiom can be positively interpreted as a requirement of informational efficiency of collective decision making. Furthermore, it should be mentioned that to restrict the amount of information utilized in the collective decision making may reduce an individual's incentive to strategically manipulate her/his preference.[2] As in the discussion about UD, for any $f \in \mathscr{A}^f$, this binary independence axiom can be considered as the same requirement as the pairwise independence condition

[2]On this issue see, for example, Craven (1992) chapter 5.

of the Arrovian framework. This axiom is, however, a considerably weakened independence condition for the rules of $\mathscr{E}^f \backslash \mathscr{A}^f$ because the coincidence of the individualistic preference information across two different profiles is no longer sufficient to ensure that we obtain the same social preferences in these two profiles.

Next, we consider the following Paretian axioms.

Extensive Weak Pareto (EWP)
$\forall \tilde{\mathbf{R}} \in \mathscr{D}$, $\forall x, y \in X$, if $(x, j)\tilde{P}_i(y, j)$ $\forall i, j \in N$, then $xP_{f(\tilde{\mathbf{R}})}y$.

Weak Pareto (WP)
$\forall \tilde{\mathbf{R}} \in \mathscr{D}$, $\forall x, y \in X$, if $(x, i)\tilde{P}_i(y, i)$ $\forall i \in N$, then $xP_{f(\tilde{\mathbf{R}})}y$.

Strong Pareto (SP)
$\forall \tilde{\mathbf{R}} \in \mathscr{D}$, $\forall x, y \in X$, if $(x, i)\tilde{R}_i(y, i)$ $\forall i \in N$, then $xR_{f(\tilde{\mathbf{R}})}y$. Moreover, if there exists an individual $j \in N$ such that $(x, j)\tilde{P}_j(y, j)$, then $xP_{f(\tilde{\mathbf{R}})}y$.

Each of WP and SP is, respectively, a direct reformulation of the corresponding Pareto criterion considered in the Arrovian social choice. On the other hand, EWP is much weaker unanimity requirement than WP for each rule of $\mathscr{E}^f \backslash \mathscr{A}^f$. But, both of EWP and WP are logically equivalent for the rules in $\mathscr{A}^f$.

Finally, we introduce a requirement of anti-paternalism. Under the assumption of UD, there is no guarantee that an individual's, say j's, sympathetic preference information about the welfare of some other individual, say i, will completely coincide with i's preference information concerning her/his own position, i.e. i's individualistic preference information. Thus, a conflict between these two preference information may occur in this extended framework. Kamaga (2007) formulated the following axiom as the resolution to such a conflicting situation.

Anti-paternalistic priority to Concerned Individual (APCI)
$\forall \tilde{\mathbf{R}} \in \mathscr{D}$, $\forall x, y \in X$, if there exist two distinct individuals $i, j \in N$ such that $(x, i)\tilde{P}_i(y, i)$, $(y, i)\tilde{P}_j(x, i)$, and $(x, l)\tilde{I}_k(y, l)$ $\forall (k, l) \in N \times N \backslash \{(i, i), (j, i)\}$, then $xP_{f(\tilde{\mathbf{R}})}y$.

APCI is based on the idea of anti-paternalism. In the antecedent of APCI, we have a conflict of opinions solely between i's preference information about her/his own welfare and j's sympathetic preference information about i's welfare. APCI asserts that, from a view point of anti-paternalism, the concerned individual's own opinion, i.e. i's own opinion, should be given priority to determine the social ranking as s/he prefers.

In this paper, we will explore a logically admissible collective choice rule that satisfies UD, BIIA, APCI, and one of the Paretian axioms, EWP, WP, or SP. Thus, we have three cases to be analyzed in accordance with which of the Pareto criteria is actually imposed. The following lemma, however, tells that we only have to consider only two of these three cases.[3]

Lemma 1. *Suppose that a collective choice rule, f, has the range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X, and that it satisfies UD, BIIA, and APCI. Then, f satisfies EWP if and only if it satisfies WP.*

Proof of Lemma 1.

It is obvious that WP implies EWP. We will prove that EWP now implies WP in the presence of the other three axioms. Fix two distinct individuals $i,m \in N$ arbitrarily. Since $\sharp N \geq 2$, we can always find such two individuals. For any pair of distinct alternatives $x,y \in X$, we consider the following three sets of profiles, $\mathfrak{R}^{xy}$, $\bar{\mathfrak{R}}^{xy}$, and $\hat{\mathfrak{R}}^{xy}$:

$\mathfrak{R}^{xy} = \{\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n : (x,i)\tilde{P}_j(y,i)\ \forall j \in N\backslash\{m\},\ (x,l)\tilde{P}_k(y,l)\ \forall k \in N\ \forall l \in N\backslash\{i\}\}.$

$\bar{\mathfrak{R}}^{xy} = \{\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n : (x,i)\tilde{P}_i(y,i),\ (x,l)\tilde{P}_k(y,l)\ \forall k \in N\ \forall l \in N\backslash\{i\}\}.$

$\hat{\mathfrak{R}}^{xy} = \{\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n : (x,j)\tilde{P}_j(y,j)\ \forall j \in N\}.$

Note that $\hat{\mathfrak{R}}^{xy}$ is the set of all profiles that satisfy the antecedent of WP for the alternatives x and y, and also that we have $\mathfrak{R}^{xy} \subseteq \bar{\mathfrak{R}}^{xy} \subseteq \hat{\mathfrak{R}}^{xy}$. We will show, in three step, that $xP_{f(\tilde{\mathbf{R}})}y$ follows for any profile in each of the three sets of profiles. We begin with the most restricted case, i.e. the smallest set of profiles, $\mathfrak{R}^{xy}$ and complete the proof by examining the case of $\hat{\mathfrak{R}}^{xy}$.

Step 1.

In this step, we show that $xP_{f(\tilde{\mathbf{R}})}y$ follows for any profile in $\mathfrak{R}^{xy}$ and any distinct alternatives $x,y \in X$. Choose any two distinct alternatives $x,y \in X$ and any profile $\tilde{\mathbf{R}} \in \mathfrak{R}^{xy}$. Let z be an alternative such that $z \neq x,y$. We consider the profile $\tilde{\mathbf{R}}^1 \in \tilde{\mathscr{R}}$ that satisfies the following properties (i) to (vi):

(i) $(x,k)\tilde{R}_j(y,l) \Leftrightarrow (x,k)\tilde{R}^1_j(y,l)\ \forall j,k,l \in N$,

(ii) $(x,l)\tilde{P}^1_k(z,l)\ \forall k,l \in N$,

(iii) $(z,i)\tilde{P}^1_i(y,i)$,

(iv) $(y,i)\tilde{P}^1_m(z,i)$,

(v) $(y,i)\tilde{I}^1_k(z,i)\ \forall k \in N\backslash\{i,m\}$,

(vi) $(y,l)\tilde{I}^1_k(z,l)\ \forall k \in N\ \ \forall l \in N\backslash\{i\}$.

[3]This lemma is a non-welfarist analogue of Lemma 1 in Kamaga (2007).

Since f satisfies UD, we can consider such a profile $\tilde{\mathbf{R}}^1$. From (ii), we can apply EWP, and thus, we have

$$xP_{f(\tilde{\mathbf{R}}^1)}z. \tag{1}$$

From the properties (iii) to (vi), APCI gives

$$zP_{f(\tilde{\mathbf{R}}^1)}y. \tag{2}$$

Consequently, from (1) and (2), we obtain

$$xP_{f(\tilde{\mathbf{R}}^1)}y \tag{3}$$

by the quasi-transitivity of $f(\tilde{\mathbf{R}}^1)$. Since the property (i) shows that the two profiles $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{R}}^1$ together satisfy the antecedent of BIIA, we have $xP_{f(\tilde{\mathbf{R}})}y$ from (3) as desired.

Step 2.

This step proves that $xP_{f(\tilde{\mathbf{R}})}y$ follows for any profile in $\bar{\mathfrak{R}}^{xy}$ and any distinct alternatives $x, y \in X$. For each pair of distinct alternatives $x, y \in X$, we now define the following set of profiles using an integer t with $2 \leq t \leq n$;

$$\begin{aligned}\bar{\mathfrak{R}}^{xy}(t) = \{\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n :& (x,i)\tilde{P}_i(y,i),\\ &(x,l)\tilde{P}_k(y,l)\ \forall k \in N\ \forall l \in N \backslash \{i\},\\ &(x,i)\tilde{P}_j(y,i)\ \forall j \in M \subset N \backslash \{i\} \text{ with } \sharp M = n-t\},\end{aligned}$$

where the individual i is the same as defined at the beginning of the proof. Note that, by definition, $\mathfrak{R}^{xy} \subseteq \bar{\mathfrak{R}}^{xy}(t) \subseteq \bar{\mathfrak{R}}^{xy}$ for all t with $2 \leq t \leq n$, $\bar{\mathfrak{R}}^{xy}(2) = \mathfrak{R}^{xy}$, and $\bar{\mathfrak{R}}^{xy}(n) = \bar{\mathfrak{R}}^{xy}$. Thus, we have to show that, $\forall \tilde{\mathbf{R}} \in \mathfrak{R}^{xy}(n)\ \forall x, y \in X$, $xP_{f(\tilde{\mathbf{R}})}y$ follows. In Step 1, we have just shown that $xP_{f(\tilde{\mathbf{R}})}y$, $\forall \tilde{\mathbf{R}} \in \mathfrak{R}^{xy}(2)$ $\forall x, y \in X$. We use the method of induction of t. Assume that we have $xP_{f(\tilde{\mathbf{R}})}y\ \forall \tilde{\mathbf{R}} \in \mathfrak{R}^{xy}(t)$ with $2 \leq t < n$, $\forall x, y \in X$, and consider any distinct $x, y \in X$ and any $\tilde{\mathbf{R}}^1 \in \mathfrak{R}^{xy}(t+1)$. We want to show $xP_{f(\tilde{\mathbf{R}}^1)}y$. Let $\bar{m} \in N \backslash M$ and $\bar{m} \neq i$, where M is the same set as in the definition of $\mathfrak{R}^{xy}(t+1)$. By UD, we can find the following profile $\tilde{\mathbf{R}}^2 \in \tilde{\mathscr{R}}^n$ and $z \neq x, y$ such that;

(i) $(x,k)\tilde{R}^1_j(y,l) \Leftrightarrow (x,k)\tilde{R}^2_j(y,l)\ \forall j, k, l \in N$,
(ii) $(x,i)\tilde{P}^2_i(z,i)$,
(iii) $(x,i)\tilde{P}^2_j(z,i)\ \forall j \in M \cup \{\bar{m}\}$,
(iv) $(x,l)\tilde{P}^2_k(z,l)\ \forall k \in N\ \forall l \in N \backslash \{i\}$,
(v) $(z,i)\tilde{P}^2_i(y,i)$,
(vi) $(y,i)\tilde{P}^2_{\bar{m}}(z,i)$,

(vii) $(y,i)\tilde{I}^2_k(z,i)\ \forall k \in N\backslash\{i,\bar{m}\}$,

(viii) $(y,l)\tilde{I}^2_k(z,l)\ \forall k \in N\ \ \forall l \in N\backslash\{i\}$.

From the properties (v) to (viii), $\tilde{\mathbf{R}}^2$ satisfies the antecedent of APCI over the pair (y,z). Thus, we have

$$zP_{f(\tilde{\mathbf{R}}^2)}y. \tag{4}$$

Notice that, form the properties (ii) to (iv), the profile $\tilde{\mathbf{R}}^2$ belongs to $\mathfrak{R}^{xz}(t)$. Thus, by the assumption of the induction method, we obtain

$$xP_{f(\tilde{\mathbf{R}}^2)}z. \tag{5}$$

Since $f(\tilde{\mathbf{R}}^2)$ is quasi-transitive, we have

$$xP_{f(\tilde{\mathbf{R}}^2)}y \tag{6}$$

from (4) and (5). From the property (i) and (6), we obtain $xP_{f(\tilde{\mathbf{R}}^1)}y$ by BIIA. By the induction method, we can complete this step.

Step 3.

In this step, we complete the proof. We define the following set of profiles for each pair of distinct alternatives, $x,y \in X$, using an integer t with $1 \leq t \leq n$;

$$\hat{\mathfrak{R}}^{xy}(t) = \{\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n : (x,i)\tilde{P}_i(y,i)\ \forall i \in M \subset N \text{ with } \sharp M = t,$$
$$(x,l)\tilde{P}_k(y,l)\ \forall k \in N\ \forall l \in N\backslash M\}.$$

By definition, $\bar{\mathfrak{R}}^{xy} \subseteq \hat{\mathfrak{R}}^{xy}(t) \subseteq \hat{\mathfrak{R}}^{xy}$ for any t with $1 \leq t \leq n$, $\bar{\mathfrak{R}}^{xy} = \hat{\mathfrak{R}}^{xy}(1)$, and $\hat{\mathfrak{R}}^{xy}(n) = \hat{\mathfrak{R}}^{xy}$. Thus, we have to show that $\forall \tilde{\mathbf{R}} \in \hat{\mathfrak{R}}^{xy}(n)\ \forall x,y \in X$, $xP_{f(\tilde{\mathbf{R}})}y$ follows. Notice that this is true for the case of $t = 1$ from Step 2 because i and m were arbitrarily chosen at the beginning of the proof. As in Step 2, we use the method of induction of t. Suppose that $xP_{f(\tilde{\mathbf{R}})}y\ \forall \tilde{\mathbf{R}} \in \hat{\mathfrak{R}}^{xy}(t)$ with $1 \leq t < n$, $\forall x,y \in X$, and consider any distinct $x,y \in X$ and any $\tilde{\mathbf{R}}^1 \in \hat{\mathfrak{R}}^{xy}(t+1)$. We want to show $xP_{f(\tilde{\mathbf{R}}^1)}y$. Let $\hat{m} \in M$, where M is the same set as in the definition of $\hat{\mathfrak{R}}^{xy}(t+1)$. We now consider the profile $\tilde{\mathbf{R}}^2 \in \tilde{\mathscr{R}}^n$ and $z \neq x,y$ such that;

(i) $(x,k)\tilde{R}^1_j(y,l) \Leftrightarrow (x,k)\tilde{R}^2_j(y,l)\ \forall j,k,l \in N$,

(ii) $(x,j)\tilde{P}^2_j(z,j)\ \forall j \in M\backslash\{\hat{m}\}$,

(iii) $(x,l)\tilde{P}^2_k(z,l)\ \forall k \in N\ \forall l \in \bar{N} \cup \{\hat{m}\}$, where $\bar{N} = N\backslash M$,

(iv) $(z,m)\tilde{P}^2_m(y,m)$,

(v) $(z,l)\tilde{P}^2_k(y,l)\ \forall k \in N\ \forall l \in N\backslash\{\hat{m}\}$.

By UD, such a profile $\tilde{\mathbf{R}}^2$ is admissible now. From (ii) and (iii), $\tilde{\mathbf{R}}^2$ belongs to $\hat{\mathfrak{R}}^{xz}(t)$. Moreover, from (iv) and (v), it also be an element of $\hat{\mathfrak{R}}^{zy}(1)$. Therefore, by the assumption of the induction method, we have

$$xP_{f(\tilde{\mathbf{R}}^2)}z \tag{7}$$

and

$$zP_{f(\tilde{\mathbf{R}}^2)}y. \tag{8}$$

Since $f(\tilde{\mathbf{R}}^2)$ is quasi-transitive, it follows from (7) and (8) that

$$xP_{f(\tilde{\mathbf{R}}^2)}y. \tag{9}$$

From (i) and (9), BIIA gives $xP_{f(\tilde{\mathbf{R}}^1)}y$. By the induction method, the proof is completed now. ■

From Lemma 1 and also the fact that SP implies APCI, we can focus our attention to the following two cases;

Case 1: a collective choice rule f satisfies UD, BIIA, APCI, and EWP;

Case 2: a collective choice rule f satisfies UD, BIIA, and SP.

4 Transitive social preferences

In order to make clear our motivation to analyze the case of quasi-transitive social preferences, in this section we show some results in the case of transitive social preferences. Throughout this section, we assume that a collective choice rule f has a range $\mathscr{B}$ equal to the set of orderings of X. In this case, Roberts (1980) obtained the following impossibility result.

Theorem 1. *(Roberts (1980), Theorem 6)*
Suppose that a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of orderings of X. If it satisfies UD, BIIA, and WP, then there exists a dictator, i.e. there exists $d \in N$ such that;

$$\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n,\ \forall x, y \in X,\ \ (x,d)\tilde{P}_d(y,d) \Rightarrow xP_{f(\tilde{\mathbf{R}})}y.$$

Although WP is the direct reformulation of Arrow's weak Pareto condition, it is not straightforward that we obtain the same result as the Arrow's impossibility in this extended framework because, under the assumption of UD, BIIA is logically much weaker condition than the direct extension of the corresponding independence condition considered in the Arrovian framework. Roberts' result, however, shows that also in this extended framework we are led to the Arrow-like impossibility.

From our Lemma 1 and Roberts' impossibility theorem, we now immediately obtain the result that if we relax WP into EWP but require the collective decision making to be anti-paternalistic in the sense that APCI is satisfied, i.e. in Case 1, we inevitably have a dictatorial individual.

Theorem 2. *Suppose that a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of orderings of X. If it satisfies UD, BIIA, APCI, and EWP, then there exists a dictator.*

These two impossibility theorems motivate us to explore a resolution to these results. This is one reason why we should examine the case of quasi-transitive social preferences.

We have another reason to explore the case of quasi-transitive social preferences. To make clear that, we next consider Case 2. Since SP implies APCI and EWP, it is straightforward that also in this case we have a dictatorial individual. Moreover, what is even worse, we obtain more serious impossibility result than in Case 1. The following theorem shows that in Case 2 the logically admissible rule will be not only dictatorial, but also informationally parsimonious in the sense that individualistic preference information is solely utilized to determine the social ranking. To state the theorem, let $\underline{N}$ be the set of natural numbers such that $\underline{N} = \{1, 2, \ldots, n\}$.

Theorem 3. *If a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of orderings of X, then it satisfies UD, BIIA, and SP if and only if it is the lexical dictatorship, i.e. $\forall\, \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, $\forall x, y \in X$, there exists a bijection $\tau : \underline{N} \to N$ such that;*

$$xP_{f(\tilde{\mathbf{R}})}y \Leftrightarrow \exists k \in \underline{N} : (x, \tau(k))\tilde{P}_{\tau(k)}(y, \tau(k)) \text{ and } (x, \tau(l))\tilde{I}_{\tau(l)}(y, \tau(l))\ \forall l < k,$$
$$xI_{f(\tilde{\mathbf{R}})}y \Leftrightarrow (x, i)\tilde{I}_i(y, i)\ \ \forall i \in N.$$

Theorem 3 tells that in Case 2 the admissible collective choice rule is solely the lexical dictatorship that is a member of the Arrovian subclass $\mathscr{A}^f$. Thus, in this case we can never make use of the sympathetic preference information, and thus, the extension of the informational basis completely loses its significance.

To prove Theorem 3, we introduce the following property.

Pareto Indifference (PI)
$\forall \tilde{\mathbf{R}} \in \mathscr{D}$, $\forall x, y \in X$, if $(x, i)\tilde{I}_i(y, i)\ \forall i \in N$, then $xI_{f(\tilde{\mathbf{R}})}y$.

It is obvious that SP implies PI. We now state the following lemma which shows that, in the presence of UD and PI, the binary independence condition BIIA becomes logically equivalent to the direct reformulation of the independence condition considered in the Arrovian framework.

Lemma 2. *Suppose a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of orderings of X, and satisfies UD, BIIA, and PI. Then, it satisfies the following property;* $\forall \tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \tilde{\mathscr{R}}^n$, $\forall x, y \in X$,

$$[(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}'_i(y,i)\ \forall i \in N] \Rightarrow [xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y].$$

Proof of Lemma 2.

Choose any $x, y \in X$, and any $\tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \tilde{\mathscr{R}}^n$ such that $(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}'_i(y,i)\ \forall i \in N$. Consider the alternative $z \neq x, y$ and the following two profiles $\tilde{\mathbf{R}}^1$ and $\tilde{\mathbf{R}}^2$ such that;

(i) $(x, j)\tilde{R}_i(y,k) \Leftrightarrow (x, j)\tilde{R}^1_i(y,k)\ \ \forall i, j, k \in N$,

(ii) $(x, j)\tilde{R}^1_i(z,k) \Leftrightarrow (x, j)\tilde{R}^2_i(z,k)\ \ \forall i, j, k \in N$,

(iii) $(x, j)\tilde{R}^2_i(y,k) \Leftrightarrow (x, j)\tilde{R}'_i(y,k)\ \ \forall i, j, k \in N$,

(iv) $(y,i)\tilde{I}^1_i(z,i)\ \ \forall i \in N$,

(v) $(y,i)\tilde{I}^2_i(z,i)\ \ \forall i \in N$.

Note that, by (i) and (iv), $(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}^1_i(z,i)\ \forall i \in N$. Moreover, (ii) and (v) ensure that $(x,i)\tilde{R}^1_i(z,i) \Leftrightarrow (x,i)\tilde{R}^2_i(y,i)\ \forall i \in N$. Thus, by UD, we can find $\tilde{\mathbf{R}}^1$ and $\tilde{\mathbf{R}}^2$. By BIIA,

$$xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}^1)}y.$$

PI gives $yI_{f(\tilde{\mathbf{R}}^1)}z$. Thus, by the transitivity of $f(\tilde{\mathbf{R}}^1)$, we have

$$xR_{f(\tilde{\mathbf{R}}^1)}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}^1)}z.$$

Then, by BIIA,

$$xR_{f(\tilde{\mathbf{R}}^1)}z \Leftrightarrow xR_{f(\tilde{\mathbf{R}}^2)}z.$$

By PI, $yI_{f(\tilde{\mathbf{R}}^2)}z$. The transitivity of $f(\tilde{\mathbf{R}}^2)$ gives

$$xR_{f(\tilde{\mathbf{R}}^2)}z \Leftrightarrow xR_{f(\tilde{\mathbf{R}}^2)}y.$$

By BIIA, we have

$$xR_{f(\tilde{\mathbf{R}}^2)}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y.$$

Combining the equivalence assertions, we obtain

$$xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y. \blacksquare$$

In view of Lemma 2, Theorem 3 will immediately follow from Gevers' (1979) characterization of the lexical dictatorship (his Theorem 2) in the Arrovian framework. Thus, we omit the proof of the theorem.

As shown in the proof of Lemma 2, the transitivity of social indifference relations is definitely crucial to establish this lemma. This observation motivates us to examine the case of quasi-transitive social preferences where social indifference relations are no longer transitive. This is the second reason why we should consider quasi-transitive social preferences.

5 Quasi-transitive social preferences

In this section we consider the case of quasi-transitive social preferences. In the Arrovian social choice, it is well known that if we relax the requirement of transitivity into the quasi-transitivity we no longer necessarily have a dictatorial individual but still have an oligarchy. An oligarchy is a decisive coalition such that (i) the unanimous strict preferences of the members of the coalition will determine the social strict preference, and (ii) every member of the coalition has a veto. In the framework of extensive social choice, it is defined as the following subset $\mathscr{O} \subseteq N$ such that $\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, $\forall x, y \in X$,

$$\text{(i) } [\forall i \in \mathscr{O} : (x,i)\tilde{P}_i(y,i)] \Rightarrow xP_{f(\tilde{\mathbf{R}})}y, \text{ and (ii) } [\exists i \in \mathscr{O} : (x,i)\tilde{P}_i(y,i)] \Rightarrow xR_{f(\tilde{\mathbf{R}})}y.$$

Our interest lies on whether or not we still necessarily have an oligarchy in Case 1 and/or in Case 2. The following theorem shows that in Case 1, thus also in Case 2, we still inevitably have an oligarchy. This means that it might be possible to resolve Roberts' impossibility, but the resolution, if any, could hardly utilize sympathetic preference information.

Theorem 4. *Suppose that a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X. If it satisfies UD, BIIA, EWP, and APCI, then there exists a unique oligarchy.*

To prove the theorem, we will use the extended version of Sen's (1986) *field expansion lemma*. To state our extended field expansion lemma, we need to introduce some additional definitions. We define the set of individuals *decisive* over the ordered pair $(x, y) \in X \times X$, denoted D, if and only if, $\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, $xP_{f(\tilde{\mathbf{R}})}y$ follows whenever $(x,i)\tilde{P}_i(y,i)$ $\forall i \in D$. The set of individuals is *almost decisive* over the ordered pair $(x, y) \in X \times X$, denoted $\bar{D}$, if and only if, $\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, $xP_{f(\tilde{\mathbf{R}})}y$ follows whenever $(x,i)\tilde{P}_i(y,i)$ $\forall i \in \bar{D}$ and $(y, j)\tilde{P}_j(x, j)$ $\forall j \in N \backslash \bar{D}$. Note that both of the two properties are defined only in terms of individualistic preference information. Thus, the decisive individuals have such a strong power that their unanimous strict preferences about their own positions will determine the social strict preference independently not only of individualistic preference information of the rest of the individuals but also of every sympathetic preference information. This means, in turn, that the antecedent of this property can be considered to be more demanding than the antecedent of the corresponding property defined in the Arrovian framework. The same argument can be applied to the property of almost decisiveness. Now, we are ready to state the extended field expansion lemma.

Lemma 3. *Suppose that a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X, and satisfies UD, BIIA, EWP, and APCI. For any $(x,y) \in X \times X$, if a set of individuals is almost decisive over (x,y), then this set must be decisive over every ordered pair $(a,b) \in X \times X$.*

Although Sen's original field expansion lemma is stated in the Arrovian framework, we can prove our extended field expansion lemma by the basically same argument as in Sen's proof. The basic idea is as follows. In the framework of extensive social choice, the profiles considered in Sen's proof can be reformulated by specifying only the individualistic preference information of extended preference orderings. By definition, in each individual's extended preference ordering there exists at most one individual who belongs to the group that is almost decisive over some arbitrary ordered pair. Thus, we can always find the profile of extended preference orderings to which we can apply Sen's original proof. To make sure of this point, we now provide the proof for the case of four distinct alternatives, a, b, x, and y, i.e. $\{a,b\} \cap \{x,y\} = \emptyset$.

Proof of Lemma 3.

Let $\bar{D}$ is the set of individuals that is almost decisive over (x,y). Choose any $\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$ such that $(a,i)\tilde{P}_i(b,i)$ $\forall i \in \bar{D}$. We want to show that $aP_{f(\tilde{\mathbf{R}})}b$ follows. We now consider the following profile $\tilde{\mathbf{R}}^1$;

(i) $(a,j)\tilde{R}_i(b,k) \Leftrightarrow (a,j)\tilde{R}_i^1(b,k)$ $\forall i,j,k \in N$,

(ii) $(a,i)\tilde{P}_i^1(x,i)\tilde{P}_i^1(y,i)\tilde{P}_i^1(b,i)$ $\forall i \in \bar{D}$, and

(iii) $(a,j)\tilde{P}_j^1(x,j)$, $(y,j)\tilde{P}_j^1(b,j)$, and $(y,j)\tilde{P}_j^1(x,j)$ $\forall j \in N \backslash \bar{D}$

It is obvious that we can find the profile satisfying both of the conditions (i) and (ii). We now check that we can also find the profile that satisfies both of (i) and (iii). Since every individual's extended preference satisfies completeness, we have the following;

$\forall j \in N \backslash \bar{D}$, $(a,j)\tilde{P}_j(b,j)$ or $(b,j)\tilde{R}_j(a,j)$.

If we have $(a,j)\tilde{P}_j(b,j)$, we can find $\tilde{P}_j^1$ such that

(iv) $(a,k)\tilde{R}_j(b,l) \Leftrightarrow (a,k)\tilde{R}_j^1(b,l)$ $\forall k,l \in N$, and

(v) $(a,j)\tilde{P}_j^2(y,j)\tilde{P}_j^2(x,j)\tilde{P}_j^2(b,j)$.

Such an ordering $\tilde{R}_j^1$ is well-defined and it satisfies the conditions (i) and (iii). Next, we consider the case where we obtain $(b,j)\tilde{R}_j(a,j)$. In this case, we can find $\tilde{R}_j^1$ such that

(vi) $(a,k)\tilde{R}_j(b,l) \Leftrightarrow (a,k)\tilde{R}^1_j(b,l)\ \forall k,l \in N$, and

(vii) $(y,j)\tilde{P}^1_j(b,j)\tilde{R}^1_j(a,j)\tilde{P}^1_j(x,j)$.

It is easily checked that $\tilde{R}^1_j$ is well-defined and it satisfies both of (i) and (iii). Therefore, we can always find the profile $\tilde{\mathbf{R}}^1$. From Lemma 1, f now satisfies WP. Thus, by WP, we obtain $aP_{f(\tilde{\mathbf{R}}^1)}x$ and $yP_{f(\tilde{\mathbf{R}}^1)}b$. Since $\bar{D}$ is almost decisive over (x,y), we have $xP_{f(\tilde{\mathbf{R}}^1)}y$. By the quasi-transitivity of $f(\tilde{\mathbf{R}}^1)$, $aP_{f(\tilde{\mathbf{R}}^1)}b$. By BIIA, we obtain $aP_{f(\tilde{\mathbf{R}})}b$. ■

The key in the above proof is that we only have to specify the individualistic preference information in each individual's extended preference ordering to invoke WP or the property of almost decisiveness. This observation is still valid in the other cases where $\{a,b\} \cap \{x,y\} \neq \emptyset$, and we omit the proofs of these cases. We now provide the proof of Theorem 4. Making use of our extended field expansion lemma, we can prove Theorem 4 by a similar argument to the proof of Theorem 1 in Weymark (1984).

Proof of Theorem 4.

From Lemma 1, f satisfies WP. Thus, N is now the coalition that is decisive over any pair of alternatives. Since N is the finite set of individuals, we must have the smallest decisive coalition $G \subseteq N$. We want to show that G is an oligarchy. If $\sharp G = 1$, it is trivial. We prove the case of $G \geq 2$. Because of G's decisive property, we only have to show that every member of G has a veto. In other words, we want to show that

$$\forall \tilde{\mathbf{R}} \in \mathscr{R}^n,\ \forall x,y \in X,\ [\exists i \in G : (x,i)\tilde{P}_i(y,i)] \Rightarrow xR_{f(\tilde{\mathbf{R}})}y.$$

To show this, choose any profile $\tilde{\mathbf{R}}$ and any alternatives x,y that satisfy the following condition; $\exists i \in G$ such that $(x,i)\tilde{P}_i(y,i)$. Since $f(\tilde{\mathbf{R}})$ is complete, it is sufficient to show that $\neg yP_{f(\tilde{\mathbf{R}})}x$. We now consider the following profile $\tilde{\mathbf{R}}'$ and $z \neq x,y$ such that;

(i) $(x,j)\tilde{R}_i(y,k) \Leftrightarrow (x,j)\tilde{R}'_i(y,k)\ \ \forall i,j,k \in N$,

(ii) $(x,i)\tilde{P}_i(y,i) \Rightarrow (x,i)\tilde{P}'_i(z,i)\tilde{P}'_i(y,i),\ \forall i \in G$,

(iii) $(x,i)\tilde{I}_i(y,i) \Rightarrow (x,i)\tilde{I}'_i(y,i)\tilde{P}'_i(z,i),\ \forall i \in G$,

(iv) $(y,i)\tilde{P}_i(x,i) \Rightarrow (y,i)\tilde{P}'_i(x,i)\tilde{P}'_i(z,i),\ \forall i \in G$,

(v) $(z,i)\tilde{P}'_i(y,i),\ \forall i \in N \backslash G$.

By UD, we can find $\tilde{\mathbf{R}}'$. If there is no individuals in G who satisfies the condition (iii) or (iv), $\neg y\tilde{P}_{f(\tilde{\mathbf{R}})}x$ will travially follow from the decisiveness of G. Thus, we consider the case where such an individual in G certainly exists. We prove by contradiction. Suppose $yP_{f(\tilde{\mathbf{R}})}x$. Then, by BIIA, we have $yP_{f(\tilde{\mathbf{R}}')}x$. Since G is a decisive coalition, $xP_{f(\tilde{\mathbf{R}}')}z$ follows. The quasi-transitivity of $f(\tilde{\mathbf{R}}')$ gives $yP_{f(\tilde{\mathbf{R}}')}z$. Notice that we have no specification on $\tilde{\mathbf{R}}'$ over (y,z) except the individualistic preference information over (y,z). Thus, from

Lemma 3, it must be that the coalition of the individuals who satisfies the condition (iii) or (iv) is decisive. This contradicts the assumption that G is the smallest decisive coalition. Therefore, we obtain $\neg y P_{f(\tilde{\mathbf{R}})} x$. The uniqueness of an oligarchy can be easily checked as follows. If we have two oligarchies $\mathscr{O}$ and $\mathscr{O}'$, then, assuming $\tilde{\mathbf{R}}$ be such that $(x,i)\tilde{P}_i(y,i)\ \forall i \in \mathscr{O}$ and $(y,j)\tilde{P}_j(x,j)\ \forall j \in \mathscr{O}' \backslash \mathscr{O}$, we will obtain $xP_{f(\tilde{\mathbf{R}})}y$ and $yR_{f(\tilde{\mathbf{R}})}x$, which contradicts the completeness of $f(\tilde{\mathbf{R}})$. ■

Although an oligarchy has a strong decisive power, it is compatible with a requirement of anonymity if the coalition is the whole set of the individuals, i.e. $\mathscr{O} = N$. An axiom of anonymity can be formalized as follows.

Anonymity (A)
$\forall \tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \mathscr{D}, \forall x, y \in X$, if there exists a permutation of N, denoted σ, such that $(x,j)\tilde{R}_i(y,k) \Leftrightarrow (x,\sigma(j))\tilde{R}'_{\sigma(i)}(y,\sigma(k))$ $\forall i,j,k \in N$, then $xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y$.

This axiom asserts that names of the individuals have no affect on social decision making.

Sen's (1970) well-established non-dictatorial rule, called *the Pareto extension rule*, can be directly reformulated as an anonymous collective choice rule of the Arrovian subclass. We provide the following example.

Example 1. The Pareto extension rule.
Sen's Pareto extension rule can be reformulated as the following collective choice rule, $f \in \mathscr{A}^f$;

$$\forall\, \tilde{\mathbf{R}} \in \mathscr{D},\ \forall x, y \in X,\ \ xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow [\exists i \in N:\ (x,i)\tilde{P}_i(y,i)] \text{ or } [\forall i \in N: (x,i)\tilde{R}_i(y,i)].$$

By definition, f has a range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X. The decisive coalition is now the whole set of the individuals N. It is easily checked that the Pareto extension rule satisfies UD, BIIA, APCI, EWP, A, and moreover, SP. ■

Since the Pareto extension rule belongs to the Arrovian subclass, it determines the social ranking by solely utilizing the individualistic preference information of the decisive coalition N. Although this rule can be the resolution to Roberts' impossibility result, it still be considered as an impossibility result in the sense that this rule eliminates the possible influence of the sympathetic preference information altogether and thus the extension of the informational basis now completely loses its significance. Now, the question to be answered is whether or not we can obtain any other resolution in this extended framework. While

Theorem 4 tells that it is hardly possible to utilize the sympathetic preference information, we can find the resolution that makes use not only of the individualistic preference information but also of the sympathetic one, i.e. $f \in \mathscr{E}^f \backslash \mathscr{A}^f$. We now provide the following proposition.

Proposition 1. *There exists a collective choice rule* $f \in \mathscr{E}^f \backslash \mathscr{A}^f$ *that has a range* $\mathscr{B}$ *equal to the set of reflexive, complete, and quasi-transitive binary relations of X, and satisfies UD, BIIA, APCI, EWP, and A.*

This proposition can be proved by the following example.

Example 2. The lexical Pareto extension rule.
We can define the following collective choice rule $f \in \mathscr{E}^f \backslash \mathscr{A}^f$;
$\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, $\forall x, y \in X$,
(i) if there exists $i \in N$ such that $(x,i)\tilde{P}_i(y,i)$ or $(y,i)\tilde{P}_i(x,i)$, then;

$$xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow [\exists i \in N \text{ such that } (x,i)\tilde{P}_i(y,i)] \text{ or } [(x,i)\tilde{R}_i(y,i)\ \forall i \in N],$$

(ii) if $(x,i)\tilde{I}_i(y,i)$, $\forall i \in N$, then;

$$xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow [\exists j \in N \text{ such that } (x,j)\tilde{P}_i(y,j),\ j \neq i] \text{ or } [(x,j)\tilde{R}_i(y,j)\ \forall i,j \in N,\ j \neq i].$$

It is easily checked that $f(\tilde{\mathbf{R}})$ satisfies reflexivity, completeness, and quasi-transitivity, and also that f satisfies UD, BIIA, APCI, EWP, and A, but violates SP. ■

By definition, the lexical Pareto extension rule is more likely to generate social strict preferences than the Pareto extension rule. Thus, this new rule shows a slight improvement on the shortcoming of the Pareto extension rule that social indifference relations are obtained in most case. Although it is just a slight improvement, it will allow us to consider the refinement of socially best alternatives, usually called a maximal set. For each collective choice rule f, each non-empty set of alternatives S, and each profile $\tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n$, the maximal set denoted $M(S, f(\tilde{\mathbf{R}}))$ is defined as follows:[4]

$$M(S, f(\tilde{\mathbf{R}})) = \{x \in S : \neg y P_{f(\tilde{\mathbf{R}})} x\ \forall y \in S\}.$$

[4]One related but slightly different notion of socially best alternatives is the greatest set G defined as, given a non-empty set of alternatives S and a social preference relation $f(\tilde{\mathbf{R}})$, $G(S, f(\tilde{\mathbf{R}})) = \{x \in S : xR_{f(\tilde{\mathbf{R}})}y\ \forall y \in S\}$. For any subset $S \subseteq X$ and any binary relation R on X, the maximal set contains the greatest set but the converse assertion does not necessarily hold in general. However, if the binary relation R is reflexive and complete, these two sets always coincide (on this see, for example, Suzumura (1983)). Thus, it does not matter which of the two different notions we adopt here. In the case of reflexive and complete binary relations, the greatest set is sometimes referred to as maximal set (for example, in Austen-Smith and Banks (2000)).

Let f^{PE} be the Pareto extension rule, and also f^{LPE} be the lexical Pareto extension rule. On the maximal sets of these two rules, we obtain the following result;

$$\forall \tilde{\mathbf{R}} \in \tilde{\mathscr{R}}^n,\ \forall S \subseteq X\ (S \neq \emptyset), \quad M(S, f^{LPE}(\tilde{\mathbf{R}})) \subseteq M(S, f^{PE}(\tilde{\mathbf{R}})).$$

Thus, it can be said that the lexical Pareto extension rule is more selective than the Pareto extension rule.

Next, we move to Case 2. In the case of transitive social preferences, we know from our Lemma 2 that it is impossible to utilize the sympathetic preference information. But, as we noted earlier, the transitivity of indifference relations is crucial to establish the lemma. Thus, this lemma can not be directly applied to the case of quasi-transitive social preferences. The following lemma, however, tells that if we relax transitivity into quasi-transitivity but strengthen PI to SP, we must obtain the same result.

Lemma 4. *Suppose that a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X, and satisfies UD, BIIA, and SP. Then,* $\forall \tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \tilde{\mathscr{R}}^n$, $\forall x, y \in X$,

$$[(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}'_i(y,i)\ \forall i \in N] \Rightarrow [xR_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xR_{f(\tilde{\mathbf{R}}')}y].$$

Proof of Lemma 4.

Consider any two profiles $\tilde{\mathbf{R}}, \tilde{\mathbf{R}}' \in \tilde{\mathscr{R}}^n$ such that $(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}'_i(y,i)\ \ \forall i \in N$. Since the social preferences satisfy completeness, it is sufficient to show that (a) $[xP_{f(\tilde{\mathbf{R}})}y \Leftrightarrow xP_{f(\tilde{\mathbf{R}}')}y]$ and (b) $[yP_{f(\tilde{\mathbf{R}})}x \Leftrightarrow yP_{f(\tilde{\mathbf{R}}')}x]$. We first consider (a). Assume $xP_{f(\tilde{\mathbf{R}})}y$. Notice that if $xP_{f(\tilde{\mathbf{R}})}y$ holds then, by SP, $\tilde{\mathbf{R}}$ never be the situation such that $(x,i)\tilde{I}_i(y,i)\ \forall i \in N$. We will show that $xP_{f(\tilde{\mathbf{R}}')}y$ follows. Choose any $z \neq x, y$, and consider any $\tilde{\mathbf{R}}^1$ that satisfies the following conditions;

(i) $(x,j)\tilde{R}_i(y,k) \Leftrightarrow (x,j)\tilde{R}^1_i(y,k)\ \ \forall i,j,k \in N$,

(ii) $(x,i)\tilde{P}_i(y,i) \Rightarrow [(x,i)\tilde{P}^1_i(y,i)$ and $(y,i)\tilde{P}^1_i(z,i)]$, $\forall i \in N$,

(iii) $(x,i)\tilde{I}_i(y,i) \Rightarrow [(x,i)\tilde{I}^1_i(y,i)$ and $(y,i)\tilde{I}^1_i(z,i)]$, $\forall i \in N$,

(iv) $(y,i)\tilde{P}_i(x,i) \Rightarrow [(y,i)\tilde{P}^1_i(z,i)$ and $(z,i)\tilde{P}^1_i(x,i)]$, $\forall i \in N$.

By UD, $\tilde{\mathbf{R}}^1$ does certainly exist. By BIIA and SP, we have $xP_{f(\tilde{\mathbf{R}}^1)}y$ and $yP_{f(\tilde{\mathbf{R}}^1)}z$. Thus, the quasi-transitivity of $f(\tilde{\mathbf{R}}^1)$ gives $xP_{f(\tilde{\mathbf{R}}^1)}z$. Note that the conditions (ii)-(iv) and the transitivity of the individuals' extended preferences ensure that $(x,i)\tilde{R}_i(y,i) \Leftrightarrow (x,i)\tilde{R}^1_i(z,i)\ \ \forall i \in N$. Next, we consider the following profile $\tilde{\mathbf{R}}^2$ and $z \neq x, y$;

(v) $(x,j)\tilde{R}^1_i(z,k) \Leftrightarrow (x,j)\tilde{R}^2_i(z,k)\ \ \forall i,j,k \in N$,

(vi) $(x,j)\tilde{R}^2_i(y,k) \Leftrightarrow (x,j)\tilde{R}'_i(y,k)\ \ \forall i,j,k \in N$,

(vii) $(x,i)\tilde{P}^1_i(z,i) \Rightarrow [(x,i)\tilde{P}^2_i(z,i)$ and $(z,i)\tilde{P}^2_i(y,i)]$, $\forall i \in N$,

(viii) $(x,i)\tilde{I}^1_i(z,i) \Rightarrow [(x,i)\tilde{I}^2_i(z,i)$ and $(z,i)\tilde{I}^2_i(y,i)]$, $\forall i \in N$,

(ix) $(z,i)\tilde{P}_i^1(x,i) \Rightarrow [(z,i)\tilde{P}_i^2(y,i)$ and $(y,i)\tilde{P}_i^2(x,i)], \forall i \in N$.

Note that nothing is specified about $\tilde{\mathbf{R}}^1$ over (x,z) except for individualistic preference information. Hence, assume that $(x,j)\tilde{R}_i^2(z,k) \Leftrightarrow (x,j)\tilde{R}_i^2(y,k) \;\; \forall i,j,k \in N$, then it is easily checked that, by UD, we can find the profile. By BIIA and SP, we have $xP_{f(\tilde{\mathbf{R}}^2)}z$ and $zP_{f(\tilde{\mathbf{R}}^2)}y$. Since $f(\tilde{\mathbf{R}}^2)$ is quasi-transitive, we obtain $xP_{f(\tilde{\mathbf{R}}^2)}y$. Then, by BIIA, we obtain $xP_{f(\tilde{\mathbf{R}}')}y$. The same argument can be applied to the proof of the inverse direction of case (a), and also to the case (b). ■

In view of Lemma 4, f satisfies the binary independence axiom which is the same as the direct reformulation of the corresponding axiom considered in the Arrovian framework. Thus, we can directly apply Weymark's (1984) Theorem 2, and immediately obtain the following result.

Theorem 5. *If a collective choice rule, f, has a range $\mathscr{B}$ equal to the set of reflexive, complete, and quasi-transitive binary relations of X, then it satisfies UD, BIIA, SP, and A if and only if it is the Pareto extension rule.*

Theorem 5 establishes the characterization of the Pareto extension rule in this extended framework. In view of this theorem, it can be said that while in Case 1 it is possible to consider an anonymous rule other than the Pareto extension rule, in Case 2 the Pareto extension rule is the solely admissible anonymous rule. In other words, even though we relax transitivity into quasi-transitivity, we must go back to the Arrovian framework in Case 2.

6 Conclusion

This paper explored an anonymous collective choice rule under the extended informational basis involving individuals' sympathies for others. Such an informational basis is regarded as the extension of the one considered in the Arrovian framework. In this paper, we in particular focused on quasi-transitive social preferences, i.e. the logically weaker assumption of rationality of social preferences than in the usual cases where transitivity is assumed to be satisfied.

Although in this extended framework we obtained the similar result to the Arrovian framework, it was shown that it is now possible to consider a new non-dictatorial rule, the lexical Pareto extension rule, by virtue of the extended informational basis. This new rule utilizes individuals' sympathetic preference information in the second step of its lexical decision flow. This rule is more selective than the Pareto extension rule, and thus, it will generate more selected socially best alternatives. Therefore, it can be said that the extension of the informational basis will help us to improve the selectiveness of social decision making.

On the other hand, we also obtained the result that if we consider the direct extension of the strong Pareto of the Arrovian framework, the Pareto extension rule is only one admissible rule that satisfies anonymity and other moderate conditions. Hence, in this case we must go back to the Arrovian framework and the extension of the informational basis completely loses its significance.

In this paper, the axiomatization of the lexical Pareto extension rule was not explored. This complicated but much interesting task is left for the future work.

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